Passive Investing Beats Active Because Volatility Drags Down Terminal Wealth's Median but not its Mean*

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Abstract. Abstracting from fund expenses, a typical active mutual fund earns the median return, but a passive fund earns the mean return. With coin-flipping bets, median wealth is less than mean wealth. With discrete time, discrete or continuous probability, and i.i.d. investment returns, wealth is asymptotically lognormal, and the limiting distribution of wealth's median is less than its mean. In continuous time, wealth's median is also less than its mean, given additional assumptions. In neither discrete nor continuous time must one assume that one-period or instantaneous continuously-compounded returns ("log returns") are normally distributed.

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In 1991, William Sharpe wrote his famous note on "The Arithmetic of Active Management," which remains a touchstone in any discussion of the superiority of passive over active management. To recall, he wrote that

1. before costs, the return on the average actively managed dollar will equal the return on the average passively managed dollar and

2. after costs, the return on the average actively managed dollar will be less than the return on the average passively managed dollar.

However, in a multi-period setting, Sharpe's first statement may give too much credit to active managers. In such a setting, even before costs, the return of the median actively managed fund will be asymptotically less than the return on a passively fund, assuming the returns are independent and identically distributed. The reason is that single-period independent, identically distributed (i.i.d.) returns give rise to multiperiod wealth which is asymptotically lognormal, and thus right-skewed, so the mean wealth, which is what a passive fund earns, is asymptotically higher than the median wealth, which is what the median active fund earns. These are not new results, but they are scattered over many decades of literature, starting with Osborne (1959) if not earlier. In this paper we document where many of the results originated, synthesize them into a coherent narrative with uniform notation, and illustrate a theoretical example. We also add rigor in two places. Bessembinder, Cooper, and Zhang (2023) (henceforth "BCZ") note von Hippel's (2005) point that positive (right) skewness is actually not a sufficient condition for the mean to be greater than the median. Since our interest is in the latter inequality, we will emphasize directly analyzing the mean and the median, rather than analyzing skewness. Then, analyzing the mean naturally leads to questions of convergence in expectation, but the Central Limit Theorem concerns convergence in distribution, which is a weaker notion of convergence, and it turns out the difference between those types of convergence has implications even in analyzing the simplest possible setting, repeated bets on tosses of a fair coin. Overall, we show that while the mean of multi-period terminal wealth is, in many contexts, a compounding of the mean of one-period wealth (or of instantaneous change in wealth), the median of multi-period wealth is, in those contexts, a compounding of the mean of one-period (or instantaneous change in) wealth *reduced because of variance*.

The empirical study of skewness has already been significantly advanced by the admirable studies of Bessembinder (2018) and BCZ. The utility of a compact and comprehensive summary of the theoretical background is suggested by Bessembinder's remark that "To my knowledge, the statistical properties of multiple-period returns generated by successive draws. . . have not been carefully explored" (2018 p. 443).

Section 1 demonstrates that median terminal wealth is less than mean terminal wealth in the elementary coin-flipping setting. For continuous probability distributions and discrete time, section 2 sets the stage by showing that if one-period returns are i.i.d. *and normally distributed* then the distribution of terminal wealth will be lognormal, thus having positive skew and a larger mean than median. Then Section 3 shows that, assuming i.i.d. returns but dropping the assumption that the one-period returns are normally distributed, the *asymptotic* distribution of terminal wealth will again be lognormal. In other words, one does not need to assume a lognormal *model* of wealth in order to get an asymptotically lognormal *distribution* of wealth. However, the difference between convergence in distribution and convergence in mean leads to a weaker result than one might expect, and stating it requires care. Finally, Section 4 shows asymptotic lognormality in continuous time: a lognormal model of wealth will result in a lognormal distribution of wealth, but one can get a lognormal distribution of wealth without assuming a lognormal model of wealth. The continuous-time result requires assumptions beyond i.i.d. behavior; the (admittedly weaker) discrete-time result only requires i.i.d. behavior and the passage of 'enough' time, which de La Grandville's results (1998 p. 79) suggest can be as little as two time periods.

1. In Repeated Betting on Flips of a Fair Coin, Passive beats Active

BCZ (2023 p. 137) illustrate how multiple-period skewness can arise from single-period symmetry using a simple example (emphases added):

> . . . the compounding of random short-horizon returns induces positive skewness in long-horizon returns (even if short-horizon returns are symmetric), a result first formally demonstrated by Arditti and Levy (1975). Intuitively, this positive skewness arises because *reversals* of [identical] percentage magnitudes lead to compound losses (e.g., successive returns of 5% and −5% in either order compound to −0.25%), while *continuations* of [identical] percentage magnitudes lead to larger gains than losses (e.g., continuations of 5% lead to accumulated returns of 10.25%, while continuations of −5% lead to accumulated returns of −9.75%).

Table 1 goes into this example in more detail to foreshadow our upcoming results for continuous probability distributions. In the last two-thirds of the Table, with respect to future outcomes, the two-period median of the return distribution is less than the mean, even if, as in the middle third of the table, the one-period the return distribution is symmetric. This was Bessembinder et al.'s point.[1] In all three parts of the table, geometric returns are less than arithmetic returns. This can be related to a "volatility drag" on geometric returns (discussed below). In the table's example of "one past realization," realized terminal wealth can be written as a function of geometric return, but not of arithmetic return; so for this situation, volatility drag affects realized terminal wealth. However, in the table's two analysis of future outcomes, expected terminal wealth can be written as a function of expected arithmetic return, but not of expected geometric return[2]—so expected terminal wealth is unaffected by a volatility drag. Median terminal wealth is, however. These results will all be echoed in the analysis of the rest of the paper, which concerns continuous distributions.

Although this paper is about the theoretical probability distribution of *future* terminal wealth obtained by investing in *one* asset, whereas empirical work must concern the realized *past* distribution of terminal wealth obtained in the past by investing in *many* assets, the theoretical future distribution and the past empirical distribution are asymptotically equal each other, because that is what it means to say that the random variable has a particular distribution. This is why Table 2, which shows the most likely past four observations, has "ensemble averages" (of past data) that equal the "expected values" (of future outcomes) in the middle of Table 1. The point of this section is that if one has the choice, at one date, to bet on

[1]Bessembinder (2018 §2.2) shows using simulations that symmetric one-period simple returns can result in a right-skewed distribution of W_n. Arditti and Levy (1975 pp. 799–801), assuming only i.i.d. returns, analytically demonstrated that a single-period skewness of zero does not necessarily imply multiple-period skewness of zero, and single-period skewness of zero is consistent with multiple-period skewness that is positive and increasing in n. Skewness is further studied in Farago and Hjalmarsson (2023), where the i.i.d. assumption is dropped and econometric estimation procedures are considered. As mentioned near the end of Section 2, for our purposes, skewness is less important than the mean being larger than the median.

[2]One way that terminal wealth is written in the table is as a function of the four squared geometric returns ll, lw, wl, and ww; but it is never written as a function of $E[1 + \text{Geom. Ret.}]$.

For a formal statement that "arithmetic mean return, when compounded... yields the correct value of expected wealth... if the mean of R is unchanged during all n periods and returns are independent" see Skoog and Ciecks (2008 p. 8); they survey a controversy about this in the 1990's.

