Section 7:

Partial Equilibrium and Positive General Equilibrium

1. [**8 points**] In Utah:

   - Both farmers and people who live in cities use water supplied by government water companies. There are different government water companies for cities and for farms.
   - Farmers pay a much lower price for water than city dwellers do.
   - In the future, farmers might start reselling the water they buy to city dwellers. (Ignore why they do not do this now.)
   - The price that city dwellers pay for water is lower than in neighboring states because Utah city water companies also get a great deal of money from taxes which have nothing to do with water. These taxes are called "property taxes."
   - The Utah legislature is considering eliminating the flow of money from "property taxes" to the city water companies.

   Question: If Utah legislature did eliminate the flow of money from "property taxes" to the city water companies, would this make the price of water sold by farmers to city dwellers go up or down? (No water is sold by farmers to city dwellers right now, but ignore that.) In particular, analyze in detail each claim made in the following paragraph written by an economics professor at another university in Utah, and explain whether you think his analysis is correct or not:

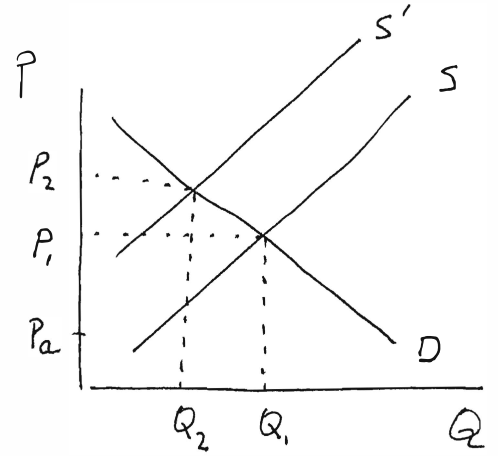
   > "I don't see why the property tax will affect much the value of water used in agriculture. A higher urban price because of the elimination of the tax subsidy will induce conservation that will reduce the urban demand for new water. This will reduce the demand for ag[ricultural] water and may reduce the equilibrium transfer price. Therefore, we might expect opposition from farmers to eliminating the [property] tax subsidy."

   Hint: It is possible to correctly answer this question by drawing a graph but using no other mathematics.

Summer 2012, Qualifying Exam, Section 2 Qu. 1

Section 2.

(1)



In a model with no sales of agricultural water, the elimination of the property tax subsidy causes the supply curve of water to shift from S to S'. This results in "a higher urban price," as the paragraph from the other professor says. However, the fall in Q from Q_1 to Q_2 is a fall in quantity demanded, not a fall in demand. If the agricultural price of water is P_a, then before, any price between P_a and P_1 would induce farmers to sell water to city dwellers and city dwellers to buy the water. After

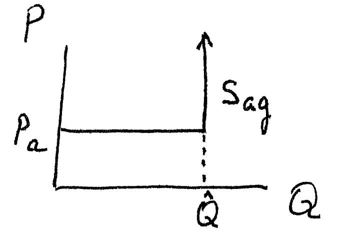
since the price is $> P_a$

since the price is $< P_1$

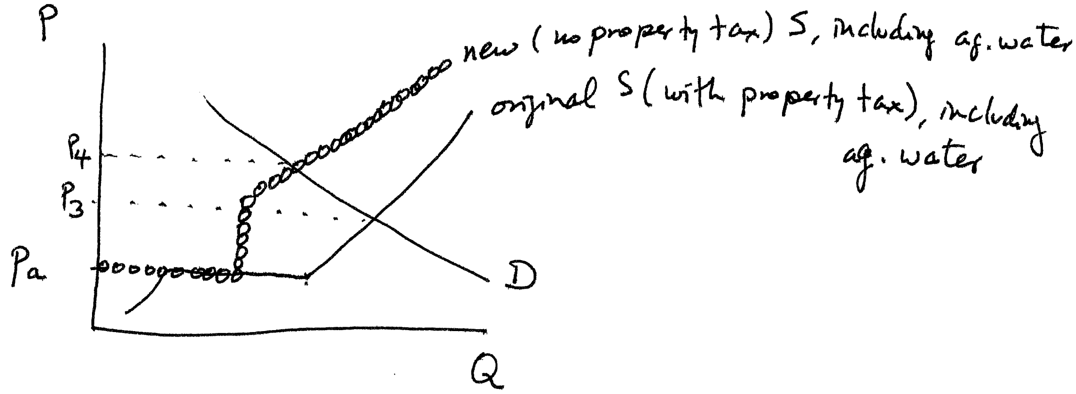
the shift in supply, any price between P_a and P_2 would induce farmers to sell water and city dwellers to buy it from them. Since $P_2 > P_1$, this means the maximum price farmers could get for their water has gone up

and the minimum price has stayed the same. Thus farmers would benefit (or at least not be hurt) by the change.

One way to model the supply of agricultural water would be

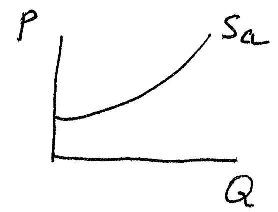


Coupled with the city water company supply curves S and S' given before, this makes the equilibrium look like

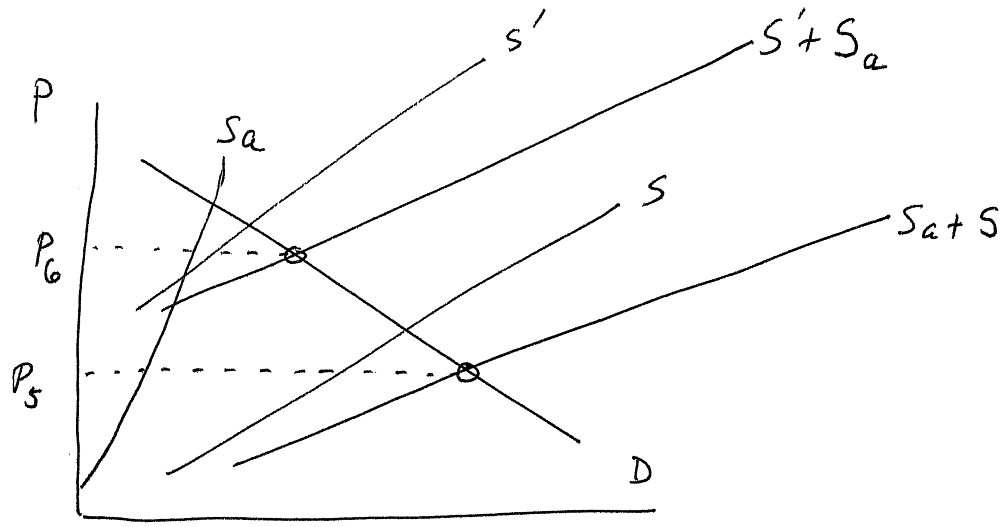


and farmers would prefer P_4 to P_3, so they would favor elimination of the property tax.

Another way to model the supply of agricultural water would be



Coupled with the city water company supply curves S and S' given before, this makes the equilibrium look like



and farmers would prefer P_6 to P_5, so they would favor elimination of the property tax.

The error in the other economist's analysis is to confuse a fall in quantity demanded with a fall in demand. The former is what happens when the property tax subsidy is removed, but he thought the latter occurred. Since in reality it is not the demand curve but the supply curve which shifted, there is no fall in "demand," only decreasing availability of water from the source farmers would be competing against — hence a greater desire for the farmers' water, not a lesser desire.

old: 2007 Final Exam Qu 6, but with an added third word, “competitive”

4. **[16 points]**

   Consider a competitive equilibrium in the market for a single commodity, such as apples. Suppose there is an increase in the price of one of the inputs used to produce apples. What effect will that have on the equilibrium price of apples? Be as rigorous and as general as you can be in answering this question (though if you wish to use the Envelope Theorem or the Slutsky Equation, you need not prove them).

2007
Final

(6)

Let p be the output price (the price of apples).

Let w be the price of the input whose price changes.

Let D be market demand for apples.

Let S be market supply for apples.

In single-market equilibrium,

$$D(p) = S(p, w; \underbrace{\text{price of other inputs}}_{\text{these remain constant so we can ignore them in what follows}})$$

Take the differential of both sides:

$$\frac{dD}{dp}\,dp = \frac{\partial S}{\partial p}\,dp + \frac{\partial S}{\partial w}\,dw$$

$$\left(\frac{dD}{dp} - \frac{\partial S}{\partial p}\right) dp = \frac{\partial S}{\partial w}\,dw$$

$$\frac{dp}{dw} = \frac{\partial S/\partial w}{\frac{dD}{dp} - \frac{\partial S}{\partial p}}.$$

As long as apples are not a Giffen good, $\frac{dD}{dp} < 0$. In particular, using the Slutsky equation, $\frac{dD}{dp} = \frac{\partial h}{\partial p} - \frac{\partial D}{\partial m} \cdot D$, so a sufficient condition for $dD/dp < 0$ is that apples are a normal good.

$\frac{\partial h}{\partial p}$: ⊖ since Hicksian demand curves slope downwards

$\frac{\partial D}{\partial m}$: ⊕ for normal goods, ⊖ for inferior goods

D: ⊕.

↳ Proved by applying the Envelope Theorem to $e(p, \bar{u}) = \min_{x} p \cdot x$ s.t. $u(x) = \bar{u}$, obtaining $\frac{\partial}{\partial p_i} = x_i$, or h_i as we usually call it; $\partial^2 e/\partial p_i^2 = \partial h_i/\partial p_i$, and LHS is ⊖ by the concavity of e in p

$\frac{\partial S}{\partial p} > 0$ since output supply curves are upward-sloping.

(Proof: apply the Envelope Theorem to $\pi = \max \tilde{p} \cdot \tilde{y}$ s.t. $y \in Y$,

obtaining $\frac{\partial \pi}{\partial p_i} = y_i$; then $\frac{\partial^2 \pi}{\partial p_i^2} = \frac{\partial y_i}{\partial p_i}$, and LHS is ⊕ by

the convexity of π.)

$$\frac{\partial S}{\partial w} = \frac{\partial S}{\partial \text{input } 1} \frac{\partial \text{ input } 1}{\partial w} + \cdots + \frac{\partial S}{\partial \text{ last input}} \frac{\partial \text{ last input}}{\partial w}$$

Presumably, $\frac{\partial S}{\partial \text{ input}} > 0$ $\forall$ inputs (otherwise the firm shouldn't be buying so much of that input).

If "w" is the price of the first input, then $\frac{\partial \text{ input } 1}{\partial w} < 0$ since input demand curves are downward-sloping (see $\partial y_i / \partial p_i > 0$ above).

So

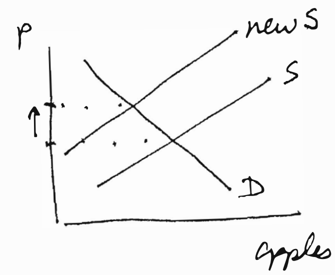
$$\frac{\partial S}{\partial w} = \underbrace{\oplus \ominus}_{\ominus} + \cdots + \oplus \frac{\partial \text{ last input}}{\partial w}.$$

A sufficient condition for $\partial S / \partial w$ to be negative would be for $\frac{\partial \text{ input}}{\partial w}$ to be negative for all inputs.

Conclusion: if apples are a normal good and if $\frac{\partial \text{ input}}{\partial w} < 0$ for all inputs,

then

$$\frac{dp}{dw} = \frac{\ominus}{\ominus - \oplus} > 0.$$



2008 Qualifier
Sec. 1

2. [**12 points**]

(a) Are individual firm's supply curves always upward-sloping? Usually? Never? Defend your answer with as general a mathematical argument as you can provide.

(b) Are individual demand curves always downward-sloping? Usually? Never? Defend your answer with as general a mathematical argument as you can provide.

(c) The US federal government imposes a tax on gasoline. Gasoline has recently greatly risen in price, and two presidential candidates, Sen. Hillary Clinton and Sen. John McCain, have proposed eliminating this tax this summer. The other major presidential candidate, Sen. Barack Obama, opposes eliminating the tax this summer. Irrelevant to this qualifying exam question are most of Sen. Obama's reasons (such as encouraging alternative energy sources), but one reason is relevant: some of Sen. Obama's supporters have said that if the tax is eliminated, the gasoline companies will just raise the price by the amount of the eliminated tax, so the "net price to consumers" will not change.

Is this true?

To answer this, derive a general mathematical equation showing how a marginal decrease (or marginal increase) of a tax changes the market price of a commodity. Will a marginal decrease of a tax increase, decrease, or have no effect on the "net price to consumers?"

Hint: Recall that 19th century Scottish historian Thomas Carlyle (who also coined the term "dismal science" to denote economics) said, "Teach a parrot the terms 'supply and demand' and you've got an economist." So, begin there (mathematically), in a market with no tax.

② a) In class, we proved that the profit function

$$\pi(\underset{\sim}{p}) = \max_{\underset{\sim}{y}} \underset{\sim}{p} \cdot \underset{\sim}{y} \quad \text{s.t. } \underset{\sim}{y} \in Y$$

is convex. By the Envelope Theorem,

$$\nabla_{\underset{\sim}{p}} \pi(\underset{\sim}{p}) = \nabla_{\underset{\sim}{p}} \mathcal{L}^* = \nabla_{\underset{\sim}{p}}(\underset{\sim}{p} \cdot \underset{\sim}{y}) = \underset{\sim}{y}$$

So

$$\nabla^2_{\underset{\sim}{p}} \pi = \nabla_{\underset{\sim}{p}} \underset{\sim}{y}.$$

The LHS is positive semi-definite because $\pi(\underset{\sim}{p})$ is convex. So the RHS is positive semi-definite. Positive semi-definite matrices have positive (or zero) terms on their main diagonal (as proven in class). So $\partial y_i / \partial p_i \geq 0$. Outputs i are positive in this framework ($y_i > 0$), so if an output price rises, the amount of output rises: an upward-sloping supply curve.

b) In class, we proved that the expenditure function

$$e(\underset{\sim}{p}, u) = \min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x} \quad \text{s.t. } u(\underset{\sim}{x}) \geq \bar{u}$$

is concave in $\underset{\sim}{p}$. By the Envelope Theorem,

$$\nabla_{\underset{\sim}{p}} e(\underset{\sim}{p}, u) = \nabla_{\underset{\sim}{p}} \mathcal{L}^* = \nabla_{\underset{\sim}{p}} (\underset{\sim}{p} \cdot \underset{\sim}{x}^*) = \underset{\sim}{x}^*$$ or, in better notation,

$\underset{\sim}{h}^*$, the Hicksian demand functions.

So

$$\nabla^2_{\underset{\sim}{p}} e(\underset{\sim}{p}, u) = \nabla_{\underset{\sim}{p}} \underset{\sim}{h}^*.$$

The LHS is negative semi-definite because $e(\underset{\sim}{p}, u)$ is concave in $\underset{\sim}{p}$. So the RHS is negative semi-definite. Negative semi-definite matrices have negative (or zero) terms on their main diagonal (as proven in class). So $\partial h_i / \partial p_i \leq 0$: downward-sloping Hicksian demand curves.

