The Hotelling Rule for Entropy-Constrained Economic Growth*

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Abstract. The entropy change of the solar system between now and its final heat death is fixed. The time to the heat death is determined by the rate of entropy increase between now and then. If this rate of entropy increase is itself increased by economic activity, then economic activity is generating a negative externality. By internalizing this, a social planner treats the fixed amount of entropy change remaining until the heat death like the stock of an exhaustible resource. This leads to an analysis along the same lines as Hotelling's neoclassical economics of exhaustible resources, forming a partial synthesis between neoclassical economics and Nicholas Georgescu-Roegen's "ecological economics" work on the entropy law.
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1. Introduction

Due to work such as Kåberger and Månsson (2001), Lozada (2004), Beard and Lozada (1999), Ayres (1998 p. 197), Floyd (2007), and Martyushev (2013), it has become clear that there is no elementary, intuitive interpretation of entropy. As Frank Lambert's article in the *Journal of Chemical Education* (2002) bluntly puts it, "Entropy is not disorder. Entropy is not a measure of disorder or chaos." For example, when metallurgical slag and matte spontaneously separate, entropy becomes higher, disorder becomes lower, and economic usefulness becomes higher.

Nevertheless, it is undoubtedly true that all spontaneous processes increase entropy. It is also true that the entropic degradation of the Earth and the rest of the solar system will eventually result in the solar system's evolution to a "heat death" equilibrium, in which entropy has been maximized and therefore no further macroscopic physical processes are possible. If economic processes, by increasing the rate of entropic degradation, are bringing forward the date of that forbidding equilibrium state, then a problem of economic interest arises. Section 3 of this paper models that problem by formulating it within the standard neoclassical exhaustible-resource economics framework due to Hotelling (1931), though the definition of the limited resource is novel.

Glucina and Mayumi (2010 p. 22) warn that "delusions of grandeur" have characterized some writing about economics and entropy. To avoid that, we do not stop with Section 3's successes in showing that the idea of a "long-run entropic problem" is conceptually valid, but instead use Section 4 to ask a further question: is the long-run entropic problem empirically important? After all, economic problems can have constraints which are interesting in theory but which in a particular empirical setting are not binding, and thus are not important in that setting. Section 4 concludes that the long-run entropy problem's constraint is probably not binding. If further investigation supports that finding, then the long-run entropic problem, while potentially important, would not be actually important in practice.

Section 2 supplies background information to help interdisciplinary audiences understand Section 3, and Section 5 asserts that using mathematical models such as in Sections 3 and 4 is methodologically appropriate. Section 6 concludes.

The impetus for this paper came from the following passage written by Nicholas Georgescu-Roegen in one of the cornerstones of Ecological Economics:

> ... let S denote the present stock of terrestrial low entropy and let r be some average annual amount of depletion. If we abstract (as we can safely do here) from the slow degradation of S, the *theoretical* maximum number of years until the complete exhaustion of that stock is S/r. This is also the number of years until the *industrial* phase in the evolution of mankind will forcibly come to its end. Give the fantastic disproportion between S and the flow of solar energy that reaches the globe annually, it is beyond question that, even with a very parsimonious use of S, the industrial phase of man's evolution will end long before the sun will cease to shine. ... the fact remains that the higher the degree of economic development, the greater

must be the annual depletion r and, hence, the shorter becomes the expected life of the human species.

The upshot is clear. Every time we produce a Cadillac, we irrevocably destroy an amount of low entropy that could otherwise be used for producing a plow or a spade. In other words, every time we produce a Cadillac, we do it at the cost of decreasing the number of human lives in the future. (Georgescu-Roegen 1980 pp. 57–58)

This excerpt has some flaws: its "beyond question" pessimism about solar energy actually is questionable. Also, its the notion of a "stock of... low entropy" is not quite right. However, merely by switching that notion to "a stock of a limited amount of entropy change," Section 3 obtains a physically-correct model of a long-run entropic problem, showing that Georgescu-Roegen's theoretical insight was mostly correct. On the other hand, Georgescu-Roegen also thought the long-run entropic problem was important in practice, which Section 4 casts doubt on. This paper makes future debate about whether Georgescu-Roegen was right or wrong on that point much easier, by showing that the question comes down to whether the shadow value of a particular constraint is close to zero.

Using the Söllner/Baumgärter classification system for papers incorporating thermodynamics into economics, this paper lies in Class 4c: "thermodynamic constraints on economic action: models incorporating entropy and entropy generation."[1]

2. Resolving Potential Interdisciplinary Misunderstandings

The first part of this preliminary Section addresses misconceptions which may otherwise cause non-physicists to misunderstand the physics used in Section 3. The rest of this section clears up misconceptions which have caused natural scientists to think the mathematical framework used in Section 3 is wrong.