	1 Period	2 Periods
$l = 0.95$ and $w = 1.05$		
One Past Realization		
(example)		$\{l, w\} = \{.95, 1.05\}$
$1 +$ Mean Arith. Ret.		$\frac{.95+1.05}{2} = 1$
$1 +$ Mean Geom. Ret.		$\sqrt{.95 \cdot 1.05} \approx .998749$
Realized Terminal Wealth		$.95 \cdot 1.05 = .9975$
		$= (1 + \text{Mean Geo. Ret.})^2$
Future Outcomes	$\{l, w\} =$	$\{ll, lw, wl, ww\} =$
	$\{0.95, 1.05\}$	$\{.9025, .9975, .9975, 1.1025\}$
	mean = 1	mean = 1
		median = .9975
$E[1 + \text{Arith. Ret.}]$	$\frac{1}{2}\frac{l}{1} + \frac{1}{2}\frac{w}{1}$	$\frac{1}{4}(\frac{l+l}{2}) + \frac{1}{4}(\frac{l+w}{2}) + \frac{1}{4}(\frac{w+l}{2})$
	$= 1$	$+ \frac{1}{4}(\frac{w+w}{2}) = 1$
$E[1 + \text{Geom. Ret.}]$	$\frac{1}{2}l^{1/1} + \frac{1}{2}w^{1/1}$	$\frac{1}{4}(ll)^{1/2} + \frac{1}{4}(lw)^{1/2} + \frac{1}{4}(wl)^{1/2}$
	$= 1$	$+ \frac{1}{4}(ww)^{1/2} \approx 0.99937$
$E[\text{terminal wealth}]$	$\frac{1}{2}l + \frac{1}{2}w$	$\frac{1}{4}(ll) + \frac{1}{4}(lw) + \frac{1}{4}(wl)$
	$= 1$	$+ \frac{1}{4}(ww) = 1$
		$= \left(\frac{l+w}{2}\right)^2 = (E[1 + \text{Arith. Ret.}])^2$
$l = 0.95$ and $w = 1.1$		
Future Outcomes	$\{0.95, 1.1\}$	$\{0.9025, 1.045, 1.045, 1.21\}$
	mean = 1.025	mean = 1.050625
		median = 1.045
$E[1 + \text{Arith. Ret.}]$	1.025	1.025
$E[1 + \text{Geom. Ret.}]$	1.025	≈ 1.02363
$E[\text{terminal wealth}]$	1.025	$1.05625 = (E[1 + \text{Arith. Ret.}])^2$

Table 1. Initial wealth of \$1; a 50% chance of losing 5% ("$l = 0.95$"), and a 50% chance of winning either 5% (top two-thirds of table, "$w = 1.05$"), or 10% (bottom third of table, "$w = 1.1$").

many flips of a fair coin, and then faces that choice again and again at future dates, then the passive strategy of wagering on all bets at each date is better than the active strategy of only wagering on some of them, because the passive strategy earns the mean, which is higher than the median of the active strategies.

The result that median wealth is affected by volatility drag but mean wealth is not affected by volatility drag can be generalized as follows.

Proposition 1. *In repeated betting on flips of a fair coin where in each round wins change wealth by a factor of 'w' and losses change wealth by a factor of 'l' (which is not necessarily less than 1), where $0 < l < w$, and where in each round all wealth is wagered, then after n rounds (n a strictly positive even integer),*

$$\textit{median wealth} = W_0 \cdot \left(\sqrt{lw}\right)^n < W_0 \cdot \left(\frac{l+w}{2}\right)^n = \textit{mean wealth}\,. \qquad (1)$$

Thus median wealth grows exponentially at the rate of the geometric mean of l and w, while mean wealth grows exponentially at the rate of the arithmetic mean of l and w. Also,

$$\lim_{n\to\infty} \frac{\textit{median}\,(W_n)}{\textit{mean}\,(W_n)} = \lim_{n\to\infty} \left(\frac{\sqrt{lw}}{\frac{l+w}{2}}\right)^n = 0\,.$$

Denote the mean of the wealth after one toss when $W_0 = 1$ as $m_1 = (l+w)/2$ and denote the variance of the wealth after one toss when $W_0 = 1$ as $var_1 = (l-w)^2/4$. Then (1) can be rewritten as

$$\textit{median wealth} = W_0 \cdot \left(\sqrt{m_1^2 - var_1}\right)^n < W_0 \cdot (m_1)^n = \textit{mean wealth}\,. \qquad (2)$$

Hence the mean of W_n is the one-period mean raised to the n th power, but its median is the one-period mean dragged down by the one-period variance, raised to the n th power.

Proof. Let W_n be the value, or "wealth level," after the n^{th} coin toss. We first prove the various components of (1).

Proof for the median: For n even, the median outcome will be half losses and half wins. The median wealth will be W_0 times the product of $n/2$ l's and $n/2$ w's. So median wealth will be $W_0 \cdot l^{n/2} w^{n/2} = W_0 \cdot (lw)^{n/2}$.

Proof for the mean: The assertion is $E(W_n) = W_0\,((l+w)/2)^n$; we prove by induction that it holds for any positive value of n. Proof that is is true

2 Periods	
$l = 0.95$ and $w = 1.05$	
Past Realization #1	$\{l, w\} = \{.95, 1.05\}$
1 + Mean Arith. Ret.	$\frac{.95+1.05}{2} = 1$
1 + Mean Geom. Ret.	$\sqrt{.95 \cdot 1.05} \approx .998749$
Realized Terminal Wealth	$.95 \cdot 1.05 = .9975$ $= (1 + \text{Mean Geo. Ret.})^2$
Past Realization #2	$\{w, l\} = \{1.05, .95\}$
1 + Mean Arith. Ret.	$\frac{1.05+.95}{2} = 1$
1 + Mean Geom. Ret.	$\sqrt{1.05 \cdot .95} \approx .998749$
Realized Terminal Wealth	$1.05 \cdot .95 = .9975$ $= (1 + \text{Mean Geo. Ret.})^2$
Past Realization #3	$\{l, l\} = \{.95, .95\}$
1 + Mean Arith. Ret.	$\frac{.95+.95}{2} = .95$
1 + Mean Geom. Ret.	$\sqrt{.95 \cdot .95} \approx .95$
Realized Terminal Wealth	$.95 \cdot .95 = .9025$ $= (1 + \text{Mean Geo. Ret.})^2$
Past Realization #4	$\{l, l\} = \{1.05, 1.05\}$
1 + Mean Arith. Ret.	$\frac{1.05+1.05}{2} = 1.05$
1 + Mean Geom. Ret.	$\sqrt{1.05 \cdot 1.05} \approx 1.05$
Realized Terminal Wealth	$1.05 \cdot 1.05 = 1.1025$ $= (1 + \text{Mean Geo. Ret.})^2$
Ensemble Arith. Average of Past Equally-likely Realizations	
	$\{lw, wl, ll, ww\}$ mean $= \frac{lw+wl+ll+ww}{4} = 1$ median $= 0.9975$
Avg.[1 + Mean Arith. Ret.]	$\frac{1+1+.95+1.05}{4} = 1$
Avg.[1 + Mean Geom. Ret.]	$\frac{\sqrt{wl}+\sqrt{lw}+\sqrt{ll}+\sqrt{ww}}{4} \approx 0.99937$
Avg.[Realized Terminal Wealth]	$\frac{lw+wl+ll+ww}{4} = \frac{(l+w)^2}{4} = 1$ $= (\text{Avg.}[1 + \text{Mean Arith. Ret.}])^2$

Table 2. Initial wealth of \$1; a 50% chance of losing 5% ("$l = 0.95$"), and a 50% chance of winning either 5% (top two-thirds of table, "$w = 1.05$"), or 10% (bottom third of table, "$w = 1.1$").

for $n = 1$: The random variable W_1 is $W_0 \cdot l$ with probability $1/2$ and $W_0 \cdot w$ with probability $1/2$. Thus

$$E(W_1) = W_0 \cdot l/2 + W_0 \cdot w/2 = W_0 \cdot \frac{l+w}{2}$$

which proves the assertion for $n = 1$. Next, suppose the assertion is true for n:

$$E(W_n) = W_0 \left(\frac{l+w}{2}\right)^n .$$

Proof that it is true for $n + 1$: The random variable W_{n+1} is $W_n \cdot l$ with probability $1/2$ and $W_n \cdot w$ with probability $1/2$. Thus

$$E(W_{n+1}) = E\left[W_n \cdot l/2 + W_n \cdot w/2\right] = E[W_n] \cdot \frac{l+w}{2} = W_n \left(\frac{l+w}{2}\right)^{n+1} .$$

This proves the assertion for $n + 1$.

Proof of the inequality: The claim, we will prove is true for all positive values of n, is equivalent to

$$\begin{aligned} [(lw)^{1/2}]^n &< [(l+w)/2]^n \iff \\ (lw)^{1/2} &< (l+w)/2 . \end{aligned}$$

The LHS is the geometric mean of l and w and the RHS is the arithmetic mean of l and w, so the inequality follows from the inequality of arithmetic and geometric means, the "AM-GM inequality." Its classic proof with $l \neq w$ is

$$0 < (l-w)^2 = (l+w)^2 - 4lw \iff 4lw \leq (l+w)^2 ; \tag{3}$$

take the positive square root of both sides and divide by two.