However, the slope of the Marshallian demand curve is

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{\partial p_i} - x_i \frac{\partial x_i}{\partial m}$$

↑ income

by the Slutsky Equation. Hence even though $\partial h_i / \partial p_i \leq 0$, $\partial x_i / \partial p_i$ could be positive, not negative, if $-x_i \frac{\partial x_i}{\partial m}$ were sufficiently positive (i.e., if $\partial x_i / \partial m$ were sufficiently negative — a very inferior good). That would be the case of a Giffen good. They are exceedingly rare (or nonexistent) in practice.

c) Begin with "supply equals demand":

$$S(p) = D(p)$$

<u>If firms are taxed</u>	<u>If consumers are taxed</u>
$S(\underbrace{p-t}_{p-t = \text{net price to firms}}) = D(p)$ ↑ tax	$S(p) = D(\underbrace{p+t}_{\text{net price to consumers}})$
$p-t$ = net price to firms	net price to consumers
p = price to consumers	p = price to firms

Comparative statics:

$$S'\,dp + S'(-1)\,dt = D'\,dp \quad (1)$$

$$(S'-D')\,dp = S'\,dt$$

$$\frac{dp}{dt} = \frac{S'}{S'-D'}$$

$$= \frac{d\,(\text{net price to consumers})}{dt}.$$

Left-hand side of (1) is

$$\underbrace{\frac{dS}{d(p-t)}}_{S'}\ \underbrace{\frac{\partial(p-t)}{\partial p}}_{1}\,dp + \underbrace{\frac{dS}{d(p-t)}}_{S'}\ \underbrace{\frac{\partial(p-t)}{\partial t}}_{-1}\,dt.$$

Right-hand side of (2) is

$$\underbrace{\frac{dD}{d(p+t)}}_{D'}\ \underbrace{\frac{\partial(p+t)}{\partial p}}_{1}\,dp + \underbrace{\frac{dD}{d(p+t)}}_{D'}\ \underbrace{\frac{\partial(p+t)}{\partial t}}_{1}\,dt.$$

Comparative statics:

$$S'\,dp = D'\,dp + D'\,dt \quad (2)$$

$$(S'-D')\,dp = D'\,dt$$

$$\frac{dp}{dt} = \frac{D'}{S'-D'}$$

$$\frac{d\,(\text{net price to consumers})}{dt} = \frac{d(p+t)}{dt}$$

$$= \frac{dp}{dt} + \frac{dt}{dt} = \frac{D'}{S'-D'} + 1$$

$$= \frac{D'}{S'-D'} + \frac{S'-D'}{S'-D'} = \frac{S'}{S'-D'}$$

So whichever framework you chose to use,

$$\frac{d\,(\text{net price to consumers})}{dt} = \frac{S'}{S'-D'} = \frac{1}{1-\frac{D'}{S'}}.$$

From parts (a) and (b), $S'>0$ and D' is usually <0, making $\frac{1}{1-\frac{D'}{S'}}$ be between 0 and 1. A decrease in a tax hence usually lowers the "net price to consumers." If $S'=0$ (a vertical supply curve ), then $\frac{S'}{S'-D'}=0$ and taxes would not affect the net price to consumers, but this is an extreme case which is unlikely to hold.

Question 2. Suppose a competitive firm transforms a single input (z) into two outputs (q_1 and q_2) according to a well-behaved, fully differentiable inverse production function.

Further suppose that the government introduces a tax (t) on each unit of q_1 sold.

a) How will this tax change:
   i. the firm's demand for the input z;
   ii. the supply of the taxed commodity q_1; and
   iii. the supply of the untaxed commodity q_2?
b) How will a change in the price of q_1 affect the supply of q_1?
c) How will a change in the price of q_1 affect the supply of q_2?
d) How will a change in the price of q_2 affect the supply of q_1?
e) How will a change in the price of q_2 affect the supply of q_2?
f) Now suppose that all competitive firms jointly producing q_1 and q_2 are subject to this tax on q_1. Derive an expression for the effect of this tax on the equilibrium price of q_1. What is the sign of this expression? What did you expect the sign of this expression to be, and why?

2000 Qualifying Exam, Question 2

Answers to 6710 Section of 2000 Qualifying Exam

Question 2, Required Section.

$$z = f(q_1, q_2)$$

z ↑ input; q_1, q_2 ↑ outputs

one could call this the inverse production function

Let w be the price of z, let p_1 be the price of q_1, and let p_2 be the price of q_2.

$$\text{Profit } \pi = p_1 q_1 + p_2 q_2 - wz - tq_1$$

Note that this is for a specific tax. For an ad valorem tax, this would instead be tp_1q_1. Either formulation is OK. These answers are for the specific tax case.

$$= (p_1 - t) q_1 + p_2 q_2 - w f(q_1, q_2).$$

Endogenous: q_1, q_2

Exogenous: p_1, t, p_2, w

F.O.C.:
$$0 = \frac{\partial \pi}{\partial q_1} = p_1 - t - w \frac{\partial f}{\partial q_1}$$
$$0 = \frac{\partial \pi}{\partial q_2} = p_2 - w \frac{\partial f}{\partial q_2}$$

For comparative statics, find the total differential:

unimportant in this problem

$$0 = (1)\, dp_1 + (-1)\, dt + 0 + (\cdot)\, dw + (-w f''_{11})\, dq_1 - w f''_{12}\, dq_2$$
$$0 = 0 + 0 + (1)\, dp_2 + (\cdot)\, dw + (-w f''_{12})\, dq_1 - w f''_{22}\, dq_2$$

or $-w f''_{21}$

$\Rightarrow$

$$\begin{bmatrix} w f''_{11} & w f''_{12} \\ w f''_{21} & w f''_{22} \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dp_1 + \begin{bmatrix} -1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dp_2 + \begin{bmatrix} (\cdot) \\ (\cdot) \end{bmatrix} dw$$

endogenous exogenous

a) Clearly, $\frac{dz}{dt} = \frac{d f(q_1, q_2)}{dt}$

$$= f_1' \frac{dq_1}{dt} + f_2' \frac{dq_2}{dt}.$$

From the last equation on p. 1, using Cramer's Rule, with $dp_1 = dp_2 = dw = 0$, we have

$$\frac{dq_1}{dt} = \frac{\begin{vmatrix} -1 & wf_{12}'' \\ 0 & wf_{22}'' \end{vmatrix}}{\begin{vmatrix} wf_{11}'' & wf_{12}'' \\ wf_{21}'' & wf_{22}'' \end{vmatrix}}$$

$\leftarrow -wf_{22}''$

$\leftarrow$ Related to $\begin{bmatrix} -wf_{11}'' & -wf_{12}'' \\ -wf_{21}'' & -wf_{22}'' \end{bmatrix}$, which is the Hessian of the objective function, $\nabla^2\pi$.

S.O.C. for a maximum are that

D_1 of $\nabla^2\pi$ is < 0

D_2 of $\nabla^2\pi$ is > 0

therefore $-wf_{11}'' < 0$ (or $f_{11}'' > 0$) (1)

$w^2 f_{11}'' f_{22}'' - w^2 f_{12}'' f_{21}'' > 0 \iff$

$$\begin{vmatrix} wf_{11}'' & wf_{12}'' \\ wf_{21}'' & wf_{22}'' \end{vmatrix} > 0. \qquad (2)$$

(1) states that $f_{11}'' > 0$. Since the designation of "1" and "2" labels is arbitrary, it is reasonable to assume that $f_{22}'' > 0$. From this and (2),

$\frac{dq_1}{dt} = \frac{-w \oplus}{\oplus} < 0$. When $t \uparrow$, the supply of the taxed commodity, which is q_1, $\downarrow$. This answers (ii).

For (iii), from the last equation on p. 1,

$$\frac{dq_2}{dt} = \frac{\begin{vmatrix} wf''_{11} & -1 \\ wf''_{21} & 0 \end{vmatrix}}{\begin{vmatrix} wf''_{11} & wf''_{12} \\ wf''_{21} & wf''_{22} \end{vmatrix}} = \frac{wf''_{21}}{\oplus}$$

. This is ambiguous because the sign of f''_{21} is unknown.

Finally, from the top of p. 2 and the answers to (ii) and (iii),

$$\frac{dz}{dt} = \left[f_1'\,(-wf''_{22}) + f_2'\,(wf''_{21})\right] \Big/ \begin{vmatrix} wf''_{11} & wf''_{12} \\ wf''_{21} & wf''_{22} \end{vmatrix}$$

$$= \frac{-w\overset{\oplus}{f_1'}\,\overset{\oplus}{f''_{22}} + w\overset{\oplus}{f_2'}\,\overset{?}{f''_{21}}}{\oplus}$$

so this is ambiguous. If f''_{21} were zero, it would be negative. Note that f_1' and f_2' are positive as long as more output requires more input, a natural assumption.

b) Let Δ be defined to be $\begin{vmatrix} wf''_{11} & wf''_{12} \\ wf''_{21} & wf''_{22} \end{vmatrix}$. From (2), $\Delta > 0$.

$$\frac{dq_1}{dp_1} = \frac{+wf''_{22}}{\Delta} = \frac{+w\,\oplus}{\oplus} > 0$$

can be derived either by applying Cramer's Rule to the last equation on p. 1, or by noting that $\frac{dq_1}{dp_1}$ has to be -1 times $\frac{dq_1}{dt}$, then using the answer to part (a)(ii).

c) $\frac{dq_2}{dp_1} = \frac{-wf''_{21}}{\Delta}$ (ambiguous sign) for similar reasons as in part (b).

d) $$\frac{dq_1}{dp_2} = \frac{\begin{vmatrix} 0 & wf''_{12} \\ 1 & wf''_{22} \end{vmatrix}}{\Delta} = \frac{-wf''_{12}}{\Delta}$$ (ambiguous sign) by Cramer's Rule and the last equation on p. 1.

e) $$\frac{dq_2}{dp_2} = \frac{\begin{vmatrix} wf''_{11} & 0 \\ wf''_{21} & 1 \end{vmatrix}}{\Delta} = \frac{wf''_{11}}{\Delta} = \frac{\oplus\ \oplus}{\oplus} > 0,$$ an upward-sloping supply curve.

f) Let "D" stand for demand and let "S" stand for supply. In market equilibrium,

$$q_1^D(p_1) = q_1^S(p_1, p_2, w, t)$$
$$q_2^D(p_2) = q_2^S(p_1, p_2, w, t)$$

From part (c), we know that a change in p_1 (because of the tax) will change q_2^S, and hence p_2. So the tax on q_1 will affect the equilibrium for q_2, and both markets must be considered.

Totally differentiating,

dw = 0 in this problem

$$\frac{dq_1^D}{dp_1}dp_1 = \frac{\partial q_1^S}{\partial p_1}dp_1 + \frac{\partial q_1^S}{\partial p_2}dp_2 + (\cdot)\,dw + \frac{\partial q_1^S}{\partial t}dt$$

$$\frac{dq_2^D}{dp_2}dp_2 = \frac{\partial q_2^S}{\partial p_1}dp_1 + \frac{\partial q_2^S}{\partial p_2}dp_2 + (\cdot)\,dw + \frac{\partial q_2^S}{\partial t}dt.$$

Rearranging,

$$\begin{bmatrix} \frac{dq_1^D}{dp_1} - \frac{\partial q_1^S}{\partial p_1} & -\frac{\partial q_1^S}{\partial p_2} \\ -\frac{\partial q_2^S}{\partial p_1} & \frac{dq_2^D}{dp_2} - \frac{\partial q_2^S}{\partial p_2} \end{bmatrix} \begin{bmatrix} dp_1 \\ dp_2 \end{bmatrix} = \begin{bmatrix} \partial q_1^S/\partial t \\ \partial q_2^S/\partial t \end{bmatrix} dt.$$

Dividing by dt and using the results of previous parts of this question,

$$\begin{bmatrix} \frac{dq_1^D}{dp_1} - \frac{wf''_{22}}{\Delta} & \frac{wf''_{12}}{\Delta} \\ \frac{wf''_{21}}{\Delta} & \frac{dq_2^D}{dp_2} - \frac{wf''_{11}}{\Delta} \end{bmatrix} \begin{bmatrix} \partial p_1/\partial t \\ \partial p_2/\partial t \end{bmatrix} = \begin{bmatrix} -wf''_{22}/\Delta \\ wf''_{21}/\Delta \end{bmatrix} \Rightarrow$$

$$\frac{\partial p_1}{\partial t} = \frac{\begin{vmatrix} -\frac{wf''_{22}}{\Delta} & \frac{wf''_{12}}{\Delta} \\ \frac{wf''_{21}}{\Delta} & \frac{dq_2^D}{dp_2} - \frac{wf''_{11}}{\Delta} \end{vmatrix}}{\begin{vmatrix} \frac{dq_1^D}{dp_1} - \frac{wf''_{22}}{\Delta} & \frac{wf''_{12}}{\Delta} \\ \frac{wf''_{21}}{\Delta} & \frac{dq_2^D}{dp_2} - \frac{wf''_{11}}{\Delta} \end{vmatrix}}$$

$$= \frac{\overbrace{\frac{-wf''_{22}}{\Delta}}^{\ominus}\left(\overbrace{\frac{dq_2^D}{dp_2}}^{\ominus} - \overbrace{\frac{wf''_{11}}{\Delta}}^{\oplus}\right) - \overbrace{\frac{wf''_{21}}{\Delta}\frac{wf''_{12}}{\Delta}}^{\oplus \text{ because it's } (wf''_{12}/\Delta)^2}}{\left(\underbrace{\frac{dq_1^D}{dp_1}}_{\ominus} - \underbrace{\frac{wf''_{22}}{\Delta}}_{\oplus}\right)\left(\underbrace{\frac{dq_2^D}{dp_2}}_{\ominus} - \underbrace{\frac{wf''_{11}}{\Delta}}_{\oplus}\right) - \underbrace{\frac{wf''_{21}}{\Delta}\frac{wf''_{12}}{\Delta}}_{\oplus \text{ (see above)}}}$$

where $dq_i^D/dp_i < 0$ assuming no Giffen goods.

$$= \frac{\ominus\ominus - \oplus}{\ominus\ominus - \oplus}$$ which is ambiguous. If f''_{12} is close to zero, then this will be close to $\frac{\ominus\ominus}{\ominus\ominus}$, which is positive, which is the intuitive result.

2016 Qualifying Exam Sec. 1 Qu. 1

1. **[14 points]** Suppose a competitive, profit-maximizing firm transforms two inputs (x_1 and x_2) into one output (y) according to a well-behaved, concave, fully differentiable production function $f(x_1, x_2)$. Let the price of the inputs be p_1 and p_2 and let the price of the output be w.

   Suppose that the government introduces a tax (t) on each unit of x_1 bought. In other words, assume this tax is a "specific tax," such as \$0.70/unit, not an "ad valorem tax," which would be expressed as a percentage such as 7%.

   Feel free to use abbreviations to simplify the answers you derive below.

   (a) Assuming no prices change (that is, p_1, p_2, and w do not change), how will this tax change:

      i. the firm's demand for the taxed input x_1;
      ii. the firm's demand for the untaxed input x_2; and
      iii. the supply of the output y?

   (b) How will a change in the price of x_1 affect the demand for x_1?

   (c) How will a change in the price of x_1 affect the demand for x_2?

   (d) How will a change in the price of x_1 affect the supply of y?

   (e) How will a change in the price of x_2 affect the demand for x_1?

   (f) How will a change in the price of x_2 affect the demand for x_2?

   (g) How will a change in the price of x_2 affect the supply of y?

   (h) How will a change in the price of y affect the demand for x_1?

   (i) How will a change in the price of y affect the demand for x_2?

   (j) How will a change in the price of y affect the supply of y?

   (k) Now suppose that all competitive firms producing y use x_1 and x_2 and are subject to this tax on x_1. Using Cramer's Rule, derive an expression for the effect of this tax on the equilibrium price of x_1 when all prices are allowed to change.

      Your answer will involve 3×3 determinants; you should leave it unevaluated to save time. Also to save time, if your answer involves quantities which you derived in parts (a)–(j), you can just write, for example, "(h)" instead of writing in the answer which you found in part (h). As a final time-saving measure, just assume the number of firms producing y is equal to one even though that is a strange assumption because the firm(s) is (are) competitive.

(A similar question appeared on a previous exam in a past year, and the answer I gave for it only involved a 2×2 determinant, but that answer should have taken one more market into account, and if it had done so, it would have involved a 3×3 determinant as well.)

① $y = f(x_1, x_2)$

specific tax rate

profit $\pi = w f(x_1, x_2) - p_1 x_1 - p_2 x_2 - t x_1$

The ~~wrong~~ wrong approach: write $\pi = wy - p_1 x_1 - p_2 x_2 - t x_1$, and don't substitute $f(x_1, x_2)$ for y, and don't use $f(x_1, x_2) = y$ as a constraint on the problem. If you set things up this way, you can't solve for any of the derivatives.* Confirm this yourself so you can recognize this symptom of this wrong approach.