Georgescu-Roegen emphasizes the dialectical nature of the entropy law with turns of phrase such as "entropic indeterminateness." However, the entropy law *can* sometimes be used to obtain precise arithmomorphic results. It is used that way in Section 3, but since that is innovative, here is a non-innovative illustration. Consider a hypothetical chemical reaction $A + B \longrightarrow 2C$ where A, B, and C are perfect gases and where the reaction occurs at "standard" pressure (one atmosphere). Most chemical reactions do not go fully "to completion"; instead, some of the reactants remain in their initial form. The entropy law can be used to determine the precise equilibrium percent of completion. Lozada (1999 pp. 330–335) shows how. Briefly, if one supposes that the reaction starts

[1]The classification system is: (1) isomorphism of formal structure between thermodynamics and economics; (2) analogies and metaphors between thermodynamics and economics; (3) energy, entropy, and exergy theories of value; (4) thermodynamic constraints on economic action: (a) models incorporating mass and the conservation of mass, either for one particular material or for a number of materials; (b) models incorporating energy and the conservation of energy, sometimes in variants such as embodied energy; (c) models incorporating entropy and entropy generation; (d) models incorporating energy and entropy, sometimes in the form of exergy; and (e) models incorporating mass, energy, and entropy. See Baumgärter (2004 pp. 112–6), who relies partially on Söllner (1997).

with 1 mole of A and 1 mole of B, and if one lets n_A denote the number of moles of A which are left when the reaction reaches chemical equilibrium, then if the reaction occurs at constant temperature and pressure and the components freely mix, Lozada shows that

$$\begin{aligned}\Delta S = -R&\left[2n_{\mathrm{A}} \ln \frac{n_{\mathrm{A}}}{2} + 2(1-n_{\mathrm{A}})\ln(1-n_{\mathrm{A}})\right]\\ &- (n_{\mathrm{A}}-1)(2S^\circ_{\mathrm{C}} - S^\circ_{\mathrm{A}} - S^\circ_{\mathrm{B}})\\ &- \frac{1}{T}\left[-(n_{\mathrm{A}}-1)(2H^\circ_{\mathrm{C}} - H^\circ_{\mathrm{A}} - H^\circ_{\mathrm{B}})\right] \end{aligned} \tag{1}$$

where ΔS is the change in entropy, R is the universal gas constant, and where S°_{A}, S°_{B}, S°_{C}, H°_{A}, H°_{B}, and H°_{C} are other constants characteristic of the substances A, B, and C. (The symbol S° denotes a substance's "standard entropy" and H° denotes its "standard enthalpy of formation"; if A, B, and C were real substances, one could look up their S° and H° in tables derived from laboratory experiments.) Lozada (op. cit., p. 334) continues (letting "J" stand for joules and °K for (degrees) Kelvin) (see also Beard and Lozada 1999 p. 94):

> Equilibrium occurs in the state of maximum entropy, since from there, any deviation would decrease entropy and thus not be allowed by the Entropy Law. The state of maximum entropy is found by maximizing ΔS with respect to n_{A}. The value of R... [is approximately 8.314 J/(mole·°K)]. If in addition we assume for illustration that $T = 500$°K, $H^\circ_{\mathrm{A}} = 2500$ J/mole, $S^\circ_{\mathrm{A}} = 1$ J/(mole °K), $H^\circ_{\mathrm{B}} = 2000$ J/mole, $S^\circ_{\mathrm{B}} = 2$ J/(mole °K), $H^\circ_{\mathrm{C}} = 1000$ J/mole, and $S^\circ_{\mathrm{C}} = 4$ J/(mole °K), then ΔS is maximized at $n_{\mathrm{A}} = 0.5229\ldots$. The reaction $\mathrm{A} + \mathrm{B} \rightarrow 2\mathrm{C}$ will therefore go to [$(1 - 0.5529) * 100 =$] 47.71 per cent completion (cf. Gaskell 1981 p. 230).

Section 3 does not try to characterize a thermodynamic equilibrium, as this example does, but it does take as given, *arithmomorphically*, that thermodynamic equilibrium is the state of maximum entropy.

Chemists and metallurgists almost always conduct calculations like those of the previous paragraph using Gibbs Free Energy instead of using entropy, but the entropy calculation is the more fundamental one—there is, tellingly, an "Entropy Law" but no "Gibbs Free Energy Law." The two calculations give exactly the same answer at constant temperature and pressure (Lozada op. cit. 346–7),[2] but as Lambert (2009) says, "the whole Gibbs relationship or function is about entropy change."

The above discussion shows that one *can* use entropy arithmomorphically, but does not address whether one *should* use entropy arithmomorphically. Section 5 addresses that.

[2]The entropy calculation in Lozada (1999 p. 334) gives the same answer as the Gibbs Free Energy calculation not only Gaskell (1981 p. 230 line 3) but also in Gaskell (1995 p. 319 line 7) and in Gaskell (2008 p. 310 second line from the end).