This proves (1) and the expression for the limit as $n \to \infty$. To prove (2), first note that the variance of wealth after one period if $W_0 = 1$ is $(1/2)(l - m_1)^2 + (1/2)(w - m_1)^2$, which simplifies after some manipulation to var_1. From (3),

$$\begin{aligned} (l+w)^2 - (l-w)^2 &= 4\,lw \text{ ; hence} \\ (2m_1)^2 - 4var_1 &= 4\,lw \\ m_1^2 - var_1 &= lw . \end{aligned}$$

∎

2. When Log Returns are Normally Distributed and i.i.d., Passive beats Active because the Wealth Distribution is Lognormal

Much of this paper concerns lognormal distributions but notation for the lognormal distribution is not as standardized as for the normal distribution; here is the notation we will use. Suppose Z is a standard normal random variable. If $Y = p + qZ$ then Y is normally distributed with mean p and variance q^2, and we write $Y \sim N(p, q^2)$. The distribution of $e^{p+qZ} = e^Y$ is called a lognormal distribution with parameters p and q, denoted $LN(p, q)$. It can be shown that $LN(p, q)$ has the following properties:[3]

$$\text{mean}(LN(p,q)) = \exp\Big(p + \frac{q^2}{2}\Big) \tag{4}$$

$$\text{median}(LN(p,q)) = \exp(p) < \text{mean} \tag{5}$$

$$\text{variance}(LN(p,q)) = (e^{q^2} - 1)\, e^{2p+q^2} \tag{6}$$

$$\text{skewness}(LN(p,q)) = \big[e^{q^2} + 2\big]\sqrt{e^{q^2} - 1}\,. \tag{7}$$

One can invert (4) and (6) to obtain[4]

$$\begin{aligned} p &= \ln \frac{(\text{mean}(LN(p,q)))^2}{\sqrt{(\text{mean}(LN(p,q)))^2 + \text{variance}(L(p,q))}} \\ &= \ln\Big[(\text{mean}(L(p,q)))\,\Big(1 + \frac{\text{variance}(LN(p,q))}{(\text{mean}(LN(p,q)))^2}\Big)^{-1/2}\Big] \end{aligned} \tag{8}$$

$$q^2 = \ln\Big(1 + \frac{\text{variance}(LN(p,q))}{(\text{mean}(LN(p,q)))^2}\Big)\,. \tag{9}$$

With preliminaries out of the way, as before let W_t be the value, or "wealth level," of a mutual fund or other asset at the end of year t. Assume without loss of generality that $W_0 = 1$. Let

$$R_{t-1,t} = \frac{W_t - W_{t-1}}{W_{t-1}} = \frac{W_t}{W_{t-1}} - 1 \quad \text{for all } t = 1, 2, \ldots, n \tag{10}$$

[3]See (11)–(13) of https://mathworld.wolfram.com/LogNormalDistribution.html; (2.11) of M. Sharpe (n.d.); https://stats.libretexts.org/Bookshelves/Probability_Theory/Probability_Mathematical_Statistics_and_Stochastic_Processes_(Siegrist)/05\%3A_Special_Distributions/5.12\%3A_The_Lognormal_Distribution.

[4]See de La Grandville's (10) and (11) or standard references such as https://en.wikipedia.org/wiki/Log-normal_distribution

be the simple return of the mutual fund between neighboring dates $t-1$ and t. Defining the log return (or "continuously compounded return") as

$$r_{t-1,t} = \ln(1 + R_{t-1,t}) \,, \tag{11}$$

it follows that $W_t/W_{t-1} = e^{r_{t-1,t}}$. Let $R_{0,n}$ be the compound annual growth rate (CAGR), also known as the geometric average rate of return, between years 0 and n:

$$(1 + R_{0,n})^n = \prod_{t=1}^{n}(1 + R_{t-1,t}) = \frac{W_n}{W_0} = W_n \,. \tag{12}$$

If we were to assume that the simple return $R_{t-1,t}$ were normally distributed, then $W_t/W_{t-1} = 1 + R_{t-1,t}$ could be negative, which we would like to rule out. If we assume that the log return $r_{t-1,t}$ is normally distributed, then $W_t/W_{t-1} = e^{r_{t-1,t}}$ could not be negative. So in this section we make the very common assumption that the log return $r_{t-1,t}$ is a normally distributed i.i.d. random variable, and denote

$$\begin{aligned} \text{mean}[\ln(1 + R_{t-1,t})] &\coloneqq \mu \\ \text{variance}[\ln(1 + R_{t-1,t})] &\coloneqq \sigma^2 \quad \text{so that} \\ \ln(1 + R_{t-1,t}) = r_{t-1,t} &\sim N(\mu, \sigma^2) \,. \end{aligned} \tag{13}$$

In order to avoid misinterpreting μ, note that $E[W_t/W_{t-1}] = E[e^{r_{t-1,t}}] > e^{E[r_{t-1,t}]} = e^{\mu}$ where the inequality comes from Jensen's Inequality; so $E[W_t] > W_{t-1}\, e^{\mu}$. On average, in one period W will grow at a faster exponential rate than μ. From (13) and the definition of a lognormal distribution,

$$e^{\ln(1+R_{t-1,t})} = 1 + R_{t-1,t} \sim LN(\mu, \sigma) \,, \tag{14}$$

and from (4)–(9),

$$M \coloneqq \text{mean}(1 + R_{t-1,t}) = \exp\left(\mu + \frac{\sigma^2}{2}\right) \tag{15}$$

$$\text{median}(1 + R_{t-1,t}) = \exp(\mu) = \frac{M}{\sqrt{M^2 + V}} \tag{16}$$

$$V \coloneqq \text{variance}(1 + R_{t-1,t}) = (e^{\sigma^2} - 1)\, e^{2\mu+\sigma^2} \tag{17}$$

$$\text{skewness}(1 + R_{t-1,t}) = \left[e^{\sigma^2} + 2\right] \sqrt{e^{\sigma^2} - 1} \tag{18}$$

$$\mu = \ln \frac{M^2}{\sqrt{M^2 + V}} \tag{19}$$

$$\sigma^2 = \ln\left(1 + \frac{V}{M^2}\right) . \tag{20}$$

Equation (15) implies that $E[W_t] = W_{t-1}e^{\mu+\sigma^2/2} > W_{t-1}e^{\mu}$, as we anticipated. From (15) and (16), the median is less than the mean because of the presence of the variance term V. From (17), $V = 0$ if and only if $\sigma = 0$.

De La Grandville was concerned with the properties of the random variables $R_{0,n}$; we are concerned with the properties of the random variables $W_n = (1 + R_{0,n})^n$. But it is useful to explain de La Grandville's result so it can later be compared with ours. Since $R_{0,n}$ is a geometric mean return, if an analogy to Table 1 holds, then the return should be less than the arithmetic mean return due to a "volatility drag"; and it is. De La Grandville proved Proposition 2 through (21).

Proposition 2. *Assume the log return $r_{t-1,t}$ is distributed as $N(\mu, \sigma^2)$, and is i.i.d. Then $1 + R_{0,n} \sim LN(\mu, \sigma/\sqrt{n})$ and therefore*

$$E[1 + R_{0,n}] = \exp\left(\mu + \frac{\sigma^2/n}{2}\right) = M\left(1 + \frac{V}{M^2}\right)^{\frac{1}{2n}-\frac{1}{2}}. \tag{21}$$

If V is strictly positive, or equivalently if σ is strictly positive, then for $n > 1$ this multi-period expected return is less than the one-period expected return,

$$\frac{E[1 + R_{0,n}]}{E[1 + R_{t-1,t}]} = e^{\frac{\sigma^2}{2}\left(\frac{1}{n}-1\right)} = \left(1 + \frac{V}{M^2}\right)^{\frac{1}{2n}-\frac{1}{2}} < 1\,; \tag{22}$$

if $V = 0$ or equivalently if $\sigma = 0$ then this ratio is equal to one. Therefore, the phenomenon illustrated by (22) is called the "volatility drag" or "volatility decay" or "variance drain" or "volatility tax" on multiperiod returns. The left-hand side of (22) is equal to

$$\frac{E\left[\sqrt[n]{\prod_{t=1}^{n}(1 + R_{t-1,t})}\right]}{E\left[\frac{1}{n}\sum_{t=1}^{n}(1 + R_{t-1,t})\right]}, \tag{23}$$

where the numerator and the denominator contain, respectively, the geometric mean and the arithmetic mean of $1 + R_{t-1,t}$ for $t = 1, 2, \ldots, n$.