* For example, FOC w.r.t. x_1 is not $0 = \frac{\partial \pi}{\partial x_1} = -p_1 \Rightarrow p_1 = 0$.

a) Maximize π over x_1 and x_2. The F.O.C.'s are:

$$0 = \frac{\partial \pi}{\partial x_1} = w \frac{\partial f}{\partial x_1} - p_1 - t$$

abbreviation: f_1'

$$0 = \frac{\partial \pi}{\partial x_2} = w \frac{\partial f}{\partial x_2} - p_2 = w f_2' - p_2 .$$

Take the differential of both sides of each F.O.C.:

$$0 = w f_{11}'' \, dx_1 + w f_{12}'' \, dx_2 - dt + f_1' \, dw - dp_1 + 0 \, dp_2 \qquad (1)$$

endogenous — exogenous

$$0 = w f_{21}'' \, dx_1 + w f_{22}'' \, dx_2 + 0 \, dt + f_2' \, dw + 0 \, dp_1 - dp_2 . \qquad (2)$$

With $dw = dp_1 = dp_2 = 0$, this is

$$\underset{\sim}{0} = \begin{bmatrix} w f_{11}'' & w f_{12}'' \\ w f_{21}'' & w f_{22}'' \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} - \begin{bmatrix} dt \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} w f_{11}'' & w f_{12}'' \\ w f_{21}'' & w f_{22}'' \end{bmatrix} \begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} .$$

(i) By Cramer's Rule,

$$\frac{dx_1}{dt} = \frac{\begin{vmatrix} 1 & w f_{12}'' \\ 0 & w f_{22}'' \end{vmatrix}}{\begin{vmatrix} w f_{11}'' & w f_{12}'' \\ w f_{21}'' & w f_{22}'' \end{vmatrix}} = \frac{w f_{22}''}{w^2 \left(f_{11}'' f_{22}'' - (f_{12}'')^2 \right)} .$$

Since f is said to be concave, you know that $f_{11}'' < 0$, $f_{22}'' < 0$, and

$$\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} > 0 \iff f''_{11} f''_{22} - (f''_{12})^2 > 0.$$ This is equivalent to

$$w^2 \left(f''_{11} f''_{22} - (f''_{12})^2 \right) > 0 \iff$$

$$\begin{vmatrix} wf''_{11} & wf''_{12} \\ wf''_{21} & wf''_{22} \end{vmatrix} > 0.$$

↳ call this "Δ" for the rest of this problem

So $$\frac{dx_1}{dt} = \frac{w f''_{22}}{\Delta} < 0$$: the firm's demand for the taxed input falls.

(ii) $$\frac{dx_2}{dt} = \frac{\begin{vmatrix} wf''_{11} & 1 \\ wf''_{21} & 0 \end{vmatrix}}{\Delta} = \frac{-wf''_{21}}{\Delta}.$$ this has the sign of $-f''_{21}$, which is unknown at this point.

(iii)

Using $y = f(x_1, x_2)$ and the Chain Rule,

$$\frac{dy^s}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} = \frac{f'_1 w f''_{22} - f'_2 w f''_{21}}{\Delta}.$$

If $f''_{21} > 0$ then $dy^s/dt < 0$.

If $f''_{21} < 0$ but f''_{21} is sufficiently small in absolute value then the numerator's first term dominates its second term and $dy^s/dt < 0$.

b) From (1) and (2), $$\underset{\sim}{0} = \begin{bmatrix} wf''_{11} & wf''_{12} \\ wf''_{21} & wf''_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} + \begin{bmatrix} -dp_1 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} wf_{11}'' & wf_{12}'' \\ wf_{21}'' & wf_{22}'' \end{bmatrix} \begin{bmatrix} dx_1/dp_1 \\ dx_2/dp_1 \end{bmatrix} \Rightarrow$$

$$\frac{dx_1}{dp_1} = \frac{\begin{vmatrix} 1 & wf_{12}'' \\ 0 & wf_{22}'' \end{vmatrix}}{\Delta} = \frac{\overset{\oplus}{w}\,\overset{\ominus}{f_{22}''}}{\underset{\oplus}{\Delta}} < 0.$$

c) $$\frac{dx_2}{dp_1} = \frac{\begin{vmatrix} wf_{11}'' & 1 \\ wf_{21}'' & 0 \end{vmatrix}}{\Delta} = \frac{-\overset{\oplus}{w}\,\overset{?}{f_{21}''}}{\underset{\oplus}{\Delta}} \text{ which is ambiguous.}$$

d) $$\frac{dy}{dp_1} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dp_1} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dp_1} = \underbrace{\underset{\oplus}{f_1'}\,\frac{\overset{\oplus}{w}\,\overset{\ominus}{f_{22}''}}{\underset{\oplus}{\Delta}}}_{\ominus} + \underbrace{\underset{\oplus}{f_2'}\left(\frac{-\overset{\oplus}{w}\,\overset{?}{f_{21}''}}{\underset{\oplus}{\Delta}}\right)}_{-\text{sign of } f_{21}''}.$$

e) From (1) and (2),

$$\underset{\sim}{0} = \begin{bmatrix} wf_{11}'' & wf_{12}'' \\ wf_{21}'' & wf_{22}'' \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -dp_2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} wf_{11}'' & wf_{12}'' \\ wf_{21}'' & wf_{22}'' \end{bmatrix} \begin{bmatrix} dx_1/dp_2 \\ dx_2/dp_2 \end{bmatrix}$$

$$\frac{dx_1}{dp_2} = \frac{\begin{vmatrix} 0 & wf_{12}'' \\ 1 & wf_{22}'' \end{vmatrix}}{\Delta} = \frac{-\overset{\oplus}{w}\,\overset{?}{f_{12}''}}{\underset{\oplus}{\Delta}} \text{ ambiguous}$$

f) $$\frac{dx_2}{dp_2} = \frac{\begin{vmatrix} wf_{11}'' & 0 \\ wf_{21}'' & 1 \end{vmatrix}}{\Delta} = \frac{\overset{\oplus}{w}\,\overset{\ominus}{f_{11}''}}{\underset{\oplus}{\Delta}} < 0$$

g) $$\frac{dy}{dp_2} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dp_2} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dp_2} = \underset{\oplus}{f_1'}\,\frac{(-\overset{\oplus}{w}\,\overset{?}{f_{12}''})}{\underset{\oplus}{\Delta}} + \underset{\oplus}{f_2'}\,\frac{\overset{\oplus}{w}\,\overset{\ominus}{f_{11}''}}{\underset{\oplus}{\Delta}} = ? + \ominus$$

h) From (1) and (2), $\underset{\sim}{0} = \begin{bmatrix} wf''_{11} & wf''_{12} \\ wf''_{21} & wf''_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} + \begin{bmatrix} f'_1\, dw \\ f'_2\, dw \end{bmatrix} \Rightarrow$

$$\begin{bmatrix} -f'_1 \\ -f'_2 \end{bmatrix} = \begin{bmatrix} wf''_{11} & wf''_{12} \\ wf''_{21} & wf''_{22} \end{bmatrix} \begin{bmatrix} dx_1/dw \\ dx_2/dw \end{bmatrix} \Rightarrow$$

$$\frac{dx_1}{dw} = \frac{\begin{vmatrix} -f'_1 & wf''_{12} \\ -f'_2 & wf''_{22} \end{vmatrix}}{\Delta} = \frac{\overset{\oplus\,\oplus\,\ominus}{-f'_1 w f''_{22}} + \overset{\oplus\,\oplus\,?}{f'_2 w f''_{12}}}{\underset{\oplus}{\Delta}} = \oplus + ?$$

i) $$\frac{dx_2}{dw} = \frac{\begin{vmatrix} wf''_{11} & -f'_1 \\ wf''_{21} & -f'_2 \end{vmatrix}}{\Delta} = \frac{\overset{\oplus\,\oplus\,\ominus}{-f'_2 w f''_{11}} + \overset{\oplus\,\oplus\,?}{f'_1 w f''_{21}}}{\underset{\oplus}{\Delta}} = \oplus + ?$$

j) $$\frac{dy}{dw} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dw} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dw} = f'_1 \frac{-f'_1 wf''_{22} + f'_2 w f''_{12}}{\Delta} + f'_2 \frac{-f'_2 wf''_{11} + f'_1 w f''_{21}}{\Delta}$$

$$= \frac{\overset{\oplus\,\oplus\,\oplus\,\ominus}{-f'_1 f'_1 w f''_{22}} \overset{\oplus\,\oplus\,\oplus\,?}{+ f'_1 f'_2 w f''_{12}} \overset{\oplus\,\oplus\,\oplus\,\ominus}{- f'_2 f'_2 w f''_{11}} \overset{\oplus\,\oplus\,\oplus\,?}{+ f'_1 f'_2 w f''_{21}}}{\underset{\oplus}{\Delta}}$$

$$= \oplus + ? + \oplus + ?$$

k) Equilibrium in the market for x_1: $x_1^{supply}(p_1) = x_1^{demand}(p_1, p_2, w, t)$

x_2: $x_2^{supply}(p_2) = x_2^{demand}(p_1, p_2, w, t)$

y: $y^{demand}(w) = y^{supply}(p_1, p_2, w, t)$.

Using abbreviations:

$$0 = x_1^S(p_1) - x_1^D(p_1, p_2, w, t)$$

$$0 = x_2^S(p_2) - x_2^D(p_1, p_2, w, t)$$

$$0 = y^D(w) - y^S(p_1, p_2, w, t)$$

Take the differential:

$$\begin{bmatrix} \frac{dx_1^S}{dp_1} - \frac{\partial x_1^D}{\partial p_1} & -\frac{\partial x_1^D}{\partial p_2} & -\frac{\partial x_1^D}{\partial w} \\ -\frac{\partial x_2^D}{\partial p_1} & \frac{dx_2^S}{dp_2} - \frac{\partial x_2^D}{\partial p_2} & -\frac{\partial x_2^D}{\partial w} \\ -\frac{\partial y^S}{\partial p_1} & -\frac{\partial y^S}{\partial p_2} & \frac{dy^D}{dw} - \frac{\partial y^S}{\partial w} \end{bmatrix} \begin{bmatrix} dp_1 \\ dp_2 \\ dw \end{bmatrix} - \begin{bmatrix} \frac{\partial x_1^D}{\partial t} \\ \frac{\partial x_2^D}{\partial t} \\ \frac{\partial y^S}{\partial t} \end{bmatrix} dt$$

$$\begin{bmatrix} (a)(i) \\ (a)(ii) \\ (a)(iii) \end{bmatrix} = \begin{bmatrix} \frac{dx_1^S}{dp_1} - (b) & -(e) & -(h) \\ -(c) & \frac{dx_2^S}{dp_2} - (f) & -(i) \\ -(d) & -(g) & \frac{\partial y^D}{\partial w} - (j) \end{bmatrix} \begin{bmatrix} dp_1/dt \\ dp_2/dt \\ dw/dt \end{bmatrix}$$

(using abbreviations refering to previous parts of this problem).

$$\frac{dp_1}{dt} = \frac{\begin{vmatrix} (a)(i) & -(e) & -(h) \\ (a)(ii) & dx_2^S/dp_2 - (f) & -(i) \\ (a)(iii) & -(g) & \partial y^D/\partial w - (j) \end{vmatrix}}{\begin{vmatrix} \frac{dx_1^S}{dp_1} - (b) & -(e) & -(h) \\ -(c) & \frac{dx_2^S}{dp_2} - (f) & -(i) \\ -(d) & -(g) & \frac{\partial y^D}{\partial w} - (j) \end{vmatrix}}$$

2017 Qualifying Exam Sec. 1 Qu. 2

2. **[20 points]** Suppose a competitive, profit-maximizing firm transforms two inputs (x_1 and x_2) into an output, which is apples (denoted by a), according to a well-behaved, concave, fully differentiable production function $f(x_1, x_2)$. Let the price of the inputs be p_1 and p_2 and let the price of apples be p_a.

   Suppose that the government introduces an *ad valorem* tax t on the price of x_1. In other words, this tax would be expressed as a percentage such as 7%, not as something like \$0.70/unit (which would be a "specific tax").

   Feel free to use abbreviations to simplify the answers you derive below.

   (a) How will a change in the price of x_1 affect the demand for x_1?

   (b) How will a change in the price of x_1 affect the demand for x_2?

   (c) How will a change in the price of x_1 affect the supply of a?

   (d) How will a change in the price of x_2 affect the demand for x_1?

   (e) How will a change in the price of x_2 affect the demand for x_2?

   (f) How will a change in the price of x_2 affect the supply of a?

   (g) How will a change in the price of a affect the demand for x_1?

   (h) How will a change in the price of a affect the demand for x_2?

   (i) How will a change in the price of a affect the supply of a?

   (j) Now suppose that all competitive firms producing a use x_1 and x_2 and are subject to this tax on x_1. Using Cramer's Rule, derive an expression for the effect of this tax on the equilibrium price of x_1 when all three prices are allowed to change.

   Your answer will involve 3×3 determinants; you should leave them unevaluated to save time. As an additional time-saving measure, just assume the number of firms producing a is equal to one even though that is a strange assumption because the firm(s) is (are) competitive.

   (A similar question appeared on a previous exam in a past year, and the answer I gave for it only involved a 2×2 determinant, but that answer should have taken one more market into account, and if it had done so, it would have involved a 3×3 determinant as well.)

Sec. 1 #2.

$$a = f(x_1, x_2)$$
$$\uparrow \quad \uparrow \quad \uparrow$$
prices $p_a \quad p_1 \quad p_2$

ad valorem tax rate

a) profit $\pi = p_a f(x_1, x_2) - p_1 (1+t) x_1 - p_2 x_2$

Maximize profit over x_1 and x_2. The F.O.C.'s are

$$0 = \frac{\partial \pi}{\partial x_1} = p_a \frac{\partial f}{\partial x_1} - p_1(1+t) \qquad = p_a f_1' - p_1(1+t)$$

$\frac{\partial f}{\partial x_1}$ abbreviation: f_1'

$$0 = \frac{\partial \pi}{\partial x_2} = p_a \frac{\partial f}{\partial x_2} - p_2 \qquad = p_a f_2' - p_2 .$$

Find the differential of both sides of the F.O.C.'s :

$$0 = p_a f_{11}'' dx_1 + p_a f_{12}'' dx_2 - (1+t)dp_1 + 0\, dp_2 + f_1' dp_a - p_1 dt$$

endogenous (dx_1, dx_2) — exogenous (dp_1, dp_2, dp_a, dt)

$$0 = p_a f_{21}'' dx_1 + p_a f_{22}'' dx_2 + 0\, dp_1 - dp_2 + f_2' dp_a + 0\, dt$$

With $dp_2 = dp_a = dt = 0$, this implies

$$\begin{bmatrix} 1+t \\ 0 \end{bmatrix} dp_1 = \begin{bmatrix} p_a f_{11}'' & p_a f_{12}'' \\ p_a f_{21}'' & p_a f_{22}'' \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \Rightarrow \begin{bmatrix} p_a f_{11}'' & p_a f_{12}'' \\ p_a f_{21}'' & p_a f_{22}'' \end{bmatrix} \begin{bmatrix} dx_1/dp_1 \\ dx_2/dp_1 \end{bmatrix} = \begin{bmatrix} 1+t \\ 0 \end{bmatrix}.$$

Use Cramer's Rule: (1)

$$\frac{dx_1}{dp_1} = \frac{\begin{vmatrix} 1+t & p_a f_{12}'' \\ 0 & p_a f_{22}'' \end{vmatrix}}{\begin{vmatrix} p_a f_{11}'' & p_a f_{12}'' \\ p_a f_{21}'' & p_a f_{22}'' \end{vmatrix}} = \frac{(1+t)\, p_a f_{22}''}{p_a^2 \left(f_{11}'' f_{22}'' - (f_{12}'')^2 \right)}$$

Since it is given that f is concave, you know that $f_{11}'' < 0$, $f_{22}'' < 0$, and

$$\begin{vmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{vmatrix} > 0 \Leftrightarrow \underbrace{f_{11}'' f_{22}'' - (f_{12}'')^2}_{\text{call this "}\Delta\text{" for the rest of this problem}} > 0 .$$

So $$\frac{dx_1}{dp_1} = \frac{\overset{\oplus}{(1+t)}\,\overset{\oplus}{p_a}\,\overset{\ominus}{f''_{22}}}{\underset{\oplus}{p_a^2}\;\underset{\oplus}{\Delta}} < 0$$ (a downward-sloping input demand curve).

