

Section 8:

Normative General Equilibrium

5. [17 points] Let E_0 be the initial state of the natural environment (clean air, clean water, and so forth).

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Suppose that there are two persons in the economy, and these two persons are indexed by “ i ” for $i = 1, 2$.

Each person is able to remove e_i units of the environment and consume them. No one owns the environment, so the person pays nothing for the right to remove e_i from the environment.

The resulting state of the natural environment, called “ E ,” is given by $E_0 - e_1 - e_2$.

Besides consuming e_i , both persons also enjoy the natural environment in which they live, and their utility functions are given by

$$u_i = 2\sqrt{e_i E} \quad \text{for } i = 1, 2.$$

- (a) In the formula for u_i , why does E have no subscript?
- (b) What action will each person take?
- (c) Find the first-order conditions characterizing Pareto Efficient allocations for this problem. Do not attempt to solve these first-order conditions.
- (d) Is the self-interested behavior of each person consistent with Pareto Efficiency in this problem?

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(5)

$$E = E_0 - e_1 - e_2$$

$$u_i = 2 \sqrt{e_i E}$$

a) Both persons "consume" the same environment.

b) Person 1 maximizes $2 \sqrt{e_1 E} = 2 \sqrt{e_1 (E_0 - e_1 - e_2)}$ taking e_2 as given.*

$$\max 2 \sqrt{E_0 e_1 - e_1^2 - e_2 e_1} = \max 2 \sqrt{(E_0 - e_2) e_1 - e_1^2}$$

$$\Rightarrow O = \frac{(E_0 - e_2) - 2e_1}{\sqrt{(E_0 - e_2) e_1 - e_1^2}}$$

$$\Rightarrow O = (E_0 - e_2) - 2e_1$$

$$2e_1 = E_0 - e_2 .$$

* Note: $\uparrow e_2 \Rightarrow \downarrow u_i$ (holding e_1 fixed). This resembles jealousy: negatively interacting utilities.

By symmetry, $e_2 = e_1$, so

$$2e_1 = E_0 - e_1$$

$$3e_1 = E_0$$

$$e_1 = E_0/3 = e_2 .$$

Optional: this results in $E = E_0 - \frac{E_0}{3} - \frac{E_0}{3} = \frac{1}{3}E_0$.

c) Find Pareto Efficiency by maximizing a social welfare function; the alternative approach, $\max u$, s.t. u_2 fixed, won't work because the utilities are interdependent.

$$\begin{aligned} W &= \alpha u_1 + (1-\alpha) u_2 \\ &= \alpha \cdot 2 \sqrt{e_1 E} + (1-\alpha) \cdot 2 \sqrt{e_2 E} \\ &= \sqrt{E} [2\alpha \sqrt{e_1} + 2(1-\alpha) \sqrt{e_2}] \end{aligned}$$

$$= \sqrt{E_0 - e_1 - e_2} \left[2\alpha \sqrt{e_1} + 2(1-\alpha) \sqrt{e_2} \right].$$

F.O.C.:

$$e_1: 0 = \frac{1}{2} \frac{-1}{\sqrt{E_0 - e_1 - e_2}} \left[2\alpha \sqrt{e_1} + 2(1-\alpha) \sqrt{e_2} \right] + \frac{\alpha \sqrt{E_0 - e_1 - e_2}}{\sqrt{e_1}} \quad (1)$$

$$e_2: 0 = \frac{1}{2} \frac{-1}{\sqrt{E_0 - e_1 - e_2}} \left[2\alpha \sqrt{e_1} + 2(1-\alpha) \sqrt{e_2} \right] + \frac{(1-\alpha) \sqrt{E_0 - e_1 - e_2}}{\sqrt{e_2}} \quad (2)$$

d) Does part (b)'s solution, $e_1 = e_2 = \frac{1}{3} E_0$, solve part (c)'s equations

(1) and (2)? Substituting $e_1 = e_2 = \frac{1}{3} E_0$ into (1) yields

$$\begin{aligned} 0 &\stackrel{?}{=} \frac{-1}{2\sqrt{\frac{1}{3}E_0}} \left[2\alpha \sqrt{\frac{1}{3}E_0} + 2(1-\alpha) \sqrt{\frac{1}{3}E_0} \right] + \frac{\alpha \sqrt{\frac{1}{3}E_0}}{\sqrt{\frac{1}{3}E_0}} \\ &= - \left[\alpha + (1-\alpha) \right] + \alpha = - [1] + \alpha = \alpha - 1 \Rightarrow \text{only true if } \alpha = 1; \end{aligned}$$

Substituting $e_1 = e_2 = \frac{1}{3} E_0$ into (2) yields

$$\begin{aligned} 0 &\stackrel{?}{=} \frac{-1}{2\sqrt{\frac{1}{3}E_0}} \left[2\alpha \sqrt{\frac{1}{3}E_0} + 2(1-\alpha) \sqrt{\frac{1}{3}E_0} \right] + (1-\alpha) \frac{\sqrt{\frac{1}{3}E_0}}{\sqrt{\frac{1}{3}E_0}} \\ &= - \left[\alpha + 1 - \alpha \right] + (1-\alpha) = -1 + 1 - \alpha = -\alpha \Rightarrow \text{only true if } \alpha = 0. \end{aligned}$$

Since α cannot simultaneously be 0 and 1, part (b)'s solution does not satisfy part (c)'s equations: self-interested behavior is not consistent with Pareto Efficiency in this problem. [See next semester's discussions of "externalities" and "public goods."]

Optional: The "symmetric" Pareto Efficient point can be found as follows.

Let $e_1 = e_2 = "e"$ in (1) and (2) :

$$\begin{aligned}
 (1) \Rightarrow O &= \frac{-1}{2\sqrt{E_0 - 2e}} \left[2\alpha\sqrt{e} + 2(1-\alpha)\sqrt{e} \right] + \alpha \frac{\sqrt{E_0 - 2e}}{\sqrt{e}} \\
 &= \frac{-\sqrt{e}}{\sqrt{E_0 - 2e}} \left[\alpha + (1-\alpha) \right] + \alpha \frac{\sqrt{E_0 - 2e}}{\sqrt{e}} \\
 &= \frac{-\sqrt{e}}{\sqrt{E_0 - 2e}} + \alpha \frac{\sqrt{E_0 - 2e}}{\sqrt{e}} \quad \xrightarrow{\text{Let } \frac{1}{2}e = E_0 - 2e} \quad \frac{1}{2}e = E_0 - 2e \\
 \frac{\sqrt{e}}{\sqrt{E_0 - 2e}} &= \alpha \frac{\sqrt{E_0 - 2e}}{\sqrt{e}} \quad \xrightarrow{\left(\frac{1}{2}+2\right)e = E_0} \quad \left(\frac{1}{2}+2\right)e = E_0 \\
 e &= \alpha(E_0 - 2e) \quad e = \frac{E_0}{\frac{1}{2}+2} \quad (3)
 \end{aligned}$$

(2) \Rightarrow

$$\begin{aligned}
 O &= \frac{-1}{2\sqrt{E_0 - 2e}} \left[2\alpha\sqrt{e} + 2(1-\alpha)\sqrt{e} \right] + \frac{(1-\alpha)\sqrt{E_0 - 2e}}{\sqrt{e}} \\
 &= \frac{-\sqrt{e}}{\sqrt{E_0 - 2e}} \left[\alpha + 1-\alpha \right] + \frac{(1-\alpha)\sqrt{E_0 - 2e}}{\sqrt{e}} \\
 \frac{\sqrt{e}}{\sqrt{E_0 - 2e}} &= (1-\alpha) \frac{\sqrt{E_0 - 2e}}{\sqrt{e}} \\
 e &= (1-\alpha)(E_0 - 2e)
 \end{aligned}$$

$$\frac{1}{1-\alpha} e = E_0 - 2e$$

$$\begin{aligned}
 \left(\frac{1}{1-\alpha} + 2\right) e &= E_0 \\
 e &= \frac{E_0}{\frac{1}{1-\alpha} + 2} \quad (4).
 \end{aligned}$$

(3) = (4) at $\alpha = \frac{1}{2}$, when $e = \frac{1}{4}E_0$, less than part (b)'s $\frac{1}{3}E_0$.

Answer all of the following four questions.

1. [17 points]

(a) We have discussed three ways of finding the contract curve of a two-person economy:

- i. Solve $\max U_1$ s.t. U_2 is fixed (and subject to feasibility constraints);
- ii. Solve $\max U_2$ s.t. U_1 is fixed (and subject to feasibility constraints);
- iii. Solve $\max \alpha U_1 + (1 - \alpha) U_2$ subject to feasibility constraints (where U_1 is the utility of person 1, U_2 is the utility of person 2, and $\alpha \in [0, 1]$).

Suppose in this two-person economy there are only two commodities, so we can use an Edgeworth-Bowley Box. Explain, using this Box, why these three approaches all generate the contract curve.

For (i), it might help to start by illustrating what allocation it results in for a few different endowments. The same comment holds for (ii).

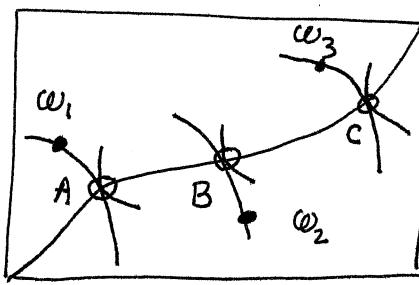
(b) Recall that we say two agents are of the same “type” if both their preferences and their initial endowments are the same.

Explain why, in the core, any two agents of the same type must receive the same bundle (the “equal treatment in the core” result). Although you can get full credit for answering this question with a correct mathematical proof, you can also get full credit if you simply give the intuition for the result, and that is what I will do on my answer sheet. You may make any assumptions you wish on the consumer’s preferences, but please explain why those assumptions are relevant to the result.

Answers to Econ 7005 Final, Fall 2011

①

a) (i)



Person 2

Person 1

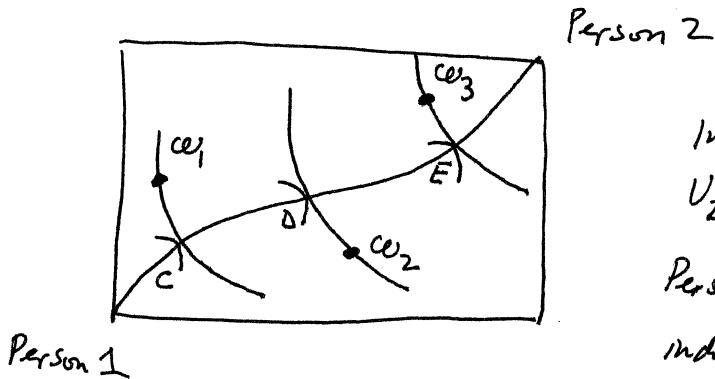
From ω_1 , holding U_2 fixed means being constrained to the indifference curve of Person 2 which passes through ω_1 and A. Maximizing U_1 with this constraint results in A. Note that A is on the contract curve.

From ω_2 , holding U_1 fixed means being constrained to the indifference curve of Person 2 which passes through ω_2 and B. Maximizing U_2 with this constraint results in B. Note that B is on the contract curve.

A similar analysis holds for ω_3 .

So if the initial endowment point $\underline{\omega}$ takes all possible values, the analog of Points A, B, or C trace out the entire contract curve.

(ii)

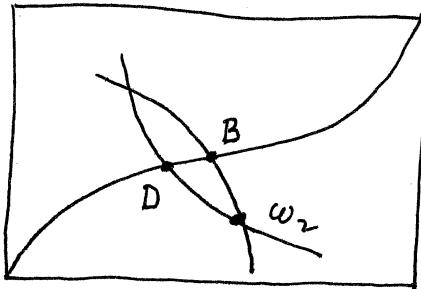


In this case, one maximizes U_2 subject to being on Person 1's original indifference curve.

Again, varying c_2 will generate the entire contract curve.

(iii)

First just consider c_2 . "B" is the same point as "B" of part (i);



"D" is the same point as "D" of part (ii). If α were equal to one, this problem would be the

same as part (i), so the answer would be B. If α were equal to zero, the answer similarly would be D. For α some fixed number between zero and one, the answer would be between B and D. So again, varying c_2 would generate the entire contract curve.

Optional. Note that neither (i) nor (ii) nor (iii) are market phenomena.

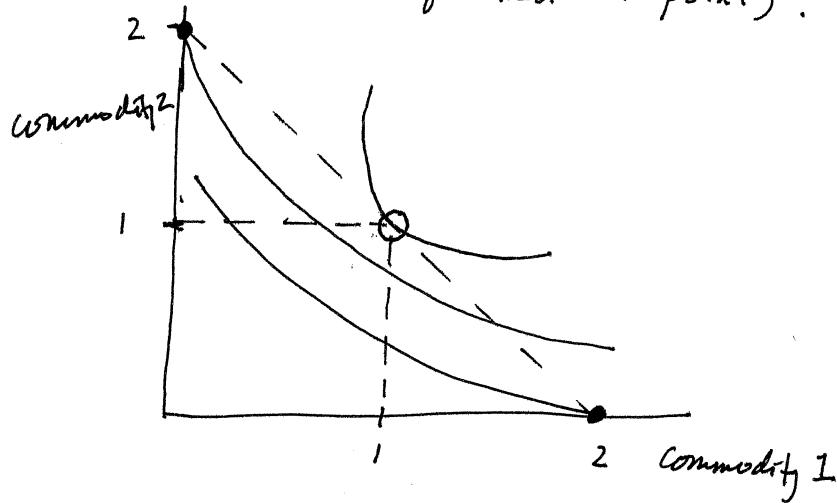
Furthermore, Person 1 would not want to do (i), he would want to max U , subject to no constraints. (i)-(iii) are done by economists or social planners, not individuals.

b) Varian has a formal proof on p. 390. The intuition, though,

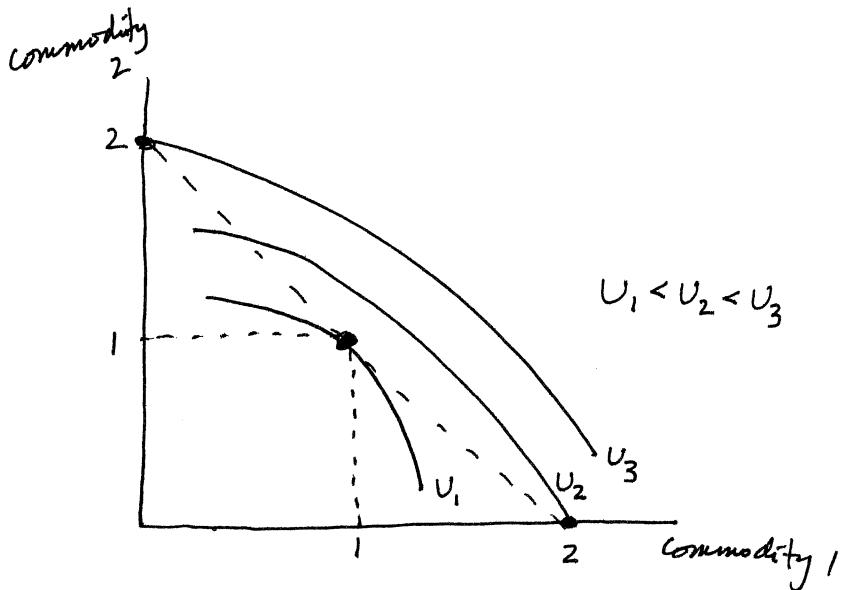
is that with convex preferences, agents prefer mixed bundles to extremes.

$\underbrace{\text{mixed bundles}}$
 $\underbrace{\text{more evenly}}$

extremes. Consider two agents of the same type, both with $\omega = (1, 1)$, so the total endowment is $(2, 2)$. Suppose they are being treated unequally, for example having $(2, 0)$ and $(0, 2)$. This can't be in the core because they would both be happier moving to $(1, 1)$ (which is an "equal treatment" point).



Optional: This result fails if nonconvexity of preferences results in concave indifference curves. For example:



Here, a move from "both persons have $(1,1)$ " to "one person has $(0,2)$ and the other person has $(2,0)$ " would not be vetoed, since $(0,2) \succ (1,1)$ and $(2,0) \succ (1,1)$, even though the new allocation has unequal treatment because $(0,2) \succ (2,0)$.

2. [12 points] Consider a two-person pure exchange economy. Suppose x_{ij} represents person i 's consumption of good j . Suppose the utility functions of the two persons are

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$$U_1 = x_{11}^{1/2} x_{12}^{1/2}$$

$$U_2 = x_{21}^{1/4} x_{22}^{3/4}$$

Find two equations in two unknowns, or three equations in three unknowns, or four equations in four unknowns, whose solution will define the contract curve of this economy. You do not have to solve the equation system nor give an explicit solution; an implicit solution will be enough.

(2)

Let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be the total amount of x_1 and x_2 in the economy.

The contract curve can be found by identifying the set of Pareto Optimal points. (Think about this in an Edgeworth Box.) So we will maximize

$\alpha U_1 + (1-\alpha)U_2$ subject to feasibility for $0 \leq \alpha \leq 1$. This is a social

planner's problem, but the solution to a (utilitarian) social planner's

problem is Pareto optimal. (We could instead solve this problem by maximizing U_1 ,

holding U_2 fixed or by maximizing U_2 holding

The feasibility constraints are

U_1 fixed.)

$$\bar{x}_1 = x_{11} + x_{21} \Rightarrow x_{11} = \bar{x}_1 - x_{21}$$

$$\bar{x}_2 = x_{12} + x_{22} \Rightarrow x_{12} = \bar{x}_2 - x_{22}.$$

The problem is to

$$\begin{aligned} \max \alpha U_1 + (1-\alpha) U_2 &= \max \alpha x_{11}^{\frac{1}{2}} x_{12}^{\frac{1}{2}} + (1-\alpha) x_{21}^{\frac{1}{4}} x_{22}^{\frac{3}{4}} \\ &= \max \alpha (\bar{x}_1 - x_{21})^{\frac{1}{2}} (\bar{x}_2 - x_{22})^{\frac{1}{2}} + (1-\alpha) x_{21}^{\frac{1}{4}} x_{22}^{\frac{3}{4}}. \end{aligned}$$

F.O.C.'s :

$$x_{21} : 0 = \frac{1}{2} \alpha (\bar{x}_1 - x_{21})^{-\frac{1}{2}} (\bar{x}_2 - x_{22})^{\frac{1}{2}} + \frac{1}{4} (1-\alpha) x_{21}^{-\frac{3}{4}} x_{22}^{\frac{3}{4}}$$

$$x_{22} : 0 = \frac{1}{2} \alpha (\bar{x}_1 - x_{21})^{\frac{1}{2}} (\bar{x}_2 - x_{22})^{-\frac{1}{2}} + \frac{3}{4} (1-\alpha) x_{21}^{\frac{1}{4}} x_{22}^{-\frac{1}{4}}$$

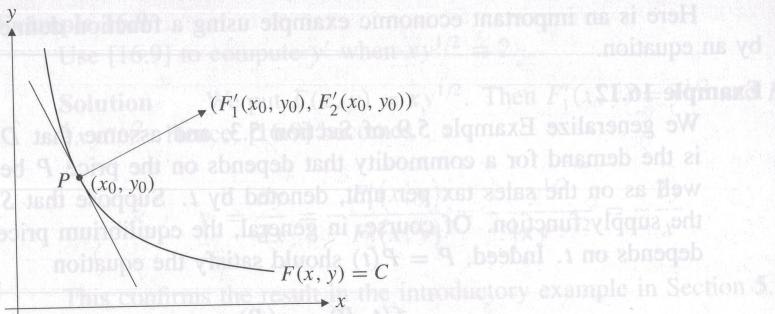
This is a set of 2 equations in the 2 unknowns x_{21} and x_{22} . They implicitly define $x_{21}(\alpha)$ and $x_{22}(\alpha)$. Then taking α between 0 and 1 would generate the contract curve.

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2. [12 points]

Consider a two-person, two-good economy with persons named Smith and Jones and goods named 1 and 2. Suppose the amount of goods 1 and 2 that Smith consumes is denoted by x_{1s} and x_{2s} , respectively. Suppose the amount of goods 1 and 2 that Jones consumes is denoted by x_{1j} and x_{2j} , respectively. Suppose Smith's utility function is $u_s = x_{1s}x_{2s}^2$ and Jones's utility function is $u_j = x_{1j}x_{2j}^2$. Suppose the total amount of good 1 in the economy is 3 and the total amount of good 2 in the economy is 4.

- (a) Find the location of the contract curve in the Edgeworth Box of this economy. Suppose the Edgeworth Box has Smith's origin in the lower-lefthand corner, and describe the contract curve as a function of Smith's consumption, x_{1s} and x_{2s} , not of Jones's consumption. Graph Smith's consumption of good 1 on the horizontal axis and Smith's consumption of good 2 on the vertical axis.
- (b) What is the slope of the contract curve? Is it positive or negative?
- (c) What is the second derivative of the contract curve? Is it positive or negative?
- (d) As pointed out on p. 554 of the text by Sydsæter and Hammond, which is attached to this exam, the gradient ∇F of a function F is orthogonal to the tangent of its level curve.
 - i. Calculate the gradient of the utility function of Smith at an arbitrary point on the contract curve.
 - ii. From this, calculate the slope of the gradient of the utility function of Smith at an arbitrary point on the contract curve.
 - iii. Is the answer to part (d)(ii) the same as the slope you found in part (b)?
 - iv. Suppose a pair of indifference curves, one for Smith and one for Jones, pass through the same point on the contract curve. At this point, is the tangent line to Smith's indifference curve perpendicular to the contract curve? At this point, is the tangent line to Jones's indifference curve perpendicular to the contract curve?

FIGURE 16.3 The gradient is orthogonal to the tangent at P .

which remains valid even if $F'_2(x_0, y_0) = 0$ and the tangent at P is vertical.

Example 16.13

Find the tangent to the curve in Example 16.10 at $(2, 1)$ by using [16.10].

Solution Here $F'_1(x, y) = 3x^2 + 2xy = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1 = 16$ at $(2, 1)$, and $F'_2(x, y) = x^2 - 4y - 10 = 2^2 - 4 \cdot 1 - 10 = -10$ at $(2, 1)$. Thus, [16.10] yields

$$16(x - 2) + (-10)(y - 1) = 0, \quad \text{or} \quad y = (1/5)(8x - 11)$$

This is the same result as in Example 16.10.

The vector $(F'_1(x_0, y_0), F'_2(x_0, y_0))$, also denoted by $\nabla F(x_0, y_0)$, is the **gradient** of $F(x, y)$ at (x_0, y_0) . Using the notation for a scalar product, [16.10] can be written as

$$(F'_1(x_0, y_0), F'_2(x_0, y_0)) \cdot (x - x_0, y - y_0) = 0 \quad [*]$$

This shows that the gradient is orthogonal to the tangent, as illustrated in Fig. 16.3.

If (h, k) is a unit vector, and $\nabla F(x_0, y_0) \neq (0, 0)$, then according to [16.2] in Section 16.1, the scalar product $D = \nabla F(x_0, y_0) \cdot (h, k)$ is the directional derivative of $F(x, y)$ at (x_0, y_0) in the direction (h, k) . A movement of one unit away from (x_0, y_0) in direction (h, k) changes the value of $F(x_0, y_0)$ by approximately D . Now, according to [12.19] in Section 12.4,

$$D = \|\nabla F(x_0, y_0)\| \cdot \|(h, k)\| \cdot \cos \phi$$

where ϕ is the angle between the vectors $\nabla F(x_0, y_0)$ and (h, k) . Hence, D attains its maximum value when $\phi = 0$, because then $\cos \phi = 1$. (Recall that $\cos \phi$ is always less than or equal to 1.) When $\phi = 0$, the vector $\nabla F(x_0, y_0)$ points in the same direction as (h, k) . Consequently, $\nabla F(x_0, y_0)$ points in the direction of

Section 1 Question 2.

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p.3

$$u_S = X_{1S} X_{2S}^2 \quad u_j = X_{1j} X_{2j}^2$$

a) The contract curve is the set of Pareto Optimal points. One could obtain this by maximizing a social welfare function with weights on the utility of Smith and Jones. Here I'll illustrate a different approach.

$$\max u_S \text{ s.t. } u_j = \bar{U}$$

$$\mathcal{L} = X_{1S} X_{2S}^2 + \lambda \left[\bar{U} - X_{1j} X_{2j}^2 \right]$$

$$= X_{1S} X_{2S}^2 + \lambda \left[\bar{U} - (3-X_{1S})(4-X_{2S})^2 \right] \text{ since}$$

$$X_{1S} + X_{1j} = 3$$

$$X_{2S} + X_{2j} = 4,$$

F.O.C.

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = \bar{U} - (3-X_{1S})(4-X_{2S})^2$$

$$0 = \frac{\partial \mathcal{L}}{\partial X_{1S}} = X_{2S}^2 + \lambda (4-X_{2S})^2 \quad (1)$$

$$0 = \frac{\partial \mathcal{L}}{\partial X_{2S}} = 2X_{1S}X_{2S} + \lambda \left[-(3-X_{1S})(2)(4-X_{2S})(-1) \right]$$

$$= 2X_{1S}X_{2S} + 2\lambda(3-X_{1S})(4-X_{2S}); \text{ dividing by 2,}$$

$$0 = X_{1S}X_{2S} + \lambda(3-X_{1S})(4-X_{2S}) \quad (2)$$

From (1) and (2),

$$\lambda = \frac{-X_{2S}^2}{(4-X_{2S})^2} = \frac{-X_{1S}X_{2S}}{(3-X_{1S})(4-X_{2S})} \Rightarrow$$

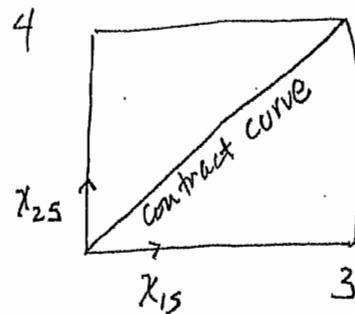
$$\frac{x_{2s}}{4 - x_{2s}} = \frac{x_{1s}}{3 - x_{1s}}$$

$$\frac{1}{\frac{4}{x_{2s}} - 1} = \frac{1}{\frac{3}{x_{1s}} - 1}$$

$$\frac{3}{x_{1s}} - 1 = \frac{4}{x_{2s}} - 1$$

$$\frac{3}{x_{1s}} = \frac{4}{x_{2s}}$$

$$x_{2s} = \frac{4}{3} x_{1s}$$



b) $\frac{4}{3} > 0$

c) the second derivative is zero

d) (i) $\nabla u_s = \left(\frac{\partial u_s}{\partial x_{1s}}, \frac{\partial u_s}{\partial x_{2s}} \right) = \left(x_{2s}^2, 2x_{1s}x_{2s} \right)$ and on the contract curve this is

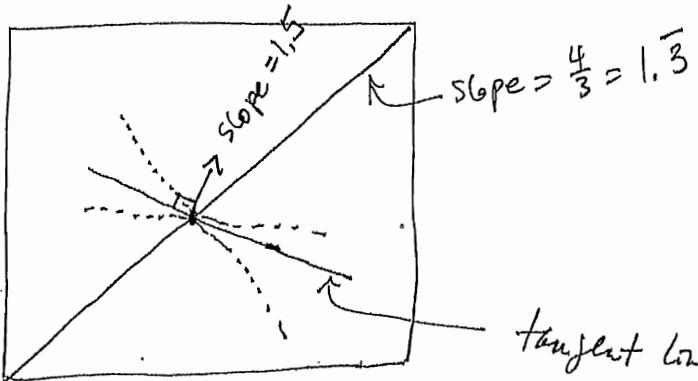
$$= \left(\left(\frac{4x_{1s}}{3} \right)^2, 2x_{1s} \cdot \frac{4}{3} x_{1s} \right)$$

$$= \left(\frac{16}{9} x_{1s}^2, \frac{8}{3} x_{1s}^2 \right)$$

(ii) Slope = $\frac{\text{rise}}{\text{run}} = \frac{\frac{8}{3} x_{1s}^2}{\frac{16}{9} x_{1s}^2} = \frac{8}{3} \cdot \frac{9}{16} = \frac{3}{2}$

(iii) no

(iv)



tangent line to both Smith's and Jones's indifference curves, which are tangent to each other on the contract curve. This tangent line

is perpendicular to the gradient, whose slope is 1.5, so it's not perpendicular to the contract curve, whose slope is $1.\overline{3}$.

Qualifying Exam

2004

Required Question

(A)

(6)

Section 1.

Answer both of the following two questions.

1. Suppose there are two consumers, Tom and Harry, and each obtains utility from hours of sleep, s , and from consuming good x . Tom and Harry's utility functions are

$$u_t(s_t, x_t) = s_t x_t \quad \text{and} \quad u_h(s_h, x_h) = s_h x_h$$

respectively. Neither Tom nor Harry has any initial endowment of x ; their initial endowments of time are

$$\omega_t = 1 \quad \text{and} \quad \omega_h = 2,$$

respectively, and they divide their time into hours of sleep and hours of work " w ". (Be careful not to confuse w and ω even though the two letters look similar.)

The total amount of good x in the economy is created from the total amount of work w in the economy according to the production function $x = 4w$.

- (a) Find the Pareto Optimal allocations of s_t , x_t , s_h , and x_h . Hint: the answer can be written in several forms; one is

$$\begin{aligned} s_t &= \frac{1}{2}\alpha & x_t &= 2\alpha \\ s_h &= \frac{3}{2} - \frac{1}{2}\alpha & x_h &= 6 - 2\alpha \end{aligned}$$

Qualifying Exam

2004

Reg. Question (B)

(6)

- (b) Suppose Tom and Harry form a two-person competitive economy. However, suppose there is government interference in this economy: the government subsidizes Tom's purchases of x and taxes Harry's purchases of x , using so-called *ad valorem* ["according to value"] taxes τ . (Note that Tom is poor and Harry is rich because of the differences in their endowments. Also, ignore any constraint concerning the government's budget balance.)