Proof. As in (6) of de La Grandville, from (12) and (11),

$$\begin{aligned} 1 + R_{0,n} &= \left(\prod_{t=1}^{n}\exp[\,r_{t-1,t}\,]\right)^{1/n} = \prod_{t=1}^{n}\exp[\,\tfrac{1}{n}\,r_{t-1,t}\,] \\ &= \exp\left[\sum_{t=1}^{n}\tfrac{1}{n}\,r_{t-1,t}\right] = \exp\left[\tfrac{1}{n}\,\sum_{t=1}^{n}r_{t-1,t}\right]. \end{aligned} \tag{24}$$

From (13), $\left[\sum_{t=1}^{n} r_{t-1,t}\right]$ is the sum of n normal random variables $N(\mu, \sigma^2)$; such a sum is distributed as $N(n\mu, n\sigma^2)$. Then $\frac{1}{n}\left[\sum_{t=1}^{n} r_{t-1,t}\right] \sim N(\mu, \sigma^2/n)$. Therefore from (24), $1 + R_{0,n} \sim LN(\mu, \sigma/\sqrt{n})$. Use (4) to obtain (21). For $n > 1$, the exponents in the second and third terms of (22) are negative.

The numerators of (22) and (23) are the same because of the definition of $R_{0,n}$ in (12). The denominator of (22) is M by the latter's definition in (15). The denominator of (23) is equal to $(1/n) \sum_{t=1}^{n} E[1 + R_{t-1,t}] = (1/n) \sum_{t=1}^{n} M = M$ as well. ▮

The main purpose of de La Grandville's paper was to point out that $M-V/2$ is not a good approximation of $E[1 + R_{0,n}]$, and to supply the exact expression (21)—though, as pointed out by Becker (2012 p. 2), de La Grandville added the assumption that $\ln(1+R_{t-1,t})$ be normally distributed. In the limit as $n \to \infty$, one could apply the binomial approximation $(1 + x)^a \approx 1 + ax$ to (21) to obtain

$$E[1 + R_{0,n}] \approx M\,(1 - \tfrac{1}{2}\tfrac{V}{M^2}) = M - \tfrac{V}{2M}$$

as an alternative approximation to $M - V/2$, although it is true that in finance usually $M \approx 1$, which would lead back to $M - V/2$. Authors obtaining $M - V/2$ without making any distributional assumptions include Messmore (1995) and Becker (op. cit.); Becker explicitly analyzes the problem through the lens of the Arithmetic Mean-Geometric Mean Inequality and Jensen's Inequality.

The main result of this section is that if returns are lognormally distributed and i.i.d. then terminal wealth is also lognormally distributed. If an analogy to Table 1 holds, then mean terminal wealth should be a compounding of the arithmetic mean return or the expected one-period return, with no influence of a "volatility drag"; and (25) shows that it is, it is simply M^n. Median terminal wealth is less than a compounding of the expected one-period return, due to V.

Proposition 3. *Assume the log return $r_{t-1,t}$ is distributed as $N(\mu, \sigma^2)$, and is i.i.d. Then $W_n = (1 + R_{0,n})^n \sim LN(n\mu, \sqrt{n}\,\sigma)$. Therefore,*

$$\textit{mean}\,(W_n) = \exp(n\mu + n\sigma^2/2) = M^n \tag{25}$$

$$\textit{median}\,(W_n) = \exp(n\mu) = \left(\tfrac{M}{\sqrt{M^2+V}}\right)^n \tag{26}$$

$$\textit{variance}\,(W_n) = (e^{n\sigma^2} - 1)\, e^{2n\mu+n\sigma^2} \tag{27}$$

$$\textit{skewness}\,(W_n) = \left(e^{n\sigma^2} + 2\right) \sqrt{e^{n\sigma^2} - 1}\,. \tag{28}$$

Hence the mean of W_n is the one-period mean raised to the n th power, but its median is the one-period mean dragged down by the one-period variance, raised to the n th power. The skewness is positive and increasing in n. The

$$\text{mean}\,(W_n) - \text{median}\,(W_n) = e^{n\mu}\,[e^{n\sigma^2/2} - 1]\,, \tag{29}$$

which is positive and increasing in n, and

$$\lim_{n\to\infty} \frac{\text{median}\,(W_n)}{\text{mean}\,(W_n)} = \lim_{n\to\infty} e^{-n\sigma^2/2} = 0\,.$$

Proof. Similarly to (6) of de La Grandville, from (12) and (11),

$$W_n = (1 + R_{0,n})^n = \prod_{t=1}^{n} \exp[\, r_{t-1,t}\,] = \exp\left[\sum_{t=1}^{n} r_{t-1,t}\right]. \tag{30}$$

From (13), $\left[\sum_{t=1}^{n} r_{t-1,t}\right]$ is the sum of n normal random variables $N(\mu, \sigma^2)$; such a sum is distributed as $N(n\mu, n\sigma^2)$. Therefore from (30), $W_n \sim LN(n\mu, \sqrt{n}\,\sigma)$. Then use (4)–(7). ▮

Bessembinder (2018 §2.2) also points out that the skewness is positive and increasing in n.

Corollary. *If the log return is normally distributed and is i.i.d., then, ignoring mutual fund expenses, an index mutual fund balance will exceed the median active mutual fund balance by the amount given by (29), which is positive and grows with n.*

Proof. As pointed out implicitly by Michaud (1981 p. 152)[5] and explicitly by Ikenberry, Shockley, and Womack (1998 pp. 14–15)[6], and by Heaton,

[5] "In skewed distributions, the median is often the descriptive parameter of choice for central tendency. In concrete investment terms, the probability that any given investor will achieve the mean may be very small. In highly right-skewed terminal wealth distributions, median terminal wealth is an estimate of the investment performance experienced by a typical individual or institution over the investment horizon."

[6] "However, a second, more subtle factor as to why managers deviate from the S&P (usually leading to underperformance) relates to the underlying statistical nature of long-run stock returns. Over short horizons such as a day or even a week, the cross-section of stock returns is close to Gaussian. . . . Yet over longer horizons, such as a year, this symmetry disintegrates. In nearly all years, the cross-section of individual stock returns exhibits considerable right-skewness. This occurs for two reasons. First, limited liability truncates equity returns (for long positions) to −100%. Second, upside returns are unbounded and, in any given year, several individual stocks will record extraordinary performance. It is not

Polson, and Witte (2017 pp. 690-691, 693)[7], ignoring expenses, an index fund grows as the mean of the distribution of W_n, and the median active fund grows to the median of the distribution of W_n. That mean minus that median is given by (29). ∎

If each observation is one stock, lognormality implies that it is better to invest in the index than in a single stock—but few people invest in a single stock. More important is that if each observation is a bundle of stocks, such as a mutual fund, it is better to invest in the index than in one mutual fund. An investor investing in more than one equity mutual fund is essentially creating their own mutual fund; the median such investor will do worse than investing in the index.

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unusual to observe the price of ten or more of the 500 issues in the S&P more than double in a year.

"This asymmetry is problematic for money managers precisely because they hold a small subset of the index's component stocks. One can imagine their portfolios coming from a limited number of "typical" draws from an underlying distribution. If the cross-section of long-horizon stock returns is positively skewed, the typical stock (or median stock) will underperform the mean of all stocks together. In short, the typical stock will appear to *underperform* an (equal-weighted) index of all the stocks together. . . . Even if active money managers *randomly* draw a subset of stocks from this pool (thus exhibiting no stock picking prowess), the median manager will tend to draw a portfolio that underperforms the index. This bias handicaps the median active fund manager even before costs and other factors are considered. . . .