Also, $dx_1/dp_1 = \dfrac{(1+t)\,f''_{22}}{p_a\,\Delta}$.

b) Using Cramer's Rule on (1) from part a,

$$\frac{dx_2}{dp_1} = \frac{\begin{vmatrix} p_a f''_{11} & 1+t \\ p_a f''_{21} & 0 \end{vmatrix}}{\begin{vmatrix} p_a f''_{11} & p_a f''_{12} \\ p_a f''_{21} & p_a f''_{22} \end{vmatrix}} = \frac{-\overset{\oplus}{(1+t)}\,\overset{\oplus}{p_a}\,\overset{?}{f''_{21}}}{\underbrace{\underset{\oplus}{p_a^2}\;\underset{\oplus}{\Delta}}} = \frac{-(1+t)\,f''_{21}}{p_a\,\Delta}.$$

This has the sign of $-f''_{21}$, which is unknown at this point.

c) Since $a = f(x_1, x_2)$,

$$da = \frac{\partial f}{\partial x_1}\,dx_1 + \frac{\partial f}{\partial x_2}\,dx_2 \text{ and}$$

$$\frac{da}{dp_1} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dp_1} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dp_1}.$$ From parts (a) and (b),

$$= f'_1\,\frac{(1+t)\ast f''_{22}}{p_a\,\Delta} + f'_2\,\frac{-(1+t)\,f''_{21}}{p_a\,\Delta}$$

$$= \underbrace{\frac{1+t}{p_a\,\Delta}}_{\oplus}\left[\underset{\oplus}{f'_1}\,\underset{\ominus}{f''_{22}} - \underset{\oplus}{f'_2}\,\underset{?}{f''_{21}}\right].$$

(Assuming $f'_1 > 0$ and $f'_2 > 0$: more input produces more output.)

If $f''_{21} > 0$ then $da/dp_1 < 0$.

If $f''_{21} < 0$ but f''_{21} is small in absolute value, then $da/dp_1 < 0$.

d) Setting $dp_1 = dp_a = dt = 0$ in (a) yields

$$\underset{\sim}{0} = \begin{bmatrix} p_a f''_{11} & p_a f''_{12} \\ p_a f''_{21} & p_a f''_{22} \end{bmatrix}\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} - \begin{bmatrix} 0 \ dp_2 \\ dp_2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_a f''_{11} & p_a f''_{12} \\ p_a f''_{21} & p_a f''_{22} \end{bmatrix} \begin{bmatrix} dx_1/dp_2 \\ dx_2/dp_2 \end{bmatrix}$$ so by Cramer's Rule,

$$\frac{dx_1}{dp_2} = \frac{\begin{vmatrix} 0 & p_a f''_{12} \\ 1 & p_a f''_{22} \end{vmatrix}}{\begin{vmatrix} p_a f''_{11} & p_a f''_{12} \\ p_a f''_{21} & p_a f''_{22} \end{vmatrix}} = \frac{-p_a f''_{12}}{p_a^2 \Delta} = \frac{-\overset{?}{f''_{12}}}{\underset{+}{p_a} \underset{+}{\Delta}}.$$

e) $$\frac{dx_2}{dp_2} = \frac{\begin{vmatrix} p_a f''_{11} & 0 \\ p_a f''_{21} & 1 \end{vmatrix}}{\begin{vmatrix} p_a f''_{11} & p_a f''_{12} \\ p_a f''_{21} & p_a f''_{22} \end{vmatrix}} = \frac{p_a f''_{11}}{p_a^2 \Delta} = \frac{\overset{-}{f''_{11}}}{\underset{+}{p_a} \underset{+}{\Delta}} < 0,$$ a downward-sloping input demand curve.

f) Since $a = f(x_1, x_2)$, $da = f'_1 dx_1 + f'_2 dx_2$ and

$$\frac{da}{dp_2} = f'_1 \frac{dx_1}{dp_2} + f'_2 \frac{dx_2}{dp_2}$$ or, from parts (d) and (e),

$$= f'_1 \frac{-f''_{12}}{p_a \Delta} + f'_2 \frac{f''_{11}}{p_a \Delta} = \frac{-\overset{+}{f'_1} \overset{?}{f''_{12}} + \overset{+}{f'_2} \overset{-}{f''_{11}}}{\underset{+}{p_a} \underset{+}{\Delta}}.$$

If $f''_{12} > 0$ then $da/p_2 < 0$.

If $f''_{12} < 0$ but f''_{12} is small in absolute value then $da/dp_2 < 0$.

g) Setting $dp_1 = dp_2 = dt = 0$ in (a) yields

$$\underset{\sim}{0} = \begin{bmatrix} p_a f''_{11} & p_a f''_{12} \\ p_a f''_{21} & p_a f''_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} + \begin{bmatrix} f'_1 \, dp_a \\ f'_2 \, dp_a \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} -f_1' \\ -f_2' \end{bmatrix} = \begin{bmatrix} p_a f_{11}'' & p_a f_{12}'' \\ p_a f_{21}'' & p_a f_{22}'' \end{bmatrix} \begin{bmatrix} dx_1/dp_a \\ dx_2/dp_a \end{bmatrix}$$

$$\frac{dx_1}{dp_a} = \frac{\begin{vmatrix} -f_1' & p_a f_{12}'' \\ -f_2' & p_a f_{22}'' \end{vmatrix}}{\begin{vmatrix} p_a f_{11}'' & p_a f_{12}'' \\ p_a f_{21}'' & p_a f_{22}'' \end{vmatrix}} = \frac{p_a(-f_1' f_{22}'' + f_2' f_{12}'')}{p_a^2 \Delta}$$

$$= \frac{\overset{+}{-f_1'}\,\overset{-}{f_{22}''} + \overset{+}{f_2'}\,\overset{?}{f_{12}''}}{\underset{+}{p_a}\ \underset{+}{\Delta}}$$

which is positive if $f_{12}'' > 0$ or if $f_{12}'' < 0$ but f_{12}'' is small in absolute value.

h) $$\frac{dx_2}{dp_a} = \frac{\begin{vmatrix} p_a f_{11}'' & -f_1' \\ p_a f_{21}'' & -f_2' \end{vmatrix}}{\begin{vmatrix} p_a f_{11}'' & p_a f_{12}'' \\ p_a f_{21}'' & p_a f_{22}'' \end{vmatrix}} = \frac{p_a(-f_2' f_{11}'' + f_1' f_{21}'')}{p_a^2 \Delta}$$

$$= \frac{\overset{+}{-f_2'}\,\overset{-}{f_{11}''} + \overset{+}{f_1'}\,\overset{?}{f_{21}''}}{\underset{+}{p_a}\ \underset{+}{\Delta}}$$

which is positive if $f_{21}'' > 0$ or if $f_{21}'' < 0$ but f_{21}'' is small in absolute value.

i) Since $a = f(x_1, x_2)$, $da = f_1' dx_1 + f_2' dx_2$ and

$$\frac{da}{dp_a} = f_1' \frac{dx_1}{dp_a} + f_2' \frac{dx_2}{dp_a}$$ or, from parts (g) and (h),

$$= f_1' \frac{-f_1' f_{22}'' + f_2' f_{12}''}{p_a \Delta} + f_2' \frac{-f_2' f_{11}'' + f_1' f_{21}''}{p_a \Delta}$$; setting $f_{21}'' = f_{12}''$, this is

$$= \frac{-f_2' f_2' f_{11}'' + (f_1' f_2' + f_1' f_2') f_{12}'' - f_1' f_1' f_{22}''}{p_a \Delta}$$

$$= \frac{\overset{\oplus}{-(f_2')^2}\,\overset{\ominus}{f_{11}''} + 2\overset{\oplus}{f_1'}\overset{\oplus}{f_2'}\,\overset{?}{f_{12}''} - \overset{\oplus}{(f_1')^2}\,\overset{\ominus}{f_{22}''}}{\underset{\oplus}{p_a}\;\underset{\oplus}{\Delta}}$$

which is positive as long as $f_{12}'' \geq 0$ or $f_{12}'' < 0$ but f_{12}'' is small in absolute value.

(A positive value would mean an upward-sloping supply curve for apples.)

j) Equilibrium in the market for

$$x_1 : \; x_1^{supply}(p_1) = x_1^{demand}(p_1, p_2, p_a, t) \quad \leftarrow \text{so } p_1 \text{ is endogenous}$$

$$x_2 : \; x_2^{supply}(p_2) = x_2^{demand}(p_1, p_2, p_a, t) \quad \leftarrow \text{so } p_2 \text{ is endogenous}$$

$$a : \; a^{demand}(p_a) = a^{supply}(p_1, p_2, p_a, t) \quad \leftarrow \text{so } p_a \text{ is endogenous}$$

Using abbreviations:

$$0 = x_1^S(p_1) - x_1^D(p_1, p_2, p_a, t)$$

The only exogenous variable is t.

$$0 = x_2^S(p_2) - x_2^D(p_1, p_2, p_a, t)$$

$$0 = a^D(p_a) - a^S(p_1, p_2, p_a, t).$$

Taking differentials:

$$\underset{\sim}{0} = \begin{bmatrix} \frac{dx_1^S}{dp_1} - \frac{\partial x_1^D}{\partial p_1} & -\frac{\partial x_1^D}{\partial p_2} & -\frac{\partial x_1^D}{\partial p_a} \\ -\frac{\partial x_2^D}{\partial p_1} & \frac{dx_2^S}{dp_2} - \frac{\partial x_2^D}{\partial p_2} & -\frac{\partial x_2^D}{\partial p_a} \\ -\frac{\partial a^S}{\partial p_1} & -\frac{\partial a^S}{\partial p_2} & \frac{da^D}{dp_a} - \frac{\partial a^S}{\partial p_a} \end{bmatrix} \begin{bmatrix} dp_1 \\ dp_2 \\ dp_a \end{bmatrix} - \begin{bmatrix} \partial x_1^D/\partial t \\ \partial x_2^D/\partial t \\ \partial a^S/\partial t \end{bmatrix} dt.$$

Simplifying and using abbreviations referring to previous parts of this problem:

$$
\begin{bmatrix} \dfrac{\partial x_1^D}{\partial t} \\[2ex] \dfrac{\partial x_2^D}{\partial t} \\[2ex] \dfrac{\partial a^s}{\partial t} \end{bmatrix}
=
\begin{bmatrix}
\overset{(a)}{\dfrac{dx_1^s}{dp_1} - \dfrac{\partial x_1^D}{\partial p_1}} & -\dfrac{\partial x_1^D}{\partial p_2}\ (d) & -\dfrac{\partial x_1^D}{\partial p_a}\ (g) \\[3ex]
-\dfrac{\partial x_2^D}{\partial p_1}\ (b) & \underset{(e)}{\dfrac{dx_2^s}{dp_2} - \dfrac{\partial x_2^D}{\partial p_2}} & -\dfrac{\partial x_2^D}{\partial p_a}\ (h) \\[3ex]
\underset{(c)}{-\dfrac{\partial a^s}{\partial p_1}} & \underset{(f)}{-\dfrac{\partial a^s}{\partial p_2}} & \underset{(i)}{\dfrac{da^D}{dp_a} - \dfrac{\partial a^s}{\partial p_a}}
\end{bmatrix}
\begin{bmatrix} dp_1/dt \\[2ex] dp_2/dt \\[2ex] dp_a/dt \end{bmatrix}
$$

To figure out these components, set $dp_1 = dp_2 = dp_a = 0$ in (a), yielding

$$
\begin{bmatrix} p_1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_a f_{11}'' & p_a f_{12}'' \\ p_a f_{21}'' & p_a f_{22}'' \end{bmatrix} \begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix},
$$

$$
\frac{\partial x_1}{\partial t} = \frac{\begin{vmatrix} p_1 & p_a f_{12}'' \\ 0 & p_a f_{22}'' \end{vmatrix}}{\begin{vmatrix} p_a f_{11}'' & p_a f_{12}'' \\ p_a f_{21}'' & p_a f_{22}'' \end{vmatrix}} = \frac{p_1 p_a f_{22}''}{p_a^2 \Delta} = \frac{p_1 f_{22}''}{p_a \Delta} \quad \text{(j1)}
$$

$$
\frac{\partial x_2}{\partial t} = \frac{\begin{vmatrix} p_a f_{11}'' & p_1 \\ p_a f_{21}'' & 0 \end{vmatrix}}{\begin{vmatrix} p_a f_{11}'' & p_a f_{12}'' \\ p_a f_{21}'' & p_a f_{22}'' \end{vmatrix}} = \frac{-p_1 p_a f_{21}''}{p_a^2 \Delta} = \frac{-p_1 f_{21}''}{p_a \Delta} \quad \text{(j2)}
$$

and $a^s = f(x_1, x_2) \Rightarrow da^s = f_1' dx_1 + f_2' dx_2$ so

$$
\frac{da^s}{dt} = f_1' \frac{dx_1}{dt} + f_2' \frac{dx_2}{dt} = f_1' \frac{p_1 f_{22}''}{p_a \Delta} + f_2' \frac{-p_1 f_{21}''}{p_a \Delta}
$$

$$
= \frac{p_1 f_1' f_{22}'' - p_2 f_2' f_{21}''}{p_a \Delta} \quad \text{(j3)}.
$$

So

$$\frac{\partial p_1}{\partial t} = \frac{\begin{vmatrix} j1 & d & -g \\ j2 & \frac{dx_2^S}{dp_2} - e & -h \\ j3 & -f & \frac{da^D}{dpa} - i \end{vmatrix}}{\begin{vmatrix} \frac{dx_1^S}{dp_1} - a & d & -g \\ b & \frac{dx_2^S}{dp_2} - e & -h \\ c & -f & \frac{da^D}{dpa} - i \end{vmatrix}}$$

2017 Qualifying Exam Sec. 3 Qu. 2

2. **[10 points]** Consider a partial equilibrium in the market for apples, with the following (aggregate) supply and demand curves:

$$
\begin{aligned}
D(p) &= 10 - p \\
S(p) &= 3p + 1\ .
\end{aligned}
$$

(a) Find the equilibrium price and quantity of apples the way most undergraduate students would do it, by solving the two equations in two unknowns.

(b) Find the equilibrium price of apples by formulating this as a fixed-point problem. Then determine the equilibrium quantity of apples. (Do not determine the equilibrium quantity of apples first.)

(c) Find the equilibrium quantity of apples by formulating this as a fixed-point problem. Then determine the equilibrium price of apples. (Do not determine the equilibrium price of apples first.)

Sec. 3 Qu 2

$D(p) = 10 - p$

$S(p) = 3p + 1$

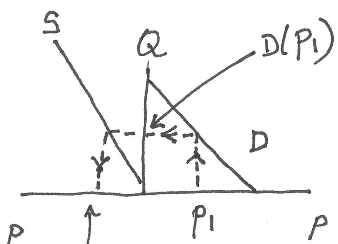
a) $D(p) = S(p)$

$10 - p = 3p + 1$

$9 = 4p \Rightarrow p = 9/4, \; D(9/4) = 10 - 9/4 = \frac{40-9}{4} = \frac{31}{4}.$

optional $\longrightarrow$ Check: $S(9/4) = 3\left(\frac{9}{4}\right) + 1 = \frac{27}{4} + \frac{4}{4} = \frac{31}{4}$, OK.

b)



$S^{-1}(D(p_1))$ Find where $S^{-1}(D(p_1)) = p_1$.

$S(p) = 3p+1 \Rightarrow \frac{S-1}{3} = p$, the S^{-1} function.

So find $\frac{D(p_1)-1}{3} = p_1$

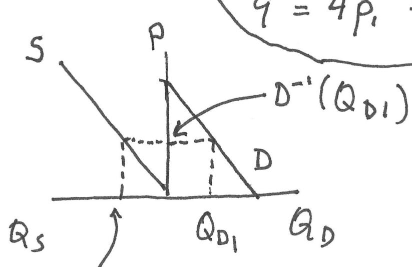
$\frac{10 - p_1 - 1}{3} = p_1$

$10 - p_1 - 1 = 3p_1$

$9 = 4p_1 \Rightarrow p_1 = 9/4$. Then quantity is $D(p) = 10 - p$

$= \frac{40}{4} - \frac{9}{4} = \frac{31}{4}.$

c)



$S(D^{-1}(Q_{D1}))$. Find where $S(D^{-1}(Q_{D1})) = Q_{D1}$.

$D(p) = 10 - p \Rightarrow p = 10 - D(p)$, the D^{-1} function.

$S(D^{-1}(Q_{D1})) = S(10 - D) = 3(10 - D) + 1 = 30 - 3D + 1 = 31 - 3D$.

Find $S(D^{-1}(Q_{D1})) = Q_{D1}$

$$31 - 3D = D$$

$$31 = 4D \Rightarrow D = 31/4$$. This is quantity.

Then price is $10 - D(p) = 10 - 31/4 = \frac{40}{4} - \frac{31}{4} = 9/4$.

1. **[17 points]** Denote a commodity by x, denote the supply of that commodity by x^S, and denote the demand for that commodity by x^D.

   Suppose that x is supplied by an industry whose firms take the price of x, which is denoted by p_x, given, and they also take the wage rate, which is denoted by w, given. Suppose that x^S depends on both p_x and on w. Just for simplicity, suppose that there is only one firm.