- i. Explain why Tom solves the problem

$$\max_{s_t, x_t} s_t x_t \quad \text{s.t. } (1 - s_t) p_w = p_x x_t (1 - \tau)$$

and why Harry solves the problem

$$\max_{s_h, x_h} s_h x_h \quad \text{s.t. } (2 - s_h) p_w = p_x x_h (1 + \tau).$$

- ii. Show that in competitive equilibrium,

$$s_t = \frac{1}{2} \quad \text{and} \quad s_h = 1.$$

- iii. Either show that in competitive equilibrium

$$x_t^* = \frac{6(1 + \tau)}{3 - \tau}$$

or show that in competitive equilibrium

$$x_h^* = \frac{12(1 - \tau)}{3 - \tau}$$

(you do not have to explain both of these).

- iv. From your answers to (a), (b-ii), and (b-iii), explain that the allocation in the two-person competitive economy with distortionary taxes can only be Pareto Optimal if the tax rate τ is zero.

Answers to 7005 Questions on the 2004 Micro Qualifying Exam

Section 1 #1.

sleep s

$$U_t = s_t x_t \quad U_h = s_h x_h$$

$$ce_t = 1$$

$$w_h = 2 \quad \text{endowments of time}$$

work " w "

$$x = 4w$$

Qualifying Exam

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Req. Answer (A)

a) Method 1: $\max U_h$ s.t. U_t is constant (or $\max U_t$ s.t. U_h is constant - either way will work)

$$\max s_h x_h \text{ s.t. } s_t x_t = \bar{u}_t, \text{ a constant.}$$

Also, $x_t + x_h = x = 4w = 4(1 - s_t + 2 - s_h)$. Solve the first

constraint for $x_t = \bar{u}_t/s_t$, substitute into the second constraint, to form:

$$\mathcal{L} = s_h x_h + \lambda \left[\frac{\bar{u}_t}{s_t} + x_h - 4(3 - s_t - s_h) \right]^{1+2}$$

$$= s_h x_h + \lambda \left[\frac{\bar{u}_t}{s_t} + x_h - 12 + 4s_t + 4s_h \right]. \text{ The FOC's are:}$$

$$x_h : 0 = s_h + \lambda [1] \longrightarrow \lambda = -s_h = -\frac{1}{4} x_h \Rightarrow s_h = \frac{1}{4} x_h$$

$$s_t : 0 = \lambda \left[-\frac{\bar{u}_t}{s_t^2} + 4 \right]. \dots$$

$$s_h : 0 = x_h + \lambda [4] \longrightarrow 4 = \frac{\bar{u}_t}{s_t^2} \Rightarrow s_t^2 = \frac{1}{4} \bar{u}_t, s_t = \frac{1}{2} \sqrt{\bar{u}_t}$$

For ease of writing, let $\alpha = \sqrt{\bar{u}_t}$. Then $s_t = \frac{1}{2} \alpha$

$$x_t = \frac{\bar{u}_t}{s_t} = \alpha^2 \div \left(\frac{\alpha}{2} \right)$$

$$= \alpha^2 \cdot \frac{2}{\alpha} = 2\alpha$$

The constraint was

$$x_t + x_h = 4(3 - s_t - s_h) \text{ so now}$$

$$2\alpha + x_h = 4\left(3 - \frac{\alpha}{2} - \frac{x_t}{4}\right)$$
$$= 12 - 2\alpha - x_t$$

$$2x_h = 12 - 4\alpha$$

$$x_h = 6 - 2\alpha. \text{ Also, } s_h = \frac{1}{4}x_h = \frac{3}{2} - \frac{1}{2}\alpha.$$

Summary: $s_t = \frac{1}{2}\alpha$ $x_t = 2\alpha$

$$s_h = \frac{3}{2} - \frac{1}{2}\alpha$$
 $x_h = 6 - 2\alpha.$

[Optional: if instead you solved $\max u_t$ s.t. u_h constant,

you would instead set (it turns out) $s_t = \frac{3}{2} - \frac{1}{2}\hat{\alpha}$ $x_t = 6 - 2\hat{\alpha}$

$$s_h = \frac{1}{2}\hat{\alpha}$$
 $x_h = 2\hat{\alpha}$

where $\hat{\alpha} = \sqrt{u_h}$. Setting $\hat{\alpha} = 3 - \alpha$ generates the previous answers.]

Method 2:

$$\max \tilde{\alpha} u_t + (1-\tilde{\alpha}) u_h \text{ s.t. } x_t + x_h = 4(1 - s_t + 2 - s_h)$$
$$= 12 - 4s_t - 4s_h$$

$\max \tilde{\alpha} s_t x_t + (1-\tilde{\alpha}) s_h x_h$ s.t. ; solve the constraint for x_h & substitute:

$$\max \tilde{\alpha} s_t x_t + (1-\tilde{\alpha}) s_h [12 - 4s_t - 4s_h - x_t]$$

$$s_t : 0 = \tilde{\alpha} x_t - 4(1-\tilde{\alpha}) s_h \Rightarrow x_t = 4 \frac{1-\tilde{\alpha}}{\tilde{\alpha}} s_h$$

$$x_t : 0 = \tilde{\alpha} s_t - (1-\tilde{\alpha}) s_h \Rightarrow s_t = \frac{1-\tilde{\alpha}}{\tilde{\alpha}} s_h$$

$$s_h : 0 = (1-\tilde{\alpha}) [12 - 4s_t - 4s_h - x_t] - 4(1-\tilde{\alpha}) s_h$$

$$0 = 12 - 4s_t - 4s_h - x_t - 4s_h = 12 - 4s_t - 8s_h - x_t$$

substitute in
to get:

$$0 = 12 - 4 \frac{1-\tilde{\alpha}}{\tilde{\alpha}} S_h - 8S_h - 4 \frac{1-\tilde{\alpha}}{\tilde{\alpha}} S_h$$

$$= 12 - 8 \frac{1-\tilde{\alpha}}{\tilde{\alpha}} S_h - 8S_h$$

$$0 = 3 - 2 \frac{1-\tilde{\alpha}}{\tilde{\alpha}} S_h - 2S_h = 3 - S_h \left[2 \frac{1-\tilde{\alpha}}{\tilde{\alpha}} + 2 \right]$$

$$= 3 - S_h \left[\frac{2-2\tilde{\alpha}}{\tilde{\alpha}} + \frac{2\tilde{\alpha}}{\tilde{\alpha}} \right] = 3 - S_h \left[\frac{2}{\tilde{\alpha}} \right] = 3 - \frac{2}{\tilde{\alpha}} S_h \Rightarrow$$

$$\frac{2}{\tilde{\alpha}} S_h = 3 \Rightarrow S_h = \frac{3}{2} \tilde{\alpha}$$

$$S_t = \frac{1-\tilde{\alpha}}{\tilde{\alpha}} S_h = \frac{1-\tilde{\alpha}}{\tilde{\alpha}} \cdot \frac{3}{2} \tilde{\alpha} = \frac{3}{2} (1-\tilde{\alpha})$$

$$x_t = 4 \frac{1-\tilde{\alpha}}{\tilde{\alpha}} S_h = 4 \frac{1-\tilde{\alpha}}{\tilde{\alpha}} \cdot \frac{3}{2} \tilde{\alpha} = 6(1-\tilde{\alpha})$$

$$x_h = 12 - 4S_t - 4S_h - x_t$$

$$= 12 - 4 \cdot \frac{3}{2} (1-\tilde{\alpha}) - 4 \cdot \frac{3}{2} \tilde{\alpha} - 6(1-\tilde{\alpha})$$

$$= 12 - 6(1-\tilde{\alpha}) - 6\tilde{\alpha} - 6(1-\tilde{\alpha})$$

$$= \underbrace{12}_{\text{II}} - \underbrace{6}_{\text{II}} + \underbrace{6\tilde{\alpha} - 6\tilde{\alpha}}_{\text{II}} - \underbrace{6}_{\text{II}} + 6\tilde{\alpha} = 6\tilde{\alpha}.$$

Summary: $S_t = \frac{3}{2} - \frac{3}{2}\tilde{\alpha}$ $x_t = 6 - 6\tilde{\alpha}$

$$S_h = \frac{3}{2}\tilde{\alpha} \qquad x_h = 6\tilde{\alpha}.$$

[Note that setting $\alpha = 3 - 3\tilde{\alpha}$ makes these answers the same as Method 1's answers.]

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2004

Req. Answer Cont...
(A)

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Req. Answer (B)

b) (i) Tom: $\max_{u_t} s_t x_t$ s.t. $\underbrace{(1-s_t)}_{\substack{\text{work hours} \\ \text{income}}} p_w = \underbrace{p_x x_t}_{\substack{\text{expenditures: } z \text{ acts to reduce } p_x \\ \text{for Tom}}} (1-z)$

$$\text{Harry: } \max_{\substack{s_h x_h \\ u_h}} \text{ s.t. } \underbrace{(2-s_h)}_{\substack{\text{work} \\ \text{hours}}} p_w = \underbrace{p_x x_h (1+\tau)}_{\substack{\text{wage} \\ \text{income}}} \quad \begin{aligned} \text{expenditures} &= \tau \text{ acts to increase} \\ p_x \text{ for Harry} & \end{aligned}$$

(ii)

$$\mathcal{L}_t = s_t x_t + \lambda_t [(1-s_t)p_w - p_x x_t (1-\tau)] \quad \mathcal{L}_h = s_h x_h + \lambda_h [(2-s_h)p_w - p_x x_h (1+\tau)]$$

$$s: 0 = x_t - \lambda_t p_w$$

$$x: 0 = s_t - \lambda_t p_x (1-\tau)$$

$$\Rightarrow \lambda_t = \frac{x_t}{p_w} = \frac{s_t}{p_x (1-\tau)}$$

$$\Rightarrow x_t = \frac{p_w}{p_x} \frac{s_t}{1-\tau}$$

into B.C.:

$$(1-s_t)p_w = p_w s_t$$

$$1-s_t = s_t$$

$$1 = 2s_t$$

$$\boxed{s_t = \frac{1}{2}}$$

(iii)

$$\Rightarrow x_t = \frac{p_w}{p_x} \frac{1}{2} \frac{1}{1-\tau}$$

$$0 = x_h - \lambda_h p_w$$

$$0 = s_h - \lambda_h p_x (1+\tau)$$

$$\Rightarrow \lambda_h = \frac{x_h}{p_w} = \frac{s_h}{p_x (1+\tau)}$$

$$\Rightarrow x_h = \frac{p_w}{p_x} \frac{s_h}{1+\tau}$$

into B.C.:

$$(2-s_h)p_w = p_w s_h$$

$$2p_w = 2p_w s_h$$

$$\boxed{s_h = 1}$$

$$\Rightarrow x_h = \frac{p_w}{p_x} \frac{1}{1+\tau}$$

$$\text{So total } x \text{ demanded is } \frac{p_w}{p_x} \frac{1}{2} \frac{1}{1-\tau} + \frac{p_w}{p_x} \frac{1}{1+\tau} = \frac{p_w}{p_x} \left[\frac{1}{2} \cdot \frac{1}{1-\tau} + \frac{1}{1+\tau} \right] =$$

$$\frac{p_w}{p_x} \left[\frac{1+\tau+2-2\tau}{2(1-\tau)(1+\tau)} \right] = \frac{p_w}{p_x} \frac{3-\tau}{2(1-\tau)(1+\tau)}$$

$$\text{Total } x \text{ supplied is } 4w = 4(1-s_t + 2-s_h) = 4(1 - \frac{1}{2} + 2 - 1) = 4 \cdot \frac{3}{2} = 6$$

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Req. Answer (B)
cont...

Equating demand and supply for x : $\frac{P_w}{P_x} \cdot \frac{3-\tau}{2(1-\tau)(1+\tau)} = 6$

$$\frac{P_w}{P_x} = 12 \cdot \frac{(1-\tau)(1+\tau)}{3-\tau}$$

$$S_0: x_t = \frac{P_w}{P_x} \cdot \frac{1}{2} \cdot \frac{1}{1-\tau} = 6 \cdot \frac{1+\tau}{3-\tau}$$

$$x_h = \frac{P_w}{P_x} \cdot \frac{1}{1+\tau} = 12 \cdot \frac{1-\tau}{3-\tau}$$

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(iv)	Pareto Optimal	Competitive	Req. Answer (B) cont...
x_t	2α	$6 \cdot \frac{1+\tau}{3-\tau}$	
x_h	$6 - 2\alpha$	$12 \cdot \frac{1-\tau}{3-\tau}$	
s_t	$\alpha/2$	$1/2$	$\Rightarrow \alpha = 1 \Rightarrow s_h = \frac{3}{2} - \frac{1}{2} = \frac{2}{2} = 1 \text{ ok}$
s_h	$\frac{3}{2} - \frac{\alpha}{2}$	1	$x_h = 6 - 2\alpha = 6 - 2(1) = 4$ $x_t = 2\alpha = 2(1) = 2$.

So equating the Pareto Optimal & the competitive:

$$x_t : 2 = 6 \cdot \frac{1+\tau}{3-\tau} \quad x_h : 4 = 12 \cdot \frac{1-\tau}{3-\tau}$$

$$\frac{1}{3} = \underbrace{\frac{1+\tau}{3-\tau}}_{\uparrow} \quad \frac{1}{3} = \underbrace{\frac{1-\tau}{3-\tau}}_{\rightarrow}$$

$$\frac{1}{3} = \frac{1}{3} \text{ so } \frac{1+\tau}{3-\tau} = \frac{1-\tau}{3-\tau}$$

$$1+\tau = 1-\tau$$

$$\tau = -\tau$$

$$\Rightarrow \tau = 0.$$

2. [14 points] Consider a two-agent two-commodity pure exchange economy, with agents named Smith (“ s ”) and Jones (“ j ”) and commodities named apples (“ a ”) and bananas (“ b ”). Suppose the total number of apples in this economy is 1 and the total number of bananas in this economy is also 1, but both apples and bananas are completely divisible (which means they can be cut into arbitrarily small pieces).

Denote Smith’s endowment by $\omega_s = (\omega_{sa}, \omega_{sb})$ and denote Jones’s endowment by $\omega_j = (\omega_{ja}, \omega_{jb})$. Suppose Smith consumes x_{sa} apples and x_{sb} bananas, and Jones consumes x_{ja} apples and x_{jb} bananas. Suppose Smith’s utility function is $u_s = x_{sa}x_{sb}$, and suppose Jones’s utility function is $u_j = x_{ja}x_{jb}$.

Suppose this economy has a social planner who wants the competitive equilibrium to be a certain arbitrary vector named

$$\bar{\mathbf{x}} = (\bar{x}_{sa}, \bar{x}_{sb}, \bar{x}_{ja}, \bar{x}_{jb}),$$

and in order to ensure that the competitive equilibrium does result in $\bar{\mathbf{x}}$, the social planner transfers T_a apples from Smith to Jones and transfers T_b bananas from Smith to Jones, all before any trading occurs. Note that T_a and T_b could take any sign (that is, they might not be positive). Also, note that because $\bar{\mathbf{x}}$ is assumed to be the outcome of a competitive equilibrium, it cannot be *completely* arbitrary, only somewhat arbitrary.

Find the values of T_a and T_b . (Note that their existence is guaranteed by the Second Theorem of Welfare Economics.)

Hint: The values of T_a and T_b may not be unique.

Summer 2013 Qualifying Exam Section 1 Question 2

Section 1 Question 2.

In total: apples $a = 1$

bananas $b = 1$

$$\text{Smith } \underline{\omega}_s = (\omega_{sa}, \omega_{sb}) \quad u_s = x_{sa} x_{sb}$$

$$\text{Jones } \underline{\omega}_j = (\omega_{ja}, \omega_{jb}) \quad u_j = x_{ja} x_{jb}$$

T_a : involuntary transfer of apples from Smith to Jones

T_b : " " " bananas" " "

Endowments after the forced transfers:

$$\omega'_s = (\omega_{sa} - T_a, \omega_{sb} - T_b)$$

$$\omega'_j = (\omega_{ja} + T_a, \omega_{jb} + T_b).$$

$$\text{Smith maximizes } u_s \text{ s.t. } p_a x_{sa} + p_b x_{sb} = p_a(\omega_{sa} - T_a) + p_b(\omega_{sb} - T_b).$$

\uparrow price of apples \downarrow price of bananas

Supposing $p_a + p_b = 1$, we'll take $p_b = 1 - p_a$. The Lagrangian for Smith's utility-maximization problem is

$$\mathcal{L} = x_{sa} x_{sb} + \lambda_s \left[p_a(\omega_{sa} - T_a) + (1-p_a)(\omega_{sb} - T_b) - p_a x_{sa} - (1-p_a)x_{sb} \right].$$

$$F.O.C. \quad 0 = \frac{\partial \mathcal{L}}{\partial x_{sa}} = \chi_{sb} - \lambda_s p_a \quad (1)$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_{sb}} = \chi_{sa} - \lambda_s (1-p_a) \quad (2)$$

and the budget constraint. (1) implies $\chi_{sb} = \lambda_s p_a \Rightarrow \lambda_s = \chi_{sb}/p_a$.

Substituting this into (2) yields

$$0 = \chi_{sa} - \frac{\chi_{sb}}{p_a} (1-p_a) = \chi_{sa} - \frac{\chi_{sb}}{p_a} + \chi_{sb} \Rightarrow$$

$$\chi_{sa} = \frac{\chi_{sb}}{p_a} - \chi_{sb}. \quad (3)$$

Then the budget constraint implies

$$p_a \left(\frac{\chi_{sb}}{p_a} - \chi_{sb} \right) + p_b \chi_{sb} = p_a (\omega_{sa} - T_a) + p_b (\omega_{sb} - T_b) \text{ or}$$

$$p_a \left(\frac{\chi_{sb}}{p_a} - \chi_{sb} \right) + (1-p_a) \chi_{sb} = \\ p_a (\omega_{sa} - T_a) + (1-p_a) (\omega_{sb} - T_b) \Rightarrow$$

$$\chi_{sb} \underbrace{\left[p_a \left(\frac{1}{p_a} - 1 \right) + 1-p_a \right]}_{1-p_a + 1-p_a} = p_a (\omega_{sa} - T_a) + (1-p_a) (\omega_{sb} - T_b)$$

$$\frac{1-p_a + 1-p_a}{2-2p_a} \Rightarrow$$

$$\chi_{sb} = \frac{p_a (\omega_{sa} - T_a) + (1-p_a) (\omega_{sb} - T_b)}{2(1-p_a)}. \quad (4)$$

By symmetry, except that the transfers go in the other direction:

$$\chi_{jb} = \frac{p_a (\omega_{ja} + T_a) + (1-p_a) (\omega_{jb} + T_b)}{2(1-p_a)} . \quad (5)$$

Since there are only two commodities, if the market of one commodity clears, then so will the other. Let's clear the banana market: $\chi_{sb} + \chi_{jb} = 1$.

Substituting for χ_{sb} and χ_{jb} , and replacing p_a with $1-p_b$ which is not necessary but makes more sense — that is, clearing the banana market will immediately solve for the price of bananas, not apples — gives

$$1 = \frac{(1-p_b)(\omega_{sa} - T_a) + p_b(\omega_{sb} - T_b)}{2p_b} + \frac{(1-p_b)(\omega_{ja} + T_a) + p_b(\omega_{jb} + T_b)}{2p_b}$$

$$2 = \left(\frac{1}{p_b} - 1\right)(\omega_{sa} - T_a) + (\omega_{sb} - T_b) + \left(\frac{1}{p_b} - 1\right)(\omega_{ja} + T_a) + (\omega_{jb} + T_b)$$

$$= \frac{1}{p_b} \left[(\omega_{sa} - T_a) + (\omega_{ja} + T_a) \right] - (\omega_{sa} - T_a) + (\omega_{sb} - T_b) \\ - (\omega_{ja} + T_a) + (\omega_{jb} + T_b)$$

$$= \frac{1}{p_b} \left[(\omega_{sa} - T_a) + (\omega_{ja} + T_a) \right] - \omega_{sa} + T_a + \omega_{sb} - T_b \\ - \omega_{ja} - T_a + \omega_{jb} + T_b$$

$$= \frac{1}{p_b} \left[(\omega_{sa} - T_a) + (\omega_{ja} + T_a) \right] - (\omega_{sa} + \omega_{ja}) + (\omega_{sb} + \omega_{jb})$$

$$\begin{aligned}
&= \frac{1}{P_b} [(\omega_{sa} - T_a) + (\omega_{ja} + T_a)] - 1 + 1 \\
&= \frac{1}{P_b} [\omega_{sa} + \omega_{ja} - T_a + T_a] = \frac{1}{P_b} [1 - T_a + T_a] = \frac{1}{P_b} \Rightarrow P_b^* = \frac{1}{2}.
\end{aligned}$$

Then $P_a^* = 1 - P_b^* = \frac{1}{2}$. (4) implies

$$x_{sb}^* = \frac{\frac{1}{2}(\omega_{sa} - T_a) + \frac{1}{2}(\omega_{sb} - T_b)}{2 \cdot \frac{1}{2}} = \frac{1}{2}(\omega_{sa} - T_a + \omega_{sb} - T_b) \quad (6)$$

and (3) implies

$$\begin{aligned}
x_{sa}^* &= \frac{x_{sb}^*}{P_a^*} - x_{sb}^* = \left(\frac{1}{P_a^*} - 1\right) x_{sb}^* = \left(\frac{1}{1/2} - 1\right) x_{sb}^* = x_{sb}^* \\
&= \frac{1}{2}(\omega_{sa} - T_a + \omega_{sb} - T_b).
\end{aligned} \quad (7)$$

Imposing $x_{sb}^* = \bar{x}_{sb}$ and $x_{sa}^* = \bar{x}_{sa}$, then substituting these into (6) and (7), respectively, yields

$$\bar{x}_{sb} = \frac{1}{2}(\omega_{sa} - T_a + \omega_{sb} - T_b) \quad (8)$$

$$\bar{x}_{sa} = \frac{1}{2}(\omega_{sa} - T_a + \omega_{sb} - T_b). \quad (9)$$

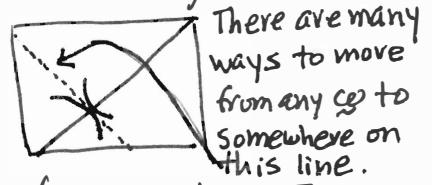
Clearly \bar{x}_{sb} must equal \bar{x}_{sa} . Also, having imposed $x_{sb}^* = \bar{x}_{sb}$ means that Smith has the amount of bananas which the social planner desired Smith to have — implying that Jones must also have the amount of bananas which the social planner wanted Jones to have. Similarly, our imposition of $x_{sa}^* = \bar{x}_{sa}$ assures it is also true that $x_{ja}^* = \bar{x}_{ja}$.

It follows that all T_a and T_b satisfying (8) will solve the problem. These transfers are

$$2\bar{x}_{sb} = \omega_{sa} + \omega_{sb} - T_a - T_b$$

$$T_a + T_b = \omega_{sa} + \omega_{sb} - 2\bar{x}_{sb}. \quad (10)$$

Neither T_a nor T_b are unique; any combination of T_a and T_b which satisfies (10) will affect Smith and Jones's incomes in the right way to achieve the social planner's objective.



Optional / Alternative: (10) can be rewritten referring only to Jones instead of only to Smith :

$$T_a + T_b = (1 - \omega_{ja}) + (1 - \omega_{jb}) - 2(1 - \bar{x}_{jb})$$

$$= 1 - \omega_{ja} + 1 - \omega_{jb} - 2 + 2\bar{x}_{jb}$$

$$= -\omega_{ja} - \omega_{jb} + 2\bar{x}_{jb}. \quad (11)$$

There are other equivalent forms as well.

Optional Note: Since \bar{x} is a competitive equilibrium, it must be Pareto Efficient due to the First Theorem of Welfare Economics. So \bar{x} is on the contract curve.

Summer
2006
Qualifiers

2. The Second Theorem of Welfare Economics states that (given certain conditions) it is possible to achieve any Pareto Efficient allocation via competitive markets, if, before the competitive markets open, lump-sum transfers can be imposed.

Suppose there are two consumers in an economy; the first has utility function $U_a = x_a y_a$ and the second has utility function $U_b = x_b y_b$, where x and y are the two goods and "a" and "b" represent the first and second consumer, respectively. The initial allocation is $(\omega_{xa}, \omega_{ya})$ for the first consumer and $(\omega_{xb}, \omega_{yb})$ for the second consumer. For an arbitrary Pareto Efficient point in this economy, what lump-sum transfers (T_x, T_y) from consumer "a" to consumer "b" are required in order for that Pareto Efficient point to be the outcome of competitive markets? (It is possible for T_x or T_y to be negative.) Hint: I found it easier to calculate the Pareto Efficient points using the "social weights" approach.

Qn. 2. $U_a = x_a y_a$

$$U_b = x_b y_b$$

To find the competitive allocation:

Suppose the post-transfer endowments are

$$\omega'_{x_a} = \omega_{x_a} - T_x$$

$$\omega'_{y_a} = \omega_{y_a} - T_y$$

$$\omega'_{x_b} = \omega_{x_b} + T_x$$

$$\omega'_{y_b} = \omega_{y_b} + T_y . \quad \begin{matrix} \text{price of } x \\ \text{price of } y \end{matrix}$$

Person a's expenditures are $p_x x_a + p_y y_a$ and his income is $p_x \omega'_{x_a} + p_y \omega'_{y_a}$, so his problem is to

$\max U_a$ s.t. income = expenditures

$$\mathcal{L} = x_a y_a + \lambda [p_x \omega'_{x_a} + p_y \omega'_{y_a} - p_x x_a - p_y y_a]$$

$$0 = \frac{\partial \mathcal{L}}{\partial x} = p_x \omega'_{x_a} + p_y \omega'_{y_a} - p_x x_a - p_y y_a$$

$$\left. \begin{array}{l} 0 = \frac{\partial \mathcal{L}}{\partial x_a} = y_a - \lambda p_x \\ 0 = \frac{\partial \mathcal{L}}{\partial y_a} = x_a - \lambda p_y \end{array} \right\} \Rightarrow \frac{p_x}{p_y} = \frac{y_a}{x_a} \Rightarrow y_a = \frac{p_x}{p_y} x_a ; \text{ substitute into the first F.O.C.}$$

$$\begin{aligned} p_x \omega'_{x_a} + p_y \omega'_{y_a} &= p_x x_a + p_y \left(\frac{p_x}{p_y} x_a \right) \\ &= 2 p_x x_a \end{aligned}$$

$$\frac{1}{2} \omega'_{x_a} + \frac{1}{2} \frac{p_y}{p_x} \omega'_{y_a} = x_a \quad \text{and since } y_a = \frac{p_x}{p_y} x_a ,$$

$$\frac{1}{2} \frac{p_x}{p_y} \omega'_{x_a} + \frac{1}{2} \omega'_{y_a} = y_a .$$

Person b's problem is symmetric to Person a's, so Person b's demands are

$$x_b = \frac{1}{2} c\omega'_{x_b} + \frac{1}{2} \frac{p_y}{p_x} c\omega'_{y_b}$$

$$y_b = \frac{1}{2} \frac{p_x}{p_y} c\omega'_{x_b} + \frac{1}{2} c\omega'_{y_b}$$

Let the total amount of x in the economy be \bar{X} and let the total amount of y in the economy be \bar{Y} . Let $p_x = 1$ (you could also have let $p_y = 1$ instead). Then for the "x" market to clear,

$$\begin{aligned}\bar{X} &= x_a + x_b \\ &= \frac{1}{2} c\omega'_{x_a} + \frac{1}{2} p_y c\omega'_{y_a} \\ &\quad + \frac{1}{2} c\omega'_{x_b} + \frac{1}{2} p_y c\omega'_{y_b} \\ &= \underbrace{\frac{1}{2} (\omega'_{x_a} + \omega'_{x_b})}_{= \omega_{xa} - T_x} + \underbrace{\frac{1}{2} p_y (\omega'_{y_a} + \omega'_{y_b})}_{= \omega_{xb} + T_x} \\ &= \omega_{xa} + \omega_{xb} \\ &= \bar{X} \qquad \text{similarly } = \bar{Y}\end{aligned}$$

$$\Rightarrow \bar{X} = \frac{1}{2} \bar{X} + \frac{1}{2} p_y \bar{Y}$$

$$\frac{1}{2} \bar{X} = \frac{1}{2} p_y \bar{Y}$$

$$\boxed{\bar{X}/\bar{Y} = p_y} \text{ and so in equilibrium}$$

$$x_a = \frac{1}{2} c\omega'_{x_a} + \frac{1}{2} \frac{\bar{X}}{\bar{Y}} c\omega'_{y_a}$$

$$y_a = \frac{1}{2} \frac{\bar{Y}}{\bar{X}} c\omega'_{x_a} + \frac{1}{2} c\omega'_{y_a}$$

$$x_b = \frac{1}{2} c\omega'_{x_b} + \frac{1}{2} \frac{\bar{X}}{\bar{Y}} c\omega'_{y_b}$$

$$y_b = \frac{1}{2} \frac{\bar{Y}}{\bar{X}} c\omega'_{x_b} + \frac{1}{2} c\omega'_{y_b}$$

To find the Pareto Optimum :

Use the "social weights" approach

$$\max \alpha U_a + (1-\alpha) U_b \text{ s.t. } x_a + x_b = \bar{X} \text{ and } y_a + y_b = \bar{Y}, \alpha \in [0,1]$$

$$\Leftrightarrow \max \alpha x_a y_a + (1-\alpha)(\bar{X}-x_a)(\bar{Y}-y_a) = \max_{\text{social welfare}} W$$

$$0 = \frac{\partial W}{\partial x_a} = \alpha y_a + (1-\alpha)(-1)(\bar{Y}-y_a)$$

$$0 = \frac{\partial W}{\partial y_a} = \alpha x_a + (1-\alpha)(\bar{X}-x_a)(-1),$$



$$(1-\alpha)(\bar{X}-x_a) = \alpha x_a$$

$$(1-\alpha)\bar{X} - (1-\alpha)x_a = \alpha x_a$$

$$(1-\alpha)\bar{X} - x_a + \alpha x_a = \alpha x_a$$

$$(1-\alpha)\bar{X} = x_a \text{ and } x_b = \bar{X} - x_a$$

$$\Rightarrow (1-\alpha)(\bar{Y}-y_a) = \alpha y_a \quad = \bar{X} - (1-\alpha)\bar{X} = \bar{X} - \bar{X} + \alpha \bar{X}$$

$$(1-\alpha)\bar{Y} - (1-\alpha)y_a = \alpha y_a \quad = \alpha \bar{X}$$

$$(1-\alpha)\bar{Y} - y_a + \alpha y_a = \alpha y_a$$

$$(1-\alpha)\bar{Y} = y_a \text{ and } y_b = \bar{Y} - y_a = \bar{Y} - (1-\alpha)\bar{Y} = \bar{Y} - \bar{Y} + \alpha \bar{Y} \\ = \alpha \bar{Y}.$$

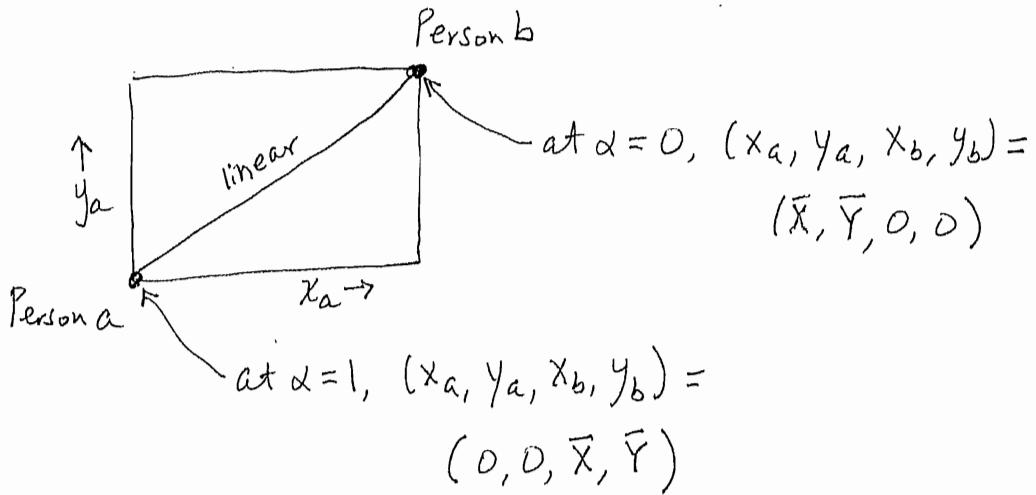
So $(x_a, y_a, x_b, y_b) = ((1-\alpha)\bar{X}, (1-\alpha)\bar{Y}, \alpha \bar{X}, \alpha \bar{Y})$ is the contract curve.