[7]"In our model, randomly selecting a small subset of securities from an index maximizes the chance of outperforming the index—the allure of active equity management—but it also maximizes the chance of underperforming the index, with the chance of underperformance being larger than the chance of overperformance. To illustrate the idea, consider an index of 5 securities, 4 of which (although it is unknown which) will return 10% over the relevant period, and 1 of which will return 50%. Suppose that active managers choose portfolios of 1 or 2 securities and that they equally weight each investment. There are 15 possible 1- or 2-security "portfolios." Of these 15, 10 will earn returns of 10%, because they will include only the 10% securities. Just 5 of the 15 portfolios will include the 50% winner.. . . Thus, in this example, the average active management return will be the same as the index (see Sharpe [1991]), but two-thirds of the actively managed portfolios will underperform the index because they will omit the 50% winner. In this example, it is a large positive skewness in returns that creates a problem for active management, illustrated here as the selection of 1 or 2 securities. The nonsymmetric shape of the distribution of returns means that random selection—which we might think of as a plausible lower bound on the quality of active management—will deliver a median return that is worse than the average of the full index of the securities. . . .

". . . stock selection itself increases the chance of underperformance relative to the chance of overperformance in many circumstances.. . . Active managers do not start out on an even playing field with passive investing. Rather, active managers must overcome an inherent disadvantage."

3. When Log Returns are i.i.d. but otherwise Arbitrarily Distributed, Passive asymptotically beats Active because the Wealth Distribution is Asymptotically Lognormal

Again, although our focus is on the behavior of W_t, we begin with a result for $1 + R_{0,n}$. Osborne (1959 pp. 154–5) has Proposition 5's main result, and Hakansson (1971 pp. 868–9) has the main result of both Propositions 4 and 5.[8] Numerous authors have alluded to these results since[9]; one of the most recent mentions is Ibbotson (2023 p. 87). He is speaking at an informal roundtable discussion, implicitly about the second parameter of the lognormal distribution:

> Even if returns were IID, what you would get, of course, is a lognormal spreading out of wealth outcomes over time—multiplied by the square root of time. And the compounded return is divided by the square root of time. So, you get two entirely different shapes, depending on whether we're talking about the compound return or just your ending wealth. Over time, ending wealth spreads out in the shape of a tulip. The compound annual return, in contrast, is averaging out and looks more like a trumpet.
>
> The tulips and trumpets apply only if returns are IID. If there's some other sort of return pattern, then the shapes will be different.

We present the results, work an example, and at the end of this section we return to Ibbotson's tulips and trumpets.

Proposition 4. *Assume the log return is i.i.d., and that its distribution has* $E[r_{t-1,t}] = m < \infty$ *and* $\text{var}[r_{t-1,t}] = s^2 < \infty$*. Then as n goes to infinity,*

[8] The Lindeberg Theorem invoked by Hakansson is a generalization of the Central Limit Theorem. For more on the early history of the lognormal distribution in finance see Cootner (1964 p. 5).

[9] Among these authors was Michaud (1981 p. 152), whose paper contains the reasoning both of Proposition 5's main result and, as pointed out in Section 2, of its corollary. Proposition 4's main result was rederived by de La Grandville (1998 p. 79). Proposition 5's main result was discussed by Kritzman (1992 p. 11) in a short article for practitioners; by Booth (2002 p. 46; 2004 pp. 5–6) in the context of retirement planning; and by Bodie, Kane, and Marcus in the eighth (2009 §5.9) and subsequent editions of their MBA-level textbook (although not in its seventh edition, even though that is more than 700 pages long). The 2021 SBBI Handbook mentions it as well (Ibbotson and Harrington 2021 p. 126).

the random variable $1 + R_{0,n}$ *converges in distribution to* $LN(m, s/\sqrt{n})$ *and therefore*

$$E[1 + R_{0,n}] \text{ converges to } \exp\left(m + \frac{s^2/n}{2}\right). \tag{31}$$

Proof. In the right-hand side of (24), the expression $\sum_{t=1}^{n} r_{t-1,t}$ is now a sum of i.i.d. random variables of unknown distribution with mean m and variance s^2. For sufficiently large n, if m and s are finite, the Central Limit Theorem for Sums states that the distribution of $\sum_{t=1}^{n} r_{t-1,t}$ converges to $N(nm, ns^2)$ (Miller and Childers 2004 Theorem 7.3). This implies that the distribution of $\frac{1}{n} \sum_{t=1}^{n} r_{t-1,t}$ converges to $N(m, s^2/n)$, so the entire right-hand side of (30) converges to $LN(m, 2/\sqrt{n})$. ∎

Given this result, one might think that similarly, under the assumptions of Proposition 4, the rest of Proposition 2 holds, asymptotically, if μ is replaced by m and if σ is replaced by s. This is plausible, and likely to be roughly true in practice, but it is not technically true; we explain the analogous complication after the next proposition.

Proposition 5. *Assume the log return is i.i.d., and that its distribution has* $E[r_{t-1,t}] = m < \infty$ *and* $\text{var}[r_{t-1,t}] = s^2 < \infty$*. Then as n goes to infinity, the random variable* $W_n = (1+R_{0,n})^n$ *converges in distribution to* $LN(nm, \sqrt{n}\, s)$*. The limiting distribution has*

$$\begin{aligned}
\textit{mean}\,(LN(nm, \sqrt{n}\, s)) &= \exp(nm + ns^2/2) = [\textit{mean}\,(LN(n, s))]^n \\
\textit{median}\,(LN(nm, \sqrt{n}\, s)) &= \exp(nm) = [\exp(m)]^n \\
&= \left[\frac{(\textit{mean}(LN(n, s)))^2}{\sqrt{(\textit{mean}(LN(n, s)))^2 + \textit{variance}(L(n, s))}} \right]^n
\end{aligned}$$

so its mean is the one-period mean raised to the n th power, but its median is the one-period mean dragged down by the one-period variance, raised to the n th power. Also,

$$\begin{aligned}
\textit{skewness}\,(LN(nm, \sqrt{n}\, s)) &= \left(e^{ns^2} + 2\right) \sqrt{e^{ns^2} - 1} \\
\textit{mean}(LN(nm, \sqrt{n}\, s)) - \textit{median}(LN(nm, \sqrt{n}\, s)) & \\
&= e^{nm}\, [e^{ns^2/2} - 1]
\end{aligned} \tag{32}$$

which is positive and increasing in n, and

$$\lim_{n\to\infty} \frac{\textit{median}(LN(nm, \sqrt{n}\, s))}{\textit{mean}(LN(nm, \sqrt{n}\, s))} = \lim_{n\to\infty} e^{-ns^2/2} = 0\,.$$

However, the mean, median, and skewness of W_n as $n \to \infty$ may not equal to the mean, median, and skewness of $LN(nm, \sqrt{n}\, s)$ because convergence in distribution does not imply convergence in expectation.

Proof. In the right-hand side of (30), the expression $\sum_{t=1}^{n} r_{t-1,t}$ is now a sum of i.i.d. random variables of unknown distribution with mean m and variance s^2. For sufficiently large n, if m and s are finite, the Central Limit Theorem for Sums states that the distribution of $\sum_{t=1}^{n} r_{t-1,t}$ converges to $N(nm, ns^2)$ (Miller and Childers 2004 Theorem 7.3). This implies that the distribution of the entire right-hand side of (30) is $LN(nm, \sqrt{n}\, s)$. Then use (4)–(7). ▮

Corollary. *If the logarithmic excess return $r_{t-1,t}$ is not normally distributed but is i.i.d., then, ignoring mutual fund expenses, wealth will for large n approach an asymptotic distribution whose mean will exceed its median by the amount given by (32), which is positive and grows with n.*

The mean of W_n, which is what an index mutual fund would earn, is not guaranteed to approach the mean of the limiting distribution, nor is the median of W_n, which is what the median active mutual fund would earn, guaranteed to approach the mean of the limiting distribution; but if they do, or come sufficiently close to doing so, then the index mutual fund wealth will exceed the median active mutual fund wealth for large N.

To explain the last sentence of the proposition and the last sentence of the corollary, note first that if $F_n(x)$, $n = 1, \ldots$ is a sequence of cumulative distribution functions and if F is a cumulative distribution function, one says that F_n converges to F *in distribution* if $F_n(x) \to F(x)$ for all x at which $F(x)$ is continuous.[10] By contrast, F_n converges to F *in expectation* if $\lim_{n\to\infty} E(F_n)$ is equal to $E(\lim_{n\to\infty} F_n) = E(F)$.[11] The problem is that convergence in distribution (which we have from the Central Limit Theorem) does not imply convergence in expectation.[12] Mistakenly asserting that if F_n approaches F then $E(F_n)$ approaches $E(F)$ is somewhat similar to

[10] McLeish (2003 p. 45).