   Suppose that x is demanded by consumers who take the price of x, which is denoted by p_x, given, and they also take the wage rate, which is denoted by w, given. Suppose that x^D depends on both p_x and on w. Just for simplicity, suppose that there is only one consumer.

   This problem is inspired by study of minimum wage legislation, so we will always suppose that the market for x clears but we will make no assumption about whether the market for labor is in equilibrium or not. We wish to study the effect of an *exogenous* increase in the wage rate w, caused, for example, by an increase in the minimum wage.

   (a) Show that
   $$\frac{dp_x}{dw} = \frac{\frac{\partial x^D}{\partial w} - \frac{\partial x^S}{\partial w}}{\frac{\partial x^S}{\partial p_x} - \frac{\partial x^D}{\partial p_x}}\,. \tag{1}$$

   (b) Suppose the firm produces x from labor ℓ using the production function $x = 2\sqrt{\ell}$. Recalling that the price of x is denoted by p_x in this problem, and that the wage rate is denoted by w, show that the firm's supply curve for x is given by $x^S = 2p/w$.

   (c) Suppose the consumer obtains utility from consuming x, from consuming another good y, and from leisure, according to
   $$u = x \cdot y \cdot \text{leisure}\,,$$
   where leisure $= 24 - \ell$. If the consumer's only income comes from supplying labor, show that the consumer's demand curve for x is given by $x^D = 8w/p_x$.

   (d) Show that, for the above firm and consumer, (1) implies
   $$\frac{dp_x}{dw} = \frac{p_x}{w}\,. \tag{2}$$

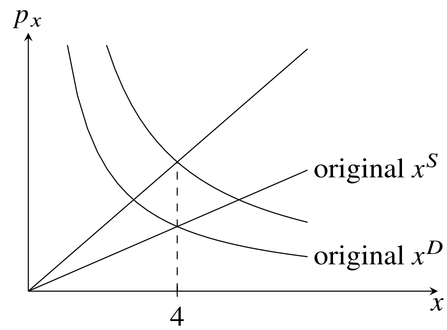


Figure 1. The market for good x. Demand curves are denoted by x^D and supply curves are denoted by x^S.

(e) Show that

$$\frac{d(p_x/w)}{dw} = 0$$

and use this fact to explain why Figure 1 illustrates what happens in the market for x when w is exogenously increased. You should label the two unlabeled curves of Figure 1 and provide an intuitive explanation of what is going on.

(f) If the original situation had $w = 1$ and $p_x = 2$, and the new situation has $w = 2$, then what are the new values for p_x, for x, and for ℓ?

Answer to 2020 Qualifier, Section 1 Question 1

a) equilibrium in the x market:

$x^D(P_x, w) = x^S(P_x, w)$. Take the differential of both sides:

$$\frac{\partial x^D}{\partial P_x} dP_x + \frac{\partial x^D}{\partial w} dw = \frac{\partial x^S}{\partial P_x} dP_x + \frac{\partial x^S}{\partial w} dw$$

$$\left(\frac{\partial x^D}{\partial P_x} - \frac{\partial x^S}{\partial P_x}\right) dP_x = \left(\frac{\partial x^S}{\partial w} - \frac{\partial x^D}{\partial w}\right) dw$$

$$\frac{dP_x}{dw} = \frac{\frac{\partial x^S}{\partial w} - \frac{\partial x^D}{\partial w}}{\frac{\partial x^D}{\partial P_x} - \frac{\partial x^S}{\partial P_x}} = \frac{\frac{\partial x^D}{\partial w} - \frac{\partial x^S}{\partial w}}{\frac{\partial x^S}{\partial P_x} - \frac{\partial x^D}{\partial P_x}}.$$

b) $\pi = P_x x^S - w\ell = P_x 2\sqrt{\ell} - w\ell$

$0 = \frac{d\pi}{d\ell} = \frac{P_x}{\sqrt{\ell}} - w \Rightarrow w = \frac{P_x}{\sqrt{\ell}} \Rightarrow \sqrt{\ell} = \frac{P_x}{w} \Rightarrow \ell^D = \frac{P_x^2}{w^2}$.

Therefore $x^S = 2\sqrt{\ell^D} = 2\sqrt{\frac{P_x^2}{w^2}} = \frac{2P_x}{w}$.

c) $u = xy(24-\ell)$. Budget constraint: $P_x x + P_y y = w\ell$

$\mathcal{L} = xy(24-\ell) + \lambda[w\ell - P_x x - P_y y]$

F.O.C. $0 = \partial\mathcal{L}/\partial\lambda = w\ell - P_x x - P_y y$

$$\left.\begin{aligned} 0 &= \partial\mathcal{L}/\partial x = y(24-\ell) - \lambda P_x \\ 0 &= \partial\mathcal{L}/\partial y = x(24-\ell) - \lambda P_y \\ 0 &= \partial\mathcal{L}/\partial \ell = -xy + \lambda w \end{aligned}\right\} \lambda = \frac{y(24-\ell)}{P_x} = \frac{x(24-\ell)}{P_y} = \frac{xy}{w}$$

$\frac{y}{P_x} = \frac{x}{P_y}$ $\qquad$ $\frac{24-\ell}{P_y} = \frac{y}{w}$

$y = \frac{P_x}{P_y} x$ $\qquad$ $24 - \ell = \frac{P_y y}{w}$

$\ell = 24 - \frac{P_y y}{w} = 24 - \frac{P_y}{w}\frac{P_x}{P_y} x$

$$\Rightarrow l = 24 - \frac{p_x}{w} x$$

Hence

$$wl = p_x x + p_y y \iff$$

$$w\left(24 - \frac{p_x}{w} x\right) = p_x x + p_y \frac{p_x}{p_y} x$$

$$24w - p_x x = p_x x + p_x x$$

$$24w = 3 p_x x$$

$$\frac{8w}{p_x} = x, \text{ the demand for } x.$$

d)

$$\frac{\partial x^D}{\partial w} = \frac{\partial}{\partial w} \frac{8w}{p_x} \text{ from (c)} \qquad \frac{\partial x^S}{\partial w} = \frac{\partial}{\partial w} \frac{2p_x}{w} \text{ from (b)}$$

$$= 8/p_x. \qquad\qquad = \frac{-2p_x}{w^2}.$$

$$\frac{\partial x^D}{\partial p_x} = \frac{\partial}{\partial p_x} \frac{8w}{p_x} \qquad \frac{\partial x^S}{\partial p_x} = \frac{\partial}{\partial p_x} \frac{2p_x}{w} = \frac{2}{w}.$$

$$= \frac{-8w}{p_x^2}$$

Substituting into (1),

$$\frac{dp_x}{dw} = \frac{\frac{8}{p_x} + \frac{2p_x}{w^2}}{\frac{2}{w} + \frac{8w}{p_x^2}} = \frac{\frac{8w^2 + 2p_x^2}{p_x w^2}}{\frac{2p_x^2 + 8w^2}{w p_x^2}} = \frac{8w^2 + 2p_x^2}{8w^2 + 2p_x^2} \frac{w p_x^2}{p_x w^2}$$

$$= \frac{p_x}{w}.$$ Optional: This is positive, so increases in w will increase p_x.

e)

$$\frac{d(p_x/w)}{dw} = \frac{1}{w}\frac{dp_x}{dw} - \frac{p_x}{w^2}\frac{dw}{dw} = \frac{1}{w}\frac{p_x}{w} - \frac{p_x}{w^2} = 0.$$

from (d)

From (b), $x^S = 2\left(\frac{p_x}{w}\right)$, so if $\frac{p_x}{w}$ does not change when w changes, which we know since $\frac{d(p_x/w)}{dw} = 0$, then x^S does not change when w changes.

Equilibrium in the x market implies

$$\underbrace{2\frac{p_x}{w}}_{\text{from (b)}} = x^S = x^D = \underbrace{\frac{8w}{p_x}}_{\text{from (c)}}$$

$$\frac{p_x^2}{w^2} = 4 \Rightarrow \frac{p_x}{w} = 2 \text{ and } x^S = 2\frac{p_x}{w} = 2 \cdot 2 = 4$$

$$x^D = 8\frac{w}{p_x} = 8/2 = 4.$$

So $x^* = 4$.

When w rises exogenously, $x^S = 2p_x/w$ falls (supply shifts left) and $x^D = 8w/p_x$ rises (demand shifts right), as shown in the figure on the next page. The equilibrium x remains at 4, so the new supply and demand curves intersect at $x = 4$, as the old ones did.

f) From (e), $\frac{p_x}{w}$ remains unchanged when w rises. Before, $\frac{p_x}{w} = \frac{2}{1} = 2$, so in the new situation, $2 = \frac{p_x}{w} = \frac{p_x}{2} \Rightarrow p_x = 4$. So p_x has risen from 2 to 4. As in (e), $x^* = 4 = 2\sqrt{\ell^*}$ so $\ell^* = 4$: x and ℓ are unchanged.

Optional: Here the increase in w increases demand for x so much that it offsets the fall in x's supply curve caused by the increase in w. One might think of this as a Keynesian aggregate demand effect, but clearly it's present in perfect-competition micro.

→ Optional comments continue on the page after next →

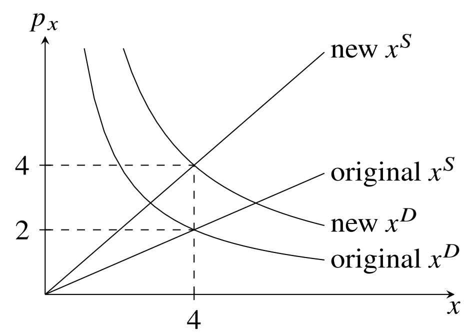


Figure 4. The market for good x. Demand curves are denoted by x^D and supply curves are denoted by x^S.

This page is optional!

However, not imposing equilibrium in the labor market raises questions about how correct this analysis is. In (f) we concluded that $x^* = 4 = 2\sqrt{\ell^*}$ so $\ell^* = 4$. However in (c), $\ell = 24 - \frac{P_x}{w}x = 24 - (2)(4) = 24 - 8 = 16$, not 4. The 16 is labor supply, the 4 is labor demand. The consumer's budget constraint is, from (c),

$$P_x x + P_y y = w\ell$$

$$\frac{P_x}{w}x + \frac{P_y}{w}y = \ell \quad ; \text{ from (c), } y = \frac{P_x}{P_y}x \text{ so}$$

$$\frac{P_x}{w}x + \frac{P_x x}{w} = \ell$$

$$\frac{2P_x x}{w} = \ell \Rightarrow \overbrace{2\cdot(2)\cdot(4) = 16}^{\text{expenditure}/w} = \ell$$

so if actually only 4 units of labor are sold, the values derived in the problem won't satisfy the budget constraint: expenditure will be more than income.

One might try to address this by retaining (b) but changing (c) to make ℓ exogenous to the consumer (and equal to the firm's ℓ^D). One would still get $y = \frac{P_x}{P_y}x$. Then from the budget constraint, $w\ell = P_x x + P_y y = P_x x + P_x x = 2P_x x \Rightarrow x^D = \frac{w\ell}{2P_x}$. Setting ℓ equal to labor demand $\ell^D = \left(\frac{P_x}{w}\right)^2$ from (b), $x^D = \frac{\ell}{2}\frac{w}{P_x} = \frac{1}{2}\frac{P_x^2}{w^2}\frac{w}{P_x} = \frac{1}{2}\frac{P_x}{w}$. However $x^S = 2\frac{P_x}{w}$ from (b), so it's not possible to set x^D equal to x^S. In general, modeling a non-equilibrium situation requires careful thought and is not easy.

Fall 2006. Final

5. [11 points] Attached to this exam is a copy of page 321 of Varian's textbook.

   (a) Why is it that the theorem at the top of the page is labeled "Existence of Walrasian Equilibrium"? In other words, what is it about the theorem that leads Varian to call it that?

   (b) The proof of the theorem ends on the next page of the textbook. That next page is not attached to this exam. Give the rest of the proof of the theorem.

Existence of Walrasian equilibria. *If* $\mathbf{z} : S^{k-1} \rightarrow R^k$ *is a continuous function that satisfies Walras' law,* $\mathbf{pz}(\mathbf{p}) \equiv 0$*, then there exists some* $\mathbf{p}^*$ *in* S^{k-1} *such that* $\mathbf{z}(\mathbf{p}^*) \leq 0$.

Proof. Define a map $g : S^{k-1} \rightarrow S^{k-1}$ by

$$g_i(\mathbf{p}) = \frac{p_i + \max(0, z_i(\mathbf{p}))}{1 + \sum_{j=1}^{k} \max(0, z_j(\mathbf{p}))} \quad \text{for } i = 1, \ldots, k.$$

Notice that this map is continuous since z and the max function are continuous functions. Furthermore, $\mathbf{g}(\mathbf{p})$ is a point in the simplex S^{k-1} since $\sum_i g_i(\mathbf{p}) = 1$. This map also has a reasonable economic interpretation: if there is excess demand in some market, so that $z_i(\mathbf{p}) \geq 0$, then the relative price of that good is increased.

By Brouwer's fixed-point theorem there is a $\mathbf{p}^*$ such that $\mathbf{p}^* = \mathbf{g}(\mathbf{p}^*)$; i.e.,

$$p_i^* = \frac{p_i^* + \max(0, z_i(\mathbf{p}^*))}{1 + \sum_j \max(0, z_j(\mathbf{p}^*))} \quad \text{for } i = 1, \ldots, k. \tag{17.1}$$

We will show that $\mathbf{p}^*$ is a Walrasian equilibrium. Cross-multiply equation (17.1) and rearrange to get

$$p_i^* \sum_{j=1}^{k} \max(0, z_j(\mathbf{p}^*)) = \max(0, z_i(\mathbf{p}^*)) \quad i = 1, \ldots, k.$$

Now multiply each of these k equations by $z_i(\mathbf{p}^*)$:

$$z_i(\mathbf{p}^*) p_i^* \left[\sum_{j=1}^{k} \max(0, z_j(\mathbf{p}^*)) \right] = z_i(\mathbf{p}^*) \max(0, z_i(\mathbf{p}^*)) \quad i = 1, \ldots, k.$$

Sum these k equations to get

$$\left[\sum_{j=1}^{k} \max(0, z_j(\mathbf{p}^*)) \right] \sum_{i=1}^{k} p_i^* z_i(\mathbf{p}^*) = \sum_{i=1}^{k} z_i(\mathbf{p}^*) \max(0, z_i(\mathbf{p}^*)).$$

Now $\sum_{i=1}^{k} p_i^* z_i(\mathbf{p}^*) = 0$ by Walras' law so we have

$$\sum_{i=1}^{k} z_i(\mathbf{p}^*) \max(0, z_i(\mathbf{p}^*)) = 0.$$

Each term of this sum is greater than or equal to zero since each term is either 0 or $(z_i(\mathbf{p}^*))^2$. But if any term were *strictly* greater than zero, the

Fall 2006 Final

(5) a) $\underset{\sim}{z}$ is excess demands.

z cannot be positive in competitive equilibrium.

z could be negative in competitive equilibrium, for free goods : excess supply (negative excess demand) is OK in equilibrium for free goods (whose price is zero).

So a price vector $\underset{\sim}{p}^*$ making $\underset{\sim}{z} \leq \underset{\sim}{0}$ is an equilibrium price vector.

b) ... the sum could not be zero because there could not be any negative terms to cancel the strictly positive term and yield a zero sum. Hence every term must be exactly zero:

$$z_i \max(0, z_i) = 0 \ \forall i$$

This is OK for $z_i = 0$, and if $z_i < 0$, it is $z_i * 0 = 0$, which is still OK. If $z_i > 0$, this is $z_i^2 > 0$, which is wrong. So $z_i > 0$ is ruled out, $z_i \leq 0 \ \forall i$ is the only possibility, so $\underset{\sim}{z} \leq \underset{\sim}{0}$.