Turning now to stumbling blocks in understanding economics: if an economist wishes to express the relationship between the amount of corn Q (in, say, liters) which is produced on a farm and the inputs water W (in liters) and fertilizer F (in kilograms) used to produce that corn, for almost a century a simple, popular choice has been the Cobb-Douglas functional form $Q = \gamma W^{\alpha} F^{\beta}$ where α, β, and γ are constants and the dimensions of γ are not discussed. All physical scientists are trained in dimensional analysis, from the perspective of which this expression for Q is incoherent: if γ has no dimensions then the left-hand side's "liters" is obviously not equal to the right-hand side's "liters to the α" times "kilograms to the β." When economists write equations like $Q = \gamma W^{\alpha} F^{\beta}$, they know that what they really mean is

$$Q = \gamma W^{\alpha} F^{\beta} * 1 \frac{\text{units of Q}}{(\text{units of W})^{\alpha}(\text{units of F})^{\beta}}$$

(assuming γ is dimensionless). It makes sense for economists to adopt the simplifying convention of never writing the last term because constants such as α and β are estimated from data and could be almost any real number (although a value between zero and one would generally have the most credibility). When for a particular farm one could obtain $\alpha = 0.2173$ and $\beta = .6894$, whereas for another farm α could be 0.8283 and β could be 0.1722, it is clear that dimensional analysis cannot be helpful in any practical way. This is the case for all of economics, and so the rest of this paper will follow economists' universal practice of writing equations in ways which are dimensionally incorrect (except by coincidence) or at least dimensionally under-specified. A reader wishing to see only dimensionally-correct expressions below is invited to insert terms of the form "one times the appropriate dimensions" in the obvious places.

In classical physics the term "Hamiltonian" typically refers to the total energy of a system, and in quantum mechanics it refers to an operator which gives the total energy of a system. However, in mathematics, the Optimal Control Theory of Lev Pontryagin, which is concerned with the (mathematical not physical) problem of maximizing a functional over a function space, uses the term "Hamiltonian" with a completely different meaning. The model of this paper requires solving an optimal control problem and therefore it only uses the term "Hamiltonian" in its second, newer, purely mathematical meaning, following standard practice in economics since the 1970's (see e.g. the economic work of mathematics professor Colin W. Clark (1976 p. 91)).

Finally, while the model in the next section is innovative in important ways, its putting technological constraints into economics is not *per se* innovative: technological constraints have been part of every school of economic thought coming after the Mercantilists, if not earlier. For example, the Physiocrats of the 17th century, with their emphasis on the productivity of nature; the Classicals, such as Adam Smith's 18th century writings on the division of labor in manufacturing and David Ricardo's early 19th century work on the law of diminishing returns, which took its modern, Neoclassical form by the time of Alfred Marshall's *Principles of Economics* in 1890, and which underlies the very idea of "a supply curve"; and the neo-Ricardian work of Piero Sraffa's *Production of Commodities by Means of Commodities* (1960)—all these works deal with understanding the technological constraints within which the economic system functions. Ecological Economics is not an attempt to add scientific or technological constraints to economics, because all respectable economists have been doing that for the last 300 years; instead Ecological Economics is an attempt to do that using more of what we know about scientific laws and theories and making sure to include externalities and the unpriced or under-priced services of nature, rather than relying mostly on empirical observations of purchased physical inputs and outputs.

3. The Model

Let the initial time be "0," the current time index be "t," and the time of the solar system's heat death be "T." The solar system's "equilibrium," "maximum-entropy," "heat death" state will occur when its temperature is uniform and is equal to the temperature of the universe's cosmic microwave background radiation (currently about 2.7 K). Calculate the entropy difference between the solar system in its current state and in its equilibrium, "heat death" state of maximum entropy (when the Sun has run out of fuel and the Earth's core becomes cold). Without loss of generality take its current entropy S_0 to be zero and choose its final entropy S_T so that $S_T - S_0$ is equal to the entropy difference just calculated.[3] Denote the resulting S_T by *HDE* for "heat death equilibrium." The initial and final conditions of the problem then are[4]

$$S_0 = 0 \text{ and} \tag{2}$$

$$S_T = HDE. \tag{3}$$

The appropriate thermodynamic model to consider has the form of a "system," on the one hand, and a "heat reservoir" (also called a "thermal reservoir"), on the other hand. A heat reservoir is defined as something whose heat capacity is so large that its temperature is forever constant. It would be inappropriate to consider the system as the Earth alone, because the Earth is strongly coupled to the Sun, which cannot play the role of a heat reservoir because its temperature will change considerably over billions

[3]The current entropy level S_0 is arbitrary not only in classical thermodynamics but also in Statistical Mechanics (Dugdale 1996 p. 99). The mathematical shortcoming in Max Planck's treatment of this issue is discussed in Beard and Lozada (1999 p. 118 fn. 12). Entropy *differences*, such as $S_T - S_0$, are not arbitrary and are cardinally measurable. (Georgescu-Roegen would say this cardinality of *entropy differences* makes entropy itself a "weakly cardinal measure.") Therefore the rate at which entropy changes, which will be a key part of the model below, is also not arbitrary and is also cardinally measurable.