[11] See Jiao (2022 p. 6) for this and for claims in the rest of this paragraph. The material is also covered in Polansky (2011) Ch. 5, leading up to his Theorem 5.13.

[12] Jiao (ibid.) points out that the expectation operator E and the limit operator lim do not always commute. He writes, "We might expect that convergence in distribution implies convergence in expectation, because 'expectation is a feature of the distribution.' However, in general, none of convergence a.s. ['almost surely'], i.p. ['in probability'], or i.d. ['in distribution'] imply convergence in expectation. . . . The convergence of sequences of functions is in general quite complex, and is explored in depth in fields such as measure theory or functional analysis."

the "fallacious manipulation of double limits" which Merton and Samuelson criticized (1974 p. 67). Whether ignoring the distinction between the types of convergence is a serious mistake in applications will depend on the situation[13]

To see how the limiting processes in Propositions 4 and 5 work in an example situation, return to the model of Section 1, flipping a fair coin. To what extent can a discrete binomial wealth distribution be approximated by a (continuous) lognormal distribution? In the coin-flipping model the log return is, from (10) and (11), $\ln(1 + R_{t-1,t}) = \ln(W_t/W_{t-1})$, which is "$\ln(l)$ with probability 1/2 and $\ln(w)$ with probability 1/2." This has a mean of $(1/2)\ln(lw) = m$ and a variance of $(1/4)(\ln(w/l))^2 = s^2$. It follows from Propositions 4 and 5 that

$$1 + R_{0,n} \text{ converges in distribution to } LN\left(\frac{\ln(lw)}{2}, \frac{\ln(w/l)}{2\sqrt{n}}\right)$$

$$W_n \text{ converges in distribution to } LN\left(\frac{n\ln(lw)}{2}, \frac{\sqrt{n}\cdot\ln(w/l)}{2}\right)$$

and from Proposition 5,

$$\text{mean}(\lim_{n\to\infty} W_n) = \exp(nm + ns^2/2) = \exp\left[n\frac{\ln(lw)}{2} + \frac{n}{2}\frac{1}{4}(\ln(w/l))^2\right] \tag{33}$$

$$\text{median}(\lim_{n\to\infty} W_n) = \exp(nm) = \exp\left[n\frac{\ln(lw)}{2}\right] = (lw)^{n/2}. \tag{34}$$

We know from (1) the mean of W_n and the median of W_n, from which we could form $\lim_{n\to\infty}\text{mean}(W_n)$ and $\lim_{n\to\infty}\text{median}(W_n)$. If these were equal to (33) and (34), respectively, then we could use the limiting distribution to discover how the mean and variance behave for large n, which would be useful in other cases when the one-period distribution is unknown.

The approximate expression for the median, (34), is not only close to the exact expression for the median from (1), but the expressions are exactly the same. So for this example, the limiting distribution's median is an

[13]Propositions 4 and 5 assume that variance is finite. This was a controversial assumption in the 1960's, but while it may be questionable for daily data, for data on time scales of a month or more, which is the subject of this paper, it is no longer controversial.See Fama (1965); Blattberg and Gonedes (1974 p. 249) for monthly data; Upton and Shannon (1979) for semi-annual and annual data; and Kon (1984 pp. 147–8.). For shorter time scales, the question was still being investigated by, for example, Grabchak and Samorodnitsky (2010).

excellent proxy for the exact median derived from knowledge of the exact one-period distribution of W_n.

To see if this is true for the mean as well, we ask whether its approximate expression, (33), is equal to the exact expression for the mean from (1). If they are, then, taking the $1/n$ th root of each, we will have

$$\begin{aligned}
\frac{l+w}{2} &\stackrel{?}{=} \exp\left[\tfrac{1}{2}\ln(lw) + \tfrac{1}{8}(\ln\tfrac{w}{l})^2\right] = \exp\left[\tfrac{1}{2}\ln(lw)\right]\cdot\exp\left[\tfrac{1}{8}(\ln\tfrac{w}{l})^2\right] \\
&= \exp\left[\ln((lw)^{1/2})\right]\cdot\exp\left[\tfrac{1}{8}(\ln\tfrac{w}{l})^2\right] = (lw)^{1/2}\cdot\exp\left[\tfrac{1}{8}(\ln\tfrac{w}{l})^2\right] \\
\frac{l+w}{2\sqrt{lw}} &\stackrel{?}{=} \exp\left[\tfrac{1}{8}(\ln\tfrac{w}{l})^2\right] \\
\frac{1+\frac{w}{l}}{2\sqrt{\frac{w}{l}}} &\stackrel{?}{=} \exp\left[\tfrac{1}{8}(\ln\tfrac{w}{l})^2\right]\ ;\ \text{letting } \lambda = w/l \geq 0, \\
\frac{1+\lambda}{2\sqrt{\lambda}} &\stackrel{?}{=} \exp\left[\tfrac{1}{8}(\ln\lambda)^2\right] \\
\ln\tfrac{1+\lambda}{2\sqrt{\lambda}} &\stackrel{?}{=} \tfrac{1}{8}(\ln\lambda)^2 \\
\ln\left[\left(\tfrac{1+\lambda}{2\sqrt{\lambda}}\right)^8\right] &\stackrel{?}{=} (\ln\lambda)^2\,.
\end{aligned}$$

If $\lambda = 1$, this is true, but it is not true in general. Thus we predict that the limit of the exact means from (1) will not be equal to the mean of the limiting distribution, (33), except in the uninteresting case when $w = l$. The good news is that they might be close to each other. The bad news is that, since for unequal constants c_1 and c_2, one is of the form c_1^n and the other is of the form c_2^n, their ratio will increasingly diverge from unity as n grows.

Table 3 shows the situation for $l = 0.91$ and $w = 1.23$. The approximate mean, as anticipated, is not converging to the exact mean as N increases. However, they are numerically close to each other, and if one did not know the exact value of the median divided by the mean, using the approximate value from the limiting distribution instead would only give an error in the fourth decimal place. The approximation looks rather good. The last column shows how quickly the wealth maximum grows; this pulls the mean above the median. For $N = 10$, Figure 1 shows the discrete, binomial cumulative probability distribution and its continuous lognormal approximation[14]; and Figure 2 shows, for the lognormal approximation, the probability density function of wealth (blue) and of return (purple). The blue

[14] Carlen (2018 pp. 1–3) uses more formal methods but also shows the convergence of a

N	exact mean	approx. mean	% diff. means	median	median/mean exact	median/mean approx.	exact max
4	1.3108	1.3110	0.02%	1.2528	0.9558	0.9556	2.29
10	1.9672	1.9680	0.04%	1.7468	0.8880	0.8876	7.93
20	3.8697	3.8730	0.09%	3.0865	0.7976	0.7969	62.82

Table 3. Summary statistics on wealth when tossing a fair coin with l = 0.91 and w = 1.23. Exact mean from (1); approximate mean from (33); and median from (1) and (33) which are identical.

PDF is Ibbotson's "tulip" and the purple PDF is Ibbotson's "trumpet." For N = 20, see Figures 3 and 4.

4. Constructing a Continuous-time Case for Passive asymptotically beating Active because the Wealth Distribution is Lognormal

We first demonstrate Proposition 6. As in discrete time, if an analogy to Table 1 holds, then mean terminal wealth should be a compounding of the mean instantaneous return, with no influence of a "volatility drag"; and (35) shows that it is. Median terminal wealth is affected by volatility drag, however, as shown by (36).

Proposition 6. *Assume wealth W_t is a continuous-time stochastic process for which dW_t: is proportional to W_t; has independent, identically distributed increments; and is normally distributed with mean $\hat{\mu}$ and variance $\hat{\sigma}^2$. Setting $W_0 = 1$ without loss of generality, one has $\ln W_t \sim N((\hat{\mu}-\hat{\sigma}^2/2)\,t,\ \hat{\sigma}^2 t)$, and hence*

$$W_t \sim LN((\hat{\mu}-\hat{\sigma}^2/2)\,t,\ \hat{\sigma}\sqrt{t})\,.$$

In addition,

$$mean(W_t) = \exp(\hat{\mu}t) \tag{35}$$

$$median(W_t) = \exp[(\hat{\mu} - \tfrac{\hat{\sigma}^2}{2})\,t] \tag{36}$$

$$variance(W_t) = (e^{\hat{\sigma}^2 t} - 1)\,e^{2\hat{\mu}t} \tag{37}$$

$$skewness(W_t) = \left[e^{\hat{\sigma}^2 t} + 2\right]\sqrt{e^{\hat{\sigma}^2 t} - 1} \tag{38}$$

discrete random variable to a continuous random variable by using their CDF's instead of their PDF's.