This is better notation than the original version. The original version is on the next page. I did not write an answer with this improved notation.

old: Fall 2009 Final, Qu. 1 (but with different notation)

2. **[17 points]**

   Consider a two-person, two-commodity economy in which the two people are named Smith (abbreviated s) and Jones (abbreviated j) and the two commodities are apples (abbreviated a) and bananas (abbreviated b). Let "x_{ij}" represent the amount of commodity i ($i \in \{a, b\}$) belonging to person j ($j \in \{s, j\}$). Suppose the utility function of Smith is

   $$\ln x_{as} + \ln x_{bs}$$

   and the utility function of Jones is

   $$\ln x_{aj} + \ln x_{bj} \, .$$

   Suppose the initial endowments of Smith and Jones are $\boldsymbol{\omega}_s = \omega_s(a_s, b_s) = (1, 1)$ and $\boldsymbol{\omega}_j = \omega_j(a_j, b_j) = (2, 1)$, respectively. Find the competitive general equilibrium of this economy. (That is, find the price of each good and find the quantities of each good consumed by each person.)

Final Exam
Fall 2009

1. **[20 points]** Consider a two-person, two-commodity economy in which "x_{ij}" represents the amount of commodity i belonging to person j. Suppose the utility function of person 1 is

$$\ln x_{11} + \ln x_{21}$$

and the utility function of person 2 is

$$\ln x_{12} + \ln x_{22}\,.$$

Suppose the initial endowments of persons 1 and 2 are $\omega_1 = (1,1)$ and $\omega_2 = (2,1)$, respectively. Find the competitive general equilibrium of this economy. (That is, find the price of each good and find the quantities of each good consumed by each person.)

Answer to Question 1, 2009 Fall Final Exam, Econ. 7005

$$u_1(x_{11}, x_{21}) = \ln x_{11} + \ln x_{21} \qquad \omega_1 = (1,1)$$

$$u_2(x_{12}, x_{22}) = \ln x_{12} + \ln x_{22} \qquad \omega_2 = (2,1)$$

We could choose a numeraire but I'll use the simplex: $p_1 + p_2 = 1$.

$$\Rightarrow \quad p_2 = 1 - p_1$$

So I won't use p_2 any more, just $1-p_1$.

Consumer 1: max u_1 s.t. expenditures = income

$$p_1 x_{11} + (1-p_1) x_{21} = p_1(1) + (1-p_1)(1)$$
$$= p_1 + 1 - p_1 = 1$$

$$\mathcal{L} = \ln x_{11} + \ln x_{21} + \lambda\left[1 - p_1 x_{11} - (1-p_1) x_{21}\right]$$

$$0 = \partial \mathcal{L}/\partial \lambda = 1 - p_1 x_{11} - (1-p_1) x_{21}$$

$$\left.\begin{aligned} 0 &= \partial \mathcal{L}/\partial x_{11} = \frac{1}{x_{11}} - \lambda p_1 \\ 0 &= \partial \mathcal{L}/\partial x_{21} = \frac{1}{x_{21}} - \lambda(1-p_1) \end{aligned}\right\} \Rightarrow \frac{1/x_{11}}{1/x_{21}} = \frac{\lambda p_1}{\lambda(1-p_1)}$$

$$\frac{x_{21}}{x_{11}} = \frac{p_1}{1-p_1}$$

$$x_{21} = \frac{p_1}{1-p_1} x_{11}.$$ Substitute into the budget constraint:

$$0 = 1 - p_1 x_{11} - (1-p_1)\frac{p_1}{1-p_1} x_{11}$$
$$= 1 - p_1 x_{11} - p_1 x_{11} = 1 - 2p_1 x_{11}$$

$$2p_1 x_{11} = 1$$

$$\left.\begin{aligned} x_{11}^D &= \frac{1}{2p_1} \quad \text{and} \\ x_{21}^D &= \frac{p_1}{1-p_1}\,\frac{1}{2p_1} = \frac{1}{2(1-p_1)} \end{aligned}\right\} \text{Person 1's demand curves.}$$

Consumer 2 : max u_2 s.t. expenditures = income

$$p_1 x_{12} + (1-p_1) x_{22} = p_1 (2) + (1-p_1)(1)$$
$$= 2p_1 + 1 - p_1 = p_1 + 1$$

$$\mathcal{L} = \ln x_{12} + \ln x_{22} + \lambda \left[p_1 + 1 - p_1 x_{12} - (1-p_1) x_{22} \right]$$

$$0 = \partial \mathcal{L} / \partial \lambda = p_1 + 1 - p_1 x_{12} - (1-p_1) x_{22}$$

$$\left.\begin{aligned} 0 &= \partial \mathcal{L} / \partial x_{12} = \frac{1}{x_{12}} - \lambda p_1 \\ 0 &= \partial \mathcal{L} / \partial x_{22} = \frac{1}{x_{22}} - \lambda (1-p_1) \end{aligned}\right\} \Rightarrow \frac{1/x_{12}}{1/x_{22}} = \frac{\lambda p_1}{\lambda (1-p_1)}$$

$$\frac{x_{22}}{x_{12}} = \frac{p_1}{1-p_1}$$

$$x_{22} = \frac{p_1}{1-p_1} x_{12}.$$ Substituting into the budget constraint :

$$0 = p_1 + 1 - p_1 x_{12} - (1-p_1) \frac{p_1}{1-p_1} x_{12}$$
$$= p_1 + 1 - p_1 x_{12} - p_1 x_{12} = p_1 + 1 - 2 p_1 x_{12}$$

$$2 p_1 x_{12} = p_1 + 1$$

$$\left.\begin{aligned} x_{12}^D &= \frac{p_1 + 1}{2 p_1} \\ x_{22}^D &= \frac{p_1}{1-p_1} \frac{p_1+1}{2p_1} = \frac{1}{2} \frac{p_1+1}{1-p_1} \end{aligned}\right\}$$ Person 2's demand curves.

Equilibrium (supply = demand) in : Market for Good 1 $x_{11}^D + x_{12}^D = 2 + 1$

" " " 2 $x_{21}^D + x_{22}^D = 1 + 1$

We only have to check one market (if it clears, the other will automatically clear). I do the market for Good 1 :

$$\frac{1}{2p_1} + \frac{p_1+1}{2p_1} = 3$$

$$\frac{p_1+2}{2p_1} = 3$$

$$p_1 + 2 = 6p_1$$

$$2 = 5p_1$$

$$\boxed{\frac{2}{5} = p_1.}$$

Optional: Check Good 2's market:

$$\frac{1}{2(1-p_1)} + \frac{p_1+1}{2(1-p_1)} = 2$$

$$\frac{1}{1-p_1} + \frac{p_1+1}{1-p_1} = 4$$

$$\frac{p_1+2}{1-p_1} = 4$$

$$p_1 + 2 = 4 - 4p_1$$

$$5p_1 = 2$$

$$p_1 = \frac{2}{5}$$ yes, it checks

So the rest of the equilibrium is

$$p_2 = 1 - p_1 = 1 - \frac{2}{5} = \frac{3}{5}$$

$$x_{11} = \frac{1}{2p_1} = \frac{1}{4/5} = \frac{5}{4}$$

$$x_{21} = \frac{1}{2(1-p_1)} = \frac{1}{2(1-\frac{2}{5})} = \frac{1}{2(3/5)} = \frac{1}{6/5} = \frac{5}{6}$$

$$x_{12} = \frac{p_1+1}{2p_1} = \frac{(2/5)+1}{2(2/5)} = \frac{7/5}{4/5} = \frac{7}{4}$$

$$x_{22} = \frac{1}{2}\,\frac{p_1+1}{1-p_1} = \frac{1}{2}\,\frac{(2/5)+1}{1-\frac{2}{5}} = \frac{1}{2}\,\frac{7/5}{3/5} = \frac{1}{2}\,\frac{7}{3} = \frac{7}{6}.$$

Optional: Check feasibility:

$$3 \stackrel{?}{=} x_{11} + x_{12}$$

$$= \frac{5}{4} + \frac{7}{4} = \frac{12}{4} \text{ OK}$$

$$2 \stackrel{?}{=} x_{21} + x_{22}$$

$$= \frac{5}{6} + \frac{7}{6} = \frac{12}{6} \text{ OK.}$$

Section 1.
Answer all of the following three questions.

1. [**14 points**] Suppose an economy has two consumers, Smith and Jones, and two commodities, x and y. Smith's utility function and initial endowment are

$$u_s = \ln x_s + \ln y_s$$
$$\boldsymbol{\omega}_s = (\omega_{sx}, \omega_{sy}) = (0, 1) .$$

Jones's initial endowment is

$$\boldsymbol{\omega}_j = (\omega_{jx}, \omega_{jy}) = (1, 0) .$$

So the only way Jones can get any of good y is to get it from Smith. Jones's true utility function is

$$u_j = \ln x_j + \ln y_j$$

but he may be unsure about the quality of the y which he gets from Smith; we will model this by assuming Jones's utility function is instead

$$u_j = \ln x_j + \psi \ln y_j$$

where $0 \leq \psi \leq 1$.

(a) What situation does $\psi = 1$ represent?
(b) What situation does $\psi = 0$ represent?
(c) Find the general equilibrium x_j, y_j, x_s, and y_s if $\psi > 0$.
(d) Find the general equilibrium x_j, y_j, x_s, and y_s if $\psi = 0$. Be sure your answer makes intuitive sense.
(e) Set up an Edgeworth Box Diagram. Show on this diagram how the allocations of x and y to Smith and Jones change as ψ changes from 1 to 0. (It makes no sense to draw indifference curves on this diagram because changing ψ implies changing utility functions.)

Summer 2012, Qualifying Exam, Section 1 Qu. 1

Answers to Prof. Lozada's portion of the

Summer 2012 Micro Qualifying Exam

①

a) Jones fully trusts that the y he gets from Smith is good.

b) Jones thinks any y he would get from Smith would be of such bad quality that it would not increase Jones's utility.

c) Smith:

Note: It's fine to choose a numeraire or to set $p_x + p_y = 1$, but I did neither and I could still find the equilibrium quantities.

$u_s = \ln x_s + \ln y_s$

$\underset{\sim}{\omega}_s = (0, 1)$

income: $0 p_x + 1 p_y = p_y$

expenditures: $p_x x_s + p_y y_s$

problem: max u_s s.t. $p_y = p_x x_s + p_y y_s$

$$\mathcal{L} = \ln x_s + \ln y_s + \lambda (p_y - p_x x_s - p_y y_s) .$$ F.O.C.'s:

$$0 = \partial\mathcal{L}/\partial x_s = \frac{1}{x_s} - \lambda p_x \quad \Rightarrow \quad \lambda = \frac{1}{p_x x_s}$$

$$0 = \partial\mathcal{L}/\partial y_s = \frac{1}{y_s} - \lambda p_y \quad \Rightarrow \quad y_s = \frac{1}{\lambda p_y} = \frac{p_x x_s}{p_y}$$

$$0 = \partial\mathcal{L}/\partial \lambda = p_y - p_x x_s - p_y y_s$$

$$\Downarrow$$

$$p_y = p_x x_s + p_y \frac{p_x x_s}{p_y} \quad \Rightarrow$$

$$p_y = p_x x_s + p_x x_s$$
$$= 2 p_x x_s$$
$$\Rightarrow x_s^* = \frac{p_y}{2p_x} \quad \text{and} \quad y_s^* = \frac{p_x}{p_y} x_s^* = \frac{p_x}{p_y} \frac{p_y}{2p_x} = \frac{1}{2}.$$

Jones:

$u_j = \ln x_j + \psi \ln y_j$

$\underset{\sim}{\omega}_j = (1, 0)$

income: $1 p_x + 0 p_y = p_x$

expenditures: $p_x x_j + p_y y_j$

problem: $\max u_j$ s.t. $p_x = p_x x_j + p_y y_j$

$\mathcal{L} = \ln x_j + \psi \ln y_j + \lambda (p_x - p_x x_j - p_y y_j).$

First Order Conditions:

$$0 = \partial\mathcal{L}/\partial x_j = \frac{1}{x_j} - \lambda p_x \Rightarrow \lambda p_x = \frac{1}{x_j} \Rightarrow \lambda = \frac{1}{p_x x_j}$$

$$\Downarrow$$

$$0 = \partial\mathcal{L}/\partial y_j = \frac{\psi}{y_j} - \lambda p_y \Rightarrow \frac{\psi}{y_j} = \lambda p_y = \frac{1}{p_x x_j} p_y \Rightarrow$$

$$0 = \partial\mathcal{L}/\partial \lambda = p_x - p_x x_j - p_y y_j$$

$$y_j = \psi \frac{p_x x_j}{p_y}.$$

$$\Downarrow$$

$$p_x = p_x x_j + p_y y_j = p_x x_j + p_y \psi \frac{p_x x_j}{p_y}$$

$$= p_x x_j + \psi p_x x_j = (1 + \psi) p_x x_j \Rightarrow$$

$$1 = (1+\psi)\,x_j \quad \Rightarrow \quad x_j^* = \frac{1}{1+\psi}$$

$$\text{and } y_j^* = \psi \frac{P_x}{P_y} x_j^* = \psi \frac{P_x}{P_y} \frac{1}{1+\psi} = \frac{\psi}{1+\psi} \frac{P_x}{P_y}.$$

Either impose market-clearing for x or for y (no need to do both).

- If you chose to impose market-clearing for x:

$S_x = D_x$ "supply equals demand"

$$1 = \underbrace{\frac{P_y}{2P_x}}_{\text{Smith } x_s^*} + \underbrace{\frac{1}{1+\psi}}_{\text{Jones } x_j^*}$$

$$1 - \frac{1}{1+\psi} = \frac{P_y}{2P_x}$$

$$\frac{1+\psi-1}{1+\psi} = \frac{P_y}{2P_x}$$

$$\frac{2\psi}{1+\psi} = \frac{P_y}{P_x} \text{, the equilibrium price ratio.}$$

$$x_j^* = \frac{1}{1+\psi}$$

$$x_s^* = \frac{P_y}{2P_x} = \frac{\psi}{1+\psi}$$

$$y_s^* = \frac{1}{2}$$

$$y_j^* = \frac{\psi}{1+\psi} \frac{P_x}{P_y} = \frac{\psi}{1+\psi} \frac{1+\psi}{2\psi} \text{ if } \psi \neq 0$$

$$= \frac{1}{2}.$$

- If you chose to impose market-clearing for y:

$$S_y = D_y \quad \text{"supply equals demand"}$$

$$1 = y_s^* + y_j^* = \frac{1}{2} + \frac{\psi}{1+\psi}\frac{P_x}{P_y}$$

$$\frac{1}{2} = \frac{\psi}{1+\psi}\frac{P_x}{P_y}$$

$$\frac{1+\psi}{2\psi} = \frac{P_x}{P_y}$$ if $\psi \neq 0$. This equilibrium price ratio gives

$$x_s^* = \frac{P_y}{2P_x} = \frac{1}{2}\frac{P_y}{P_x} = \frac{1}{2}\frac{2\psi}{1+\psi} = \frac{\psi}{1+\psi}$$

$$x_j^* = \frac{1}{1+\psi}$$

$$y_s^* = \frac{1}{2}$$

$$y_j^* = \frac{\psi}{1+\psi}\frac{P_x}{P_y} = \frac{\psi}{1+\psi}\frac{1+\psi}{2\psi} = \frac{1}{2}.$$

(Note that the answers are the same regardless of which one of the two markets you chose for market-clearing.)

d) If $\psi = 0$, the demands are

$$x_s = \frac{P_y}{2P_x}$$

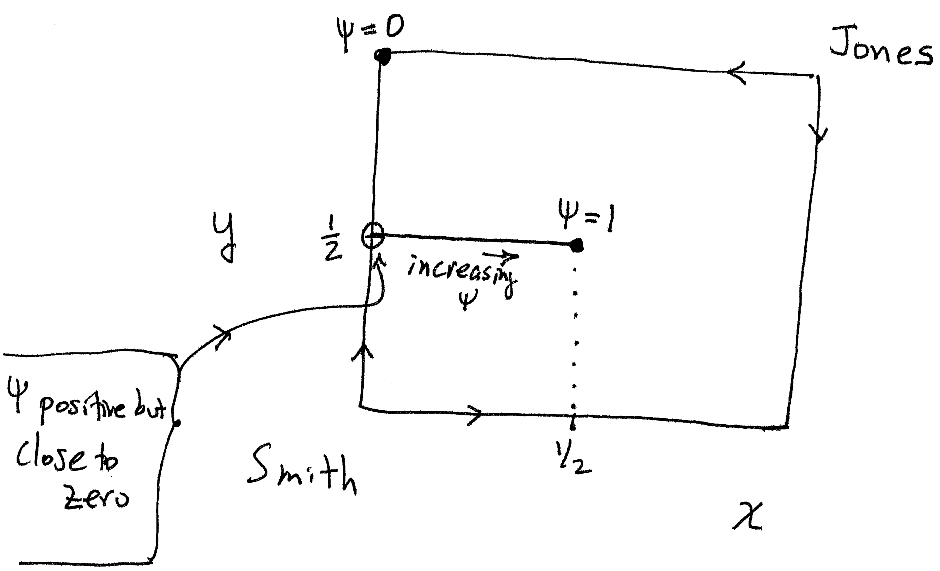
$$y_s = \frac{1}{2}$$

$$x_j = \frac{1}{1+\psi} = 1$$

$$y_j = \frac{\psi}{1+\psi}\frac{P_x}{P_y} = 0.$$

Jones gets no utility from y, so $y_j = 0$. Jones therefore keeps his entire endowment of x (why should he trade any away for the worthless y?) : $x_j = 1$. Since Jones does not want to trade, Smith has no one to trade with, so he keeps his endowment, $\underset{\sim}{\omega}_s = (0,1)$. This is actually incompatible with competitive equilibrium, since the only competitive equilibrium y_s is $\frac{1}{2}$, not 1. The failure of existence of a competitive general equilibrium when $\psi = 0$ is related to the fact that Smith has to stay at $\underset{\sim}{\omega}_s = (0,1)$, but at that point his utility is $-\infty$. optional

e)



For $\psi > 0$, $x_s = \frac{\psi}{1+\psi}$ $\quad y_s = \frac{1}{2}$. At $\psi = 1$, $x_s = \frac{1}{2}$ and $x_j = \frac{1}{2}$.