[4]Kåberger and Månsson (2001 pp. 171–2) nicely describe a mathematically equivalent procedure as well as the reason for not using it: "For any system, there is an upper limit to the amount of entropy it can contain under specific conditions. The difference between this maximum and the actual amount of entropy in the system has been given the name 'negentropy'. To determine the negentropy, one first has to determine the entropy and calculate the maximum entropy. Since the determination of these two entities provides most of the thermodynamic properties of the system, and since their difference adds nothing to the knowledge about the system, we regard negentropy as a concept of very limited usefulness for thermodynamics proper. Its use lies mainly in shortening the notation in some derivations—but the effect is, in our view, not sufficiently large to motivate the introduction of an additional concept in the thermodynamic theory."

of years. We will consider the system to be the solar system (or just the Earth and the Sun), because then we can take the heat reservoir to be the cosmic background radiation, which is what absorbs most of the Sun's light. (It is true that the cosmic background radiation's temperature will fall over billions of years as the universe expands[5], but it cannot fall much because it is already below 3K, so we will consider the cosmic background radiation to be a heat reservoir to a sufficiently close approximation.) Alternatively, we may be able to define the system as the Earth together only with that portion of the Sun's surface which illuminates the Earth, plus the spherical sector of the Sun which powers the Earth-illuminating surface part and which has a zero net energy flux with the rest of the Sun. (The portion of the Sun illuminating the Earth changes every moment, but if the Sun is sufficiently spherically symmetric that should not matter.) Which of these two definitions of "the system" is chosen is irrelevant to the theoretical model which follows, but it will influence the empirical magnitudes of the variables. Both choices of systems are closed (with respect to matter) but open (with respect to energy), not isolated, because it is thermodynamically important to take into account their outward flow of energy—that is, their relationship with their heat reservoir, which defines their final "heat death" temperature.

(The "heat death" of the solar system may not be its final equilibrium state: indeed the Sun itself is not a "first generation" star, but it (and the Earth) are formed partially from elements generated by the collapse of earlier stars. This paper takes the position that such rebirths, made possible because no system in the universe is *gravitationally* isolated, are not important for the future of the human economy.)

Let $\overline{\Delta S_t}$ denote the change in the system's entropy at time t if there were no human activity. Let $\Delta\Delta S_t$ denote the human-caused change to $\overline{\Delta S_t}$. Then the actual change in the system's entropy is

$$\dot{S}_t = \overline{\Delta S_t} + \Delta\Delta S_t\,. \tag{4}$$

In the SI (or MKS) system of units, S is measured in "joules per Kelvin," and the terms in (4) by "joules per Kelvin" per second, which is watts per Kelvin, W/K.[6]

Suppose the arguments of the social welfare function W (not to be confused with the notation for watts, W) are T, the length of time before the heat death arrives, and some measure of economic well-being before T. Assume both of these increase social welfare. If c denotes consumption, u instantaneous social welfare ("utility"), and r the social rate of discount, the simplest such social welfare function is probably

$$W\left(T, \int_0^T u(c_t)\, e^{-rt}\, dt\right)\,. \tag{5}$$

Since $T < \infty$ the reader could, if desired, set $r = 0$. The second argument is "simple" because it, which represents the ethical viewpoint of utilitarianism across generations,

[5] See for example `http://www.cv.nrao.edu/course/astr534/CMB.html`.

[6] In Section 2, S° was measured in J/(mole °K); that was because standard entropies are intensive quantities, and have to be multiplied by the number of particles (moles) in order to get the entropy of a system, which is an extensive quantity. Also, in the late 1960's, the Thirteenth Conférence Générale des Poids et Mesures (CGPM) replaced "degrees Kelvin," denoted °K, with "Kelvin," denoted K.

has been the standard way of representing intertemporal preferences in neoclassical economics since a version of it appeared in the seminal paper of Ramsey (1928 p. 547), with elaborations introduced by the 1975 Nobel Laureate in Economics, Tjalling C. Koopmans, first in (1963 p. 21). It not only underlies several subdisciplines of neoclassical microeconomics, it also underlies the late-20th-century "real business cycle" theory of macroeconomics. It does make utility cardinal rather than ordinal, but then so do standard social welfare functions, the standard economics of uncertainty, and standard game theory—as well as the new, not very standard field of happiness research.[7] In future work, (5) could be modified to account for different population sizes at different times; for different types of intergenerational altruism (or jealousy); for consumption being a vector of many goods instead of just one aggregate commodity; for the form of u changing with time; for welfare depending on relative rather than absolute income (James Duesenberry's "relative income hypothesis," which is related to the "positional goods" idea of Fred Hirsh, and to the work of Thorstein Veblen (1899) and of Herman Daly (1991, Ch. 8, "On Biophysical Equilibrium and Moral Growth")); for technological change; for explicit savings and investment behavior; for studying behavior by profit-maximizing firms instead of by a social planner; and for other effects.

The social planner's problem now is one of maximizing (5) subject to (2), (3), and a slight modification of (4),

$$\dot{S}_t = \overline{\Delta S}_t + \Delta\Delta S_t(c_t)\,, \tag{6}$$

indicating that $\Delta\Delta S_t$ depends on society's choice of consumption c_t. The form that $\Delta\Delta S_t(c_t)$ takes is crucial to the nature of the solution. Clearly $c_t \geq 0$ for all t. Georgescu-Roegen thought that $\Delta\Delta S_t > 0$; however, a negative value, at least for some values of c and t, cannot be ruled out *a priori*. If the graph of $\Delta\Delta S$ versus c has a negative slope, representing an inverse relationship between $\Delta\Delta S$ and c, then entropy does not constrain c, because increasing c would simultaneously lessen the growth in S, thus increasing T. If the graph of $\Delta\Delta S$ versus c has a positive slope, then entropy does constrain c, because increasing c would simultaneously increase the growth in S, thus decreasing T. This second case is thus the one of interest, together with mixed cases in which the slope of the graph of $\Delta\Delta S$ versus c has a slope which does not have a constant sign.