Start of Optional Discussion.

Since $x_a = (1-\alpha)\bar{X}$ and $y_a = (1-\alpha)\bar{Y}$, $\frac{y_a}{x_a} = \frac{(1-\alpha)\bar{Y}}{(1-\alpha)\bar{X}}$ and

The Pareto Optimal $y_a = \frac{\bar{Y}}{\bar{X}} x_a$. If $x_a = 0, y_a = 0$, and if $x_a = \bar{X}, y_a = \bar{Y}$.

The Pareto Optimal y_a is a linear function of x_a , so the contract curve is linear and looks like



But at $\alpha=1$, $W = 1U_a + 0U_b$, so Person a should get everything, not nothing. Something is wrong.

OPTIONAL →
PAGE ←

is

$$\begin{bmatrix} \frac{\partial^2 W}{\partial x_a^2} & \frac{\partial^2 W}{\partial x_a \partial y_a} \\ \frac{\partial^2 W}{\partial y_a \partial x_a} & \frac{\partial^2 W}{\partial y_a^2} \end{bmatrix} = \begin{bmatrix} 0 & \alpha + 1 - \alpha \\ \alpha + 1 - \alpha & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So $D_1 = 0$ and $D_2 = 0 - 1 = -1$. Not only does this violate the second-order sufficient conditions for a maximum ($D_1 < 0, D_2 > 0$), these violate the second-order necessary conditions for a maximum ($D_1 \leq 0, D_2 \geq 0$),

so our solution was not a maximum.

As an alternative, try the other approach to finding Pareto Optimal allocations:

* Or work out $\mathcal{L} = \alpha x_a y_a + (1-\alpha)x_b y_b + \lambda[\bar{x} - x_a - x_b] + \mu[\bar{y} - y_a - y_b]$ to see if it's better.

$\max U_a$ s.t. $U_b = \bar{U}$ fixed and s.t. $x_a + x_b = \bar{x}$, $y_a + y_b = \bar{y}$

$\max x_a y_a$ s.t. $x_b y_b = \bar{U}$ — " —

$\max x_a y_a$ s.t. $(\bar{x} - x_a)(\bar{y} - y_a) = \bar{U}$

$$\mathcal{L} = x_a y_a + \lambda [(\bar{x} - x_a)(\bar{y} - y_a) - \bar{U}]$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = (\bar{x} - x_a)(\bar{y} - y_a) - \bar{U}$$

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial x_a} = y_a - \lambda(\bar{y} - y_a) \\ 0 &= \frac{\partial \mathcal{L}}{\partial y_a} = x_a - \lambda(\bar{x} - x_a) \end{aligned} \quad \left\{ \Rightarrow \frac{\bar{y} - y_a}{\bar{x} - x_a} = \frac{y_a}{x_a} \Rightarrow \frac{\bar{y} - y_a}{y_a} = \frac{\bar{x} - x_a}{x_a} \right.$$

$$\frac{\bar{Y} - Y_a}{Y_a} - 1 = \frac{\bar{X} - X_a}{X_a} - 1$$

$$\frac{\bar{Y}}{Y_a} = \frac{\bar{X}}{X_a}$$

$$Y_a = \frac{\bar{Y}}{\bar{X}} X_a.$$

This is the same (linear) contract curve we had before (see 2 lines after "Start of Optional Discussion").

There is no " α " here, but the graph would be the same as on the previous page. How about second-order conditions for this problem?

$$\nabla^2 \mathcal{L} = \begin{bmatrix} 0 & y_a - \bar{Y} & x_a - \bar{X} \\ y_a - \bar{Y} & 0 & 1 + \lambda \\ x_a - \bar{X} & 1 + \lambda & 0 \end{bmatrix}$$

$m=1$; $n=2$; SOC for a max are

$D_{2m+1} \dots D_{m+n} = D_3 \dots D_3$ alternate

in sign starting with $(-1)^{m+1} = +1$. So

we want $D_3 > 0$. Expanding along the first column,

$$D_3 = (-1)^{2+1} (y_a - \bar{Y}) [0 - (1+\lambda)(x_a - \bar{X})]$$

$$+ (-1)^{3+1} (x_a - \bar{X}) [(y_a - \bar{Y})(1+\lambda) - 0]$$

or \rightarrow

$$= (y_a - \bar{Y})(1+\lambda)(x_a - \bar{x}) \\ + (x_a - \bar{x})(y_a - \bar{Y})(1+\lambda)$$

$$= 2(1+\lambda) \underbrace{(x_a - \bar{x})}_{\begin{array}{c} \downarrow \\ \text{due to constraints} \end{array}} \underbrace{(y_a - \bar{Y})}_{\begin{array}{c} \downarrow \\ \text{due to constraints} \end{array}}$$

Use $y_a = \frac{\bar{Y}}{\bar{x}} x_a$ and $(\bar{x} - x_a)(\bar{Y} - y_a) = \bar{U}$

$$(\bar{x} - x_a) \left(\bar{Y} - \frac{\bar{Y}}{\bar{x}} x_a \right) = \bar{U}; \text{ multiply both sides by } \bar{x}/\bar{Y} \Rightarrow$$

$$(\bar{x} - x_a)(\bar{x} - x_a) = \bar{U}\bar{x}/\bar{Y}$$

$$\bar{x} - x_a = \sqrt{\bar{U}\bar{x}/\bar{Y}}$$

$$x_a = \bar{x} - \sqrt{\bar{U}\bar{x}/\bar{Y}};$$

substituting this into the third F.O.C.

$$\lambda = \frac{x_a}{\bar{x} - x_a} = \frac{\bar{x} - \sqrt{\bar{U}\bar{x}/\bar{Y}}}{\sqrt{\bar{U}\bar{x}/\bar{Y}}} = \frac{\bar{x}}{\sqrt{\bar{U}\bar{x}/\bar{Y}}} - 1 = \sqrt{\frac{\bar{x}\bar{Y}}{\bar{U}}} - 1$$

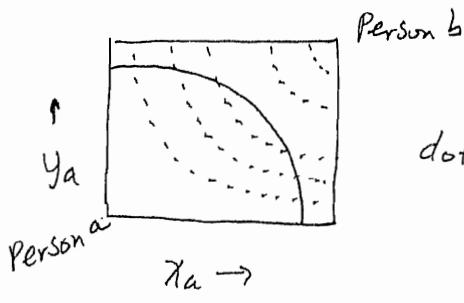
$$\Rightarrow 1+\lambda = \sqrt{\bar{x}\bar{Y}/\bar{U}} > 0.$$

So $D_3 > 0$ and the S.O.C. for this problem are O.K. This is

not surprising; in this problem we were trying to maximize U_a

subject to $U_b = x_b y_b = \bar{U}$ fixed, which graphically means

we were trying to maximize U_a subject to being on an indifference curve of Person b like the solid curve in this Edgeworth Box:



dotted lines are Person a's indifference curves; they have the form $1/x$ because

$$y_A = \frac{U_A}{x_A} \leftarrow \text{fixed}$$

It seems reasonable that the problem "max U_A s.t. being on the solid line" has a solution.

In any case, the "solution" obtained before the "Start of Optional Discussion" gives the correct set of Pareto Optimal points, even though as α goes from 1 to 0 it traces the contract curve out from left to right instead of from right to left.

End of Optional Discussion.

Setting the competitive allocations equal to the Pareto Optimal allocations:

$$x_A = \frac{1}{2} (\omega_{x_A} - T_X) + \frac{1}{2} \frac{\bar{X}}{\bar{Y}} (\omega_{y_A} - T_Y) = (1-\alpha) \bar{X}$$

$$y_A = \frac{1}{2} \frac{\bar{Y}}{\bar{X}} (\omega_{x_A} - T_X) + \frac{1}{2} (\omega_{y_A} - T_Y) = (1-\alpha) \bar{Y}$$

$$x_B = \frac{1}{2} (\omega_{x_B} + T_X) + \frac{1}{2} \frac{\bar{X}}{\bar{Y}} (\omega_{y_B} + T_Y) = \alpha \bar{X}$$

$$y_B = \frac{1}{2} \frac{\bar{Y}}{\bar{X}} (\omega_{x_B} + T_X) + \frac{1}{2} (\omega_{y_B} + T_Y) = \alpha \bar{Y}$$

The unknowns are T_X and T_Y . So we only need 2 of these equations.

If we take the first two, we have, rewriting,

$$-\frac{1}{2} T_x - \frac{1}{2} \frac{\bar{x}}{\bar{y}} T_y = (1-\alpha) \bar{x} - \frac{1}{2} \omega_{xa} - \frac{1}{2} \frac{\bar{x}}{\bar{y}} \omega_{ya} \leftarrow \text{call the RHS } "c_1"$$

$$-\frac{1}{2} \frac{\bar{y}}{\bar{x}} T_x - \frac{1}{2} T_y = (1-\alpha) \bar{y} - \frac{1}{2} \frac{\bar{y}}{\bar{x}} \omega_{xa} - \frac{1}{2} \omega_{ya} \leftarrow \text{call the RHS } "c_2"$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{-\bar{x}}{2\bar{y}} \\ \frac{-\bar{y}}{2\bar{x}} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

and so T_x and T_y can be solved by the usual methods.

optional:

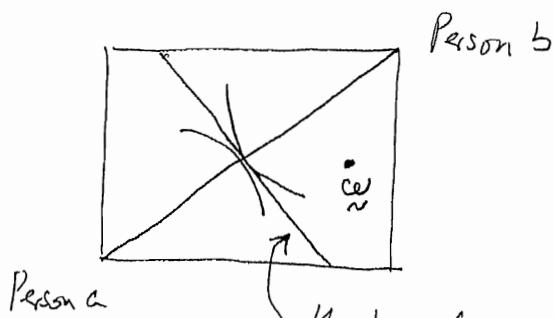
$$\begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\bar{x}/2\bar{y} \\ -\bar{y}/2\bar{x} & -\frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \frac{1}{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(-\frac{\bar{y}}{2\bar{x}}\right)\left(\frac{-\bar{x}}{2\bar{y}}\right)} \begin{bmatrix} -\frac{1}{2} & \bar{x}/2\bar{y} \\ \bar{y}/2\bar{x} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \frac{1}{\frac{1}{4} - \frac{1}{4}} \begin{bmatrix}] \\] \end{bmatrix} \begin{bmatrix}] \\] \end{bmatrix}$$

which does not work. The equations we picked were not independent.

It turns out there is no pair of equations that are independent here. So there is another degree of freedom. This makes sense:



the transfers just have to get you from e to anywhere on this straight line; the price system will take you from there to the contract curve

So take, say, T_y arbitrary, then use any of the four equations to find T_x . For example, the first equation yields

$$\frac{1}{2}(T_x - \omega_{x_A}) = \frac{1}{2} \frac{\bar{x}}{\bar{y}} (\omega_{y_A} - T_y) - (1-\alpha) \bar{X}$$

$$T_x = \frac{\bar{x}}{\bar{y}} (\omega_{y_A} - T_y) - 2(1-\alpha) \bar{X} + \omega_{x_A}.$$

Any T_y will work as long as it and its associated T_x keep all the ω' amounts positive.

End of optional comments.

2018 Final Exam Qu. 4

4. [17 points]

Suppose a pure exchange economy consists of two consumers, Smith “S” and Jones “J,” and two commodities, apples “a” and bananas “b.” Suppose the total number of apples in the economy is 3 and the total number of bananas in the economy is 2. Suppose the utility functions for Smith and for Jones are

$$u_S(a_S, b_S) = \ln a_S + 3 \ln b_S \quad \text{and} \\ u_J(a_J, b_J) = \ln a_J + \ln b_J,$$

respectively.

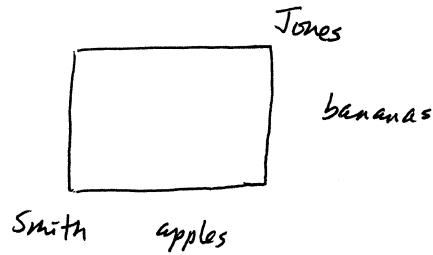
Consider an Edgeworth Box with Smith’s origin being in the lower left-hand corner of the box and with apples graphed on the horizontal axis.

- (a) Find an expression for the contract curve of this economy. You may leave it in implicit form.
- (b) Show that the point on the contract curve corresponding to Smith getting two apples is $(a_S, b_S) = (2, 12/7)$.
- (c) Find the equation of the tangent line to the indifference curve of Smith which passes through the point given in part (b). (Hint 1: marginal rate of substitution. Hint 2: one way to write the equation of a generic straight line is $y = mx + b$ (where “b” stands for the general mathematical intercept, not for “bananas,” and “y” stands for a general mathematical dependent variable, not for a particular economic concept); another is $y - y_1 = m(x - x_1)$.)
- (d) Describe in the context of the Second Theorem of Welfare Economics the economic interpretation of the tangent line you derived in part (c).

Answers to Qn. 4 of Econ. 7005 Final Exam, Fall 2018

$$U_S = \ln a_S + 3 \ln b_S \quad a_S + a_J = 3$$

$$U_J = \ln a_J + \ln b_J \quad b_S + b_J = 2$$



- a) A social planner may want to maximize, for some $0 \leq \alpha \leq 1$ weighting, social welfare
- (Alternatively, maximize one agent's welfare keeping the other agent's welfare constant. See the end of this answer for an example of this method.)

The solutions to this problem will be the set of Pareto Efficient points (the contract curve).

$$W = \alpha (\ln a_S + 3 \ln b_S) + (1-\alpha) (\ln a_J + \ln b_J)$$

$$= \alpha (\ln a_S + 3 \ln b_S) + (1-\alpha) (\ln (3-a_S) + \ln (2-b_S))$$

using the endowments equations to eliminate two of the unknowns, a_J and b_J ;

$$= \alpha \ln a_S + 3\alpha \ln b_S + (1-\alpha) \ln (3-a_S) + (1-\alpha) \ln (2-b_S).$$

The F.O.C. for maximizing W are

$$0 = \frac{\partial W}{\partial a_S} = \frac{\alpha}{a_S} - \frac{1-\alpha}{3-a_S} \Rightarrow \frac{1-\alpha}{3-a_S} = \frac{\alpha}{a_S} \Rightarrow (1-\alpha)a_S = \alpha(3-a_S)$$

$$a_S - \alpha a_S = 3\alpha - \alpha a_S$$

$$0 = \frac{\partial W}{\partial b_S} = \frac{3\alpha}{b_S} - \frac{1-\alpha}{2-b_S}$$

\Downarrow

$$\frac{3\alpha}{b_S} = \frac{1-\alpha}{2-b_S} \Rightarrow 3\alpha(2-b_S) = b_S(1-\alpha)$$

$$6\alpha - 3\alpha b_S = b_S - \alpha b_S$$

$$6\alpha = b_S - \alpha b_S + 3\alpha b_S = b_S + 2\alpha b_S = (1+2\alpha)b_S$$

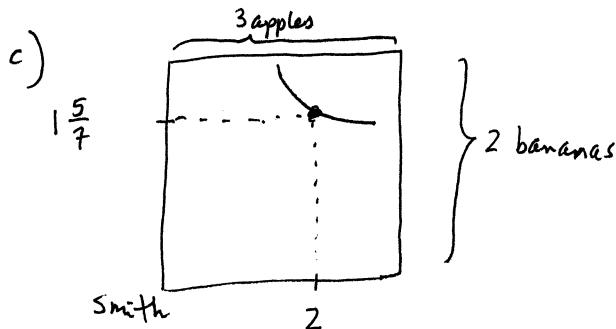
$$\Rightarrow b_S = 6\alpha / (1+2\alpha).$$

So the contract curve is $(a_s, b_s) = \left(3\alpha, \frac{6\alpha}{1+2\alpha}\right)$ for $0 \leq \alpha \leq 1$.

b) If $a_s = 2$ then since $a_s = 3\alpha$ we have $2 = 3\alpha$

$$\frac{2}{3} = \alpha \text{ and}$$

$$b_s = \frac{6\alpha}{1+2\alpha} = \frac{6 \cdot \frac{2}{3}}{1+2 \cdot \frac{2}{3}} = \frac{6 \cdot 2}{3+4} = \frac{12}{7}.$$



Find Smith's marginal rate of substitution at $(a_s, b_s) = (2, 1\frac{5}{7})$.

$$U_s = \ln a_s + 3 \ln b_s$$

$$0 = dU_s = \underbrace{\frac{1}{a_s} da_s}_{\text{so along an}} + \frac{3}{b_s} db_s$$

indifference curve

$$-\frac{1}{a_s} da_s = \frac{3}{b_s} db_s$$

$-\frac{b_s}{3a_s} = \frac{db_s}{da_s}$ is the marginal rate of substitution in general.

In particular, at $(a_s, b_s) = (2, 1\frac{12}{7})$, Smith's marginal rate of

$$\text{substitution is } MRS_s = -\frac{b_s}{3a_s} = -\frac{12/7}{3 \cdot 2} = -\frac{6/7}{3} = -\frac{2}{7}.$$

So that's the slope of the tangent line to Smith's indifference curve at

$(a_s, b_s) = (2, 1\frac{12}{7})$. This tangent line has the form

$$b_s = -\frac{2}{7} a_s + b \quad \text{the intercept (not "bananas")}$$

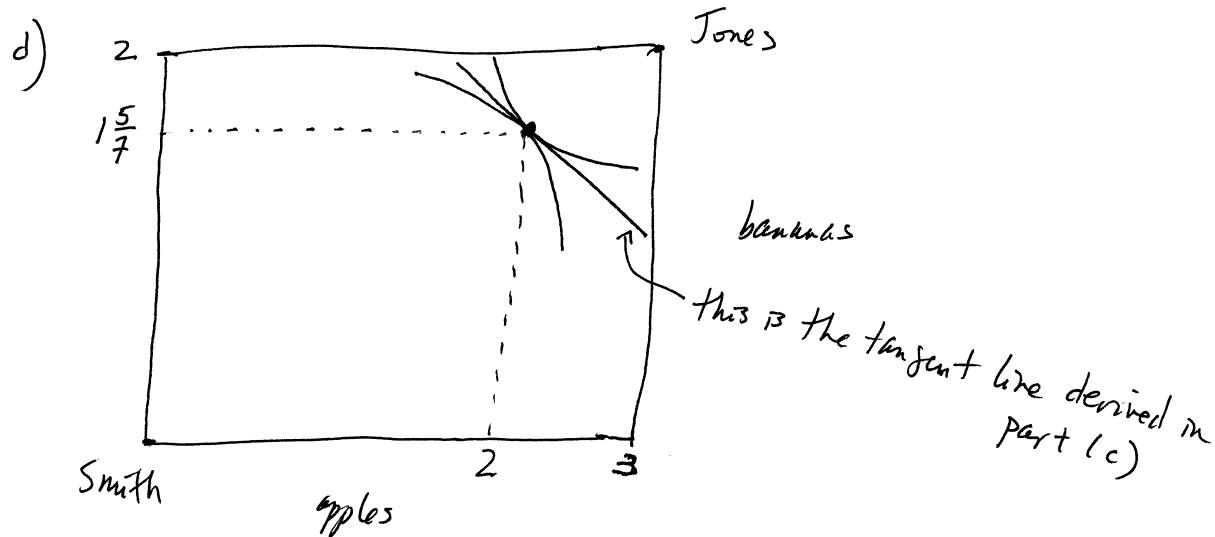
$y = m x + b$ and passes through $(2, 1\frac{12}{7})$ so

$$\frac{12}{7} = -\frac{2}{7} \cdot 2 + b$$

$$\frac{12}{7} + \frac{4}{7} = b \Rightarrow b = \frac{16}{7} \text{ and the tangent line is } b_s = -\frac{2}{7} a_s + \frac{16}{7}.$$

ask before.

$$\begin{aligned} b_s &= -\frac{2}{7} a_s + b \\ &= -\frac{2}{7} a_s + \frac{12}{7} + \frac{4}{7} \\ &= -\frac{2}{7} a_s + \frac{16}{7} \end{aligned}$$



The Second Theorem of Welfare Economics says that any Pareto Optimal point, such as $(2, 1\frac{5}{7})$ here in this economy, can be achieved by a competitive market after possibly forcibly redistributing initial endowments. As long as the endowments (after possible forcible redistribution) end up somewhere on the tangent line derived in part (c), a competitive market would lead from that endowment to the Pareto Optimal point $(2, 1\frac{5}{7})$.

Part (a) alternative →

Alternative Method for Part (a)

Here I show how to max u_s s.t. u_J fixed. One could also max u_J s.t. u_s fixed.

$$\max u_s \text{ s.t. } u_J = \bar{u}_J$$

$\max \ln a_s + 3 \ln b_s + \lambda (\ln a_J + \ln b_J - \bar{u}_J)$. From feasibility:

$$\max_{a_s, b_s} \ln a_s + 3 \ln b_s + \lambda [\ln(3-a_s) + \ln(2-b_s) - \bar{u}_J]$$

F.O.C.:

$$0 = \frac{\partial \mathcal{L}}{\partial a_s} = \frac{1}{a_s} - \frac{\lambda}{3-a_s} \Rightarrow \frac{\lambda}{3-a_s} = \frac{1}{a_s} \Rightarrow \lambda = \frac{3-a_s}{a_s}$$

$$0 = \frac{\partial \mathcal{L}}{\partial b_s} = \frac{3}{b_s} - \frac{\lambda}{2-b_s} \Rightarrow \frac{1}{2-b_s} = \frac{3}{b_s} \Rightarrow \lambda = \frac{3(2-b_s)}{b_s} = \frac{6-3b_s}{b_s}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = \ln(3-a_s) + \ln(2-b_s) - \bar{u}_J \quad \Downarrow \quad \frac{3-a_s}{a_s} = \lambda = \frac{6-3b_s}{b_s}$$

$$\frac{3}{a_s} - 1 = \frac{6}{b_s} - 3$$

$$\frac{3}{a_s} + 2 = \frac{6}{b_s}$$

$$b_s = \frac{6}{\frac{3}{a_s} + 2} = \frac{6a_s}{3+2a_s}$$

So the contract curve is $(a_s, \frac{6a_s}{3+2a_s})$, $0 \leq a_s \leq 3$.

Proof that this is the same as the previous answer, $(3\alpha, \frac{6\alpha}{1+2\alpha})$ for $0 \leq \alpha \leq 1$. "Set

$$a_s = 3\alpha. \text{ Then } b_s = \frac{6a_s}{3+2a_s} = \frac{6 \cdot 3\alpha}{3+2 \cdot 3\alpha} = \underline{\underline{\frac{6\alpha}{1+2\alpha}}} \quad \blacksquare$$

new: 2019 Final Exam, Qu. 1.
Resembles 2018 Final Exam Question 4

1. [17 points]

Suppose a pure exchange economy consists of two consumers, Smith “S” and Jones “J,” and two commodities, apples “ a ” and bananas “ b .” Suppose the total number of apples in the economy is 10 and the total number of bananas in the economy is 10. Suppose the utility functions for Smith and for Jones are

$$u_S(a_S, b_S) = 2 \ln a_S + \ln b_S \quad \text{and}$$
$$u_J(a_J, b_J) = \ln a_J + \ln b_J ,$$

respectively.

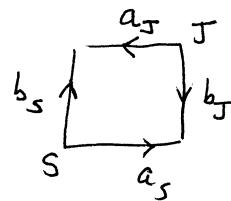
Consider an Edgeworth Box with Smith’s origin being in the lower left-hand corner of the box and with apples graphed on the horizontal axis.

- (a) Find an expression for the contract curve of this economy. You may leave it in implicit form. (I did not derive it by maximizing the utility of one person holding the other person’s utility fixed; I did it a different way.)
- (b) Show that the point on the contract curve corresponding to Smith getting four apples is $(a_S, b_S) = (4, 5/2)$.
- (c) Find the slope of the contract curve, and using this result, make a rough sketch of the contract curve in the Edgeworth Box.
- (d) Taking the price of bananas to be the numéraire, what would the price of apples have to be in order for the point $(4, 5/2)$ on the contract curve to be a competitive general equilibrium?
- (e) Tell me everything you know about the indifference curves of Smith and of Jones which pass through the point $(4, 5/2)$ on the contract curve. Make a rough sketch of these two indifference curves in your Edgeworth Box diagram, including information gained from part (d).

Answer to Econ. 7005 Final, Fall 2019, Question 1

a) $u_S = 2 \ln a_S + \ln b_S$

$u_J = \ln a_J + \ln b_J$



$$a_S + a_J = 10$$

$$b_S + b_J = 10$$

A social planner with a utilitarian social welfare function would

$\max \alpha u_S + (1-\alpha) u_J$ with $0 \leq \alpha \leq 1$ being the social weight on Smith.

Using the feasibility conditions, this is equivalent to

$$\max \alpha (2 \ln a_S + \ln b_S) + (1-\alpha) (\ln a_J + \ln b_J)$$

$$= \max \alpha (2 \ln a_S + \ln b_S) + (1-\alpha) [\ln (10-a_S) + \ln (10-b_S)] \Rightarrow$$

$$0 = \frac{\partial \text{social welfare}}{\partial a_S} = \frac{2\alpha}{a_S} - \frac{1-\alpha}{10-a_S} \quad (1)$$

$$0 = \frac{\partial \text{social welfare}}{\partial b_S} = \frac{\alpha}{b_S} - \frac{1-\alpha}{10-b_S} \quad (2)$$

Method 1: Solve (1) and (2) for a_S and b_S , respectively.

$$(1) \Rightarrow \frac{2\alpha}{a_S} = \frac{1-\alpha}{10-a_S} \quad (2) \Rightarrow \frac{\alpha}{b_S} = \frac{1-\alpha}{10-b_S}$$

$$20\alpha - 2\alpha a_S = a_S - \alpha a_S$$

$$10\alpha - \alpha b_S = b_S - \alpha b_S$$

$$20\alpha = a_S + \alpha a_S$$

$$10\alpha = b_S$$

$$= (1+\alpha) a_S$$

$$\frac{20\alpha}{1+\alpha} = a_S$$

Then the contract curve (a_S, b_S) is

$$\left(\frac{20\alpha}{1+\alpha}, 10\alpha \right), \text{ parameterized on } 0 \leq \alpha \leq 1.$$

Method 2. Solve (1) for α : $\frac{2\alpha}{a_s} = \frac{1-\alpha}{10-a_s}$

$$20\alpha - 2\alpha a_s = a_s - \alpha a_s$$

$$20\alpha - a_s = \alpha a_s$$

$$20\alpha - \alpha a_s = a_s$$

$$(20-a_s)\alpha = a_s$$

$$\alpha = \frac{a_s}{20-a_s}.$$

Substitute this into (2) :

$$0 = \frac{\left(\frac{a_s}{20-a_s}\right)}{b_s} - \frac{1 - \left(\frac{a_s}{20-a_s}\right)}{10-b_s}. \text{ Solve for } b_s :$$

$$\frac{1 - \left(\frac{a_s}{20-a_s}\right)}{10-b_s} = \frac{\left(\frac{a_s}{20-a_s}\right)}{b_s}$$

$$b_s - \frac{a_s b_s}{20-a_s} = (10-b_s) \frac{a_s}{20-a_s}$$

$$= \frac{10a_s}{20-a_s} - \frac{a_s b_s}{20-a_s}$$

$$b_s = \frac{10a_s}{20-a_s}, \text{ the contract curve.}$$

b) Part (a)'s Method 1: $a_s = 4$ and $a_s = \frac{20\alpha}{1+\alpha} \Rightarrow 4 = \frac{20\alpha}{1+\alpha}$

$$4 + 4\alpha = 20\alpha$$

$$4 = 16\alpha \Rightarrow \alpha = \frac{4}{16} = \frac{1}{4}$$

$$\text{and hence } b_s = 10\alpha = \frac{10}{4} = \frac{5}{2}.$$

Part (a)'s Method 2:

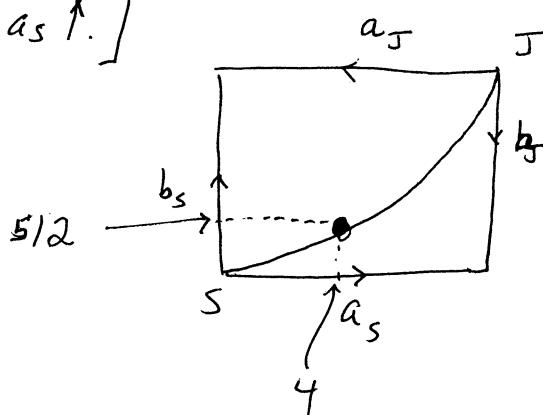
$$b_s = \frac{10a_s}{20-a_s} = \frac{10 \cdot 4}{20-4} = \frac{40}{16} = \frac{5}{2}.$$

c) Part (a)'s Method 1.

$$\begin{aligned} \frac{db_s}{da_s} &= \frac{\partial b_s / \partial \alpha}{\partial a_s / \partial \alpha} = \frac{\frac{\partial}{\partial \alpha}(10\alpha)}{\frac{\partial}{\partial \alpha}\left(\frac{20\alpha}{1+\alpha}\right)} = \frac{10}{\frac{20}{1+\alpha} - \frac{20\alpha}{(1+\alpha)^2}} \\ &= \frac{10}{\frac{20+20\alpha-20\alpha}{(1+\alpha)^2}} = \frac{10}{\frac{20}{(1+\alpha)^2}} = \frac{10(1+\alpha)^2}{20} = \frac{(1+\alpha)^2}{2}. \end{aligned}$$

Qualitatively this resembles α^2 , whose graph is $\sqrt{\alpha}$ for positive α ,

so the contract curve will look something like that because as α increases, the social weight on Smith increases, so a_s (the Edgeworth Box's horizontal axis) should increase. [Optional proof: $a_s = \frac{20\alpha}{1+\alpha} = \frac{20}{\frac{1}{\alpha} + 1}$ so as $\alpha \uparrow$, $\frac{1}{\alpha} \downarrow$, $\frac{1}{\alpha} + 1 \downarrow$, so $a_s \uparrow$.]