$x_j = \frac{1}{1+\psi}$ $\quad y_j = \frac{1}{2}$

Final Exam
2004
Question 3

5

√ 3. [19 points] Suppose an economy consists of two persons, "a" and "b". Person "a" has available 1 unit of "time" which he divides between rest R_a and labor l_a. Person "b" has available 1 unit of "time" which he divides between rest R_b and labor l_b.

Good "x" is produced by one competitive firm according to the production function

$$x = \sqrt{\text{labor hired}}\,.$$

This firm is completely owned by Person a.

The utility functions of the two individuals are

$$u_a = x_a\, R_a$$
$$u_b = x_b^2\, R_b\,.$$

Take the price of labor as the numeraire. Find one equation in one unknown, with the unknown being the competitive general equilibrium price of x. Do not try to solve this equation for the equilibrium price of x.

Final Exam

2004

Answer 3

(3) $R_a + \ell_a = 1$ $\quad u_a = x_a R_a$ $\quad$ price of labor $= 1$

$R_b + \ell_b = 1$ $\quad u_b = x_b^2 R_b$ $\quad$ price of x : call this just "p"

Person a's firm : $x = \sqrt{\text{labor}}$

Firm : $\pi = px - 1\ell$

$$= p\sqrt{\ell} - \ell$$

$$\max \pi \Rightarrow 0 = \tfrac{1}{2} p \ell^{-1/2} - 1 \Rightarrow 1 = \tfrac{1}{2} p \ell^{-1/2}$$

$$\frac{2}{p} = \ell^{-1/2}$$

$$\frac{p}{2} = \ell^{1/2} \Rightarrow \frac{p^2}{4} = \ell \quad \text{labor demand}$$

$$\pi^* = p\sqrt{\ell^*} - \ell^* = p\left(\frac{p}{2}\right) - \frac{p^2}{4}$$

$$= \frac{p^2}{2} - \frac{p^2}{4} = \frac{p^2}{4}.$$

$$x \text{ supplied} = \sqrt{\ell^{D*}} = \sqrt{\frac{p^2}{4}} = \underline{\underline{\frac{p}{2}}}.$$

Person a. budget constraint : income $\pi^* + 1\, l_a = \frac{p^2}{4} + l_a$

expenditures $p x_a$

$$U_a = x_a R_a = x_a (1 - l_a)$$

$$\mathcal{L} = x_a(1-l_a) + \lambda\left(\tfrac{1}{4}p^2 + l_a - p x_a\right)$$

$$0 = \partial\mathcal{L}/\partial\lambda = \tfrac{1}{4}p^2 + l_a - p x_a$$

$$\left.\begin{aligned} 0 &= \partial\mathcal{L}/\partial x_a = 1 - l_a - \lambda p \\ 0 &= \partial\mathcal{L}/\partial l_a = -x_a + \lambda \end{aligned}\right\} \quad \frac{1-l_a}{-x_a} = \frac{\lambda p}{-\lambda}$$

$$\frac{l_a - 1}{x_a} = -p$$

Final Exam
2004
Answer 3 cont...

$$\frac{1-l_a}{x_a} = p$$

$$1 - l_a = p x_a$$

$1 - p x_a = l_a$ into budget constraint :

$$0 = \tfrac{1}{4}p^2 + (1 - p x_a) - p x_a = \tfrac{1}{4}p^2 + 1 - 2 p x_a \Rightarrow 2 p x_a = \frac{p^2}{4} + 1$$

$$x_a = \frac{p}{8} + \frac{1}{2p}.$$

Person b budget constraint $1\, l_b = p x_b$

$$u_b = x_b^2 R_b = x_b^2 (1 - l_b)$$

$$\mathcal{L} = x_b^2 - x_b^2 l_b + \lambda(l_b - p x_b)$$

$$0 = \partial\mathcal{L}/\partial\lambda = l_b - p x_b$$

$$\left.\begin{aligned} 0 &= \partial\mathcal{L}/\partial x_b = 2x_b - 2x_b l_b - \lambda p \\ 0 &= \partial\mathcal{L}/\partial l_b = -x_b^2 + \lambda \end{aligned}\right\} \quad \frac{2x_b - 2x_b l_b}{-x_b^2} = \frac{-\lambda p}{\lambda}$$

$$\Downarrow$$

$$\frac{2 - 2 l_b}{-x_b} = -p \Rightarrow$$

$$\frac{2-2\ell_b}{x_b} = p$$

$2-2\ell_b = p x_b \Rightarrow 2 - p x_b = 2\ell_b \Rightarrow \ell_b = 1 - \frac{1}{2} p x_b$ into budget constraint

$1 - \frac{1}{2} p x_b = p x_b$

$1 = \frac{3}{2} p x_b \Rightarrow x_b = \frac{2}{3p}.$

Final Exam
2004
Answer 3 cont...

Clear $x \Rightarrow$

$$x_a^D + x_b^D = x^S$$

$\boxed{\frac{p}{8} + \frac{1}{2p} + \frac{2}{3p} = \frac{p}{2}}$ which is the answer.

Optional: $\frac{1}{p}\left(\frac{1}{2}+\frac{2}{3}\right) = p\left(\frac{1}{2}-\frac{1}{8}\right)$

$\frac{1}{p} \ \frac{3+4}{6} = p \ \frac{4-1}{8} \Leftrightarrow \frac{7}{6p} = \frac{3p}{8} \Rightarrow 7 \cdot 8 = 3 \cdot 6 p^2$

$7 \cdot 4 = 3 \cdot 3 p^2$

$p = \sqrt{\frac{7 \cdot 4}{3 \cdot 3}} = \frac{2}{3}\sqrt{7}$

One could also solve this question by getting labor supply functions and then clearing the labor market.

Checks (optional): $x_a^D = \frac{p}{8} + \frac{1}{2p} = \frac{1}{8} \cdot \frac{2}{3}\sqrt{7} + \frac{1}{2} \frac{3}{2\sqrt{7}} = \frac{\sqrt{7}}{12} + \frac{3}{4\sqrt{7}}$

$x_b^D = \frac{2}{3p} = \frac{2}{3} \cdot \frac{3}{2} \frac{1}{\sqrt{7}} = \frac{1}{\sqrt{7}}$

$x^D = x_a^D + x_b^D = \frac{\sqrt{7}}{12} + \frac{3}{4\sqrt{7}} + \frac{1}{\sqrt{7}}$

$x^S = \frac{1}{2} p = \frac{\sqrt{7}}{3}$

$x^D - x^S = \frac{\sqrt{7}}{12} + \frac{3}{4\sqrt{7}} + \frac{1}{\sqrt{7}} - \frac{\sqrt{7}}{3} = \sqrt{7}\left(\frac{1}{12} - \frac{1}{3}\right) + \frac{1}{\sqrt{7}}\left(\frac{3}{4}+1\right)$

$= \sqrt{7} \ \frac{1-4}{12} + \frac{1}{\sqrt{7}} \ \frac{3+4}{4} = \frac{-3}{12}\sqrt{7} + \frac{7}{4}\frac{1}{\sqrt{7}} = \frac{-1}{4}\sqrt{7} + \frac{\sqrt{7}}{4} = 0$ or

x clears

Continuation of optional checks

Does l clear? $l_a^s = 1 - p x_a$

$$= 1 - \frac{2}{3}\sqrt{7}\left(\frac{\sqrt{7}}{12} + \frac{3}{4\sqrt{7}}\right)$$

$$= 1 - \frac{14}{36} - \frac{1}{2} = \frac{36-14-18}{36} = \frac{4}{36} = \frac{1}{9}$$

Final Exam
2004
Answer 3 cont...

$$l_b^s = 1 - \frac{1}{2} p x_b = 1 - \frac{1}{2} \frac{2}{3}\sqrt{7} \frac{1}{\sqrt{7}} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$l^D = \frac{p^2}{4} = \frac{1}{4} \frac{4}{9} \cdot 7 = \frac{7}{9}$$

$$l^D - l_a^s - l_b^s = \frac{7}{9} - \frac{1}{9} - \frac{2}{3} = \frac{7}{9} - \frac{1}{9} - \frac{6}{9} = 0$$ OK, l clears

Person a's budget constraint:

$$\pi = p x^s - 1 l^D = \frac{2}{3}\sqrt{7} \cdot \frac{\sqrt{7}}{3} - \frac{7}{9} = \frac{2 \cdot 7}{9} - \frac{7}{9} = \frac{7}{9}$$

$$\text{income} = \pi + 1 l_a = \frac{7}{9} + \frac{1}{9} = 8/9$$ OK!

$$\text{expenditures } p x_a = \frac{2}{3}\sqrt{7} \cdot \left(\frac{\sqrt{7}}{12} + \frac{3}{4\sqrt{7}}\right)$$

$$= \frac{\cancel{2}^{1}}{3} \cdot \frac{7}{\cancel{12}_{6}} + \frac{1}{2} = \frac{7}{18} + \frac{9}{18} = \frac{16}{18} = \frac{8}{9}$$

Person b's budget constraint:

$$\text{income} = 1 l_b^s = \frac{2}{3}$$

$$\text{expenditures} = p x_b = \frac{2\sqrt{7}}{3} \cdot \frac{1}{\sqrt{7}} = \frac{2}{3}$$ OK!

2015 Final Exam Qu. 1

1. [18 points] Suppose an economy consists of two persons, "a" and "b". Person "a" has available 1 unit of "time" which he divides between rest R_a and labor l_a. Person "b" has available 1 unit of "time" which he divides between rest R_b and labor l_b.

   Good "x" is produced by one competitive firm according to the production function

   $$x = \sqrt{\text{labor hired}}\,.$$

   This firm is completely owned by Person a.

   The utility functions of the two individuals are

   $$u_a = x_a\, R_a$$
   $$u_b = x_b^2\, R_b\,.$$

   Take the price of labor as the numéraire. Find the competitive general equilibrium of this economy (that is, find the prices of all the goods, the amount produced of any produced goods, and the consumption bundle of each person).

Answer to Final Exam, Econ. 7005, Fall 2015

①

From the previous question (Final Exam 2004 Answer 3), including parts optional on it which were required here, we have:

price of labor $= 1$

labor supply by a : $\frac{1}{9}$

labor supply by b : $\frac{2}{3}$

total labor supply : $\frac{1}{9} + \frac{2}{3} = \frac{1+6}{9} = \frac{7}{9}$

labor demand : $\frac{7}{9}$

Rest of a $= 1 - \frac{1}{9} = \frac{8}{9}$

Rest of b $= 1 - \frac{2}{3} = \frac{1}{3}$

price of x, called "p" : $\frac{2}{3}\sqrt{7}$

supply of x : $\sqrt{7}/3$

demand for x by a : $\frac{\sqrt{7}}{12} + \frac{3}{4\sqrt{7}} = \frac{4\sqrt{7}}{21}$ *

demand for x by b : $\frac{1}{\sqrt{7}}$

total demand for x : $\sqrt{7}/3 = \frac{\sqrt{7}}{12} + \frac{3}{4\sqrt{7}} + \frac{1}{\sqrt{7}}$, as shown earlier.

Optional : $(x_a, R_a) = \left(\frac{\sqrt{7}}{12} + \frac{3}{4\sqrt{7}}, \frac{8}{9}\right) \approx (0.50, 0.89)$

$(x_b, R_b) = \left(\frac{1}{\sqrt{7}}, \frac{1}{3}\right) \approx (0.38, 0.33)$

"a" and "b" have the same endowments of time, but "a" owns the firm (which makes strictly positive profit), so in equilibrium, "a" gets more of both goods.

$$* \quad \frac{\sqrt{7}}{12} + \frac{3}{4\sqrt{7}}\left(\frac{\sqrt{7}}{\sqrt{7}}\right) = \frac{\sqrt{7}}{12} + \frac{3\sqrt{7}}{4\cdot 7} = \frac{\sqrt{7}}{4}\left(\frac{1}{3} + \frac{3}{7}\right) = \frac{\sqrt{7}}{4}\,\frac{7+9}{21} = \frac{16\sqrt{7}}{4\cdot 21} = \frac{4\sqrt{7}}{21}$$

2017 Final Exam Qu. 3

3. **[17 points]**

Suppose an economy consists of two persons, "a" and "b". Person "a" has available 1 unit of "time" which he divides between rest R_a and labor l_a. Person "b" has available 1 unit of "time" which he divides between rest R_b and labor l_b.

Good "x" is produced by one competitive firm according to the production function

$$x = \sqrt{\text{labor hired}} \,.$$

This firm is completely owned by Person a.

The utility functions of the two individuals are

$$u_a = x_a R_a \quad \text{and}$$
$$u_b = x_b R_b \,.$$

Take the price of labor as the numéraire. Find the competitive general equilibrium of this economy (that is, find the prices of all the goods, the amount produced of any produced goods, and the consumption bundle of each person).

(3)

$R_a + l_a = 1 \qquad u_a = x_a R_a$

$R_b + l_b = 1 \qquad u_b = x_b R_b$

Take the price of labor to be 1. Let the price of x be "p."

Person a's firm: $x = \sqrt{labor}$

Firm: $\pi = px - 1l$ ← total labor, $l_a + l_b$

$= p\sqrt{l} - l$

$\max \pi \Rightarrow 0 = \frac{d\pi}{dl} = \frac{1}{2} p l^{-1/2} - 1 \Rightarrow 1 = \frac{1}{2} p l^{-1/2}$

$\frac{2}{p} = l^{-1/2}$

$\frac{p}{2} = l^{1/2} \Rightarrow l = p^2/4$

labor demand, l^D

$x \text{ supplied} = \sqrt{l^D} = \sqrt{p^2/4} = p/2.$

$\pi^* = p\sqrt{l^*} - l^*$

$= p\sqrt{p^2/4} - p^2/4 = p\left(\frac{p}{2}\right) - \frac{p^2}{4} = \frac{p^2}{2} - \frac{p^2}{4} = \frac{p^2}{4}.$

Person a. budget constraint: income is $\pi^* + 1 l_a = \frac{p^2}{4} + l_a$

expenditure is $p x_a$

$u_a = x_a R_a = x_a(1 - l_a).$

The same as 2004 Final Exam Question 3 for this "Person a" part, but not for the rest of the problem.