There seems to be no mathematical theory that would enable us to solve problems as general as maximizing (5) subject to (2), (3), and (6). One solvable alternative would be to use discrete instead of continuous time. A second solvable alternative, which is pursued in this paper, is to assume that W is linear, and in particular that $W(x, y) = \alpha x+\beta y$

[7] Hirschauer, Lehberger, and Musshoff (2015): " 'In the last 35 years, however, psychologists and economists in growing numbers have tried to overcome the problems of measuring happiness by the simple device of asking people directly how pleasant or disagreeable they find particular activities throughout their day or by inquiring how satisfied [. . .] they are with the lives they are leading' (Bok 2010: 5). Self-reported well-being is either qualitatively assessed or—more commonly—quantitatively measured via Likert scales (psychometric scales)." The citation to Bok is from D.C. Bok (2010), *The politics of happiness: what government can learn from the new research on well-being*, Princeton: Princeton University Press.

where α and β are the weights the social planner puts on the two objectives.[8] Then from (5) one has

$$W = \alpha T + \beta \int_0^T u(c_t)e^{-rt}\, dt \tag{7}$$

$$= \int_0^T \alpha\, dt + \beta \int_0^T u(c_t)e^{-rt}\, dt \tag{8}$$

$$= \int_0^T \left(\alpha + \beta u(c_t)e^{-rt}\right)\, dt \tag{9}$$

and the problem becomes one of maximizing (9) over T and the time path of c subject to (2), (3), and (6). This is a standard problem of optimal control theory.

The solution to this problem is obtained by forming the Hamiltonian

$$\mathscr{H} = \alpha + \beta u(c_t)\, e^{-rt} + \mu_t(\overline{\Delta S}_t + \Delta\Delta S_t(c_t)) \tag{10}$$

where μ_t is the adjoint, or costate, variable. The necessary conditions for optimality are then

$$0 = \frac{\partial H}{\partial c} = \beta u' e^{-rt} + \mu_t\, \Delta\Delta S_t' \tag{11}$$

$$-\dot{\mu}_t = \frac{\partial H}{\partial S} = 0 \tag{12}$$

$$0 = \mathscr{H}(T) \tag{13}$$

where differentiation with respect to c is denoted by the prime symbol, $'$, and differentiation with respect to t is denoted by a raised dot. Using the standard interpretation of costate variables (see Léonard and Long 1992 section 4.5.1), the variable μ is the "shadow value" of the entropy constraint, and one would expect $\mu < 0$ because increases in S decrease rather than increase welfare. (In a model of profit-maximizing firms, $-\mu$ would be related to the socially-optimal "entropy tax" to be levied on firms so that they appropriately internalize their entropy externality, because without government intervention, profit-maximizing firms would treat μ as zero, since they do not care about their effect on how long society lives.) From (11) and (12) one obtains

$$\frac{u'}{\Delta\Delta S'} = \frac{-\mu}{\beta} e^{rt}\,. \tag{14}$$

In other words, in contrast to the standard Hotelling Rule result that an extractive firm's marginal profit rises at the rate of interest, here what rises at the rate of interest is marginal utility (or marginal instantaneous social welfare) divided by marginal $\Delta\Delta S$.

For example, suppose that $u(c) = \sqrt{c}$ and $\Delta\Delta S(c) = c^2$, so there is diminishing marginal utility of consumption and an ever-increasing marginal entropy externality as

[8] Writing $W(T, y) = \phi(T) + \beta y$ and interpreting $\phi(T)$ as a "scrap value" function will not work because since the terminal value of S is fixed, ϕ would not imply any new necessary condition for an optimum. See p. 227 of Léonard and Long (1992).

c rises—quite reasonable second-derivative conditions in the present context. From (11) and (12) we find that

$$c_t \propto e^{-\frac{2}{3}rt}$$

whereas if there were no entropy effect ($\Delta\Delta S' = 0$),

$$c_t \propto e^{-2rt}\,. \tag{15}$$

So in this example the effect of the entropy constraint is to slow the rate at which consumption falls (at every date including the present).

This model can be extended in interesting ways. In one extension, there are two production processes available to produce the consumption good, and each of the production processes has a different functional form for consumption's effect on entropy. For example, "production process one" produces output c_1, "production process two" produces output c_2, and c_1 and c_2 are perfect substitutes in consumption, so the model becomes

$$\max \int_0^T [\alpha + \beta u(c_1 + c_2)e^{-rt}]\, dt \tag{16}$$

subject to (2), (3), and

$$\dot{S}_t = \overline{\Delta S}_t + \Delta\Delta S_1(c_1) + \Delta\Delta S_2(c_2)\,. \tag{17}$$

The necessary conditions for a maximum become (12), (13), and $0 = \partial\mathscr{H}/\partial c_1$ and $0 = \partial\mathscr{H}/\partial c_2$ for the new $\mathscr{H}$. Combining the last two of these conditions results in

$$\Delta\Delta S_1' = \Delta\Delta S_2' \tag{18}$$

so at the optimum, the marginal effect which increasing c_1 has on increasing ΔS must be the same as the marginal effect which increasing c_2 has on increasing ΔS, a very intuitively appealing result.