Part (a)'s Method 2.

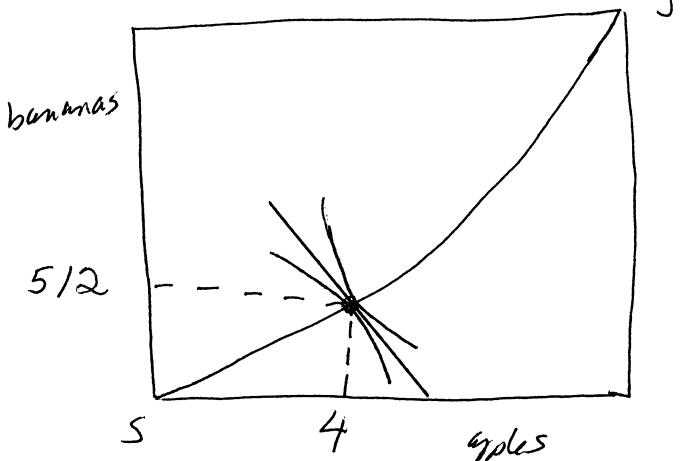
$$\begin{aligned} \frac{db_s}{da_s} &= \frac{d}{da_s} \frac{10a_s}{20-a_s} = \frac{d}{da_s} \frac{10}{\frac{20}{a_s} - 1} = -\frac{10}{\left(\frac{20}{a_s} - 1\right)^2} \cdot \frac{-20}{a_s^2} \\ &= \frac{200}{(20-a_s)^2}. \quad \text{Since } 0 \leq a_s \leq 10, \text{ this is positive.} \end{aligned}$$

Optionally, one could check its convexity:

$$\frac{d^2 b_s}{da_s^2} = \frac{-2(200)}{(20-a_s)^3} (-1) = \frac{400}{(20-a_s)^3}, \text{ which is positive for } 0 \leq a_s \leq 10.$$

$0 \leq a_s \leq 10$. This results in the same sketch as Method 1 does.

d)



If $(4, 5/2)$ is a competitive equilibrium, the slope of Smith's budget constraint in equilibrium would have to equal the slope of his indifference curve at $(4, 5/2)$.

Smith's budget constraint is $P_a a + P_b b = m_s \Leftrightarrow$

$$P_b b = -P_a a + m_s$$

$$b = \frac{-P_a}{P_b} a + \frac{m_s}{P_b}$$

So the slope of Smith's budget constraint is $-P_a/P_b$, or $-P_a$ using $P_b = 1$, bananas as the numeraire.

On an indifference curve of Smith, $0 = du_s = \frac{\partial u_s}{\partial a_s} da_s + \frac{\partial u_s}{\partial b_s} db_s$

$$= \frac{2}{a_s} da_s + \frac{1}{b_s} db_s \Rightarrow \frac{1}{b_s} db_s = \frac{2}{a_s} da_s \text{ so the slope of Smith's}$$

indifference curve is $\frac{db_s}{da_s} = -\frac{2}{a_s}$. At $(4, 5/2)$ this is

$$-\frac{2}{4} = -\frac{10}{8} = -\frac{5}{4}. \text{ Equating this to the slope of the budget constraint, } -\frac{5}{4} = -P_a \text{ so}$$

$$\boxed{P_a = 5/4.}$$

It is also possible to solve part (d) by finding the generic competitive equilibrium prices and quantities in this economy, but that method takes longer. I do it on the next page.

- e) The two indifference curves drawn in part (d) must be tangent at $(4, 5/2)$ and their slope should be $-5/4$. Part (d) already showed this was true for Smith. For Jones:

$$0 = du_J = \frac{\partial u_J}{\partial a_J} da_J + \frac{\partial u_J}{\partial b_J} db_J \Rightarrow$$

$$-\frac{\partial u_J}{\partial b_J} db_J = \frac{\partial u_J}{\partial a_J} da_J$$

$$\frac{db_J}{da_J} = -\frac{\partial u_J / \partial a_J}{\partial u_J / \partial b_J} = -\frac{\frac{1}{a_J}}{\frac{1}{b_J}} = -\frac{b_J}{a_J} = -\frac{10-b_s}{10-a_s}$$

$$\text{so at } (a_s, b_s) = (4, 5/2), \quad \frac{db_J}{da_J} = -\frac{10-5/2}{10-4} = -\frac{15/2}{6} = -\frac{15}{12} = -\frac{5}{4},$$

as we wanted to show, although to confirm this in Smith's coordinate system,

$$\frac{-5}{4} = \frac{db_J}{da_J} = \frac{d(10-b_s)}{d(10-a_s)} = \frac{-db_s}{-da_s} = \frac{db_s}{da_s}.$$

Alternative method for part (d)

Smith $\omega_s = (a_{so}, b_{so})$

$$\max u_s \text{ s.t. } p_a a_s + p_b b_s = p_a a_{so} + p_b b_{so}$$

$$\mathcal{L} = 2 \ln a_s + \ln b_s + \lambda (p_a a_{so} + p_b b_{so} - p_a a_s - p_b b_s)$$

$$\begin{aligned} 0 = \mathcal{L}'_{a_s} &= \frac{2}{a_s} - \lambda p_a \\ 0 = \mathcal{L}'_{b_s} &= \frac{1}{b_s} - \lambda p_b \end{aligned} \quad \left\{ \begin{array}{l} \lambda = \frac{2}{p_a a_s} = \frac{1}{p_b b_s} \Rightarrow b_s = \frac{p_a a_s}{2 p_b} \\ \end{array} \right.$$

$$0 = \mathcal{L}'_\lambda = p_a a_{so} + p_b b_{so} - p_a a_s - p_b b_s$$

$$p_a a_{so} + p_b b_{so} = p_a a_s + p_b \left(\frac{p_a a_s}{2 p_b} \right)$$

$$\begin{aligned} &= p_a a_s + \frac{1}{2} p_a a_s \\ &= \frac{3}{2} p_a a_s \end{aligned}$$

$$\Rightarrow a_s = \frac{2}{3 p_a} (p_a a_{so} + p_b b_{so}),$$

$$b_s = \frac{p_a}{2 p_b} \cdot \frac{2}{3 p_a} (p_a a_{so} + p_b b_{so}) = \frac{1}{3 p_b} (p_a a_{so} + p_b b_{so}). \xrightarrow{\text{continues}}$$

Jones. $\omega_j \Rightarrow (a_{jo}, b_{jo})$. Skipping a few steps,

$$\mathcal{L} = \ln a_j + \ln b_j + \lambda (p_a a_{jo} + p_b b_{jo} - p_a a_j - p_b b_j)$$

$$\dots \Rightarrow \dots \Rightarrow b_j = \frac{p_a a_j}{p_b} \Rightarrow p_a a_{jo} + p_b b_{jo} = p_a a_j + p_b \frac{p_a a_j}{p_b} = 2 p_a a_j \Rightarrow$$

unneeded
(although correct) \rightarrow

$$a_j = \frac{1}{2 p_a} (p_a a_{jo} + p_b b_{jo}) \text{ and}$$

$$b_j = \frac{1}{2 p_b} (p_a a_{jo} + p_b b_{jo}).$$

We want $(4, \frac{5}{2})$ to be the competitive equilibrium (a_s, b_s) .

Hence $\frac{5}{2} = \frac{1}{3P_b} (P_a a_{s0} + P_b b_{s0})$, and taking $P_b = 1$ (bananas are the numeraire), this implies that Smith's income is

$$P_a a_{s0} + P_b b_{s0} = \frac{5}{2} \cdot 3 = \frac{15}{2}.$$

Then from Smith's demand curve for apples,

$$a_s = \frac{2}{3P_a} \underbrace{(P_a a_{s0} + P_b b_{s0})}_{\frac{15}{2}}$$

$$\Rightarrow P_a = \frac{2}{3} \cdot \frac{15}{2} \cdot \frac{1}{4} = \frac{15}{12} = \frac{5}{4}, \text{ as with the other method.}$$

2. [18 points] Suppose an economy consists of two price-taking persons, “ a ” and “ b ”. Person a has available 1 unit of “time” which he divides between rest R_a and labor l_a . Person b has available 1 unit of “time” which he divides between rest R_b and labor l_b .

Good “ x ” is produced by one competitive firm according to the production function

$$x = \text{labor hired}.$$

This firm is completely owned by Person a .

Let the amount of good x consumed by Person a be x_a and the amount of good x consumed by Person b be x_b . Let $X = x_a + x_b$. (Although the symbol for the good, “ x ,” and for the total amount of it produced, “ X ,” are easily confused, only the latter plays an important role in the equations needed to work out this problem.)

Suppose production of x causes air pollution which decreases the utility of Person b but not of Person a (perhaps because Person a lives far away from the source of the air pollution). The utility functions of the two individuals are

$$\begin{aligned} u_a &= x_a R_a \\ u_b &= x_b R_b - X. \end{aligned}$$

Suppose that Person b always sees “ X ” as exogenous; he never thinks of X as being $x_a + x_b$ (even though it is), and therefore he never considers the effect of his own consumption of x on the amount of air pollution.

To combat pollution, the government may intervene in this economy by imposing a tax of TX on the firm, where $T \geq 0$. (So the marginal tax rate on the firm’s output is T and the total tax revenue is TX .) If it does so, it gives half of the money it receives to each consumer. Neither consumer ever knows where this money comes from; each consumer thus considers it a “lump sum” gift. Call the amount of money each consumer receives from the government “ t .” Clearly $t \geq 0$.

Take the price of labor as the numéraire.

- (a) Does the production function have increasing, decreasing, or constant returns to scale? Why?
- (b) Show that, in equilibrium, $2t = TX$. (This is very easy.)

- (c) Find the demand for x , supply of l , and demand for R of Person a , assuming nothing about the values of T and t except that they are not negative.
- (d) Find the demand for x , supply of l , and demand for R of Person b , assuming nothing about the values of T and t except that they are not negative.
- (e) Using your results so far, show that

$$T = \frac{2t}{1-t}.$$

(Hint: use the production function.)

- (f) Find the general equilibrium price of x and find the amount of firm profit.
- (g) Find the equilibrium demand for x of Person a , and find the equilibrium demand for x of Person b . It is fine if they are functions of t , but they should not be functions of T (you can use the result of part (e) to eliminate T).
- (h) Show that in general equilibrium,

$$u_a = \frac{1-t^2}{4} \quad \text{and}$$

$$u_b = \frac{-3+4t-t^2}{4}.$$

(You can think of these as being the indirect utility functions.)

- (i) Using the results of part (h), suppose the government is controlled by a social planner who wishes to maximize

$$W = \alpha u_a + (1-\alpha) u_b$$

for $\alpha \in [0, 1]$. Find the social planner's optimal value of t . Why does it make sense?

- (j) In this problem, is the competitive equilibrium with no government intervention Pareto Optimal?

2014 Qualifying Exam, Section 1

(1)

$$R_a + l_a = 1$$

$$R_b + l_b = 1$$

Production function $X = l_a + l_b$.

This is constant returns to scale :

$$f(\underline{l}) = l_a + l_b$$

$$f(2\underline{l}) = 2l_a + 2l_b = 2f(l).$$

Therefore the firm will make equilibrium profits of zero.

$$X = X_a + X_b$$

$$u_a = X_a R_a$$

$$u_b = X_b R_b - X$$

tax on firm TX

$\frac{1}{2}TX = t$, the tax going to each consumer is half
of the total tax

Price of labor = 1.

Let price of X be " p ".

a) Answered above.

b) Follows from $\frac{1}{2}TX = t$, above.

$$c) u_a = X_a R_a = X_a (1-l_a).$$

income = profit income + labor income + government gift

$$= 0 + (1)l_a + t$$

due to constant
returns to scale

$$\text{expenditures} = p X_a$$

Budget constraint $\ell_a + t = px_a$

$$\mathcal{L} = x_a(1-\ell_a) + \lambda(\ell_a + t - px_a)$$

$$0 = \mathcal{L}'_\lambda = \ell_a + t - px_a \quad (1)$$

$$0 = \mathcal{L}'_{x_a} = 1 - \ell_a - \lambda p = 1 - \ell_a - x_a p \Rightarrow \ell_a = 1 - px_a; \text{ substituting}$$

$$0 = \mathcal{L}'_{\ell_a} = -x_a + \lambda \Rightarrow \lambda = x_a$$

$$0 = 1 - px_a + t - px_a$$

$$= 1 + t - 2px_a$$

$$2px_a = 1 + t$$

$$x_a = \frac{1+t}{2p}, x_a \text{ demanded.}$$

$$\Rightarrow \ell_a = 1 - px_a = 1 - \frac{1+t}{2} = \frac{2}{2} - \frac{1+t}{2} = \frac{1-t}{2}$$

$$R_a = 1 - \ell_a = px_a = \frac{1+t}{2}$$

d) $u_b = x_b R_b - X = x_b(1-\ell_b) - X$

income = $(1)\ell_b + t$ expenditures px_b

Budget constraint $\ell_b + t = px_b$

$$\mathcal{L} = x_b(1-\ell_b) - X + \lambda(\ell_b + t - px_b)$$

$$0 = \mathcal{L}'_\lambda = \ell_b + t - px_b$$

$$0 = \mathcal{L}'_{x_b} = 1 - \ell_b - \lambda p$$

$$0 = \mathcal{L}'_{\ell_b} = -x_b + \lambda$$

which are the same first-order conditions
as for Person a. Hence by symmetry:
(except switching "a" subscripts for "b")

$$x_b = \frac{1+t}{2p}$$

$$l_b = \frac{1-t}{2}$$

$$R_b = \frac{1+t}{2} .$$

e) $2t = TX = T(l_a + l_b)$
 $= T \left(\frac{1-t}{2} + \frac{1-t}{2} \right) = T(1-t) \Rightarrow$
 $\frac{2t}{1-t} = T.$

f) profit $\pi = p(l_a + l_b) - (1)\ell - TX$
 $= p(l_a + l_b) - (l_a + l_b) - T(l_a + l_b)$
 $= (p-1-T)(l_a + l_b)$

and since $\pi = 0$ due to constant returns to scale,

$$0 = p - 1 - T \Rightarrow \boxed{p = 1 + T.}$$

g) $x_a^D = \frac{1+t}{2p} = \frac{1+t}{2(1+T)} = \frac{1+t}{2} \cdot \frac{1}{1+T} = \frac{1+t}{2} \cdot \frac{1}{1 + \frac{2t}{1-t}}$
 $= \frac{1+t}{2} \cdot \frac{1}{\frac{1-t}{1-t} + \frac{2t}{1-t}} = \frac{1+t}{2} \cdot \frac{1}{\frac{1+t}{1-t}} = \frac{1-t}{2}$

x_b^D is the same as x_a^D .

$$h) u_a = x_a (1 - l_a)$$

$$= \frac{1-t}{2} \left(1 - \frac{1-t}{2}\right) = \frac{1-t}{2} \left(\frac{2}{2} - \frac{1-t}{2}\right) = \frac{1-t}{2} \cdot \frac{1+t}{2} = \frac{1-t^2}{4}$$

$$u_b = x_b (1 - l_b) - X$$

$$= \frac{1-t}{2} \left(1 - \frac{1-t}{2}\right) - X ; \text{ by symmetry with } u_a,$$

$$= \frac{1-t^2}{4} - X = \frac{1-t^2}{4} - (l_a + l_b) = \frac{1-t^2}{4} - \left(\frac{1-t}{2} + \frac{1-t}{2}\right)$$

$$= \frac{1-t^2}{4} - (1-t) = \frac{1-t^2 - 4 + 4t}{4} = \frac{-t^2 + 4t - 3}{4} .$$

$$i) W = \alpha \frac{1-t^2}{4} + (1-\alpha) \frac{-t^2 + 4t - 3}{4} .$$

\uparrow
 Social
 welfare

Maximize W over t :

$$0 = \frac{\partial W}{\partial t} = \alpha \frac{-2t}{4} + (1-\alpha) \frac{-2t+4}{4}$$

$$0 = \alpha (-2t) + (1-\alpha) (-2t+4)$$

$$= -\underline{2t\alpha} + (-2t+4) + \underline{2t\alpha} - 4\alpha$$

$$2t = 4 - 4\alpha$$

$$t = 2 - 2\alpha$$

$\boxed{t = 2(1-\alpha)}$. If $\alpha = 1$, the social planner does not care about

the pollution victim (Person b), and so sets $t = 0$, thus $T = 0 : u_0$ pollution tax. For all other values of α (that is, $\alpha \in [0, 1]$),

the social planner does care about Person b to some extent, and sets $t \neq 0$, hence $T \neq 0$, imposing some pollution tax.

j) The competitive equilibrium with no government intervention is $t = 0$. From part (i), this is only Pareto Optimal when $\alpha = 1$.

optional
When $\alpha \neq 1$, the First Theorem of Welfare Economics fails to apply because of the "external" effect of pollution.

2008 Qualifier

Section 2.

Answer one of the following two questions.

1. [12 points]

Baby Alice has two grandfathers, Grandpa Smith and Grandpa Jones. Suppose there is one good, called “ x ”, and suppose the consumption of that good by Baby Alice, Grandpa Smith, and Grandpa Jones is x_A , x_S , and x_J , respectively.

Grandpa Smith has an endowment of 3 units of good x , of which he consumes x_S and gives x_{SA} to Baby Alice:

$$x_S + x_{SA} = 3.$$

Grandpa Jones has an endowment of 3 units of good x , of which he consumes x_J and gives x_{JA} to Baby Alice:

$$x_J + x_{JA} = 3.$$

Baby Alice has no endowment of good x ; all of her consumption consists of gifts from her grandfathers:

$$x_A = x_{SA} + x_{JA}.$$

Baby Alice's utility function is

$$u_A(x_A) = \ln x_A.$$

Grandpa Smith's utility function is

$$u_S(x_S, u_A) = \ln x_S + u_A.$$

Grandpa Jones's utility function is

$$u_J(x_J, u_A) = \ln x_J + u_A.$$

(So the grandfathers care about their own consumption and the utility of Baby Alice.)

- (a) Find the values of x_{SA} , x_{JA} , x_A , x_S , x_J , u_S , u_J , and u_A . Assume that each grandfather takes the other grandfather's actions as fixed. (So this is technically an equilibrium in the sense of Nash.)

- (b) Denote the values found in part (a) as x_{SA}^* , x_{JA}^* , x_A^* , x_S^* , x_J^* , u_S^* , u_J^* , and u_A^* . Now suppose that instead of giving x_{SA}^* and x_{JA}^* , each grandfather increases his gift by $\epsilon > 0$. Show that for sufficiently small ϵ , this change increases *everyone's* utility compared with the situation of part (a).
- (c) Is the Nash Equilibrium efficient?
- (d) Briefly speculate about the welfare implications of this example if, instead of Baby Alice having two grandfathers, she had four great-grandfathers.

Section 2 #1 2008 Qualifier

a) Grandpa Smith: $\max(\ln x_S + \ln x_A) = \max(\ln x_S + \ln x_J)$

Grandpa Jones: $\max(\ln x_J + \ln x_A)$

$$x_S + x_{SA} = 3$$

$$x_J + x_{JA} = 3$$

$$x_A = x_{SA} + x_{JA}$$

So Grandpa Smith's problem is to $\max \ln x_S + \ln(x_{SA} + x_{JA})$

$$\max_{x_{SA}} \ln(3 - x_{SA}) + \ln(x_{SA} + x_{JA})$$

$$0 = \frac{-1}{3 - x_{SA}} + \frac{1}{x_{SA} + x_{JA}}$$

$$\frac{1}{3 - x_{SA}} = \frac{1}{x_{SA} + x_{JA}} \Rightarrow x_{SA} + x_{JA} = 3 - x_{SA} \Rightarrow$$

$$2x_{SA} = 3 - x_{JA}$$

$$x_{SA} = \frac{3}{2} - \frac{1}{2}x_{JA}$$

The solution to Grandpa Jones's problem can be found by maximizing $\ln(3 - x_{JA}) + \ln(x_{SA} + x_{JA})$ over x_{JA} , or it can be deduced by symmetry with Grandpa Smith's.

The answer is $x_{JA} = \frac{3}{2} - \frac{1}{2}x_{SA}$.

Substituting, $x_{JA} = \frac{3}{2} - \frac{1}{2}(\frac{3}{2} - \frac{1}{2}x_{SA})$
 $= \frac{3}{2} - \frac{3}{4} + \frac{1}{4}x_{SA}$

$$\frac{3}{4}x_{SA} = \frac{3}{4}$$

$$x_{SA} = 1 \Rightarrow x_{SA} = \frac{3}{2} - \frac{1}{2}(1) = 1.$$

$$\Downarrow \\ x_J = 3 - 1 = 2$$

$$x_S = 3 - x_{SA} = 2$$

$$x_A = x_{SA} + x_{JA} = 1+1=2$$

$$u_A = \ln x_A = \ln 2$$

$$u_S = \ln x_S + \ln x_A = \ln 2 + \ln 2 = 2 \ln 2$$

$$u_J = \ln x_J + \ln x_A = \ln 2 + \ln 2 = 2 \ln 2$$

b) Now $x_{JA} = 1+\epsilon$

$$x_{SA} = 1+\epsilon$$

$$x_J = 2-\epsilon$$

$$x_S = 2-\epsilon$$

$$x_A = 2+2\epsilon$$

$$u_A = \ln(2+2\epsilon)$$

$$u_S = \ln(2-\epsilon) + \ln(2+2\epsilon)$$

$$u_J = \ln(2-\epsilon) + \ln(2+2\epsilon)$$

(just like) \downarrow
Qualifying Exam
2004

Answer 3 cont...

As $\epsilon \uparrow$, u_A obviously \uparrow .

$$\frac{\partial u_S}{\partial \epsilon} = \frac{\partial u_J}{\partial \epsilon} = \frac{-1}{2-\epsilon} + \frac{2}{2+2\epsilon} = \frac{-2+2\epsilon+4-2\epsilon}{(2-\epsilon)(2+2\epsilon)}$$

$$= \frac{2-4\epsilon}{(2-\epsilon)2(1+\epsilon)} = \frac{1-2\epsilon}{(2-\epsilon)(1+\epsilon)} \underset{\text{for small } \epsilon}{\approx} \frac{1}{2 \cdot 1} = \frac{1}{2} > 0$$

so as $\epsilon \uparrow$, $u_S \uparrow$ and $u_J \uparrow$ (as long as ϵ stays small).

c) No, the Nash Equilibrium is not efficient, because from part (b), it is possible to move away from the Nash Equilibrium and make all agents better off.

d) The inefficiency in this problem comes from the positive externality of one grandfather's gifts (to Baby Alice) on the other grandfather.

The other grandfather benefits from such gifts (since they raise u_A , which he cares about), but he pays nothing for the benefit, so it's a positive externality, which the free market supplies inefficiently little of.

With four great-grandfathers instead of two grandfathers, the free-market inefficiency will be worse, I conjecture, because more agents are affected by the market failure.

The upshot is that our bequests to future generations, especially generations far in the future, will be inefficiently small, basically due to the fact that humans reproduce sexually and so family lines mix. (I think this argument was first made by the ecological economist Herman Daly.)

2017 Final Exam Qu. 2

2. [17 points]

Baby Alice has two grandfathers, Grandpa Smith and Grandpa Jones. Suppose there is one good, called “ x ”, and suppose the consumption of that good by Baby Alice, Grandpa Smith, and Grandpa Jones is x_A , x_S , and x_J , respectively.

Grandpa Smith has an endowment of 3 units of good x , of which he consumes x_S and gives x_{SA} to Baby Alice:

$$x_S + x_{SA} = 3.$$

Grandpa Jones has an endowment of 3 units of good x , of which he consumes x_J and gives x_{JA} to Baby Alice:

$$x_J + x_{JA} = 3.$$

Baby Alice has no endowment of good x ; all of her consumption consists of gifts from her grandfathers:

$$x_A = x_{SA} + x_{JA}.$$

Baby Alice’s utility function is

$$u_A(x_A) = \ln x_A.$$

Grandpa Smith’s utility function is

$$u_S(x_S, u_A) = \ln x_S + u_A.$$

Grandpa Jones’s utility function is

$$u_J(x_J, u_A) = \ln x_J + u_A.$$

(So the grandfathers care about their own consumption and the utility of Baby Alice.)

- (a) Find the values of x_{SA} , x_{JA} , x_A , x_S , x_J , u_S , u_J , and u_A . Assume that each grandfather takes the other grandfather’s actions as fixed. (So, as you’ll study next semester, technically this is an equilibrium in the sense of Nash.) Denote the values you find as x_{SA}^* , x_{JA}^* , x_A^* , x_S^* , x_J^* , u_S^* , u_J^* , and u_A^* .

- (b) Suppose a social planner maximizes a social welfare function which weights the utility of Grandpa Smith and Grandpa Jones equally but puts no weight on Baby Alice's utility. Denote the social planner's optimal transfers, allocations, and the resulting utility levels as \hat{x}_{SA} , \hat{x}_{JA} , \hat{x}_A , \hat{x}_J , \hat{u}_S , \hat{u}_J , and \hat{u}_A . Find these values.
- (c) Using the answers to parts (a) and (b), determine whether the allocation in part (a) is efficient. If it is not efficient, are the transfers from the grandfathers to Baby Alice in part (a) too large or too small? If you do not have a calculator, you may leave logarithms unevaluated and explain what procedure you would follow if you had a calculator. Note that $2 \ln 2 \approx 1.39$, $\ln 2 \approx 0.69$, $\ln(9/2) \approx 1.51$, and $\ln 3 \approx 1.10$.

(2)

(a) Identical to part (a) of 2008 Qualifying Exam's Section 2 Question 1. (See previous question.)

(b) The utilitarian social welfare function in general is

$$W = \alpha u_S + \beta u_J + \gamma u_A$$

with weights α , β , and γ . In this problem you are told to set $\gamma = 0$ and $\alpha = \beta$.

Without loss of generality we can set $\alpha = \beta = 1$. So the social planner wants to maximize

↑ More typically, $\alpha = \beta = 1/2$, but either one is OK.

$$\begin{aligned} W &= u_S + u_J = (\ln x_S + u_A) + (\ln x_J + u_A) \\ &= \ln x_S + \ln x_J + 2 \ln x_A \\ &= \ln(3 - x_{SA}) + \ln(3 - x_{JA}) + 2 \ln(x_{SA} + x_{JA}) \end{aligned}$$

where we use the question's equations $x_S + x_{SA} = 3$, $x_J + x_{JA} = 3$, and $x_A = x_{SA} + x_{JA}$.

We maximize W with respect to x_{SA} and x_{JA} :

$$0 = \frac{\partial W}{\partial x_{SA}} = \frac{-1}{3 - x_{SA}} + \frac{2}{x_{SA} + x_{JA}}$$

$$0 = \frac{\partial W}{\partial x_{JA}} = \frac{-1}{3 - x_{JA}} + \frac{2}{x_{SA} + x_{JA}}$$

Clearly $\frac{2}{x_{SA} + x_{JA}}$ is equal both to $\frac{1}{3 - x_{SA}}$ and to $\frac{1}{3 - x_{JA}}$, so $\frac{1}{3 - x_{SA}} = \frac{1}{3 - x_{JA}}$

and therefore $x_{SA} = x_{JA}$. The first F.O.C. then becomes

$$0 = \frac{-1}{3 - x_{SA}} + \frac{2}{2x_{SA}} = \frac{-1}{3 - x_{SA}} + \frac{1}{x_{SA}}$$

$$\frac{1}{3 - x_{SA}} = \frac{1}{x_{SA}} \Rightarrow x_{SA} = 3 - x_{SA} \Rightarrow \boxed{x_{SA} = 3/2}, \boxed{x_{JA} = 3/2},$$

$$\hat{x}_S = 3 - x_{SA} = 3 - \frac{3}{2} = \frac{3}{2},$$

$$\hat{x}_J = 3 - x_{JA} = 3 - \frac{3}{2} = \frac{3}{2},$$

$$\hat{x}_A = x_{SA} + x_{JA} = \frac{3}{2} + \frac{3}{2} = 3, \quad \hat{u}_A = \ln \hat{x}_A = \ln 3,$$

$$\hat{u}_S = \ln \hat{x}_S + \hat{u}_A = \ln \frac{3}{2} + \ln 3 = \ln \left(\frac{3}{2} \cdot 3 \right) = \ln \frac{9}{2}$$

$$\hat{u}_J = \ln \hat{x}_J + \hat{u}_A = \ln \frac{3}{2} + \ln 3 = \ln \left(\frac{3}{2} \cdot 3 \right) = \ln \frac{9}{2}$$

② From part (a), $u_A^* = \ln 2 < \hat{u}_A$

$$u_S^* = 2 \ln 2 = \ln 4 < \hat{u}_S$$

$$u_J^* = 2 \ln 2 = \ln 4 < \hat{u}_J$$

Clearly an allocation in (a) is not efficient because everyone would be better off with the allocation in (b).