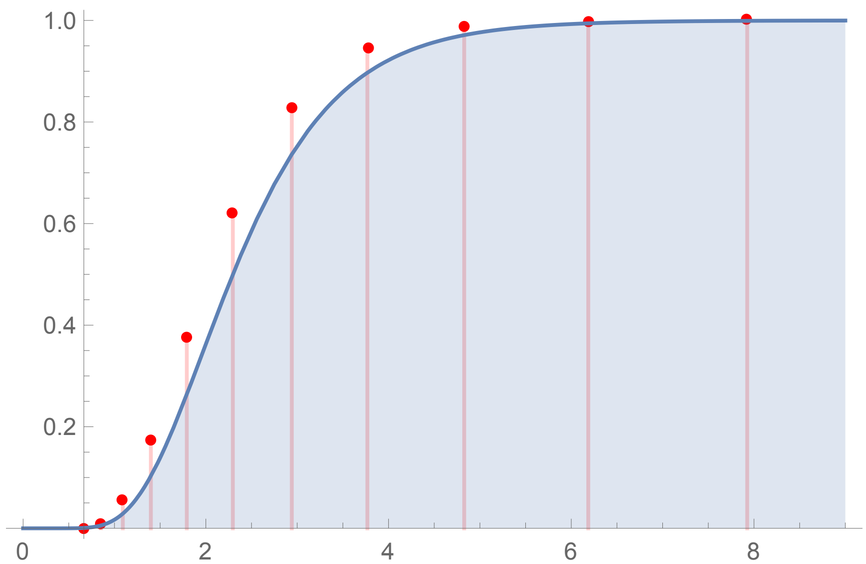

Figure 1. The exact cumulative density function (red) and the approximate log-normal cumulative density function (blue) for $N = 10$.

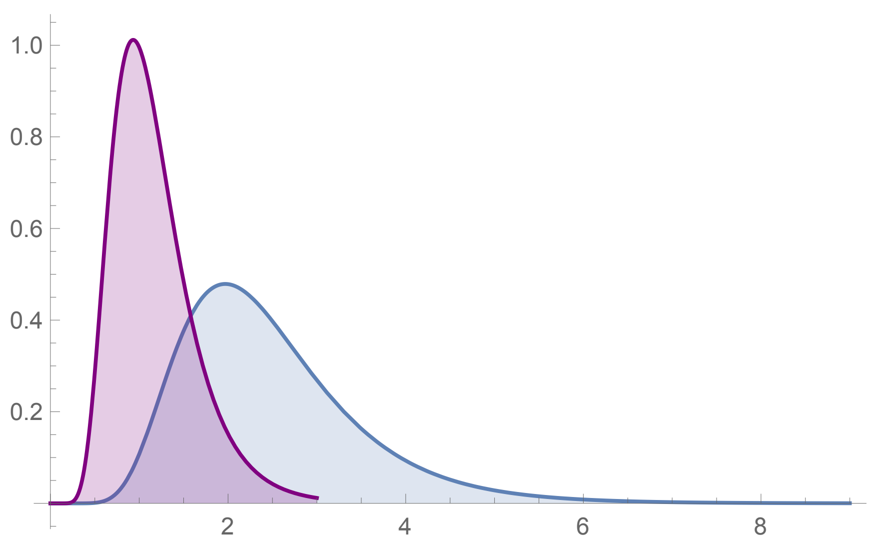

Figure 2. The approximate probability density function for wealth (blue) and for return (purple).

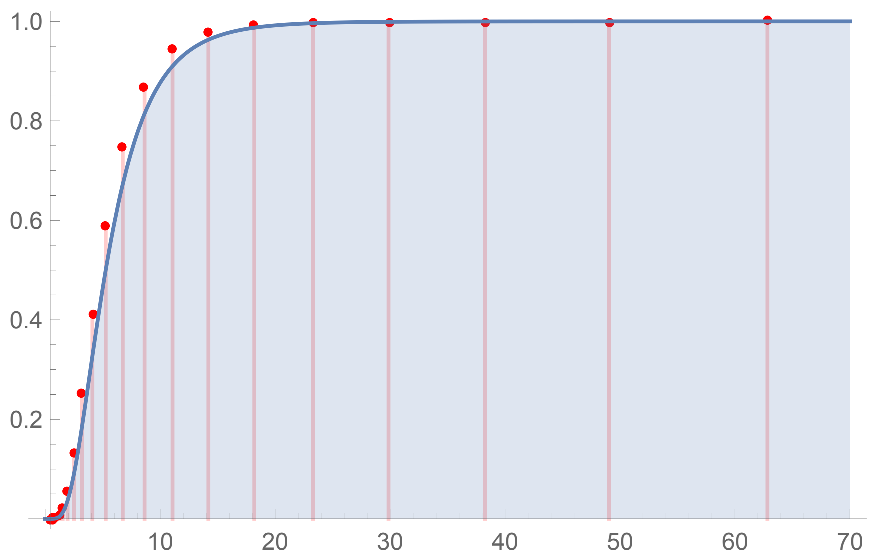

Figure 3. The exact cumulative density function (red) and the approximate log-normal cumulative density function (blue) for $N = 10$.

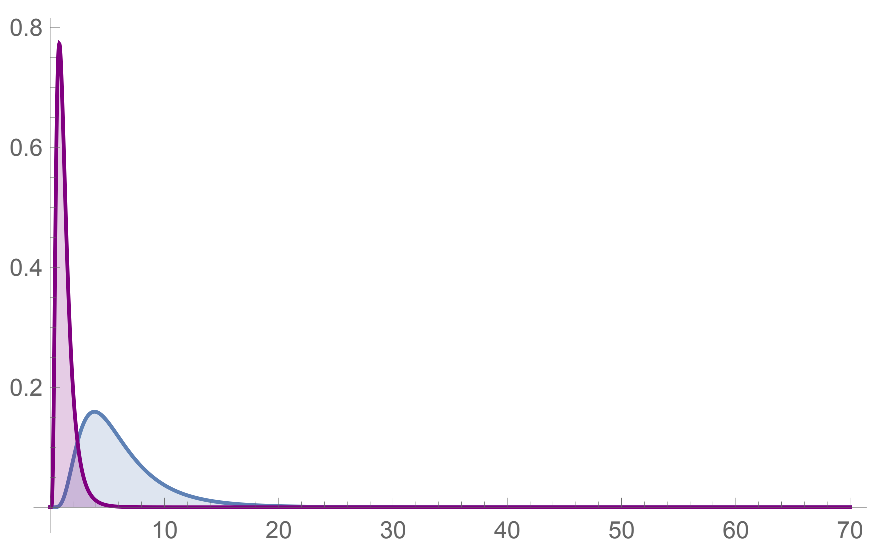

Figure 4. The approximate probability density function for wealth (blue) and for return (purple).

so the mean of W_t is e raised to the mean $\hat{\mu}$ of the instantaneous change dW_t, but the median of W_t is e raised to the mean $\hat{\mu}$ of the instantaneous change dW_t dragged down by the variance $\hat{\sigma}$ of the instantaneous change. Also,

$$mean(W_t) - median(W_t) = e^{\hat{\mu} t}\,[1 - e^{-\hat{\sigma}^2 t/2}]\,, \tag{39}$$

which is positive and increasing in t, and its

$$\lim_{t\to\infty} \frac{median(W_t)}{mean(W_t)} = \lim_{t\to\infty} e^{-t\,\hat{\sigma}^2/2} = 0\,.$$

Proof. We first need to show that these assumptions imply that wealth follows the stochastic process first suggested by Samuelson (1965 (9)), which is called geometric Brownian motion process with drift parameter $\hat{\mu}$ and volatility parameter $\hat{\sigma}$, and written

$$dW_t = \hat{\mu} W_t\, dt + \hat{\sigma} W_t\, dz_t \tag{40}$$

where dz_t is standard Brownian motion (dz_t is a "Wiener process") explained in the next paragraph.[15] Samuelson (pp. 13, 15) explains that (40) is better than the 1900 "arithmetic Brownian motion" proposal of Bachelier,[16]

$$dW_t = \hat{\mu}\, dt + \hat{\sigma}\, dz_t\,, \tag{41}$$

because the latter can lead to negative values of W_t, whereas the former cannot: if $W_t = 0$ in (40), $dW_t = 0$, so W_t cannot fall into the negative region. However, many other formulations would have this property, for example, $dW_t = \hat{\mu} W_t^{\alpha}\, dt + \hat{\sigma} W_t^{\beta}\, dz_t$ for positive α and β. The $\alpha = \beta = 1$ case makes the most sense because if W_t doubled, it would make sense for dW_t to double. For example, if a stock experienced a 2-for-1 reverse split, W_t would double and one would expect dW_t to double as well, so that chances of a given percent increase in price would be the same as it was before the reverse stock split.[17] This is the assumption made in the proposition, that "dW_t is proportional to W." Unfortunately for (40), Andersen

[15](18.4.1) of Siegrist 2022b.