As a final extension of the original model, suppose the "stuff" consumed is an exhaustible resource whose supply is fixed. Denote the stock of this resource at any given time by x_t and denote the resource flow by q_t. The problem in full becomes

$$\max_{q_t, T} \int_0^T [\alpha + \beta u(q_t)e^{-rt}]\, dt \quad \text{such that} \tag{19}$$

$$S_0 = 0 \tag{2}$$

$$S_T = HDE \tag{3}$$

$$\dot{S}_t = \overline{\Delta S}_t + \Delta\Delta S_t(q) \tag{20}$$

$$x_0 = \text{fixed} \tag{21}$$

$$\dot{x}_t = -q_t \tag{22}$$

$$x_t \geq 0 \text{ and } q_t \geq 0 \text{ for all } t\,. \tag{23}$$

The Hamiltonian becomes

$$\mathscr{H} = \alpha + \beta u(q_t)e^{-rt} + \mu_t(\overline{\Delta S}_t + \Delta\Delta S_t(q_t)) - \lambda_t\, q_t \tag{24}$$

with λ_t being the second adjoint variable. The first-order conditions are

$$0 = \frac{\partial H}{\partial q} = \beta u' e^{-rt} + \mu_t \Delta\Delta S_t' - \lambda_t \tag{25}$$

$$-\dot{\mu} = \frac{\partial H}{\partial S} = 0 \tag{26}$$

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = 0 \tag{27}$$

$$0 = \mathscr{H}(T)\,. \tag{28}$$

From (25), (26), and (27),

$$\beta u' = (\lambda - \mu\,\Delta\Delta S')\, e^{rt}\,. \tag{29}$$

The contrast with the standard Hotelling Rule result is that the latter lacks the $\Delta\Delta S'$ term, representing the new entropy constraint. Since one expects $\mu < 0$, the right-hand side of (29) represents a larger wedge between $u' = 0$ (which is the unconstrained optimum) and the optimal u' than would be the case without the entropy constraint. In other words, adding the entropy constraint would have the same effect as decreasing the initial stock of the exhaustible resource: at each date (including the present), consumption of the resource will be less than it would otherwise have been.

4. How Important is this Entropy Constraint?

The previous section has proven that the question "how important is this entropy constraint suggested by Georgescu-Roegen?" is equivalent to the question "what is the magnitude of the shadow value μ on the entropy constraint," which in turn depends on the answer to the question "what is the form and magnitude of $\Delta\Delta S(c)$?"

Some qualitative conclusions are easily reached: the more often $\Delta\Delta S_t$ is positive, the more likely it is for the entropy constraint to be binding. Also, the ratio of $\Delta\Delta S(c)$ to $\overline{\Delta S}_t$ will be much smaller if the system is defined to be the Earth and the entire Sun, instead of being defined as the Earth and its coupled spherical sector of the Sun.

Quantitative estimates of $\overline{\Delta S}_t$ for a system including just the Earth go back at least to Aoki (1983). Kåberger and Månsson (2001 p. 168, 175) write that the power output of the Sun per unit area of its surface is σT_s^4 where T_s is the temperature of the Sun and σ is the Stefan-Boltzmann constant; the area of the Sun's surface is $4\pi R_s^2$ where R_s is the radius of the Sun; and "the fraction of that radiation impinging the Earth's atmosphere is given by $\pi R_e^2/(4\pi R^2)$" [using slightly different notation] where R_e is the radius of the Earth and R is the distance from the Sun to the Earth. Using m for meters, $\sigma = 5.6710 \times 10^{-8}$ W/(m^{2}K^4), $T_s = 5760$ K, $R_s = 6.9 \times 10^8$ m, $R_e = 6.4 \times 10^6$ m, and $R = 1.5 \times 10^{11}$ m, Kåberger and Månsson (op. cit. p. 175) continue in the following way, where I correct the typographical error in the left-hand side of their equation (5):

> Approximating further, by assuming that the entropy flow is the energy flow divided by the temperature, the flow of entropy impinging on the atmosphere is given by
>
> $$\sigma T_s^3 \frac{R_s^2}{R^2} \pi R_e^2 = 0.03\,\mathrm{PW/K}\,, \tag{KM-5}$$

that is, 3×10^{13} W/K. They continue (with the temperature of the Earth $T_e = 278$ K):

> Assuming also that the Earth radiates as a black body. . . , and assuming the same relation between energy and entropy flow as above, we get the flow of entropy from the Earth as
>
> $$\sigma T_e^3 4\pi R_e^2 = 0.63\,\mathrm{PW/K} \quad [= 6.3 \times 10^{14}\,\mathrm{W/K}] \tag{KM-6}$$
>
> . . . . Note that the temperature of the Earth used in this expression must be compatible with the energy balance requirement for the model. . . . We see that the Earth emits more entropy than it receives. The difference, 0.6 PW/K [= 6×10^{14} W/K], corresponds to the rate of entropy production on the Earth.