From part (a), $x_{JA}^* = 1$ whereas $\hat{x}_{JA} = \frac{3}{2}$

$$x_{SA}^* = 1 \quad \hat{x}_{SA} = \frac{3}{2}$$

So the transfers from the grandfathers to Baby Alice in part (a) are too small.

Optional

- In part (b), instead of solving for socially-optimal transfers, we could consider the social planner directly choosing the allocation (x_S, x_J, x_A) in order to maximize W , which we saw was equal to $\ln x_S + \ln x_J + 2 \ln x_A$. Maximizing this subject to $x_S + x_J + x_A = 6$ yields $\mathcal{L} = \ln x_S + \ln x_J + 2 \ln x_A + \lambda(6 - x_S - x_J - x_A)$, with F.O.C.'s:

$$0 = \mathcal{L}'_{x_S} = \frac{1}{x_S} - \lambda \quad \left\{ \Rightarrow x_S = x_J \right.$$

$$0 = \mathcal{L}'_{x_J} = \frac{1}{x_J} - \lambda \quad \left\{ \frac{1}{x_J} = \frac{2}{x_A} \Rightarrow x_A = 2x_J \right.$$

$$0 = \mathcal{L}'_{x_A} = \frac{2}{x_A} - \lambda \quad \downarrow$$

and $0 = 6 - x_S - x_J - x_A \rightarrow 6 = x_J + x_J + 2x_J = 4x_J, \hat{x}_J = \frac{6}{4} = \frac{3}{2} = \hat{x}_S, \hat{x}_A = 3$, as above.

Optional

- What would the answer to part (b) be with general, unspecified values of the social weights α , β , and γ ?

$$\begin{aligned}
 W &= \alpha u_S + \beta u_J + \gamma u_A = \alpha(\ln x_S + u_A) + \beta(\ln x_J + u_A) + \gamma u_A \\
 &= \alpha(\ln x_S + \ln x_A) + \beta(\ln x_J + \ln x_A) + \gamma \ln x_A \\
 &= \alpha \ln x_S + \beta \ln x_J + (\alpha + \beta + \gamma) \ln x_A \\
 &= \alpha \ln x_S + \beta \ln x_J + (\alpha + \beta + \gamma) \ln(x_{SA} + x_{JA}) \\
 &= \alpha \ln(3 - x_{SA}) + \beta \ln(3 - x_{JA}) + (\alpha + \beta + \gamma) \ln(x_{SA} + x_{JA})
 \end{aligned}$$

The first-order conditions are

$$0 = \frac{\partial W}{\partial x_{SA}} = \frac{-\alpha}{3 - x_{SA}} + \frac{\alpha + \beta + \gamma}{x_{SA} + x_{JA}}$$

$$0 = \frac{\partial W}{\partial x_{JA}} = \frac{-\beta}{3 - x_{JA}} + \frac{\alpha + \beta + \gamma}{x_{SA} + x_{JA}}$$

Clearly $\frac{\alpha + \beta + \gamma}{x_{SA} + x_{JA}}$ is equal both to $\frac{\alpha}{3 - x_{SA}}$ and to $\frac{\beta}{3 - x_{JA}}$, so $\frac{\alpha}{3 - x_{SA}} = \frac{\beta}{3 - x_{JA}}$,

$3 - x_{JA} = \frac{\beta}{\alpha}(3 - x_{SA})$, and $x_{JA} = 3 - \frac{\beta}{\alpha}(3 - x_{SA}) = 3 + \frac{\beta}{\alpha}(x_{SA} - 3)$.

From the first F.O.C.,

$$\frac{\alpha}{3 - x_{SA}} = \frac{\alpha + \beta + \gamma}{x_{SA} + 3 + \frac{\beta}{\alpha}(x_{SA} - 3)}$$

$$\alpha x_{SA} + 3\alpha + \beta(x_{SA} - 3) = 3(\alpha + \beta + \gamma) - (\alpha + \beta + \gamma)x_{SA}$$

$$\underbrace{\alpha x_{SA} + \beta x_{SA} + 3\alpha - 3\beta}_{\alpha x_{SA} + \beta x_{SA} + 3\alpha - 3\beta} = 3\alpha + 3\beta + 3\gamma - \alpha x_{SA} - \beta x_{SA} - \gamma x_{SA}$$

Entire page is optional

$$\chi_{SA} (2\alpha + 2\beta + \gamma) = 6\beta + 3\gamma$$

$$\chi_{SA} = \frac{6\beta + 3\gamma}{2\alpha + 2\beta + \gamma}$$

and

$$\begin{aligned}\chi_{JA} &= 3 + \frac{\beta}{\alpha} (\chi_{SA} - 3) = 3 - \frac{3\beta}{\alpha} + \frac{\beta}{\alpha} \frac{6\beta + 3\gamma}{2\alpha + 2\beta + \gamma} \\ &= \frac{3\alpha - 3\beta}{\alpha} + \frac{\beta}{\alpha} \frac{6\beta + 3\gamma}{2\alpha + 2\beta + \gamma} = \frac{(3\alpha - 3\beta)(2\alpha + 2\beta + \gamma) + 6\beta^2 + 3\beta\gamma}{\alpha(2\alpha + 2\beta + \gamma)} \\ &= \frac{6\alpha^2 + 6\alpha\beta + 3\alpha\gamma - 6\alpha\beta - 6\beta^2 - 3\beta\gamma + 6\beta^2 + 3\beta\gamma}{\alpha(2\alpha + 2\beta + \gamma)} \\ &= \frac{6\alpha^2 + 3\alpha\gamma}{\alpha(2\alpha + 2\beta + \gamma)} = \frac{6\alpha + 3\gamma}{2\alpha + 2\beta + \gamma}.\end{aligned}$$

In the special case of $\alpha = \beta = 1$ and $\gamma = 0$,

$$\chi_{SA} = \frac{6}{2+2} = \frac{6}{4} = \frac{3}{2} \text{ and}$$

$$\chi_{JA} = \frac{6}{2+2} = \frac{3}{2},$$

which are $\hat{\chi}_{SA}$ and $\hat{\chi}_{JA}$, as expected.

1. [12 points]

We wish to construct an Edgeworth Box for a case in which the two agents have altruism for each other.

Suppose Person 1 consumes x_1 units of good x and y_1 units of good y , and suppose Person 2 consumes x_2 units of good x and y_2 units of good y . Suppose the total amount of x available to the two persons is 1 unit and suppose the total amount of y available to the two persons is also 1 unit.

- (a) First suppose the utility functions of the two persons are given by

$$U_1(x_1, y_1, U_2) = x_1 + y_1 + \frac{1}{2}U_2$$

$$U_2(x_2, y_2, U_1) = x_2 + y_2 + \frac{1}{2}U_1.$$

By solving for U_1 as only a function of x_1 and y_1 , and solving for U_2 as only a function of x_2 and y_2 , argue that the contract curve would be the same as if the two persons were not altruistic, indeed the same as if the utility functions of the two persons were instead given by

$$U_1(x_1, y_1) = x_1 + y_1$$

$$U_2(x_2, y_2) = x_2 + y_2.$$

- (b) Re-work part (a) supposing that the utility functions of the two persons are *not* given by

$$U_1(x_1, y_1, U_2) = x_1 + y_1 + \frac{1}{2}U_2$$

$$U_2(x_2, y_2, U_1) = x_2 + y_2 + \frac{1}{2}U_1.$$

but instead are given by

$$U_1(x_1, y_1, U_2) = x_1y_1 + \frac{1}{2}U_2$$

$$U_2(x_2, y_2, U_1) = x_2y_2 + \frac{1}{2}U_1.$$

You should obtain

$$U_1(x_1, y_1) = 2x_1y_1 - \frac{2}{3}x_1 - \frac{2}{3}y_1 + \frac{2}{3}$$

$$U_2(x_2, y_2) = 2x_2y_2 - \frac{2}{3}x_2 - \frac{2}{3}y_2 + \frac{2}{3}.$$

In this case, will the outcome of the agents' behavior be affected by their altruism?

- (c) The rest of this question concerns the persons of part (b). Draw an Edgeworth Box (put x_1 on the horizontal axis and y_1 on the vertical axis). In this Box, indicate the regions where:

- $\partial U_1 / \partial x_1 > 0$;
- $\partial U_1 / \partial x_1 < 0$;
- $\partial U_1 / \partial y_1 > 0$;
- $\partial U_1 / \partial y_1 < 0$.

Also in this Box, indicate the regions where:

- $\partial U_2 / \partial x_2 > 0$;
- $\partial U_2 / \partial x_2 < 0$;
- $\partial U_2 / \partial y_2 > 0$;
- $\partial U_2 / \partial y_2 < 0$.

You may reason by analogy with the results for Person 1.

- (d) In each region of the Box, draw a small dot, and from this dot draw an arrow showing the direction (“northwest,” “northeast,” “southwest,” or “southeast”) of increasing utility for Person 1. Also, from each dot draw an arrow showing the direction of increasing utility for Person 2.
- (e) Considering U_1 as a function of x_1 and y_1 , and considering U_2 as a function of x_1 and y_1 (*not* of x_2 nor y_2), it is trivial to show that

$$\begin{aligned} U_1(0,0) &= \frac{2}{3} & U_2(0,0) &= \frac{4}{3} \\ U_1\left(\frac{1}{3}, \frac{1}{3}\right) &= \frac{4}{9} & U_2\left(\frac{1}{3}, \frac{1}{3}\right) &= \frac{2}{3} \\ U_1\left(\frac{2}{3}, \frac{2}{3}\right) &= \frac{2}{3} & U_2\left(\frac{2}{3}, \frac{2}{3}\right) &= \frac{4}{9} \\ U_1(1,1) &= \frac{4}{3} & U_2(1,1) &= \frac{2}{3} \end{aligned}$$

so I do not want you to prove that they are true. Use these facts to argue that the only Pareto Efficient points (x_1, y_1) in this two-person economy are $(0,0)$ and $(1,1)$.

- (f) The conclusion for this economy is that in it, altruism leads to complete inequality as the only efficient allocation. This may be counterintuitive; the explanation can only be that the rich people are happy because they are rich, and the poor people are happy because the rich people are rich. Briefly conjecture what parameter values of the utility functions would have to be changed in

order to reverse the conclusion that “altruism leads to complete inequality as the only efficient allocation.” (One or two sentences will suffice here; do not work out anything mathematically, because I am asking only for a reasonable conjecture, not a proof.)

Answers to Micro Qualifying Exam, Summer 2009

Section 1.

$$(1) \text{a)} U_1 = x_1 + y_1 + \frac{1}{2} U_2$$

$$U_2 = x_2 + y_2 + \frac{1}{2} U_1$$

$$x_1 + x_2 = 1$$

$$y_1 + y_2 = 1$$

$$U_1 = x_1 + y_1 + \frac{1}{2} (x_2 + y_2 + \frac{1}{2} U_1)$$

$$= x_1 + y_1 + \frac{1}{2} x_2 + \frac{1}{2} y_2 + \frac{1}{4} U_1$$

$$\frac{3}{4} U_1 = x_1 + y_1 + \frac{1}{2} (1 - x_1) + \frac{1}{2} (1 - y_1)$$

$$= \frac{1}{2} x_1 + \frac{1}{2} y_1 + 1$$

$$U_1 = \frac{2}{3} x_1 + \frac{2}{3} y_1 + \frac{4}{3}$$

$= \frac{2}{3} (x_1 + y_1) + \frac{4}{3}$ which is a monotonically increasing function of $x_1 + y_1$,

and so describes the same behavior as $x_1 + y_1$.

In this completely symmetric problem, it is acceptable to reason by symmetry.

So $U_2 = \frac{2}{3} (x_2 + y_2) + \frac{4}{3}$, which describes the same behavior as $x_2 + y_2$.

$$b) U_1 = x_1 y_1 + \frac{1}{2} U_2$$

$$U_2 = x_2 y_2 + \frac{1}{2} U_1.$$

Substituting,

$$U_2 = x_2 y_2 + \frac{1}{2} (x_1 y_1 + \frac{1}{2} U_2) = x_2 y_2 + \frac{1}{2} x_1 y_1 + \frac{1}{4} U_2 \Rightarrow$$

$$\frac{3}{4} U_2 = x_2 y_2 + \frac{1}{2} x_1 y_1. \text{ Now using the fact that } x_1 + x_2 = 1 \text{ and } y_1 + y_2 = 1,$$

$$\frac{3}{4} U_2 = x_2 y_2 + \frac{1}{2} (1 - x_2)(1 - y_2)$$

$$= x_2 y_2 + \frac{1}{2} (1 - y_2 - x_2 + x_2 y_2) = \frac{3}{2} x_2 y_2 - \frac{y_2}{2} - \frac{x_2}{2} + \frac{1}{2}$$

$$= \frac{1}{2} (3 x_2 y_2 - y_2 - x_2 + 1) \text{ and}$$

$$U_2 = \frac{2}{3} (3 x_2 y_2 - x_2 - y_2 + 1). \text{ Then}$$

$$U_1 = x_1 y_1 + \frac{1}{2} U_2 = x_1 y_1 + \frac{1}{2} \cdot \frac{2}{3} (3 x_2 y_2 - x_2 - y_2 + 1)$$

$$= x_1 y_1 + \frac{1}{3} [3(1 - x_1)(1 - y_1) - (1 - x_1) - (1 - y_1) + 1]$$

$$= \overset{\downarrow}{x_1 y_1} + \frac{1}{3} [3 - 3y_1 - 3x_1 + \overset{\downarrow}{3x_1 y_1} - 1 + x_1 - 1 + y_1 + 1]$$

$$= 2x_1 y_1 + \frac{1}{3} [-2x_1 - 2y_1 + 2]$$

$$= \frac{2}{3} [3x_1 y_1 - x_1 - y_1 + 1].$$

U_2 is not a monotonically increasing function of $\overset{\text{or}}{x_2 y_2}$, nor is U_1 a monotonically increasing function of $\overset{\text{or}}{x_1 y_1}$, so agents' behavior in (b) will be affected by their altruism.

(c)

$$U_1 = 2x_1 y_1 - \frac{2}{3} x_1 - \frac{2}{3} y_1 + \frac{2}{3}.$$

$$\frac{\partial U_1}{\partial x_1} = 2y_1 - \frac{2}{3}, \text{ so } \frac{\partial U_1}{\partial x_1} < 0 \text{ when } 2y_1 - \frac{2}{3} < 0$$

$$\Leftrightarrow y_1 - \frac{1}{3} < 0$$

$$\Leftrightarrow y_1 < \frac{1}{3}.$$

$$\frac{\partial U_1}{\partial y_1} = 2x_1 - \frac{2}{3}, \text{ so } \frac{\partial U_1}{\partial y_1} < 0 \text{ when } 2x_1 - \frac{2}{3} < 0$$

$$\Leftrightarrow x_1 - \frac{1}{3} < 0$$

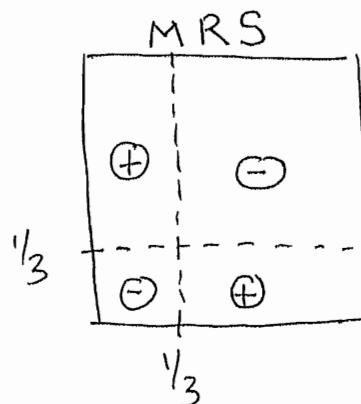
$$(c) \text{ continues on the next page. } x_1 < \frac{1}{3}.$$

Optional

In the y_1 | plane (or the Edgeworth Box), the slope of

$\frac{dy_1}{dx_1}$, holding U_1 constant) is Person 1's indifference curve (that is, $\frac{dy_1}{dx_1}$, holding U_1 constant) is his marginal rate of substitution, $- \frac{\partial U_1 / \partial x_1}{\partial U_1 / \partial y_1}$. In the standard case,

indifference curves are downward-sloping, but in this case we have



and in the "+" regions the indifference curves are positively sloped.

Finally, we observe that

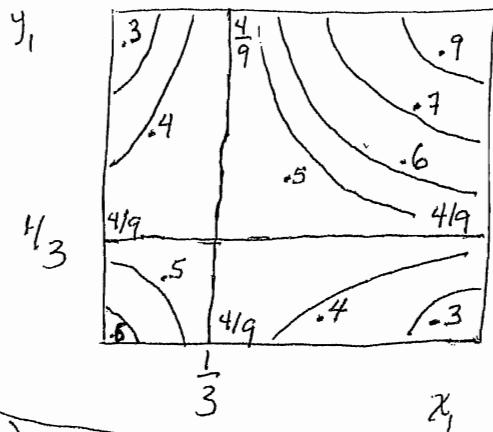
$$U_1\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{3} \left(3 \cdot \frac{1}{3} \cdot \frac{1}{3} - \frac{1}{3} - \frac{1}{3} + 1\right) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$$

$$U_1\left(\frac{1}{3}, y_1\right) = \frac{2}{3} \left(3 \cdot \frac{1}{3} y_1 - y_1 - \frac{1}{3} + 1\right) = \frac{2}{3} \left(\frac{2}{3}\right) = \frac{4}{9}$$

$$U_1\left(x_1, \frac{1}{3}\right) = \frac{2}{3} \left(3 x_1 \cdot \frac{1}{3} - \frac{1}{3} + 1\right) = \frac{2}{3} \left(\frac{2}{3}\right) = \frac{4}{9}$$

Optimal

So it is not surprising that a numerical analysis on a computer gives

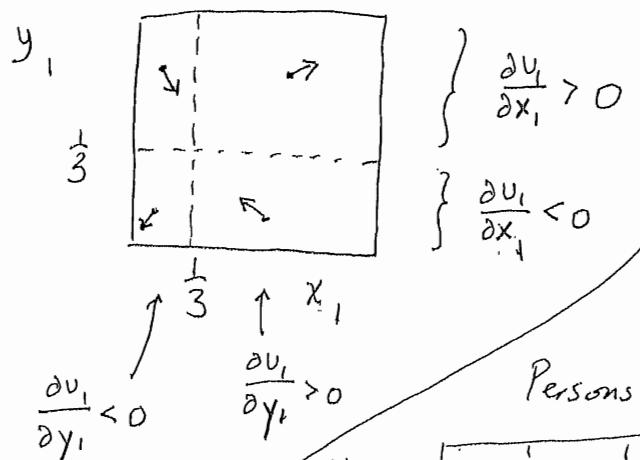


$$\frac{4}{9} = 0.\overline{4}$$

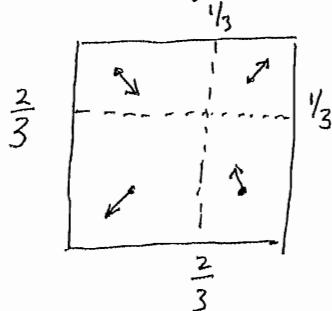
which means
0.44444...

(c)(d)

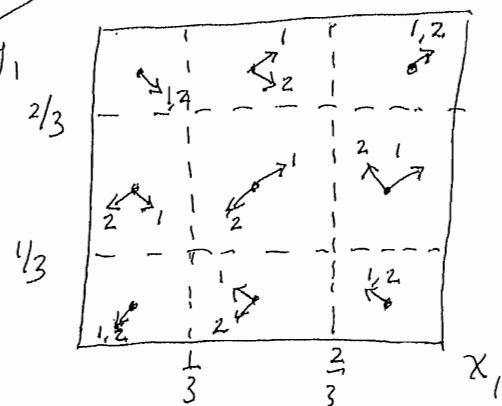
Person 1



Analogously for Person 2



Persons 1 & 2



Optional

For the Box edges:

$$\text{If } x_1 = 0, U_1 = 0 - \frac{2}{3}y_1 - 0 + \frac{2}{3} = -\frac{2}{3}y_1 + \frac{2}{3} \text{ so } \uparrow y_1 \Rightarrow \downarrow U_1.$$

$$\text{If } y_1 = 0, U_1 = 0 - 0 - \frac{2}{3}x_1 + \frac{2}{3} = -\frac{2}{3}x_1 + \frac{2}{3} \text{ so } \uparrow x_1 \Rightarrow \downarrow U_1.$$

$$\text{If } x_1 = 0, U_2 = 2x_2y_2 - \frac{2}{3}x_2 - \frac{2}{3}y_2 + \frac{2}{3}$$

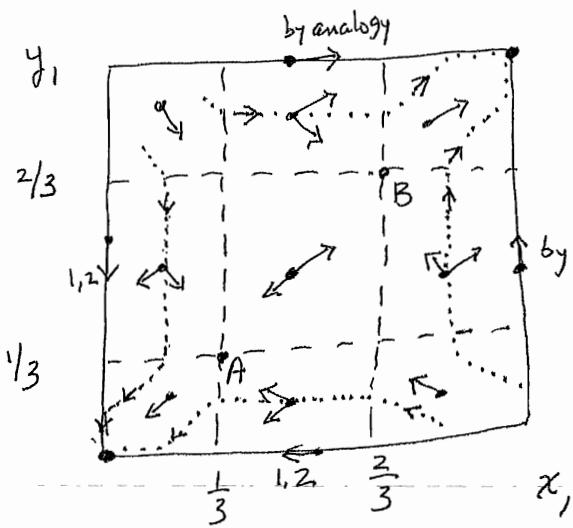
$$= 2(1)y_2 - \frac{2}{3}(1) - \frac{2}{3}y_2 + \frac{2}{3} = 2y_2 - \frac{2}{3}y_2 = \frac{4}{3}y_2 \text{ so } \uparrow y_2 \Rightarrow \uparrow U_2,$$

or $\downarrow y_1 \Rightarrow \uparrow U_2$.

$$\text{If } y_1 = 0, U_2 = 2x_2(1) - \frac{2}{3}x_2 - \frac{2}{3}(1) + \frac{2}{3} = 2x_2 - \frac{2}{3}x_2 = \frac{4}{3}x_2 \text{ so } \uparrow x_2 \Rightarrow \uparrow U_2,$$

or $\downarrow x_1 \Rightarrow \uparrow U_2$.

(e)



If the initial allocation is outside of the middle $\frac{1}{9}$, a succession of freely accepted offers will end up either at $(0,0)$ or at $(1,1)$; which of these will be the result is ambiguous only if the initial allocation is in the NW or SE one-ninth of the Box.

Outside of the middle $\frac{1}{9}$ of the Edgeworth Box, negotiations will lead either to $(0,0)$ or to $(1,1)$. These result in

$$\left. \begin{array}{ll} \text{optional } & U_1(0,0) = \frac{2}{3} \quad U_2(0,0) = U_2(x_2=1, y_2=1) = \frac{2}{3}(3-1-1+1) = \frac{4}{3} \\ & U_1(1,1) = \frac{2}{3}(3-1-1+1) = \frac{4}{3} \quad U_2(1,1) = U_2(x_2=0, y_2=0) = \frac{2}{3}. \end{array} \right.$$

The middle $\frac{1}{9}$ of the Box has typical indifference curves. At point A,

$$U_1(A) = U_1\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{4}{9} \text{ from earlier}$$

$$\left. \begin{array}{l} \text{optional } \\ - \end{array} \right. \begin{aligned} U_2(A) &= U_2\left(x_2 = \frac{2}{3}, y_2 = \frac{2}{3}\right) = \frac{2}{3}\left(3, \frac{2}{3}, \frac{2}{3} - \frac{2}{3} - \frac{2}{3} + 1\right) \\ &= \frac{2}{3}\left(\frac{4}{3} - \frac{4}{3} + 1\right) = \frac{2}{3}. \end{aligned}$$

$$\left. \begin{aligned}
 U_1(B) &= U_1\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{2}{3} \left(3 \cdot \frac{2}{3} \cdot \frac{2}{3} - \frac{2}{3} - \frac{2}{3} + 1\right) = \frac{2}{3} \\
 U_2(B) &= U_2\left(x_2 = \frac{1}{3}, y_2 = \frac{1}{3}\right) = \frac{2}{3} \left(3 \cdot \frac{1}{3} \cdot \frac{1}{3} - \frac{1}{3} - \frac{1}{3} + 1\right) \\
 &= \frac{2}{3} \left(\frac{1}{3} - \frac{1}{3} - \frac{1}{3} + 1\right) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}.
 \end{aligned} \right\} \text{optional}$$

In this middle $\frac{1}{9}$, the worst Person 1 can do is at point A, where $U_1 = \frac{4}{9}$, and the best he can do is at Point B, where $U_1 = \frac{2}{3} = \frac{6}{9}$. But comparing point B with $(0,0)$, Person 1 is indifferent, while Person 2 gets utility of $\frac{4}{9}$ at B and $\frac{4}{9} = \frac{12}{9}$ at $(0,0)$, so $(0,0)$ Pareto-dominates Point B. We already saw that Point B dominates any ^{other} point in the middle $\frac{1}{9}$. So if the initial allocation is in the middle $\frac{1}{9}$, Person 1 will not refuse an offer to go to $(0,0)$, and $(0,0)$ gives Person 2 more utility ($\frac{4}{9} = \frac{12}{9}$) than Person 2 can get at Point A ($\frac{2}{3}$) or anywhere else. Analogously, if the initial allocation is in the middle $\frac{1}{9}$, Person 2 will not refuse an offer to go to $(1,1)$, and Person 1 will want to make such an offer since $(1,1)$ gives him more utility than any other point.

f) Decreasing these parameters,

$$\begin{aligned}
 U_1 &\approx x_1 y_1 + \left(\frac{1}{2}\right) U_2 \\
 U_2 &\approx x_2 y_2 + \left(\frac{1}{2}\right) U_1
 \end{aligned}$$

might help. This decreases altruism, so the poor people won't be so happy that the rich people are rich.

Final Exam

2004

Question 4

(6)

- ✓ 4. [19 points] Suppose there are two persons in a pure-exchange economy; they are named "a" and "b". Their utility functions, which depend on the amounts of the two goods "x" and "y" which they consume, are:

$$u_a = \ln x_a + \ln y_a$$

$$u_b = 2 \ln x_b + \ln y_b.$$

(So " x_a " is the amount of good x which consumer "a" consumes.) The consumers' initial endowments are given by the following vectors, where the first entry of the vector is the initial amount of good x and the second entry of the vector is the initial amount of good y :

$$\omega_a = (10, 10)$$

$$\omega_b = (1, 1)$$

Suppose the social planner's social welfare function is

$$\min(u_a, u_b).$$

Suppose the social planner is unable to directly allocate commodities to individuals; all the social planner is able to do is to make lump-sum transfers of good y but not good x from one person to the other.

What should the social planner do to achieve his objective? Stop when you find one equation in one unknown whose solution would enable you to solve the rest of the problem by simple substitutions. Do not try to solve that equation!

(4)

$$u_a = \ln x_a + \ln y_a \quad w_a = (10, 10)$$

$$u_b = 2 \ln x_b + \ln y_b \quad w_b = (1, 1)$$

Final Exam
2004

Answer 4

Social welfare function ("SWF") is $\min(u_a, u_b)$; maximizing this requires $u_a = u_b$. (A Rawlsian, or maximin, SWF.)

$$\ln x_a + \ln y_a = 2 \ln x_b + \ln y_b$$

$$\ln x_a y_a = \ln x_b^2 y_b$$

$$x_a y_a = x_b^2 y_b \quad \text{to maximize the SWF. Clearly } x_a + x_b = 11$$

$$y_a + y_b = 11$$

by feasibility. So we obtain

$$x_a y_a = (11 - x_a)^2 (11 - y_a)$$

Person a: First let the price of y be 1 (so y is the numeraire).

Let the price of x be "p".

$$\text{Income: } 10p + (10 - T)(1)$$

↑ ↑ ↑
 price of price of price of
 x y y
 transferred

$$\text{Expenditures: } x_a p + y_a (1)$$

$$\text{Budget constraint: } 10p + 10 - T = p x_a + y_a$$

$$\mathcal{L} = \ln x_a + \ln y_a + \lambda (10p + 10 - T - px_a - y_a)$$

$$0 = \partial \mathcal{L} / \partial \lambda = 10p + 10 - T - px_a - y_a$$

$$0 = \partial \mathcal{L} / \partial x_a = \frac{1}{x_a} - \lambda p \quad p = \frac{y_a}{x_a} \Rightarrow y_a = px_a \text{ into}$$

$$0 = \partial \mathcal{L} / \partial y_a = \frac{1}{y_a} - \lambda \quad \text{budget constraint:}$$

$$10p + 10 - T = px_a + px_a \\ \Rightarrow 2px_a \Rightarrow x_a = \frac{10p + 10 - T}{2p}$$

Final Exam

2004

Person b : income is $1 \cdot p + (1+T)(1)$

expenditure is $x_b p + y_b (1)$

Answer 4 cont...