[16]Pollock (n.d.) (8).

[17]For other arguments see Aldrich 2016 `https://ealdrich.github.io/Teaching/Econ236/LectureNotes/wiener.html` and Hull (2009 pp. 265–6), who writes (emphasis mine): "...the expected *percentage* return required by investors from a stock is independent of the stock's price.... the expected drift rate in S should be assumed to be μS for some constant parameter μ.... A reasonable assumption is that the variability of the *percentage* return in a short period of time...is the same regardless of the stock price.... This suggests that the standard deviation of the change in a short period of time should be proportional to the stock price...."

et al. (2010 p. 86) point out that "the model is overwhelmingly rejected for moderately frequently sampled data (say, daily, weekly, or monthly), as it fails to accommodate the well-documented strong intertemporal volatility dependencies," and those authors discuss many other models currently in use (ibid. §3.1).[18] So proportionality is a strong assumption.

The Wiener process z_t in (40) is usually[19] characterized by four traits:

starting point: $z_0 = 0$

independent increments: if $0 < t_1 < t_2 < t_3 < t_4$ then $z_{t_2} - z_{t_1}$ and $z_{t_4} - z_{t_3}$ are independent random variables[20]

identically distributed increments: for any $s < t$, $z_t - z_s$ is equal in distribution to z_{t-s} (so that the distribution depends only on the length $t - s$, not on the individual value of t or s: increments on equally long time intervals are identically distributed)

normal: z_t has the normal distribution with mean $\hat{\mu}t$ and variance $\hat{\sigma}^2 t$ for $t \geq 0$.

These are the assumptions we have used (the starting point assumption is easily replaced).

It can be shown that the Wiener process is continuous, and Bachelier's form (41) implies that $W_t \sim N(\hat{\mu}, \hat{\sigma}^2)$.[21] Using Ito's Lemma we can differentiate (40) to give

$$\begin{aligned} d(\ln W_t) &= (\hat{\mu} - \hat{\sigma}^2/2)\, dt + \hat{\sigma}\, dz \quad \text{which implies} \qquad (42) \\ \ln(W_t/W_0) &= (\hat{\mu} - \hat{\sigma}^2/2)\, t + \hat{\sigma} z_t \quad \text{and} \\ W_t &= W_0\, e^{(\hat{\mu} - \frac{\hat{\sigma}^2}{2})\, t + \hat{\sigma} z_t} \,. \end{aligned}$$

Equation (42) has the form of arithmetic Brownian motion with drift, as in (41) above, so $\ln(W_t/W_0) \sim N((\hat{\mu} - \hat{\sigma}^2/2)\, t,\ \hat{\sigma}^2 t)$. ▮

The main thrust of the following corollary is due to Heaton, Polson, and Witte (2017).[22]

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[18] One of the simplest alternatives is $dW_t = \hat{\mu} W_t\, dt + \hat{\sigma}\, dz_t$, which exhibits mean reversion and is known as the Ornstein-Uhlenbeck Process. See also Rao and Jelvis, Appendix C.8 p. 511.

[19] See Bass (2011 p. 6); Siegrist (2022a); see generally Hull (2009 §12.2).

[20] This is true not only for pairwise comparisons.

[21] The references for this paragraph are Kasa (2018, pp. 32, 35), Keng (2022) (4.4), and Siegrist (2022a, 2022b).

[22] In particular, in the passage starting at the very bottom of their p. 601.

Corollary. *Under the assumptions of Proposition 6, ignoring mutual fund expenses, an index mutual fund balance will exceed the median active mutual fund balance by the amount given by (39), which is positive and grows with t.*

The shortcoming of Proposition 6 is that it assumes normality, thus resembling the approach of Section 2, rather than the more general approach of Section 3, which used the Central Limit Theorem to relieve us from having to assume normality. Fortunately, in continuous time, there is another result which relieves us from having to assume normality: the Lévy characterisation of the Wiener process.

Proposition 7. *If in Proposition 6 the assumption that dW_t is normally distributed is replaced by the assumption that dW_t is continuous, then dW_t will be normally distributed and the rest of Proposition 6 holds.*

Proof. The Lévy characterisation states that if a random variable B_t satisfies

continuity: B_t is continuous

independent increments: if $0 < t_1 < t_2 < t_3 < t_4$ then $B_{t_2} - B_{t_1}$ and $B_{t_4} - B_{t_3}$ are independent random variables

identically distributed increments: for any $s < t$, $B_t - B_s$ is equal in distribution to B_{t-s}

then the distribution of B_t for each t must be normal.[23] Therefore, we can replace the assumption of normality of z_t with the assumption of continuity of z_t. ∎

In other words, the Lévy characterisation says that if one wants the stochastic process to be continuous with i.i.d. increments, it has to be a Wiener process.

Corollary. *If in Proposition 6 the assumption that dW_t is normally distributed is replaced by the assumption that dW_t is continuous, but Proposition 6's other assumptions are retained, then, ignoring mutual fund expenses, an index mutual fund balance will exceed the median active mutual fund balance by the amount given by (39), which is positive and grows with n.*

[23] Lawler (2014 pp. 44, 222–3) and Kasa (2018 p. 17). Also see `https://en.wikipedia.org/wiki/L\%C3\%A9vy_process`.

Conclusion

Whenever median terminal wealth is less than the mean terminal wealth, passive investing will beat the median active investor. In a coin-tossing model, the median terminal wealth is indeed less than the mean terminal wealth, because variance drags down the median but does not drag down the mean. With any probability distribution of one-shot returns, in discrete time, as long as log returns are i.i.d., terminal wealth is asymptotically lognormally distributed, so it is plausible (though in this case there are technically no guarantees) that median terminal wealth is less than the mean terminal wealth and active management asymptotically beats passive management. In continuous time, terminal wealth is lognormally distributed, so once more an index mutual fund terminal wealth will exceed the median active mutual terminal wealth, if log returns are i.i.d., if wealth is assumed to be continuous, and if dW_t is proportional to W_t. In all these cases, median terminal wealth can be described as being hindered by a volatility drag which does not hinder mean terminal wealth.

None of the assumptions described above holds in all situations. The i.i.d. assumption can fail: Bessembinder 2018 p. 447 implies it does for US stocks; BCZ 2023 p. 145 imply it does for US stock mutual funds; and it certainly fails for individual bonds which have finite maturity dates. Prices modeled in continuous time can take discrete jumps and require modeling using jump processes. Variance can be volatile, contradicting the proportionality of dW_t to W_t. The difference between median and mean terminal wealth in more complicated, realistic cases should be the subject of further investigation.

Nevertheless, the results of this paper probably hold approximately for many assets. This bolsters William Sharpe's case for passive management, and suggests that a fat right tail should be a starting point when thinking about the long-term distribution of the value of assets whose prices have an evolution that can be roughly assumed to be i.i.d. Bessembinder's empirical finding, that only 4% of US stocks were responsible for the large out-performance of US stocks over Treasury bills from 1926 to 2016, illustrate a fat right tail which theory tells us should be typical—not perhaps in magnitude but certainly in direction—of the long-run terminal wealth distribution of many assets.

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