Since our purposes are rough, for us, Kåberger and Månsson's results are just as good as those coming from more accurate models.[9] They continue:

> The rate of commercial energy use of the human society is ≈10 TW. If we assume that the energy is converted to heat at Earth temperature, the corresponding entropy production is 0.04 TW/K. The natural rate of entropy production is 15 000 times larger. Even considering that only about half the solar radiation avoids reflection and absorption in the atmosphere, the natural entropy production at the surface of the planet is ≈7500 times the production of entropy by the human society.

Limiting anthropogenic entropy production to that coming from "commercial energy use" is restrictive; a more complete mathematical description of anthropogenic entropy production would include more sources. However, a more inclusive analysis may not change the numbers much.

[9]If E stands for energy per unit time then Aoki (1983), following the work of Planck (1959) on the entropy of radiation (that is, of photons), calculates the entropy flow due to the sun shining on the earth correctly as $(4/3)\,E/T$ instead of Kåberger and Månsson's E/T (for example, there is radiation pressure to consider; see Appendices A1 and A1.1 of Wu and Liu (2010)). Aoki also imposes "the energy balance requirement for the model" which Kåberger and Månsson do not impose; but since Aoki retains the assumption that the Sun and the Earth are blackbodies, imposing the energy balance requirement forces Aoki to use a less realistic $T_e = 254$ K. Aoki obtains "the net amount of radiation entropy absorbed by the Earth per unit time" as 6.055×10^{14} W/K. A much more complicated, modern calculation by Wu and Liu (op. cit.), which among many other advances does not assume the Earth to be a blackbody, gives "the overall Earth's entropy production rate from 6.481×10^{14} to 6.547×10^{14} W/K" (ibid., abstract). Even Wu and Liu's calculations leave some things out, such as the entropy produced by processes in the Earth's molten core.

In our model, $\overline{\Delta S_t}$ is the natural entropy change of the Earth *plus* the natural entropy change of part of or all of the Sun, making our $\Delta\Delta S_t/\overline{\Delta S_t}$ even smaller than Kåberger and Månsson's.[10] This suggests that $\Delta\Delta S_t/\overline{\Delta S_t}$ is so small as to be unimportant.

The model of Section 3 includes no constraints on the final time T besides $0 < T < \infty$, but astronomers tell us that changes in the Sun may make life on the Earth impossible well before the Sun's "heat death." This may not happen (see Appell 2008), and even if it does, life on other planets in the solar system may be possible, in which case no additional constraints on T are needed as long as the system is extended to include those other planets; but if evolutionary changes in the Sun do constrain T in this additional way, it is possible that that constraint will be binding and therefore that the shadow value μ of the entropy constraint will be zero.

Although the magnitude of μ can still be considered to be an open empirical question, as of now, evidence points to μ's absolute value being so small that its practical importance is negligible.

5. The Methodology

By approaching the question of entropy's constraint on economic growth using a Hotelling model I have adopted an approach scorned by Georgescu-Roegen (1979), who wrote:

> [concerning] the famous 1931 article of Harold Hotelling. Beautiful mathematical piece though that article is, it set a fallacious pattern of approach to the economics of exhaustible resources. (p. 101)

Georgescu-Roegen was right that Hotelling's mechanistic approach is limited, and that arithmomorphic models are incapable of capturing important aspects of reality.[11] One can only welcome broader ways of thinking about environmental problems, as pointed

[10] The difficulty with limiting an analysis to the Earth is that the heat death condition for the Earth might be reached while the Sun was still far from its heat death; in other words, such a model would imply that industrial civilization cannot exist on the flow of solar energy alone (together with materials on Earth that will exist as long as the planet does). Georgescu-Roegen felt this was true. However, such a skeptical outlook on the possibilities of technological progress, while it might be prudent, is not demanded by science. Georgescu-Roegen's (1971 pp. 299, 428–9) skeptical predictions about technological progress in bioengineering have been largely refuted in the years since. For example, while we still have no "nanotweezers," we can do genetic engineering, and we do have "molecular motors" (see Astumian 2001)—so predictions of any kind (skeptical or hopeful) about technological progress are hazardous. Since therefore industrial civilization may be able to exist on the flow of solar energy alone (with whatever permanently-available materials Earth will have), the Earth alone is not the appropriate system to analyze when investigating entropic constraints definitely demanded by science.