B.C. is $p + 1 + T = x_b p + y_b$

$$\mathcal{L} = 2 \ln x_b + \ln y_b + \lambda (p + 1 + T - px_b - y_b)$$

$$0 = \partial \mathcal{L} / \partial \lambda = p + 1 + T - px_b - y_b$$

$$\begin{aligned} 0 = \partial \mathcal{L} / \partial x_b &= \frac{2}{x_b} - \lambda p \\ 0 = \partial \mathcal{L} / \partial y_b &= \frac{1}{y_b} - \lambda \end{aligned} \quad \left. \begin{aligned} p &= \frac{2y_b}{x_b} \Rightarrow y_b = \frac{1}{2} px_b \text{ into} \\ &\text{budget constraint} \end{aligned} \right.$$

$$p + 1 + T = px_b + \frac{1}{2} px_b$$

$$= \frac{3}{2} px_b \Rightarrow x_b = \frac{2}{3p} (p + 1 + T)$$

$$\text{Clear } x : x_a^D + x_b^D = x^S$$

$$\frac{10p + 10 - T}{2p} + \frac{2}{3p} (p + 1 + T) = 11 \quad \text{Multiply both sides by } 6p :$$

$$3(10p + 10 - T) + 4(p + 1 + T) = 6p \cdot 11$$

$$\begin{array}{c} 30p + 30 - 3T + 4p + 4 + 4T = 66p \\ \uparrow \quad \uparrow \quad \uparrow \end{array}$$

$$34 + T = 32 \Rightarrow p = \frac{34 + T}{32}$$

Therefore

$$x_a = \frac{10 \left(\frac{34 + T}{32} \right) + 10 - T}{2 \left(\frac{34 + T}{32} \right)}$$

$$y_a = px_a = \frac{34 + T}{32} \cdot \frac{10 \left(\frac{34 + T}{32} \right) + 10 - T}{2 \left(\frac{34 + T}{32} \right)}$$

Substitute these into

the social planner's $x_a y_a = (11 - x_a)^2 (11 - y_a)$ and one obtains one equation in the one unknown, "T". Solving for that would complete the problem.

Optional :

$$x_a = \frac{10(34+T) + 320 - 32T}{2(34+T)} = \frac{340 + 10T + 320 - 32T}{68 + 2T} = \frac{660 - 22T}{68 + 2T}$$

$$= \frac{330 - 11T}{34 + T} \quad \text{--- --- } \downarrow$$

$$y_a = p x_a = \frac{34+T}{32} \cdot \frac{330 - 11T}{34+T} = \frac{330 - 11T}{32}$$

$$x_a y_a = (11 - x_a)^2 (11 - y_a) \Leftrightarrow$$

$$\frac{330 - 11T}{34 + T} \cdot \frac{330 - 11T}{32} = \left[11 - \frac{330 - 11T}{34 + T} \right]^2 \left[11 - \frac{330 - 11T}{32} \right]$$

The rest of the problem is worked out in the computer printout which follows.

Final Exam

2004

Answer 4 cont...

Mathematica printout

Newnb-2

```
In[1]:= 
xa := (330-11 T)/(34+T)
ya := ((34+T)/32) * xa
possibleTs = Solve[ xa*ya == (11-xa)^2 * (11-ya), T];
N[possibleTs]

Out[4]=
{{T -> 6.05199}, {T -> -6.39809 + 8.67742 I}, {T -> -6.39809 - 8.67742 I} }

In[5]:= 
(* So the first possible T is the right one (the others are imaginary numbers. *)

(* Assign "T" to the correct value.*)
T=T/.possibleTs[[1]]
```

exact T

$$\frac{\frac{290}{129} - \frac{131072}{129} \cdot \frac{2}{(1648492544 + 287440896 \sqrt{33})^{1/3}}}{129 \cdot (1648492544 + 287440896 \sqrt{33})^{1/3}} + \frac{(1648492544 + 287440896 \sqrt{33})^{1/3}}{129^2}$$

```
In[6]:= 
{N[T], N[xa], N[ya]}
```

```
Out[6]=
{6.05199, 6.57715, 8.23213}
```

```
In[7]:= 
xb=11-xa; yb=11-ya;
{N[xb], N[yb]}
```

```
Out[8]=
{4.42285, 2.76787}
```

```
In[9]:=
```

```
{"a's utility", N[ Log[xa]+Log[ya]]}
 {"b's utility", N[2Log[xb]+Log[yb]]}
```

```
Out[10]=
```

```
{a's utility, 3.99165}
{b's utility, 3.99165}
```

```
In[11]:=
```

```
{ "a's utility", N[ Log[xa]+Log[ya]]}
 {"b's utility", N[2Log[xb]+Log[yb]]}
```

```
Out[12]=
```

```
{a's utility, 3.99165}
{b's utility, 3.99165}
```

approximate T, xa, ya. Note that T > 0 so

the transfer is from the richer person ("a") to the poorer person ("b")

approximate xb, yb

Verification that the people have the same utility

Answer all of the following five questions.

1. [12 points] Consider a two-person, two-commodity economy in which " x_{ij} " represents the amount of commodity i belonging to person j . Suppose the utility function of person 1 is

$$\ln x_{11} + \ln x_{21}$$

and the utility function of person 2 is

$$\ln x_{12} + \ln x_{22} .$$

Suppose the initial endowments of persons 1 and 2 are $\omega_1 = (1, 1)$ and $\omega_2 = (2, 1)$, respectively. Find the core of this economy.

Answers to ECON 7005 Final Exam,
Fall 2007

$$\textcircled{1} \quad U^1 = \ln x_{11} + \ln x_{21} \quad \underline{\omega}_1 = (1, 1)$$

$$U^2 = \ln x_{12} + \ln x_{22} \quad \underline{\omega}_2 = (2, 1)$$

To find the core, first find the contract curve (set of Pareto optimal points).

$$\max \alpha (\ln x_{11} + \ln x_{21}) + (1-\alpha) (\ln x_{12} + \ln x_{22}) \text{ s.t. } x_{11} + x_{12} = 3$$

$$x_{21} + x_{22} = 2$$

$$\Leftrightarrow \max \alpha (\ln x_{11} + \ln x_{21}) + (1-\alpha) [\ln(3-x_{11}) + \ln(2-x_{21})]$$

F.O.C.

$$0 = \frac{\alpha}{x_{11}} - \frac{1-\alpha}{3-x_{11}} \Rightarrow \frac{1-\alpha}{3-x_{11}} = \frac{\alpha}{x_{11}} \Rightarrow (1-\alpha)x_{11} = 3\alpha - \alpha x_{11}$$

$$x_{11} - \alpha x_{11} = 3\alpha - \alpha x_{11}$$

$$0 = \frac{\alpha}{x_{21}} - \frac{1-\alpha}{2-x_{21}}$$

$$x_{11} = 3\alpha$$

$$\frac{x_{11}}{x_{12}} = \frac{3\alpha}{3-3\alpha} = \frac{3\alpha}{3(1-\alpha)} = 1-\alpha$$

$$(1-\alpha)x_{21} = \alpha(2-x_{21})$$

$$x_{21} - \alpha x_{21} = 2\alpha - \alpha x_{21}$$

$$\underline{x_{21}} = 2\alpha$$

$$x_{22} = 2 - x_{21} = 2 - 2\alpha$$

$$= 2(1-\alpha)$$

So this is the contract curve, parameterized by α . On this curve,

$$U^1 = \ln x_{11} + \ln x_{21} = \ln 3\alpha + \ln 2\alpha = \ln 6\alpha^2$$

$$U^2 = \ln x_{12} + \ln x_{22} = \ln 3(1-\alpha) + \ln 2(1-\alpha) = \ln 6(1-\alpha)^2$$



The initial utility of Person 1 is

$$\begin{aligned} U^1(\underline{\omega}_1) &= \ln 1 + \ln 1 \\ &= 0 + 0 = 0. \end{aligned}$$

Person 1 will not veto a Pareto Optimal allocation if its U^1 is $\geq U^1(\underline{\omega}_1) = 0$:

$$0 \leq \ln 6\alpha^2 \Leftrightarrow 6\alpha^2 \geq 1$$

$$\alpha^2 \geq \frac{1}{6} \Leftrightarrow \alpha > \frac{1}{\sqrt{6}} \text{ or } \alpha < \underbrace{-\frac{1}{\sqrt{6}}}_{\text{violates } \alpha \text{ between 0 and 1}}.$$

violates α between 0 and 1

The initial utility of Person 2 is

$$U^2(\underline{\omega}_2) = \ln 2 + \ln 1 = \ln 2 + 0 = \ln 2.$$

Person 2 will not veto a Pareto Optimal allocation if its U^2 is $\geq U^2(\underline{\omega}_2) = \ln 2$:

$$\ln 2 \leq \ln 6(1-\alpha)^2$$

$$2 \leq 6(1-\alpha)^2$$

$$\frac{1}{3} \leq (1-\alpha)^2 \Leftrightarrow 1-\alpha \geq \frac{1}{\sqrt{3}} \text{ or } 1-\alpha \leq -\frac{1}{\sqrt{3}}$$

$$1 - \frac{1}{\sqrt{3}} \geq \alpha \quad \underbrace{1 + \frac{1}{\sqrt{3}} \leq \alpha}_{\text{violates } \alpha \text{ between 0 and 1}}$$

violates α between 0 and 1

So we need $\alpha > \frac{1}{\sqrt{6}}$ and $\alpha < 1 - \frac{1}{\sqrt{3}}$.

$$\downarrow \sim 0.408 \quad \downarrow \sim 0.423$$

This implies x_{11} between $3\alpha = \frac{3}{\sqrt{6}} = \frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2}$ and

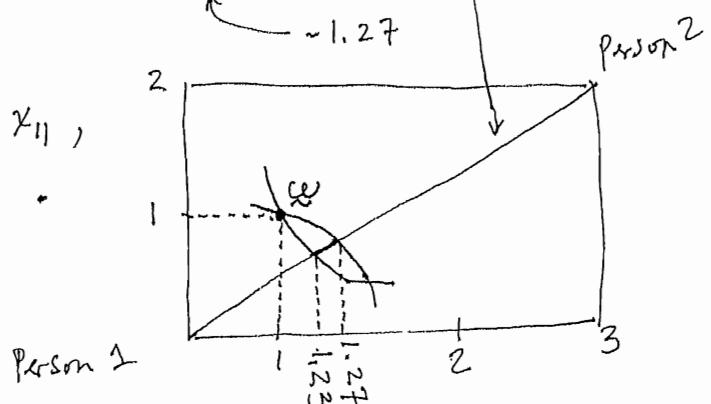
$$3(1 - \frac{1}{\sqrt{3}}) = 3 - \sqrt{3},$$

$$x_{12} = 3 - x_{11},$$

$$x_{21} = 2\alpha = \frac{2}{3} \cdot 3\alpha = \frac{2}{3} x_{11},$$

$$x_{22} = 2 - x_{21} = 2 - \frac{2}{3} x_{11}.$$

$$\begin{aligned} \text{Contract Curve:} \\ \text{slope} &= \frac{x_{21}/x_{11}}{2\alpha/3\alpha} \\ &= 2/3 \end{aligned}$$



Alternative solution method:

$$\max U^1 \text{ s.t. } U^2 = \bar{U}$$

$$L = \ln x_{11} + \ln x_{21} + \lambda [\bar{U} - \ln(3-x_{11}) - \ln(2-x_{21})]$$

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x_{11}} = \frac{1}{x_{11}} + \lambda \frac{1}{3-x_{11}} \\ 0 &= \frac{\partial L}{\partial x_{21}} = \frac{1}{x_{21}} + \lambda \frac{1}{2-x_{21}} \end{aligned} \quad \left. \begin{array}{l} \frac{1/x_{11}}{1/x_{21}} = \frac{-\lambda}{\frac{3-x_{11}}{2-x_{21}}} \\ \frac{x_{21}}{x_{11}} = \frac{2-x_{21}}{3-x_{11}} \end{array} \right\}$$

$$\frac{x_{21}}{x_{11}} = \frac{2-x_{21}}{3-x_{11}}$$

$$x_{21}(3-x_{11}) = x_{11}(2-x_{21})$$

$$3x_{21} - x_{21}x_{11} = 2x_{11} - x_{11}x_{21}$$

$$3x_{21} = 2x_{11}$$

$$x_{21} = \frac{2}{3}x_{11}$$

$$\text{and also } x_{12} = 3 - x_{11}$$

$$x_{22} = 2 - x_{21} = 2 - \frac{2}{3}x_{11}$$

which describes the same contract curve as before. The rest of the problem follows as before.

Summer 2010 Qualifying Exam

2. [13 points, divided as: 3 points for (a), (e), and (f) taken together; 3 points for (b) and (c) taken together; 3 points for (d); 4 points for (g).]

In an exam from a previous year, I (Prof. Lozada) asked the following question:

Consider a two-person, two-commodity economy in which " x_{ij} " represents the amount of commodity i belonging to person j . Suppose the utility function of person 1 is

$$\ln x_{11} + \ln x_{21}$$

and the utility function of person 2 is

$$\ln x_{12} + \ln x_{22}.$$

Suppose the initial endowments of persons 1 and 2 are $\omega_1 = (1, 1)$ and $\omega_2 = (2, 1)$, respectively. Find the core of this economy.

I have attached the two-page handwritten answer to this question right after page ~~9 of your exam~~
3 of this question. Please look at those two pages.

- (a) After the old answer sheets are two pages with computer-drawn graphs. You should ignore all of the typewritten computer commands on this page, which are written in a language I do not

This question has 7 pages.

expect you to be familiar with. Find Fig. 1. Its horizontal axis is, as labeled, $U^1 = \ln(x_{11} \cdot \frac{2}{3}x_{11})$, and its vertical axis is, as labeled, $U^2 = \ln[(3-x_{11}) \cdot (2-\frac{2}{3}x_{11})]$. As indicated, the graph shows what happens in this (U^1, U^2) space as x_{11} goes from $\sqrt{6}/2$ to $3-\sqrt{3}$, in other words, as x_{11} goes approximately from 1.23 to 1.27. Give an economic interpretation of this curve in (U^1, U^2) space. Hint 1: I have never shown you a graph in (U^1, U^2) space before, so I am not asking you to remember something, I am asking you to interpret something which is new to you. Hint 2: Look back to the old answer sheet, especially the bottom of its second page, for help.

- (b) All the remaining parts below pertain to finding the core of the “2-replica” of the above economy, in other words, finding the core of the economy with two people who are identical to Person 1 above and two people who are identical to Person 2 above.

To begin, explain why one of the first steps in finding the core of the 2-replica might be to solve

$$\max[\alpha \ln(x_{11} x_{21}) + (1-\alpha) \ln(x_{12} x_{22})]$$

for $\alpha \in [0, 1]$ such that

$$\begin{aligned} 2x_{11} + x_{12} &= 4 \quad \text{and} \\ 2x_{21} + x_{22} &= 3. \end{aligned}$$

Read part (c) before you answer part (b).

- (c) Next, explain why one of the first steps in finding the core of the 2-replica might be to solve

$$\max[\alpha \ln(x_{11} x_{21}) + (1-\alpha) \ln(x_{12} x_{22})]$$

for $\alpha \in [0, 1]$ such that

$$\begin{aligned} x_{11} + 2x_{12} &= 5 \quad \text{and} \\ x_{21} + 2x_{22} &= 3. \end{aligned}$$

- (d) Solve the problem in part (b). Hint: one way of expressing the answer is

$$\begin{aligned} x_{11} &= 2\alpha \\ x_{12} &= 4 - 4\alpha \\ x_{21} &= \frac{3}{2}\alpha \\ x_{22} &= 3 - 3\alpha. \end{aligned}$$

Next, argue that not all $\alpha \in [0, 1]$ are economically relevant, and that the economically relevant α 's are

$$\alpha \in \left[\frac{1}{\sqrt{3}}, 1 - \frac{1}{\sqrt{6}} \right] \approx [0.58, 0.59].$$

Hint: When I worked the last sentence of this part, one of my intermediate steps was to solve $6\alpha^2 - 12\alpha + 5 > 0$. In solving that it was helpful to remember that if $ax^2 + bx + c = 0$ then $x = (-b \pm \sqrt{b^2 - 4ac})/2a$.

- (e) Find Fig. 2 on the first sheet of computer-drawn graphs. It is related to parts (b) and (d) above. Its horizontal axis is, as labeled, $U^1 = \ln(2\alpha \cdot \frac{3}{2}\alpha)$, and its vertical axis is, as labeled, $U^2 = \ln[(4-4\alpha) \cdot (3-3\alpha)]$. As indicated, the graph shows what happens in this (U^1, U^2) space as α goes from $\frac{1}{\sqrt{3}}$ to $1 - \frac{1}{\sqrt{6}}$. Give an economic interpretation of this curve in (U^1, U^2) space.
- (f) Find Fig. 3 on the first sheet of computer-drawn graphs. It is related to part (c) above. Guess what Fig. 3 shows. Hint: I give part of the answer away in the first two sentences of part (g), so you might want to read them before answering.
- (g) Find Fig. 4, on the second sheet of computer-drawn graphs. Fig. 4 just superimposes Figs. 1, 2, and 3. I have indicated what values of x_{11} correspond to two important points in Fig. 4.
 - i. Explain the economic conclusions which come from Fig. 4. Then:
 - ii. draw an Edgeworth Box, as on the second page of the old exam answer;
 - iii. in this Edgeworth Box, show the core of the 2-replica of the economy; and
 - iv. in this Edgeworth Box, show how the core of the 2-replica compares to the core of the original economy; and finally,
 - v. explain why this result (the result of subpart (iv) which you just completed) is just what one would expect.

Answers to ECO 7005 Final Exam, [Attached to the
Fall 2007
2000 Qualifying
Exam.]

$$\textcircled{1} \quad U^1 = \ln x_{11} + \ln x_{21} \quad \underline{\omega}_1 = (1, 1)$$

$$U^2 = \ln x_{12} + \ln x_{22} \quad \underline{\omega}_2 = (2, 1)$$

To find the cone, first find the contract curve (set of Pareto optimal points).

$$\max \alpha (\ln x_{11} + \ln x_{21}) + (1-\alpha) (\ln x_{12} + \ln x_{22}) \text{ s.t. } x_{11} + x_{12} = 3$$

$$x_{21} + x_{22} = 2$$

$$\Leftrightarrow \max \alpha (\ln x_{11} + \ln x_{21}) + (1-\alpha) [\ln(3-x_{11}) + \ln(2-x_{21})]$$

F.O.C.

$$0 = \frac{\alpha}{x_{11}} - \frac{1-\alpha}{3-x_{11}} \Rightarrow \frac{1-\alpha}{3-x_{11}} = \frac{\alpha}{x_{11}} \Rightarrow (1-\alpha)x_{11} = 3\alpha - \alpha x_{11}$$

$$x_{11} - \alpha x_{11} = 3\alpha - \alpha x_{11}$$

$$0 = \frac{\alpha}{x_{21}} - \frac{1-\alpha}{2-x_{21}}$$

$$\underline{x_{11} = 3\alpha}$$

$$x_{12} = 3 - x_{11} = 3 - 3\alpha$$

$$\underline{= 3(1-\alpha)}.$$

$$\frac{1-\alpha}{2-x_{21}} = \frac{\alpha}{x_{21}}$$

$$(1-\alpha)x_{21} = \alpha(2-x_{21})$$

$$x_{21} - \alpha x_{21} = 2\alpha - \alpha x_{21}$$

$$\underline{x_{21} = 2\alpha}$$

$$x_{22} = 2 - x_{21} = 2 - 2\alpha$$

$$\underline{= 2(1-\alpha)}.$$

So this is the contract curve, parameterized by α . On this curve,

$$U^1 = \ln x_{11} + \ln x_{21} = \ln 3\alpha + \ln 2\alpha = \ln 6\alpha^2$$

$$U^2 = \ln x_{12} + \ln x_{22} = \ln 3(1-\alpha) + \ln 2(1-\alpha) = \ln 6(1-\alpha)^2$$

\Rightarrow

The initial utility of Person 1 is

$$U^1(\underline{\omega}_1) = \ln 1 + \ln 1 \\ = 0 + 0 = 0.$$

Person 1 will not veto a Pareto Optimal allocation if its U^1 is $\geq U^1(\underline{\omega}_1) = 0$:

$$0 \leq \ln 6\alpha^2 \Leftrightarrow 6\alpha^2 \geq 1$$

$$\alpha^2 \geq \frac{1}{6} \Leftrightarrow \alpha > \frac{1}{\sqrt{6}} \text{ or } \alpha < \underbrace{-\frac{1}{\sqrt{6}}}_{\text{violates } \alpha \text{ between 0 and 1}}$$

violates α between 0 and 1

The initial utility of Person 2 is

$$U^2(\underline{\omega}_2) = \ln 2 + \ln 1 = \ln 2 + 0 = \ln 2.$$

Person 2 will not veto a Pareto Optimal allocation if its U^2 is $\geq U^2(\underline{\omega}_2) = \ln 2$:

$$\ln 2 \leq \ln 6(1-\alpha)^2$$

$$2 \leq 6(1-\alpha)^2$$

$$\frac{1}{3} \leq (1-\alpha)^2 \Leftrightarrow 1-\alpha \geq \frac{1}{\sqrt{3}} \text{ or } 1-\alpha \leq -\frac{1}{\sqrt{3}}$$

$$1 - \frac{1}{\sqrt{3}} \geq \alpha \quad \underbrace{1 + \frac{1}{\sqrt{3}} \leq \alpha}_{\text{violates } \alpha \text{ between 0 and 1}}$$

violates α between 0 and 1

So we need $\alpha > \frac{1}{\sqrt{6}}$ and $\alpha < 1 - \frac{1}{\sqrt{3}}$.

$$\hookrightarrow \alpha \approx 0.408$$

$$\hookrightarrow \alpha \approx 0.423$$

This implies x_{11} between $3\alpha = \frac{3}{\sqrt{6}} = \frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2}$ and

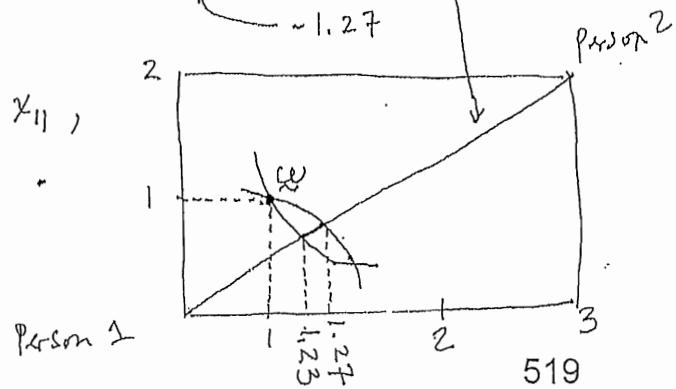
$$3(1 - \frac{1}{\sqrt{3}}) = 3 - \sqrt{3},$$

$$x_{12} = 3 - x_{11},$$

$$x_{21} = 2\alpha = \frac{2}{3} \cdot 3\alpha = \frac{2}{3} x_{11},$$

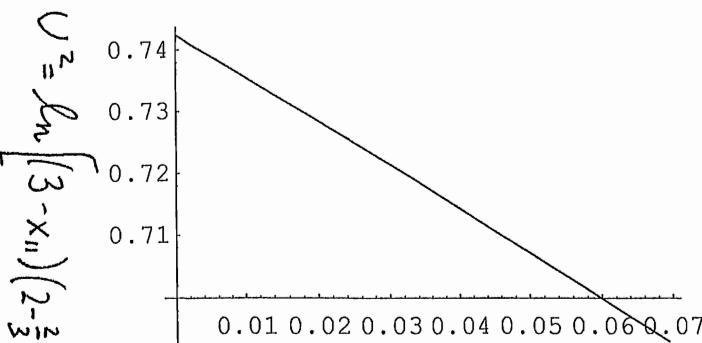
$$x_{22} = 2 - x_{21} = 2 - \frac{2}{3} x_{11}.$$

Contract curve:
slope = $\frac{x_{21}/x_{11}}{= 2\alpha/3\alpha} = \frac{2}{3}$



In[1]:=

```
ParametricPlot[ {Log[xoneone*(2/3)xoneone], Log[(3-xoneone)(2-(2/3)xoneone
{xoneone, Sqrt[6]/2, 3-Sqrt[3]})}]
```



$$x_{11} \in \left[\frac{\sqrt{6}}{2}, 3 - \sqrt{3} \right] \approx [1.23, 1.27]$$

(Fig. 1)

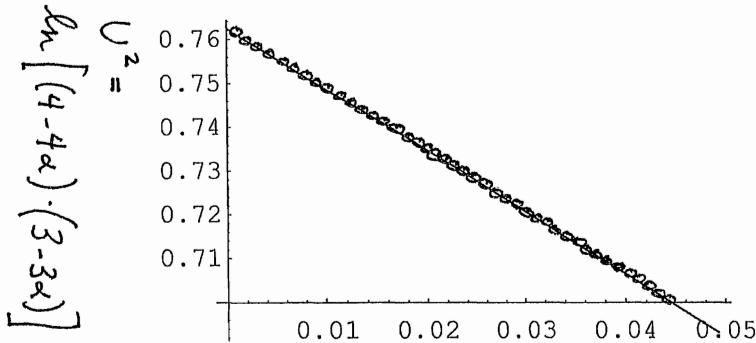
$$U^2 = \ln(x_{11} \cdot \frac{2}{3}x_{11})$$

Out[1]=

-Graphics-

In[2]:=

```
ParametricPlot[ {Log[3 alpha^2], Log[(4-4alpha)(3-3alpha)]},
{alpha, 1/Sqrt[3], 1-(1/Sqrt[6])}]
```



$$\alpha \in \left[\frac{1}{\sqrt{3}}, 1 - \frac{1}{\sqrt{6}} \right]$$

(Fig. 2)

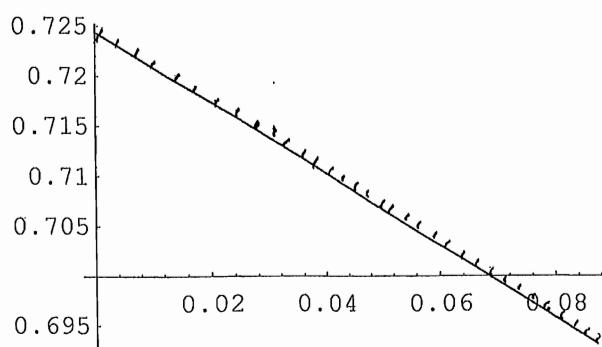
$$U^2 = \ln(2\alpha \cdot \frac{3}{2}\alpha)$$

Out[2]=

-Graphics-

In[3]:=

```
ParametricPlot[ {Log[(5 alpha/(2-alpha))*(3 alpha/(2-alpha))],
Log[((5(1-alpha))/(2-alpha)) ((3(1-alpha))/(2-alpha))]},
{alpha, (-1+Sqrt[15])/7, (11-Sqrt[30])/13}]
```



(Fig. 3)

Out[3]=

-Graphics-

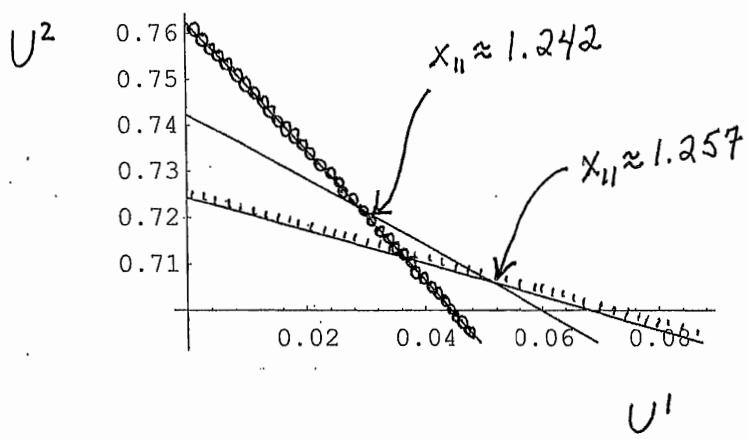


Fig.4

Answers to 2010 Qualifying Exam, Section I Question 2.

a) $U^1 = \ln x_{11} + \ln x_{21}$

$$U^2 = \ln x_{12} + \ln x_{22}$$

but along the contract curve for the 2-person economy, as given at the bottom of p. 2 of the old answer sheet,

$$x_{12} = 3 - x_{11}$$

$$x_{21} = \frac{2}{3} x_{11} \text{ and}$$

$$x_{22} = 2 - \frac{2}{3} x_{11},$$

so substituting into the utility functions results in

$$U^1 = \ln x_{11} + \ln \frac{2}{3} x_{11} = \ln \left(x_{11} \cdot \frac{2}{3} x_{11} \right)$$

$$U^2 = \ln (3 - x_{11}) + \ln \left(2 - \frac{2}{3} x_{11} \right) = \ln \left[(3 - x_{11}) \left(2 - \frac{2}{3} x_{11} \right) \right]$$

for the utility of persons 1 and 2 along the contract curve. Hence the graph shows the "utility possibility frontier" for the 2-person economy along the contract curve. The question says the graph shows

$$x_{11} \in \left[\frac{\sqrt{6}}{2}, 3 - \sqrt{3} \right] \approx [1.23, 1.27], \text{ which, according to the old}$$

answer sheet's p. 2, is where x_{11} is in the core of the 2-person economy.

So Fig. 1 shows the utility possibilities for the core of the 2-person economy.

b) $\omega_1 = (1, 1)$ and $\omega_2 = (2, 1)$.

In the 2-replica of the economy, there are four people, two of type 1 and two of type 2. We need to find which allocations in the 4-person economy (that is, the economy of the "grand coalition", $(1, 1, 2, 2)$) will not be blocked by other coalitions. By symmetry, the $(1, 2)$ coalition will not block any allocation in the core of the $(1, 1, 2, 2)$ coalition. So the only subcoalitions which could block some of $(1, 1, 2, 2)$'s core allocations would be the $(1, 1, 2)$ subcoalition and the $(1, 2, 2)$ subcoalition.

If the $(1, 1, 2)$ subcoalition broke away from the grand coalition, its feasible allocations would be constrained by its endowment $\omega_1 + \omega_1 + \omega_2 =$

$$\begin{array}{r} (1, 1) \\ + (1, 1) \\ + (2, 1) \\ \hline (4, 3) \end{array}$$

Due to the "equal treatment in the core" result, both Type 1 people would consume the same amount. This explains why the feasibility constraint for commodity 1 is $2x_{11} + x_{12} = 4$, and for commodity 2 is $2x_{21} + x_{22} = 3$.

To find the Pareto Optimal allocations for the $(1, 1, 2)$ coalition, you could $\max U^2$ s.t. $2U^1 = \text{const.}$, or $\max 2U^1$ s.t. $U^2 = \text{const.}$, or $\max 2\alpha U^1 + \alpha U^2$. But since

$$\max U^2 \text{ s.t. } 2U^1 = \text{const.} \Leftrightarrow$$

$$\max U^2 \text{ s.t. } U^1 = (\frac{1}{2} \text{ const.}) = (\text{new const.}) \Leftrightarrow$$

$$\max \alpha U^1 + (1-\alpha) U^2$$

or

$$\max 2U^1 \text{ s.t. } U^2 = \text{const.} \Leftrightarrow$$

$2 \max U^1 \text{ s.t. } U^2 = \text{const.}$ which has the same maximizing point as

$$\max U^1 \text{ s.t. } U^2 = \text{const.} \Leftrightarrow$$

$$\max \alpha U^1 + (1-\alpha) U^2,$$

one can also just maximize $\alpha U^1 + (1-\alpha) U^2$. $\nearrow (U^1 = \ln x_{11} x_{21} \text{ and } U^2 = \ln x_{12} x_{22}$
are elementary.)

c) The feasible allocations for the $(1, 2, 2)$ coalition would be constrained by

$$\begin{aligned} \text{its endowment } & e_1 + e_2 + e_2 = (1, 1) \\ & + (2, 1) \\ & + (2, 1) \\ & \hline (5, 3). \end{aligned}$$

Since this coalition has two people of type 2, who consume the same amount as each other due to "equal treatment," feasibility for good 1 implies $x_{11} + 2x_{12} = 5$, and for good 2 it is $x_{21} + 2x_{22} = 3$.

d) $\max \alpha \ln x_{11} x_{21} + (1-\alpha) \ln x_{12} x_{22}$ s.t. $2x_{11} + x_{12} = 4$
 $2x_{21} + x_{22} = 3 \iff$

$$\max \alpha \ln x_{11} x_{21} + (1-\alpha) \ln [(4-2x_{11})(3-2x_{21})].$$

If we call the objective function "W" (for "social welfare") then *

$$O = \frac{\partial W}{\partial x_{11}} = \alpha \frac{1}{x_{11} x_{21}} x_{21} + (1-\alpha) \frac{1}{(4-2x_{11})(3-2x_{21})} (-2)(3-2x_{21})$$

$$= \alpha \frac{1}{x_{11}} + (1-\alpha) \frac{(-2)}{4-2x_{11}} \Rightarrow \frac{1-\alpha}{2-x_{11}} = \frac{\alpha}{x_{11}}$$

$$x_{11} - \alpha x_{11} = 2\alpha - \alpha x_{11}$$

$x_{11} = 2\alpha$

 and

$$O = \frac{\partial W}{\partial x_{21}} = \alpha \frac{1}{x_{11} x_{21}} x_{11} + (1-\alpha) \frac{1}{(4-2x_{11})(3-2x_{21})} (-2)(4-2x_{11})$$

$$= \frac{\alpha}{x_{21}} + (1-\alpha) \frac{(-2)}{3-2x_{21}} \Rightarrow \frac{2-2\alpha}{3-2x_{21}} = \frac{\alpha}{x_{21}}$$

$$2x_{21} - 2\alpha x_{21} = 3\alpha - 2\alpha x_{21}$$

$x_{21} = \frac{3}{2}\alpha$

Then from feasibility,

$$\boxed{x_{12}} = 4 - 2x_{11} = 4 - 2(2\alpha) = \boxed{4-4\alpha} \text{ and}$$

$$\boxed{x_{22}} = 3 - 2x_{21} = 3 - 2\left(\frac{3}{2}\alpha\right) = \boxed{3-3\alpha}.$$

* (Expressing the objective function as $\alpha \ln x_{11} + \alpha \ln x_{21} + (1-\alpha) \ln x_{12} + (1-\alpha) \ln x_{22}$ would have made some of the calculations on this page easier.)

Optional: If instead one had maximized $2\alpha U^1 + (1-\alpha)U^2$ subject to the same constraints, one would have gotten

$$x_{11} = \frac{4\alpha}{1+\alpha}$$

$$x_{21} = \frac{3\alpha}{1+\alpha}$$

together with $x_{12} = 4 - 2x_{11}$ and $x_{22} = 3 - 2x_{12}$ from feasibility as before.

This looks different than our original answer

$$x_{11} = 2\alpha$$

$$x_{21} = \frac{3}{2}\alpha.$$

However, the answers are the same : as α goes from 0 to 1, both

$x_{11} = 2\alpha$ and $x_{11} = \frac{4\alpha}{1+\alpha}$ go from 0 to 2, and as x_{11} is going from

0 to 2, $x_{21} = \frac{3}{4}x_{11}$ for both the $x_{21} = \frac{3\alpha}{1+\alpha}$ form and the $\frac{3}{2}\alpha$ form.

(Because $x_{21} = \frac{3\alpha}{1+\alpha}$ is $\frac{3}{4}$ of $x_{11} = \frac{4\alpha}{1+\alpha}$ and

$x_{21} = \frac{3}{2}\alpha$ is $\frac{3}{4}$ of $x_{11} = 2\alpha$.)

This confirms that one can use the simpler objective function

$\alpha U^1 + (1-\alpha)U^2$ even when the coalition has an unequal number of Type 1 and Type 2 people.

End of Optional Comment.

[(d) continues]

This gives us the Pareto Optimal set for the $(1,1,2)$ coalition. To find its core, impose :

$$U^2 \geq U_0^2 \quad \text{and}$$

$$U^1 \geq U_0^1$$

$$\ln x_{12} x_{22} \geq \ln(2 \cdot 1)$$

$$\ln x_{11} x_{21} \geq \ln(1 \cdot 1)$$

$$x_{12} x_{22} \geq 2 \quad \text{since } \ln x \text{ is increasing in } x$$

$$x_{11} x_{21} \geq 1$$

$$2\alpha \cdot \frac{3}{2}\alpha \geq 1$$

$$\alpha^2 \geq \frac{1}{3}$$

$$(4-4\alpha)(3-3\alpha) \geq 2$$

$$12 - 24\alpha + 12\alpha^2 \geq 2$$

$$6 - 12\alpha + 6\alpha^2 \geq 1$$

$$\alpha \geq \frac{1}{\sqrt{3}} \approx 0.58$$

$$6\alpha^2 - 12\alpha + 5 \geq 0$$

Equality at

$$\alpha = \frac{12 \pm \sqrt{144 - 4(6)(5)}}{12}$$

$$= \frac{12 \pm \sqrt{144 - 120}}{12}$$

$$= \frac{12 \pm \sqrt{24}}{12} = \frac{12 \pm 2\sqrt{6}}{12}$$

$$= 1 \pm \frac{\sqrt{6}}{6}$$

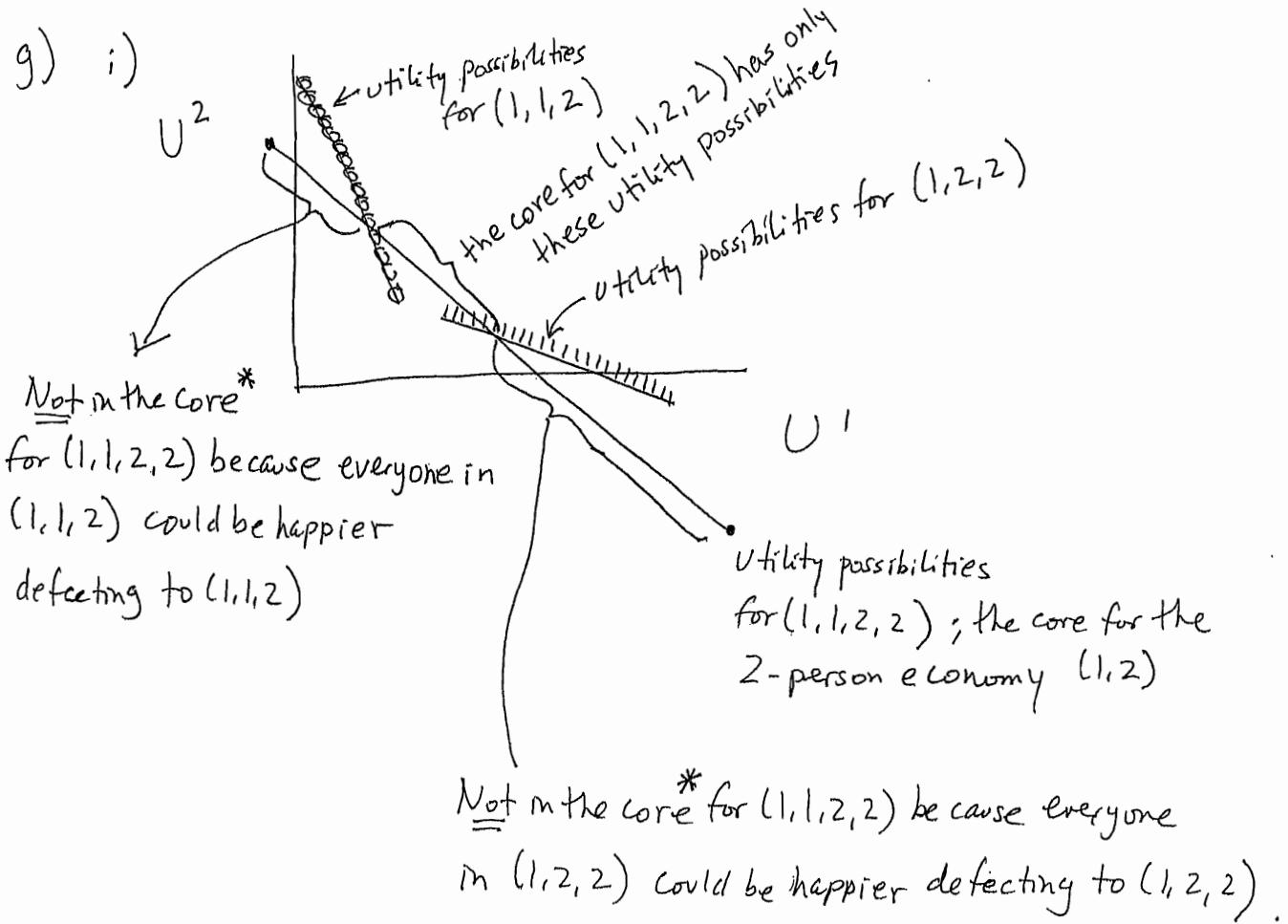
$b\alpha^2 - 12\alpha + 5 \geq 0$; ignore $\alpha > 1$
since $\alpha \in [0, 1]$

Sketch

So for both this ↑
and this to be true, α must be
between $\frac{1}{\sqrt{3}}$ and $1 - \frac{1}{\sqrt{6}}$.

$$\alpha < 1 - \frac{\sqrt{6}}{6} \approx 0.59$$

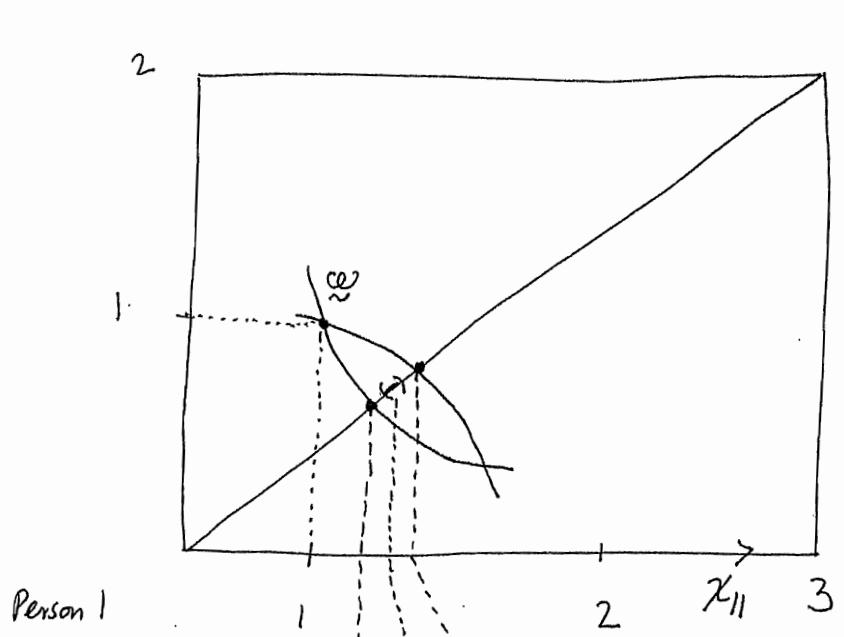
- e) This is the utility possibility frontier for the $(1, 1, 2)$ coalition. It shows what utility levels $(1, 1, 2)$ can reach by themselves. It comes from part (d).
- f) It is the utility possibility frontier for the $(1, 2, 2)$ coalition.
 (An optional proof follows my answer to part (g).)



So the core of the $(1, 1, 2, 2)$ economy has x_{11} between 1.242 and 1.257.

* "Not in the core" here means "Utility levels not reachable from the core."

ii) iii) iv)



i.2.7 the core of the two-person economy has
 $1.23 \leq x_{11} \leq 1.27$

the core of the 2-replica has

$$1.242 \leq x_{11} \leq 1.257$$

v) This illustrates the "shrinking core" result : the core shrinks with replications of the economy. Optional: In the "positive general equilibrium" section there is a question proving that the competitive general equilibrium value of x_{11} is $5/4 = 1.25$. The core shrinks towards that.

OPTIONAL

As promised in my answer to (f), here is the solution to (c).

$$\max \alpha \ln x_{11} x_{21} + (1-\alpha) \ln \left(\frac{5-x_{11}}{2} \frac{3-x_{21}}{2} \right)$$

$$0 = \frac{\partial W}{\partial x_{11}} = \frac{\alpha}{x_{11}} + (1-\alpha) \frac{(-2)}{5-x_{11}} \Rightarrow \frac{2-2\alpha}{5-x_{11}} = \frac{\alpha}{x_{11}}$$

$$2x_{11} - 2\alpha x_{11} = 5\alpha - \alpha x_{11}$$

$$2x_{11} - \alpha x_{11} = 5\alpha \Rightarrow x_{11} = \frac{5\alpha}{2-\alpha}$$

$$0 = \frac{\partial w}{\partial x_{21}} = \frac{\alpha}{x_{21}} + (1-\alpha) \frac{(-2)}{3-x_{21}} \Rightarrow \frac{2-2\alpha}{3-x_{21}} = \frac{\alpha}{x_{21}}$$

$$2x_{21} - 2\alpha x_{21} = 3\alpha - \alpha x_{21}$$

$$2x_{21} - \alpha x_{21} = 3\alpha$$

$$x_{21} = \frac{3\alpha}{2-\alpha}$$

Feasibility \Rightarrow $x_{12} = \frac{5-x_{11}}{2} = \frac{5}{2} - \frac{1}{2}x_{11} = \frac{5}{2} - \frac{1}{2} \frac{5\alpha}{2-\alpha} = \frac{10-5\alpha-5\alpha}{2(2-\alpha)}$

$$= \frac{5-5\alpha}{2-\alpha} = \boxed{\frac{5(1-\alpha)}{2-\alpha}} \text{ and}$$

$$x_{22} = \frac{3-x_{21}}{2} = \frac{3}{2} - \frac{1}{2}x_{21} = \frac{3}{2} - \frac{1}{2} \frac{3\alpha}{2-\alpha} = \frac{6-3\alpha-3\alpha}{2(2-\alpha)}$$

$$= \frac{3-3\alpha}{2-\alpha} = \boxed{\frac{3(1-\alpha)}{2-\alpha}}.$$

No blocking:

$$U^1 \geq U_0^1$$

and

$$U^2 \geq U_0^2$$

$$\ln x_{11} x_{21} \geq \ln 1 \cdot 1$$

$$\ln x_{12} x_{22} \geq \ln 2 \cdot 1$$

$$x_{11} x_{21} \geq 1$$

$$x_{12} x_{22} \geq 2$$

$$\frac{5\alpha}{2-\alpha} \frac{3\alpha}{2-\alpha} \geq 1$$

$$\frac{5(1-\alpha)}{2-\alpha} \frac{3(1-\alpha)}{2-\alpha} \geq 2$$

$$15\alpha^2 \geq (2-\alpha)^2$$

$$15(1-\alpha)^2 \geq 2(2-\alpha)^2$$

$$15\alpha^2 \geq 4-4\alpha+\alpha^2$$

$$15(1-2\alpha+\alpha^2) \geq 2(4-4\alpha+\alpha^2)$$

$$14\alpha^2+4\alpha-4 \geq 0.$$

$$15-30\alpha+15\alpha^2 \geq 8-8\alpha+2\alpha^2$$

Equality at:
 $\alpha = \frac{-4 \pm \sqrt{16 - 4(14)(-4)}}{2 \cdot 14}$

Equality at:
 $\alpha = \frac{22 \pm \sqrt{22^2 - 4(13)(7)}}{2 \cdot 13}$ continues

continues

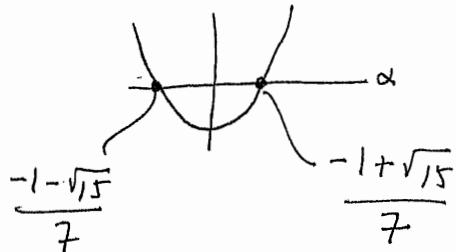


$$\alpha = \frac{-4 \pm \sqrt{16+224}}{28} = \frac{-4 \pm \sqrt{240}}{28}$$

$$= \frac{-4 \pm 4\sqrt{15}}{4 \cdot 7} = \frac{-1 \pm \sqrt{15}}{7}.$$

Note $\frac{-1+\sqrt{15}}{7} \approx +0.41$

Sketch of $14\alpha^2 + 4\alpha - 4$



For $14\alpha^2 + 4\alpha - 4 > 0$

and $\alpha \in [0, 1]$,

we need $\alpha > \frac{-1 + \sqrt{15}}{7} \approx 0.41$.

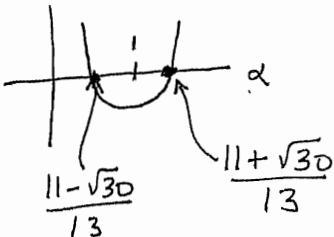
$$\alpha = \frac{22 \pm \sqrt{484-364}}{2 \cdot 13} = \frac{22 \pm \sqrt{120}}{2 \cdot 13}$$

$$= \frac{22 \pm 2\sqrt{30}}{2 \cdot 13} = \frac{11 \pm \sqrt{30}}{13}$$

Note $\frac{11-\sqrt{30}}{13} \approx 0.42$,

$$\frac{11+\sqrt{30}}{13} > 1$$

Sketch of $13\alpha^2 - 22\alpha + 7$



For $13\alpha^2 - 22\alpha + 7 > 0$

and $\alpha \in [0, 1]$,

we need $\alpha \leq \frac{11-\sqrt{30}}{13} \approx 0.42$.

So to satisfy both \uparrow and \nearrow , we need

$$\alpha \in \left[\frac{-1 + \sqrt{15}}{7}, \frac{11 - \sqrt{30}}{13} \right] \approx [0.41, 0.42];$$

and

$$(v^1, v^2) = \left(\ln \frac{5\alpha}{2-\alpha} \frac{3\alpha}{2-\alpha}, \ln \frac{5(1-\alpha)}{2-\alpha} \frac{3(1-\alpha)}{2-\alpha} \right).$$

Finally, incase you are curious, here is how I found the two intersection points in Figure 4. The first point is the intersection of the utility possibility curve for $(1, 1, 2)$ with that for $(1, 1, 2, 2)$. Look at Figures 1 and 2. We can think of the problem as one of finding an (x_{11}, α) pair which simultaneously makes U^1 in Fig. 1 equal to U^1 in Fig. 2 and makes U^2 in Fig. 1 equal to U^2 in Fig. 2. This implies a system of two equations in two unknowns,

$$x_{11} \cdot \frac{2}{3} x_{11} = 2\alpha \cdot \frac{3}{2} \alpha$$

$$(3 - x_{11}) (2 - \frac{2}{3} x_{11}) = (4 - 4\alpha) (3 - 3\alpha).$$

The first equation implies $\frac{2}{3} x_{11}^2 = 3\alpha^2 \Rightarrow \alpha^2 = \frac{2}{9} x_{11}^2 \Rightarrow \alpha = \pm \frac{\sqrt{2}}{3} x_{11}$, but since $\alpha \in [0, 1]$, and $x_{11} > 0$, we can rule out $\alpha = -\frac{\sqrt{2}}{3} x_{11} \leq 0$, so $\alpha = +\frac{\sqrt{2}}{3} x_{11}$. The second equation simplifies to

$$3 - 2x_{11} + \frac{1}{3} x_{11}^2 = 6 - 12\alpha + 6\alpha^2 ;$$

Substituting $\alpha = \frac{\sqrt{2}}{3} x_{11}$ into this eventually gives

$$0 = x_{11}^2 + (2 - 4\sqrt{2}) x_{11} + 3,$$

which eventually gives

$$x_{11} = -1 + 2\sqrt{2} \pm \sqrt{6 - 4\sqrt{2}}.$$

The “-” sign gives $x_{11} \approx 1.242$, which is the answer. The “+” sign gives $x_{11} \approx 2.4$, which would not be in the core of the $(1, 2)$ economy.

The second intersection point arises from an (x_{11}, α) pair which

simultaneously makes U' in Fig. 1 equal to U' on (the horizontal axis of) Fig. 3 and makes U^2 in Fig. 1 equal to U^2 on (the vertical axis of) Fig. 3. Using the (optional) solution to part (c), the two-by-two system is

$$\chi_{11} \cdot \frac{2}{3} \chi_{11} = \frac{5\alpha}{2-\alpha} \cdot \frac{3\alpha}{2-\alpha}$$

$$(3 - \chi_{11})(2 - \frac{2}{3} \chi_{11}) = \frac{5(1-\alpha)}{2-\alpha} \cdot \frac{3(1-\alpha)}{2-\alpha}.$$

In this case, a numerical solution is easier than an analytical one to obtain, if you have the right software. For an analytical solution, the first equation implies

$$\frac{2}{3} \chi_{11}^2 = \frac{15\alpha^2}{(2-\alpha)^2}$$

$$\chi_{11}^2 = \frac{45\alpha^2}{2(2-\alpha)^2}$$

$$\chi_{11} = \pm \sqrt{\frac{45}{2}} \frac{\alpha}{2-\alpha} \quad \text{but since } \alpha \in [0, 1] \text{ and}$$

$$\text{we need } \chi_{11} > 0, \quad \chi_{11} = + \sqrt{\frac{45}{2}} \frac{\alpha}{2-\alpha}.$$

Substituting into the second equation,

$$\left[3 - \sqrt{\frac{45}{2}} \frac{\alpha}{2-\alpha} \right] \left[2 - \frac{2}{3} \sqrt{\frac{45}{2}} \frac{\alpha}{2-\alpha} \right] = \frac{15(1-\alpha)^2}{(2-\alpha)^2}.$$

Multiply both sides by $(2-\alpha)^2$:

$$\left[3(2-\alpha) - \sqrt{\frac{45}{2}} \alpha \right] \left[2(2-\alpha) - \frac{2}{3} \sqrt{\frac{45}{2}} \alpha \right] = 15(1-\alpha)^2$$

$$\left[6 - \left(3 + \sqrt{\frac{45}{2}} \right) \alpha \right] \left[4 - \left(2 + \frac{2}{3} \sqrt{\frac{45}{2}} \right) \alpha \right] = 15(1-\alpha)^2$$

$$24 - 6 \left(2 + \frac{2}{3} \sqrt{\frac{45}{2}} \right) \alpha - 4 \left(3 + \sqrt{\frac{45}{2}} \right) \alpha + \left(3 + \sqrt{\frac{45}{2}} \right) \left(2 + \frac{2}{3} \sqrt{\frac{45}{2}} \right) \alpha^2 = 15 - 30\alpha + 15\alpha^2$$

which after a while simplifies to

$$0 = \left[-6 - 4\sqrt{\frac{45}{2}} \right] \alpha^2 + \left[-6 + 8\sqrt{\frac{45}{2}} \right] \alpha - 9.$$

So

$$\alpha = \frac{6 - 8\sqrt{\frac{45}{2}} \pm \sqrt{(-6 + 8\sqrt{\frac{45}{2}})^2 - 4(-6 - 4\sqrt{\frac{45}{2}})(-9)}}{2(-6 - 4\sqrt{\frac{45}{2}})} \Rightarrow$$

$$= \frac{6 - 8\sqrt{\frac{45}{2}} \pm 2\sqrt{315 - 60\sqrt{\frac{45}{2}}}}{2(-6 - 4\sqrt{\frac{45}{2}})} = \frac{-3 + 4\sqrt{\frac{45}{2}} \mp \sqrt{315 - 60\sqrt{\frac{45}{2}}}}{6 + 4\sqrt{\frac{45}{2}}}$$

and since $\sqrt{\frac{45}{2}} = \sqrt{\frac{9 \cdot 5}{2}} = 3\sqrt{\frac{5}{2}}$,

$$\alpha = \frac{-3 + 12\sqrt{\frac{5}{2}} \mp \sqrt{315 - 180\sqrt{\frac{5}{2}}}}{6 + 12\sqrt{\frac{5}{2}}} = \frac{-1 + 4\sqrt{\frac{5}{2}} \mp \sqrt{35 - 20\sqrt{\frac{5}{2}}}}{2 + 4\sqrt{\frac{5}{2}}}.$$

Choosing the "+" sign, calculating α , then calculating $x_{11} = \sqrt{\frac{45}{2}} \frac{\alpha}{2-\alpha}$, gives an $x_{11} > 3$, which is not feasible. So the correct α uses the "-" sign, and it leads to the $x_{11} = 1.257$ given in Fig. 4.

4. [17 points] Consider a pure exchange economy with two persons named "a" and "b" and two commodities named "1" and "2." The utility function and endowment of person "a" are given by

$$u_a = \ln(x_{1a}x_{2a}) \quad \omega_a = (1, 0)$$

and the utility function and endowment of person "b" are given by

$$u_b = \ln(x_{1b}x_{2b}) \quad \omega_b = (0, 2).$$

- (a) Find the competitive general equilibrium prices and quantities in this economy.
- (b) Find the Pareto Optimal allocations in this economy.
- (c) Is the competitive general equilibrium in this economy Pareto Optimal? Should it be?
- (d) What is the core of this economy? (Sketching an Edgeworth Box might help in answering this.)

(4)

$$u_a = \ln(x_{1a} x_{2a}) = \ln x_{1a} + \ln x_{2a} \quad w_a = (1, 0)$$

$$u_b = \ln(x_{1b} x_{2b}) = \ln x_{1b} + \ln x_{2b} \quad w_b = (0, z)$$

a) Person a:

Instead of using a numeraire (either $p_1 = 1$ or $p_2 = 1$), I'll work on the simplex, so $p_1 + p_2 = 1$.

$$\mathcal{L} = \underbrace{\ln x_{1a} + \ln x_{2a}}_{\text{easier to work with than }} + \lambda \left(\underbrace{p_1 1 + p_2 0}_{\text{income}} - \underbrace{p_1 x_{1a} - p_2 x_{2a}}_{\text{expenditures}} \right) \\ \ln(x_{1a} x_{2a})$$

F.O.C.

$$x_{1a}: D = \frac{1}{x_{1a}} - \lambda p_1 \quad \left. \begin{array}{l} \lambda = \frac{1}{p_1 x_{1a}} = \frac{1}{p_2 x_{2a}} \Rightarrow x_{2a} = \frac{p_1}{p_2} x_{1a}, \\ x_{2a} = \frac{p_1}{1-p_1} x_{1a} \text{ using the price simplex} \end{array} \right\}$$

$$x_{2a}: D = \frac{1}{x_{2a}} - \lambda p_2$$

$$\lambda: D = p_1 - p_1 x_{1a} - p_2 x_{2a}$$

$$D = p_1 - p_1 x_{1a} - (1-p_1) \frac{p_1}{1-p_1} x_{1a}$$

$$= p_1 - 2p_1 x_{1a}$$

$$= 1 - 2x_{1a} \Rightarrow 2x_{1a} = 1 \Rightarrow x_{1a}^* = \frac{1}{2} \text{ and}$$

$$x_{2a}^* = \frac{p_1}{1-p_1} \frac{1}{2}.$$

The fact that the demand for x_{1a} does not depend on price may seem odd, but Person "a" might supply Good 1; he has an endowment of it.