[11] This realization has also been expressed in the popular press. Daniel Yankelovich coined the term "the McNamara fallacy" (after US President Lyndon Johnson's Secretary of Defense Robert McNamara) to describe the following reasoning: "The first step is to measure what can be easily measured. This is okay as far as it goes. The second step is to disregard that which cannot be measured, or give it an arbitrary quantitative value. This is artificial and misleading. The third step is to presume that what cannot be measured really isn't important. This is blindness. The fourth step is to say that what can't be easily measured really doesn't exist. This is suicide." (Quotation from an interview quoted in "Adam Smith" [pseudonym of George J. W. Goodman], *Supermoney*, New York: Random House, 1972, p. 290.)

out by Norgaard (2010) and Kosoy and Corbera (2010). However, Georgescu-Roegen's insistence that the Entropy Law is a qualitative, not quantitative, statement is not correct, as shown by the example at the beginning of Section 2. Furthermore, Georgescu-Roegen's valid point about entropy which was quoted in the introduction *can* be captured, more or less in its entirety, in the mechanistic, Hotelling-type model of Section 3, and Georgescu-Roegen quite rightly praised "the immense satisfaction which Understanding derives from arithmomorphic models" when they are appropriate (Georgescu-Roegen 1971 p. 332). In addition, nonarithmomorphic models are imperfect as well, being less precise and therefore often harder to interpret, as Georgescu-Roegen (1971 p. 331) himself readily admitted. Finally, constructing an arithmomorphic model of Georgescu-Roegen's ideas seems to be a fruitful path towards increasing interest in his ideas by making them clearer to understand, particularly since one of Georgescu-Roegen's (1979 p. 101) primary objections to Hotelling's model can be fixed in the framework of this paper by taking the rate of discount to be zero.[12]

At first glance this paper may not seem to have much in common with the work of Raine et al. (2006), Foster (2011), or Hermann-Pillath (2011, 2015). However, the key variable of concern in Section 3, $\dot{S}_t$, the change in entropy, could be called *the rate of entropy production*.[13] Thus, Section 3 is unequivocally a work of nonequilibrium thermodynamics, following in the tradition of Ilya Prigogine (see e.g. Prigogine and Stenger (1984)), which was carried forward by Wicken (1987 p. 115) and Fry (1995), and in which Raine et al., Foster, and Hermann-Pillath work. A recent treatment is given in Martyushev (2013), in which the question is whether, in nonequilibrium systems, nature might maximize $\dot{S}_t$ under certain constraints and minimize $\dot{S}_t$ under other constraints, but in the context of "some small element of the system volume in a relatively small time interval" (p. 1162), so that, whatever the result of this line of inquiry and however useful it might be to understand not only simple physical systems but also biological evolution, it does not contradict the freedom which Section 3 assumes humans might have to influence $\dot{S}_t$ in a global context.

6. Conclusion

Nicholas Georgescu-Roegen thought that one of the ways in which the Second Law of Thermodynamics was important for economics was that the economy has a "long-run entropic problem," namely that human activity hastens the pace at which the Earth approaches the forbidding state of thermodynamic equilibrium. However, he thought that it was not useful to express Entropy-Law constraints arithmomorphically, so he never expressed the long-run entropic problem in a mathematical model. Since we have fewer methodological compunctions, in Section 3 we did construct such a model. The

[12] For a somewhat skeptical view of Georgescu-Roegen's methodological innovations see Samuelson (1999, pp. xii, xv).

[13] I do not call it that in Section 3 to avoid criticisms such as that of Lucia and Grazzini (2015 p. 7788): "Moreover, we must foreground how the thermodynamicists usually use the terms 'entropy generation' and 'entropy production'. However, nothing is really produced or generated; entropy varies in relation to energy and mass fluxes and to irreversibility, but it is not produced or generated."

model is physically correct and economically closely related to standard Hotelling-type optimal control analyses. The economic cost of the long-run entropic constraint is the constraint's shadow value. By the end of Section 3, by rejecting Georgescu-Roegen's methodological compunctions, we had confirmed Georgescu-Roegen's theoretical conception of a long-run entropic problem.

While an issue may be interesting in theory, it is also important to ascertain if it is important in practice. Section 4 took this next step, asking how large Section 3's shadow value of the long-run entropic constraint is likely to be. It concluded that that shadow value is probably indistinguishable from zero. That clearly contradicts Georgescu-Roegen's feelings about the importance of the long-run entropic problem.

Georgescu-Roegen's work is of foundational importance for Ecological Economics, but it is not flawless. This paper covered his "entropy as a scarce stock" conception, and even correcting that to "entropy change as a scarce stock," this paper still only partially validates Georgescu-Roegen's thoughts on the matter. In the Introduction we mentioned that framing "entropy as disorder," which Georgescu-Roegen and many others did, is an even less useful idea. However, there is more in Georgescu-Roegen's work than just these two ideas, and there are more connections between entropy and economics than just these two ideas. The latter are being increasingly well understood[14], but there is more careful work to be done in 'separating the wheat from the chaff' in the former.

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[14]Within the broad area of the connection between entropy and economics, besides the "long-run entropic problem" studied in this paper, entropy and the Second Law of Thermodynamics form the basis for calculating feasible technologies in much of chemical, metallurgical, and mechanical engineering. In this way the economy's ultimate production possibilities frontier—the frontier that will limit production regardless of future technical progress—is influenced by the Second Law. Engineers are aware of many impossible processes, but with the occasional exception of the impossibility of recycling energy and of running production processes both forwards and backwards (for which see Baumgärtner (2005)), few of these impossibilities have been captured in existing economic analyses. As Baumgärtner (2004 p. 122) writes, "Thermodynamics is necessary to identify which options and scenarios of resource use, economic production, and waste generation are feasible and which are not. It, thereby, contributes to making informed choices about the future."

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