Person b :

$$\mathcal{L} = \ln x_{1b} + \ln x_{2b} + \lambda (p_1 0 + p_2 2 - p_1 x_{1b} - p_2 x_{2b})$$

$$\begin{array}{l} F.O.C. \quad 0 = \frac{1}{x_{1b}} - \lambda p_1 \\ 0 = \frac{1}{x_{2b}} - \lambda p_2 \end{array} \quad \left\{ \quad \frac{p_1}{p_2} = \frac{x_{2b}}{x_{1b}} \Rightarrow x_{2b} = \frac{p_1}{p_2} x_{1b} \right.$$

$$0 = 2p_2 - p_1 x_{1b} - p_2 x_{2b} \quad \leftarrow \quad = \frac{p_1}{1-p_1} x_{1b}$$

$$= 2p_2 - p_1 x_{1b} - p_2 \frac{p_1}{1-p_1} x_{1b}$$

$$= 2(1-p_1) - p_1 x_{1b} - (1-p_1) \frac{p_1}{1-p_1} x_{1b}$$

$$= 2(1-p_1) - 2p_1 x_{1b}$$

$$= 1 - p_1 - p_1 x_{1b} \Rightarrow x_{1b}^D = \frac{1-p_1}{p_1}$$

$$x_{2b}^D = \frac{p_1}{1-p_1} x_{1b} = \frac{p_1}{1-p_1} \frac{1-p_1}{p_1} = 1$$

Supply = Demand for Good I :

$$\text{Supply} = 1$$

$$\text{Demand} = x_{1a}^D + x_{1b}^D = \frac{1}{2} + \frac{1-p_1}{p_1}$$

$$S=D \Rightarrow 1 = \frac{1}{2} + \frac{1-p_1}{p_1}$$

$$\frac{1}{2} = \frac{1-p_1}{p_1}$$

$$p_1 = 2 - 2p_1$$

$$3p_1 = 2$$

$$\underline{p_1 = \frac{2}{3}} \Rightarrow p_2 = 1 - p_1 = \underline{\frac{1}{3}}$$

$$x_{1a}^* = \frac{1}{2}, \quad x_{1b}^* = \frac{1-2/3}{2/3} = \frac{1/3}{2/3} = \frac{1}{2}, \quad x_1^{S*} = 1$$

$$\text{Good 2: } x_2^* = 2$$

$$x_{2a}^* = \frac{p_1}{1-p_1} \cdot \frac{1}{2} = \frac{2/3}{1/3} \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1$$

$x_{2b}^* = 1$. (As long as the Good 1 market clears, so will the Good 2 market.)

b)

$$x_{1a} + x_{1b} = 1 \Rightarrow x_{1b} = 1 - x_{1a}$$

$$x_{2a} + x_{2b} = 2 \Rightarrow x_{2b} = 2 - x_{2a}$$

We'll maximize u_a s.t. u_b is constant:

$$\begin{aligned} \mathcal{L} &= \ln(x_{1a} x_{2a}) + \lambda [u_b - \ln(1-x_{1a})(2-x_{2a})] \\ &= \ln x_{1a} + \ln x_{2a} + \lambda [u_b - \ln(1-x_{1a}) - \ln(2-x_{2a})] \end{aligned}$$

F.O.C.

$$x_{1a}: 0 = \frac{1}{x_{1a}} + \lambda \frac{1}{1-x_{1a}} \Rightarrow \lambda = -\frac{1-x_{1a}}{x_{1a}}$$

$$x_{2a}: 0 = \frac{1}{x_{2a}} + \lambda \frac{1}{2-x_{2a}} \Rightarrow 0 = \frac{1}{x_{2a}} - \frac{1-x_{1a}}{x_{1a}} \frac{1}{2-x_{2a}}$$

$$\frac{1-x_{1a}}{x_{1a}(2-x_{2a})} = \frac{1}{x_{2a}}$$

$$\frac{1-x_{1a}}{x_{1a}} = \frac{2-x_{2a}}{x_{2a}}$$

$$\frac{1}{x_{1a}} - 1 = \frac{2}{x_{2a}} - 1$$

$$\frac{1}{x_{1a}} = \frac{2}{x_{2a}}$$

$$x_{2a} = 2x_{1a}$$

So the contract curve, described parametrically, is

$$x_{1a}$$

$$x_{2a} = 2x_{1a}$$

$$x_{1b} = 1 - x_{1a}$$

$$x_{2b} = 2 - x_{2a} = 2 - 2x_{1a},$$

as $x_{1a} \in [0, 1]$.

- c) In competitive equilibrium, $x_{1a}^* = \frac{1}{2}$. The corresponding point on the contract curve (which is the set of Pareto Optimal points) is

$$x_{1a} = \frac{1}{2}$$

$$x_{2a} = 2x_{1a} = 1$$

$$x_{1b} = 1 - x_{1a} = \frac{1}{2}$$

$$x_{2b} = 2 - 2x_{1a} = 1.$$

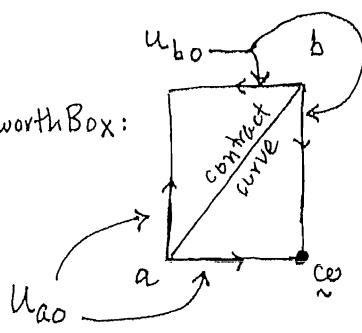
Since this is equal to the competitive equilibrium of part (a), the C.E. is Pareto Optimal.

This is as expected from the First Theorem of Welfare Economics.

- d) The core is the set of all Pareto Optimal points which cannot be blocked by either person; in this case, it's all the Pareto Optimal points whose utility for Person a is more than that person's initial utility, and



Edgeworth Box:



At the initial point,

$$U_a = \ln(1 \cdot 0) = \ln 0 = -\infty \text{ and}$$

$U_b = \ln(0 \cdot 2) = \ln 0 = -\infty$. Nothing is worse than this, so nothing will be blocked: the whole contract curve is the core.

*Fall 2006
Final*

6. [11 points]

- (a) Suppose an economy consists of two persons, A, and B. They have initial endowments ω_a and ω_b , respectively. They have consumption bundles x_a and x_b , respectively. They have utility functions $u_a(x_a)$ and $u_b(x_b)$, respectively.

What procedure would you use to calculate the core of this economy? List all the steps. If there are optimization problems involved, give the Lagrangians and state what the unknowns are. You need not calculate nor state any first-order conditions. I just want you to state a step-by-step procedure which someone else who knew multidimensional optimization but knew nothing about the core could follow to calculate the core, if that person knew what the utility functions were.

- (b) Suppose an economy consists of three persons, A, B, and C. They have initial endowments ω_a , ω_b , and ω_c , respectively. They have consumption bundles x_a , x_b , and x_c , respectively. They have utility functions $u_a(x_a)$, $u_b(x_b)$, and $u_c(x_c)$, respectively.

What procedure would you use to calculate the core of this economy? List all the steps. If there are optimization problems involved, give the Lagrangians and state what the unknowns are. You need not calculate nor state any first-order conditions. I just want you to state a step-by-step procedure which someone else who knew multidimensional optimization but knew nothing about the core could follow to calculate the core, if that person knew what the utility functions were.

⑥ a)

$$\text{Feasibility } \underline{x}_a + \underline{x}_b = \underline{w}_a + \underline{w}_b$$

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Pareto optimality $\max u_b(\underline{x}_b)$ s.t. $u_a(\underline{x}_a)$ fixed at \bar{u}_a ; or

$$\max \alpha u_a(\underline{x}_a) + (1-\alpha) u_b(\underline{x}_b) \quad (\text{either approach works})$$

No Blocking: $u_a(\underline{x}_a) \geq u_a(\underline{w}_a)$

$$u_b(\underline{x}_b) \geq u_b(\underline{w}_b).$$

Lagrangians, first approach:

$$\mathcal{L} = u_b(\underline{x}_b) + \lambda \left[u_a(\underline{x}_a) - \bar{u}_a \right] + \mu \cdot \left[\underline{x}_a + \underline{x}_b - \underline{w}_a - \underline{w}_b \right] \quad \text{or}$$

$$\mathcal{L} = u_b(\underline{x}_b) + \lambda \left[u_a(\underline{w}_a + \underline{w}_b - \underline{x}_b) - \bar{u}_a \right] ; \text{arrows point to the}$$

Second approach:

$$\mathcal{L} = \alpha u_a(\underline{x}_a) + (1-\alpha) u_b(\underline{x}_b) + \mu \cdot \left[\underline{x}_a + \underline{x}_b - \underline{w}_a - \underline{w}_b \right] \quad \text{or}$$

$$\mathcal{L} = \alpha u_a(\underline{w}_a + \underline{w}_b - \underline{x}_b) + (1-\alpha) u_b(\underline{x}_b)$$

b) Feasibility $\underline{x}_a + \underline{x}_b + \underline{x}_c = \underline{w}_a + \underline{w}_b + \underline{w}_c$

Pareto optimality: Either

$$\max u_b(\underline{x}_b) \text{ s.t. } u_a(\underline{x}_a) \text{ fixed at } \bar{u}_a \text{ and } u_b(\underline{x}_b) \text{ fixed at } \bar{u}_b ; \text{ or}$$

$$\max \alpha_a u_a(\underline{x}_a) + \alpha_b u_b(\underline{x}_b) + (1-\alpha_a-\alpha_b) u_c(\underline{x}_c).$$

Lagrangians, first approach:

$$\begin{aligned} \mathcal{L} &= u_b(\underline{x}_b) + \lambda_a \left[u_a(\underline{x}_a) - \bar{u}_a \right] \\ &\quad + \lambda_c \left[u_c(\underline{x}_c) - \bar{u}_c \right] + \mu \cdot \left[\underline{x}_a + \underline{x}_b + \underline{x}_c - w_a - w_b - w_c \right] \quad \text{or} \\ \mathcal{L} &= u_b(\underline{x}_b) + \lambda_a \left[u_a(w_a + w_b + w_c - \underline{x}_b - \underline{x}_c) - \bar{u}_a \right] \\ &\quad + \lambda_c \left[u_c(\underline{x}_c) - \bar{u}_c \right]. \end{aligned}$$

Second approach:

$$\begin{aligned} \mathcal{L} &= \alpha_a u_a(\underline{x}_a) + \alpha_b u_b(\underline{x}_b) + (1-\alpha_a-\alpha_b) u_c(\underline{x}_c) + \mu \cdot \left[\underline{x}_a + \underline{x}_b + \underline{x}_c - w_a - w_b - w_c \right] \\ \text{or} \\ \mathcal{L} &= \alpha_a u_a(w_a + w_b + w_c - \underline{x}_b - \underline{x}_c) + \alpha_b u_b(\underline{x}_b) + (1-\alpha_a-\alpha_b) u_c(\underline{x}_c) \end{aligned}$$

No Blocking: Clearly we need $u_a(\underline{x}_a) \geq u_a(w_a)$,
 $u_b(\underline{x}_b) \geq u_b(w_b)$,
 $u_c(\underline{x}_c) \geq u_c(w_c)$. But this is not all. Let

the notation \underline{x}_i^{ij} be the consumption bundle which agent i could get if agent i were in a two-person coalition with person j . The \underline{x}_i^{ij} can be calculated using the procedure in part (a), for two-person coalitions. Then we also need $u_a(\underline{x}_a) \geq u_a(\underline{x}_a^{ab})$, $u_a(\underline{x}_a) \geq u_a(\underline{x}_a^{ac})$,
 $u_b(\underline{x}_b) \geq u_b(\underline{x}_b^{ab})$, $u_b(\underline{x}_b) \geq u_b(\underline{x}_b^{bc})$,
 $u_c(\underline{x}_c) \geq u_c(\underline{x}_c^{ac})$, $u_c(\underline{x}_c) \geq u_c(\underline{x}_c^{bc})$.

2020 Qualifying Exam Sec. 1 Qu. 2 [Open-book exam due to the pandemic]

2. [17 points]

- (a) Suppose an economy consists of two consumers, Jones “ j ” and Smith “ s ,” and two commodities, apples “ a ” and bananas “ b .” Suppose that the total amounts of apples and bananas in this economy are given by

$$\begin{aligned} a_j + a_s &= 1 \quad \text{and} \\ b_j + b_s &= 1 \end{aligned}$$

and the utility functions for Jones and Smith are

$$\begin{aligned} u_j(a_j, b_j) &= a_j \cdot b_j \\ u_s(a_s, b_s) &= a_s + \ln(b_s + 1). \end{aligned}$$

Derive the equation or equations which I must have used to draw Figure 2 (this economy’s “utility possibility frontier”), which shows all the possible utility levels for Jones and for Smith for all the Pareto Optimal allocations possible in this economy. Do this by:

- i. eliminating a_s and b_s from your calculations;
- ii. either giving the answer as one equation containing u_j and u_s , or giving the answer as two equations, each expressing u_j or u_s as a function of a_j (or b_j), and give the relevant upper and lower bounds of a_j (or b_j).

(I add these requirements because there are many equally-good ways of solving this problem and if students choose among these ways arbitrarily, it will take me a very long time to grade the varying answers.)

- (b) What is the u_j -intercept and the u_s -intercept of the curve in Figure 2?
- (c) If the initial endowments are $a_j = 0$, $b_j = 1$, $a_s = 1$, and $b_s = 0$, indicate in Figure 2 which part of the utility possibility frontier would be in the core of this economy.

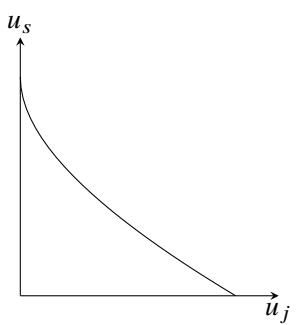


Figure 2. The utility possibility frontier.

Answer to 2020 Qualifier, Section 1 Question 2

a) $a_j + a_s = 1 \quad u_j = a_j b_j$
 $b_j + b_s = 1 \quad u_s = a_s + \ln(b_s + 1)$

\Downarrow

$a_s = 1 - a_j$
 $b_s = 1 - b_j \Rightarrow u_s = 1 - a_j + \ln(1 - b_j + 1) = 1 - a_j + \ln(2 - b_j)$

Find the Pareto Optimal points:

Method 1: $\max u_j$ s.t. u_s fixed

Method 2: $\max u_s$ s.t. u_j fixed

Method 3: $\max \alpha u_j + (1-\alpha) u_s$.

You can use any of these methods. I'll illustrate Method 3.

$$\max_{a_j, b_j} \alpha u_j + (1-\alpha) u_s \Rightarrow \max_{a_j, b_j} \alpha a_j b_j + (1-\alpha) [1 - a_j + \ln(2 - b_j)]$$

s.t. $0 \leq a_j \leq 1$
 $0 \leq b_j \leq 1$.

F.O.C.

$$0 = \frac{\partial \text{maximand}}{\partial a_j} = \alpha b_j + (1-\alpha) [-1] \Rightarrow \alpha b_j = 1 - \alpha$$

$$0 = \frac{\partial \text{maximand}}{\partial b_j} = \alpha a_j + (1-\alpha) \left[\frac{-1}{2-b_j} \right] \Rightarrow \alpha a_j = \frac{1-\alpha}{2-b_j}$$

\Downarrow

$$\frac{\alpha b_j}{\alpha a_j} = \frac{1-\alpha}{1} \cdot \frac{2-b_j}{1-\alpha} \Leftrightarrow \frac{b_j}{a_j} = 2 - b_j$$

$$\text{so } a_j = \frac{b_j}{2-b_j} \text{ and } u_j = a_j b_j = \frac{b_j}{2-b_j} b_j = \frac{b_j^2}{2-b_j}$$

$$u_s = 1 - a_j + \ln(2 - b_j) = 1 - \frac{b_j}{2-b_j} + \ln(2 - b_j)$$

$$\text{with } 0 \leq b_j \leq 1$$

} one answer

Or, expressed with a_j : $a_j = \frac{b_j}{2a_j + b_j}$

$$2a_j - a_j b_j = b_j$$

$$2a_j = (a_j + 1)b_j$$

$$\frac{2a_j}{a_j + 1} = b_j$$

$$u_j = a_j b_j = a_j \frac{2a_j}{a_j + 1} = \frac{2a_j^2}{a_j + 1}$$

$$u_s = 1 - a_j + \ln(2 - b_j) = 1 - a_j + \ln\left(2 - \frac{2a_j}{a_j + 1}\right)$$

$$= 1 - a_j + \ln \frac{2a_j + 2 - 2a_j}{a_j + 1} = 1 - a_j + \ln \frac{2}{a_j + 1}$$

second answer
 $(0 \leq a_j \leq 1)$

Or, another expression: $u_j = \frac{2a_j^2}{a_j + 1} \Rightarrow u_j a_j + u_j = 2a_j^2 \Rightarrow$

$$2a_j^2 - u_j a_j - u_j = 0$$

$$a_j = \frac{+u_j \pm \sqrt{u_j^2 - 4 \cdot 2 \cdot (-u_j)}}{4} = \frac{u_j \pm \sqrt{u_j^2 + 8u_j}}{4} \quad \begin{matrix} \text{we need } a_j > 0 \text{ so} \\ \text{pick the "+" sign:} \end{matrix}$$

$$a_j = \frac{u_j + \sqrt{u_j^2 + 8u_j}}{4}$$

Then from above, yet another expression can be obtained:

$$u_s = 1 - a_j + \ln \frac{2}{a_j + 1} = 1 - \frac{u_j + \sqrt{u_j^2 + 8u_j}}{4} + \ln \frac{2}{\frac{u_j + \sqrt{u_j^2 + 8u_j}}{4} + 1}$$

$$= \frac{4 - u_j - \sqrt{u_j^2 + 8u_j}}{4} + \ln \frac{2}{\frac{u_j + \sqrt{u_j^2 + 8u_j}}{4} + 4}$$

$$= \frac{4 - u_j - \sqrt{u_j^2 + 8u_j}}{4} + \ln \frac{8}{u_j + \sqrt{u_j^2 + 8u_j} + 4}$$

This directly relates u_j and u_s .

third answer

b) Giving Jones all the apples and bananas implies $a_j = 1, b_j = 1, a_s = 0, b_s = 0$:

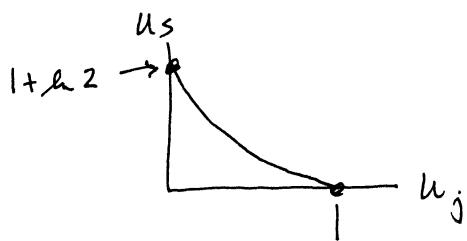
$$u_j = a_j \cdot b_j = 1 \cdot 1 = 1$$

$$u_s = a_s + \ln(b_s + 1) = 0 + \ln(0 + 1) = 0.$$

Giving Smith all the apples and bananas, $a_s = 1, b_s = 1, a_j = 0, b_j = 0$:

$$u_j = a_j \cdot b_j = 0 \cdot 0 = 0$$

$$u_s = a_s + \ln(b_s + 1) = 1 + \ln(1 + 1) = 1 + \ln 2 \approx 1.69.$$

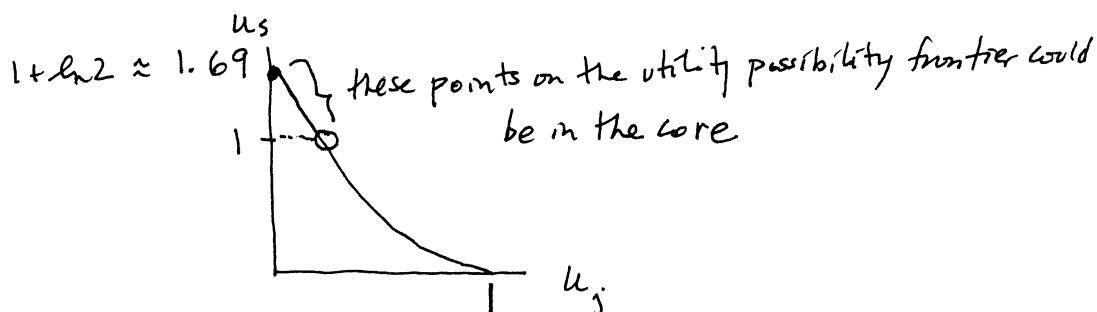


Note: A utility possibilities curve can be in any quadrant or quadrants; I designed the minimum utilities to be zero just for convenience. A more natural way to answer this question does not start with an allocation, but rather with $u_j = 0$ or $u_s = 0$.

c) $a_j = 0 \quad b_j = 1 \Rightarrow u_j = a_j \cdot b_j = 0 \cdot 1 = 0$

$$a_s = 1 \quad b_s = 0 \Rightarrow u_s = a_s + \ln(b_s + 1) = 1 + \ln(0 + 1) = 1 + 0 = 1.$$

So any point with $u_j > 0$ and $u_s > 1$ would be in the core.



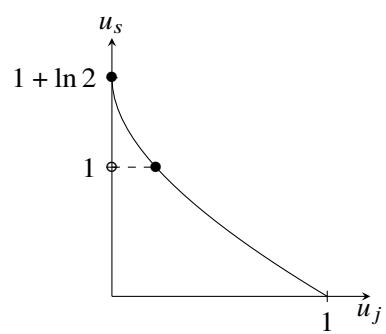


Figure 5. The utility possibility frontier.

2016 Final Exam Qu. 4

4. [17 points] Suppose an economy consists of two persons, denoted “1” and “2,” and two commodities, apples “ a ” and bananas “ b . ” Suppose there is no production in this economy (that is, apples and bananas are available for free—though in limited quantities, as explained below). Suppose the utility functions of persons 1 and 2 are

$$u_1 = a_1 + b_1 \quad \text{and}$$

$$u_2 = a_2 + \frac{1}{4}b_2$$

respectively.

- (a) Explain why one of the lines in Figure 1 represents all possible combinations for the utility of the two people, (u_1, u_2) , if the society has available 16 apples and no bananas. Confirm that the coordinates given for the four points on that line are their correct coordinates. (Figure 1 is *not* an Edgeworth Box.)
 - (b) Explain why one of the lines in Figure 1 represents all possible combinations for the utility of the two people, (u_1, u_2) , if the society has available no apples and 32 bananas. Confirm that the coordinates given for the four points on that line are their correct coordinates.
 - (c) Suppose a social planner is trying to decide if this society should have “16 apples and no bananas” or “no apples and 32 bananas.” Call the first option “a $(16, 0)$ economy” and the second option “a $(0, 32)$ economy.”
- Is a move from $(u_1, u_2) = (4, 7)$ to $(u_1, u_2) = (12, 4)$ an actual Pareto Improvement?
- (d) The definition of a “Potential Pareto Improvement” is:

Economy E' is a “Potential Pareto Improvement” over economy E if there exists at least one (re)allocation of commodities in E' which is preferred by everyone to a given initial allocation of commodities in E .

Explain why a move from the economy generating $(u_1, u_2) = (4, 7)$ to the economy generating $(u_1, u_2) = (12, 4)$ is a Potential Pareto Improvement. Use the points given on the diagram in your explanation.

- (e) Is a move from $(u_1, u_2) = (12, 4)$ to $(u_1, u_2) = (4, 7)$ an actual Pareto Improvement?

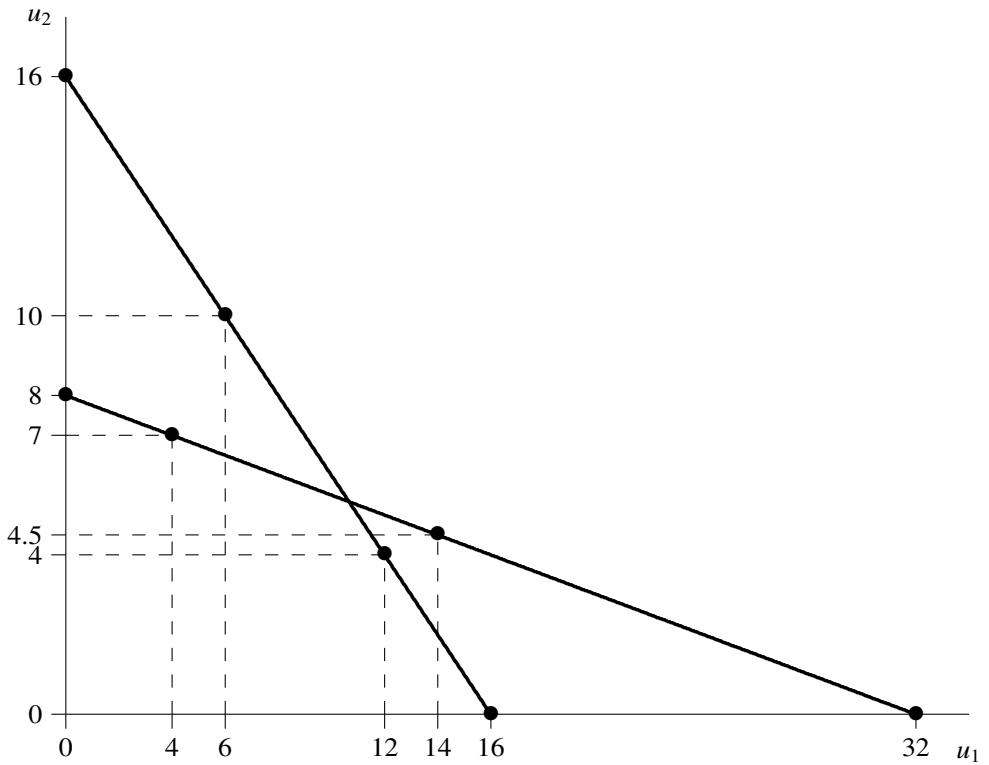


Figure 1

- (f) Explain why a move from the economy generating $(u_1, u_2) = (12, 4)$ to the economy generating $(u_1, u_2) = (4, 7)$ is a Potential Pareto Improvement. Use the points given on the diagram in your explanation.
- (g) In light of the results in (d) and (f) of this question, criticize the decision to use the criterion of “Potential Pareto Improvement” to make social decisions. In other words, attack the following position:
 “A social planner should choose economy E' over economy E if E' is a Potential Pareto Improvement over E .”

Answer to Question 4, Final Exam, Econ. 7005, Fall 2016

- a. If a social planner were to give 100% weight to Person 1, then Person 1 would get all of this society's resources: 16 apples and no bananas. In that case, $u_1 = 16 + 0 = 16$ and $u_2 = 0 + \frac{1}{4}0 = 0$, giving rise to the point $(16, 0)$ on the horizontal axis of Figure 1.

If a social planner were to give 100% weight to Person 2, then Person 2 would get all of this society's resources: 16 apples and no bananas. In that case, $u_1 = 0 + 0 = 0$ and $u_2 = 16 + \frac{1}{4}0 = 16$, giving rise to the point $(0, 16)$ on the vertical axis of Figure 1.

Since this is a pure exchange economy and the utility functions are linear, social weights between 0% and 100% for either person would generate a straight line between the endpoints. The equation of this line is $u_2 = -u_1 + 16$. Points $(6, 10)$ and $(12, 4)$ lie on this line because

$$10 = -6 + 16 \quad \text{and}$$
$$4 = -12 + 16.$$

- b. If a social planner were to give 100% weight to Person 1, then Person 1 would get all of this society's resources: 0 apples and 32 bananas. In that case, $u_1 = 0 + 32 = 32$ and $u_2 = 0 + \frac{1}{4} \cdot 0 = 0$, giving rise to the point $(32, 0)$ on the horizontal axis of Figure 1.

If a social planner were to give 100% weight to Person 2, then Person 2 would get all of this society's resources: 0 apples and 32 bananas. In that case, $u_1 = 0 + 0 = 0$ and $u_2 = 0 + \frac{1}{4} \cdot 32 = 8$, giving rise to the point $(0, 8)$ on the vertical axis of Figure 1.

Since this is a pure exchange economy and the utility functions are linear, social weights between 0% and 100% for either person would generate a straight line between the endpoints. The equation of this line is $u_2 = -\frac{1}{4}u_1 + 8$. Points $(4, 7)$ and $(14, 4.5)$ lie on this line because

$$7 = -\frac{1}{4} \cdot 4 + 8 \quad \text{and}$$
$$4.5 = -\frac{1}{4} \cdot 14 + 8.$$

- c. No. An actual Pareto Improvement requires all agents to be better off (or at least no worse off). In the case of this move, Person 2's utility falls from 7 to 4, so it is not an actual Pareto Improvement.
- d. The initial point $(4, 7)$ is in the " $(0, 32)$ economy." The other economy, the " $(16, 0)$ economy," would be a Potential Pareto Improvement over the

$(0, 32)$ economy if there exists at least one (re)allocation of commodities in the $(16, 0)$ economy which is preferred by everyone to the initial point $(4, 7)$ in the $(0, 32)$ economy. There is such a point: Figure 1 shows that the point $(6, 10)$ in the $(16, 0)$ economy is preferred by everyone to the initial point $(4, 7)$ in the $(0, 32)$ economy. Hence the $(16, 0)$ economy is a Potential Pareto Improvement over the $(0, 32)$ economy.

Optional: Another way to express this is: starting at $(4, 7)$ in the $(0, 32)$ economy, we should move to the $(16, 0)$ economy because—thinking about moving to the point $(12, 4)$ in the $(16, 0)$ economy—the ‘winner’ (Person 1) could compensate the ‘loser’ (Person 2) in such a way that (at $(6, 10)$) both people are better off in the $(16, 0)$ economy than they were in the original $(0, 32)$ economy at point $(4, 7)$. Economists who advocate that in this situation a social planner should change from the $(0, 32)$ economy to the $(16, 0)$ economy even if the winner does not *actually* compensate the loser, just because the winner *could* compensate the loser, are using the “Kaldor compensation criterion.” This is the same as the “Potential Pareto Improvement” criterion described in the exam question. See https://en.wikipedia.org/wiki/Kaldor_compensation_criteria.

- e. No. An actual Pareto Improvement requires all agents to be better off (or at least no worse off). In the case of this move, Person 1’s utility falls from 12 to 4, so it is not an actual Pareto Improvement.
- f. The initial point $(12, 4)$ is in the “ $(16, 0)$ economy.” The other economy, the “ $(0, 32)$ economy,” would be a Potential Pareto Improvement over the $(16, 0)$ economy if there exists at least one (re)allocation of commodities in the $(0, 32)$ economy which is preferred by everyone to the initial point $(12, 4)$ in the $(16, 0)$ economy. There is such a point: Figure 1 shows that the point $(14, 4.5)$ in the $(0, 32)$ economy is preferred by everyone to the initial point $(12, 4)$ in the $(16, 0)$ economy. Hence the $(0, 32)$ economy is a Potential Pareto Improvement over the $(16, 0)$ economy.

Optional: Another way to express this is: starting at $(12, 4)$ in the $(16, 0)$ economy, we should move to the $(0, 32)$ economy because—thinking about moving to the point $(4, 7)$ in the $(0, 32)$ economy—the ‘winner’ (Person 2) could compensate the ‘loser’ (Person 1) in such a way that (at $(14, 4.5)$) both people are better off in the $(0, 32)$ economy than they were in the original $(16, 0)$ economy at point $(12, 4)$.

- g. Part (d) showed that the $(16, 0)$ economy is a Potential Pareto Improvement over the $(0, 32)$ economy. Part (f) showed that the $(0, 32)$ economy

is a Potential Pareto Improvement over the (16,0) economy. So each economy is a Potential Pareto Improvement over the other. Clearly a social planner using the decision rule given in the question would not know which economy to prefer.

Note: This phenomenon is called the “Scitovsky Paradox.” See Tibor Scitovsky, “A Note on Welfare Propositions in Economics,” *Review of Economic Studies*, 1941: 77.