$\max x_a(1-l_a) \text{ s.t. } p^2/4 + l_a = p x_a$

$\mathcal{L} = x_a(1-l_a) + \lambda\left(\frac{p^2}{4} + l_a - p x_a\right)$

$0 = \partial\mathcal{L}/\partial\lambda = p^2/4 + l_a - p x_a$

$0 = \partial\mathcal{L}/\partial x_a = 1 - l_a - \lambda p \Rightarrow \lambda = \frac{l_a - 1}{-p} = \frac{1-l_a}{p}$

$0 = \partial\mathcal{L}/\partial l_a = -x_a + \lambda \Rightarrow \lambda = x_a$

$\left.\right\} \frac{1-l_a}{p} = x_a$

Substituting into the budget constraint,

$$\frac{p^2}{4} + l_a = p\left(\frac{1-l_a}{p}\right)$$

$$\frac{p^2}{4} + l_a = 1 - l_a$$

$$2l_a = 1 - \frac{p^2}{4}$$

$$l_a = \frac{1}{2} - \frac{p^2}{8} \text{ and } x_a = \frac{1-l_a}{p} = \frac{1}{p} - \frac{1}{p}l_a = \frac{1}{p} - \frac{1}{2p} + \frac{p}{8}$$

$$= \frac{p}{8} + \frac{2}{2p} - \frac{1}{2p} = \frac{p}{8} + \frac{1}{2p}.$$

Person b.

income is $1\,l_b$; expenditure is $p x_b$; $u_b = x_b R_b = x_b(1-l_b)$

Maximizing u_b s.t. $1\,l_b = p x_b \Rightarrow$

$$\mathcal{L} = x_b(1-l_b) + \lambda(l_b - p x_b)$$

$$0 = \partial\mathcal{L}/\partial\lambda = l_b - p x_b$$

$$0 = \partial\mathcal{L}/\partial x_b = 1 - l_b - \lambda p \Rightarrow \lambda = \frac{1-l_b}{p}$$

$$0 = \partial\mathcal{L}/\partial l_b = -x_b + \lambda \Rightarrow \lambda = x_b$$

$\left.\right\}$ $x_b = \dfrac{1-l_b}{p}$. Substituting into the budget constraint,

$$l_b = p\left(\frac{1-l_b}{p}\right)$$

$$= 1 - l_b$$

$$2l_b = 1 \Rightarrow l_b = 1/2 \text{ and}$$

$$x_b = \frac{1-l_b}{p} = \frac{1}{2p}.$$

Method 1: Clear the market for x.

supply $x^S = p/2$

demand $x^D = x_a^D + x_b^D = \left(\frac{p}{8} + \frac{1}{2p}\right) + \frac{1}{2p} = \frac{p}{8} + \frac{2}{2p} = \frac{p}{8} + \frac{1}{p}$. Now impose equilibrium:

$$x^S = x^D \Rightarrow \frac{p}{2} = \frac{p}{8} + \frac{1}{p}$$

$$\frac{p}{2} - \frac{p}{8} = \frac{1}{p} \Rightarrow \frac{1}{p} = \frac{4}{4}\cdot\frac{p}{2} - \frac{p}{8} = \frac{4p-p}{8} = \frac{3p}{8}$$

$$8 = 3p^2$$

$$8/3 = p^2 \Rightarrow p = \sqrt{8/3} = 2\sqrt{2/3}.$$

$$x_a^D = \frac{p}{8} + \frac{1}{2p} = \frac{1}{8}\sqrt{\frac{8}{3}} + \frac{1}{2}\sqrt{\frac{3}{8}} = \frac{2}{8}\sqrt{\frac{2}{3}} + \frac{1}{2}\frac{1}{2}\sqrt{\frac{3}{2}} = \frac{1}{4}\left(\sqrt{\frac{2}{3}} + \sqrt{\frac{3}{2}}\right) = \frac{1}{4}\frac{2+3}{\sqrt{3}\cdot\sqrt{2}}$$

$$= \frac{5}{4\sqrt{6}} = \frac{5}{4\sqrt{3}\sqrt{2}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{12\sqrt{2}}$$

$$x_b^D = \frac{1}{2p} = \frac{1}{2}\frac{1}{2}\sqrt{\frac{3}{2}} = \frac{1}{4}\sqrt{\frac{3}{2}}$$

$$x^D = x_a^D + x_b^D = \frac{1}{4}\sqrt{\frac{2}{3}} + \frac{1}{4}\sqrt{\frac{3}{2}} + \frac{1}{4}\sqrt{\frac{3}{2}} = \frac{1}{4}\left(\sqrt{\frac{2}{3}} + 2\sqrt{\frac{3}{2}}\right) = \frac{1}{4}\frac{2+2\cdot 3}{\sqrt{3}\sqrt{2}}$$

$$= \frac{8}{4}\frac{1}{\sqrt{3}\sqrt{2}} = \frac{2}{\sqrt{3}\sqrt{2}} = \sqrt{2/3}.$$

$x^S = p/2 = \sqrt{2/3}$, checks with ↑: the market for x clears

$$l_a^S = \frac{1}{2} - \frac{p^2}{8} = \frac{1}{2} - \frac{1}{8}\left(\frac{8}{3}\right) = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$

$$l_b^S = \frac{1}{2}$$

$$l^S = l_a^S + l_b^S = \frac{1}{6} + \frac{1}{2} = \frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}$$

$l^D = p^2/4 = \frac{1}{4}\cdot\frac{8}{3} = \frac{2}{3}$ ← checks: the market for labor clears

Optional note: $x_b < x_a$ (the difference is $\frac{1}{4}\sqrt{\frac{2}{3}}$)

$l_b > l_a$ (the difference is $\frac{1}{2} - \frac{1}{6} = \frac{3}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$). So the poorer person, who does not own the firm, works longer hours and consumes less x than the owner of the firm, although their endowments are otherwise the same.

Method 2: Clear the market for l.

supply $l^s = l_a^s + l_b^s = \left(\frac{1}{2} - \frac{p^2}{8}\right) + \frac{1}{2} = 1 - \frac{p^2}{8}$

demand $l^D = p^2/4$. Now impose equilibrium:

$l^s = l^D \Rightarrow 1 - \frac{p^2}{8} = \frac{p^2}{4}$

$$1 = \frac{p^2}{8} + \frac{p^2}{4} = p^2\left(\frac{1}{8} + \frac{2}{8}\right) = \frac{3}{8}p^2$$

$8/3 = p^2 \Rightarrow p = \sqrt{8/3}$, as in Method 1. The rest continues as before. Note that p is the price of x, not the price of l, which was assumed to be equal to 1 (l is the numéraire). So even though we were clearing the market for l, the price we found that cleared the l market was the price of x.

1. Consider a two-person two-commodity competitive general equilibrium in which the two people are denoted by 1 and 2, the two goods by x and y, and person i consumes x_i of good x and y_i of good y. Suppose the utility functions of the two individuals are

$$U_1 = x_1^{\alpha} y_1^{\beta} \quad \text{and}$$
$$U_2 = x_2^{\gamma} y_2^{\delta}.$$

2005 Qualifier Sec. 1

Suppose α, β, γ, and δ are all positive. Suppose both people have 1 unit of good x and 1 unit of good y as their endowment (in other words, $\omega_1 = \omega_2 = (1, 1)$).

What effect will an increase in α have on the amount of good y which person 2 consumes?

Hint: The following answer is *wrong:* "an increase in α has no effect on the amount of good y which person 2 consumes."

Answers to 7005 portion of 2005

Microeconomics Qualifying Exam

Section 1 Question 1) Person 1 solves max U_1 s.t. $\overset{\text{expenditures}}{p_x x_1 + p_y y_1} = \overset{\text{income}}{p_x 1 + p_y 1}$

$$\max_{x_1, y_1} x_1^{\alpha} y_1^{\beta} \quad \text{s.t.} \quad ''$$

$$\mathcal{L} = x_1^{\alpha} y_1^{\beta} + \lambda \left(p_x + p_y - p_x x_1 - p_y y_1 \right)$$

F.O.C.'s:

$$0 = \partial \mathcal{L} / \partial \lambda = p_x + p_y - p_x x_1 - p_y y_1$$

$$0 = \partial \mathcal{L} / \partial x_1 = \alpha \frac{x_1^{\alpha} y_1^{\beta}}{x_1} - \lambda p_x \Rightarrow \frac{\lambda p_x x_1}{\alpha} = x_1^{\alpha} y_1^{\beta}$$

$$0 = \partial \mathcal{L} / \partial y_1 = \beta \frac{x_1^{\alpha} y_1^{\beta}}{y_1} - \lambda p_y \Rightarrow \frac{\lambda p_y y_1}{\beta} = x_1^{\alpha} y_1^{\beta}$$

so $$\frac{\lambda p_x x_1}{\alpha} = \frac{\lambda p_y y_1}{\beta}$$

$$x_1 = \frac{\alpha}{p_x} \frac{p_y y_1}{\beta} = \frac{\alpha}{\beta} \frac{p_y}{p_x} y_1 .$$

Then the budget constraint becomes

$$p_x \left[\frac{\alpha}{\beta} \frac{p_y}{p_x} y_1 \right] + p_y y_1 = p_x + p_y$$

$$p_y \frac{\alpha}{\beta} y_1 + p_y y_1 = p_x + p_y$$

$$p_y y_1 \left(\frac{\alpha}{\beta} + 1 \right) = p_x + p_y$$

$$p_y y_1 \left(\frac{\alpha + \beta}{\beta} \right) = p_x + p_y$$

$$y_1 = \frac{\beta}{\alpha + \beta} \frac{p_x + p_y}{p_y} = \frac{\beta}{\alpha + \beta} \left(\frac{p_x}{p_y} + 1 \right)$$

$$= \frac{1}{\frac{\alpha}{\beta} + 1} \left(\frac{p_x}{p_y} + 1 \right)$$

Quick checks: as $\beta \uparrow$, consumer likes y more, so demand for it should $\uparrow$, which it does. As $p_y \uparrow$, demand for y should $\downarrow$ (barring Giffen goods), which it does. So this looks OK.

Then $x_1 = \frac{\alpha}{\beta}\frac{p_y}{p_x} y_1 = \frac{\alpha}{\beta}\frac{p_y}{p_x}\frac{\beta}{\alpha+\beta}\frac{p_x+p_y}{p_y} = \frac{\alpha}{\alpha+\beta}\frac{p_x+p_y}{p_x}$.

Person 2's problem is identical to Person 1's except that γ replaces α and δ replaces β. So by analogy:

$$y_2 = \frac{\delta}{\gamma+\delta}\frac{p_x+p_y}{p_y}$$

$$x_2 = \frac{\gamma}{\gamma+\delta}\frac{p_x+p_y}{p_x}.$$

Setting demand equal to supply: $x_1+x_2=2$ and $y_1+y_2=2$. If one of these markets clears, so does the other, so we only need to analyze one of them:

$$2 = y_1+y_2 = \frac{\beta}{\alpha+\beta}\frac{p_x+p_y}{p_y} + \frac{\delta}{\gamma+\delta}\frac{p_x+p_y}{p_y}.$$ Letting y be the numéraire (so $p_y=1$):

$$2 = \frac{\beta}{\alpha+\beta}\frac{p_x+1}{1} + \frac{\delta}{\gamma+\delta}\frac{p_x+1}{1} = \left(\frac{\beta}{\alpha+\beta}+\frac{\delta}{\gamma+\delta}\right)(p_x+1)$$

$$\Rightarrow p_x^* = 2\left(\frac{\beta}{\alpha+\beta}+\frac{\delta}{\gamma+\delta}\right)^{-1} - 1.$$

Therefore

$$y_2 = \frac{\delta}{\gamma+\delta}\frac{p_x+p_y}{p_y} = \frac{\delta}{\gamma+\delta}\frac{p_x+1}{1} = \frac{\delta}{\gamma+\delta}\left[2\left(\frac{\beta}{\alpha+\beta}+\frac{\delta}{\gamma+\delta}\right)^{-1}\right].$$

If $\alpha\uparrow$, $\alpha+\beta\uparrow$, $\frac{\beta}{\alpha+\beta}\downarrow$, the term $(\ )^{-1}$ increases, so $y_2\uparrow$.

Intuition (optional): If $\alpha\uparrow$, consumer 1 buys more x so, less y, so with a fixed supply of y, consumer 2 has to buy more of y.

Optional: $$\frac{\partial y_2}{\partial\alpha} = \frac{\delta}{\gamma+\delta}(-2)\left(\frac{\beta}{\alpha+\beta}+\frac{\delta}{\gamma+\delta}\right)^{-2}\frac{(-\beta)}{(\alpha+\beta)^2} > 0.$$

Alternatively, consider

$$0 = p_x + p_y - p_x x_1 - p_y y_1$$

$$0 = \partial U_1/\partial x_1 - \lambda_1 p_x$$

$$0 = \partial U_1/\partial y_1 - \lambda_1 p_y$$

$$0 = p_x + p_y - p_x x_2 - p_y y_2$$

$$0 = \partial U_2/\partial x_2 - \lambda_2 p_x$$

$$0 = \partial U_2/\partial y_2 - \lambda_2 p_y$$

$$1 = x_1 + x_2$$

$$1 = y_1 + y_2$$

as a system of 8 equations in 8 unknowns, $x_1, y_1, \lambda_1, x_2, y_2, \lambda_2, p_x$, and p_y. Take the differential of each equation; treat the 8 unknowns given in the line above as the endogenous variables and treat α as the exogenous variable; then use matrix algebra to find $\partial y_2/\partial \alpha$. (Actually, you should set $p_y \equiv 1$ and drop one of the two last equations, because the fact that if one market clears so does the other implies that the last two equations are not independent of each other.)

1. **[20 points]** This question concerns a "Robinson Crusoe" economy.

   (a) First consider Robinson Crusoe the consumer. Assume he obtains utility from consumption of bananas, "b," and hours of leisure, "z," according to the utility function $2\sqrt{b} + 2\sqrt{z}$.

   Suppose "L" represents his number of hours working per day ("labor time"). Assume he earns wage income at a rate of w per hour. Also assume he receives income π from his ownership of the firm.

   The fact that there are only 24 hours in a day constrains z and L. Suppose the price of bananas is p. Suppose Robinson takes p and the wage rate w as given (so he acts in a perfectly competitive way).

   What is his demand for bananas?

   (b) Next consider Robinson Crusoe the firm. Suppose he produces bananas b from labor L according to the production function $b = L^2$.

      i. What kind of returns to scale is this production function?

      ii. Suppose Robinson:

         A. takes the wage rate w as given (so the labor market is competitive); but

         B. *does not* take the price of bananas p as given. Instead, he can control p. Also, he knows everything about the answer to part (a) of this question except he does not know that the π appearing in the consumer's problem is the same as the firm's profit (Robinson-the-firm treats the π in the consumer's problem as just some exogenous constant), and Robinson functions as a monopoly seller of bananas.

      Taking the wage rate w as the numéraire, express the firm's profit as a function of p, instead of taking the usual approach of expressing the firm's profit as a function of L.

      Hint 1: A monopolist's total revenue is "price times output," just as for a competitive firm. However, the firm can pick its p, and it knows the demand curve for b given in the answer to part (a) above.

      Hint 2: As an intermediate step, it might be helpful to express the firm's profit as a function of b.

iii. Implicitly find the profit-maximizing level of p. It is completely acceptable to leave this as a system of two equations in two unknowns which you do not try to solve and which you do not try to simplify very much (though you should evaluate all derivatives). Of the two unknowns, one of the unknowns should be p.

Answers to 2017 Micro Qualifying Exam

Sec. 1 #1

a) income $= wL + \pi$ utility $u = 2\sqrt{b} + 2\sqrt{z}$

leisure is z

$z + L = 24$ hours $\Rightarrow L = 24 - z$ and income $= w(24-z) + \pi$

expenditure pb

Budget constraint expenditure = income

$$pb = w(24-z) + \pi$$

Lagrangian

$$\mathcal{L} = 2\sqrt{b} + 2\sqrt{z} + \lambda\left(w(24-z) + \pi - pb\right)$$

First-order Conditions

$$0 = \mathcal{L}'_b = \frac{1}{\sqrt{b}} + \lambda(-p) \Rightarrow \lambda p = \frac{1}{\sqrt{b}} \Rightarrow \lambda = \frac{1}{p\sqrt{b}}$$

$$\Downarrow$$

$$0 = \mathcal{L}'_z = \frac{1}{\sqrt{z}} + \lambda(-w) \Rightarrow \lambda w = \frac{1}{\sqrt{z}} \Rightarrow \frac{1}{p\sqrt{b}} w = \frac{1}{\sqrt{z}}$$

$$\frac{p\sqrt{b}}{w} = \sqrt{z}$$

$$\frac{p^2}{w^2} b = z$$

and from the budget constraint,

$$pb = w\left(24 - \frac{p^2}{w^2} b\right) + \pi$$

$$= 24w - \frac{p^2}{w} b + \pi$$

$$pb + \frac{p^2}{w} b = 24w + \pi$$

$$\left(p + \frac{p^2}{w}\right) b = 24w + \pi$$

$$\frac{pw + p^2}{w} b = 24w + \pi$$

$$\boxed{b = \frac{w}{pw + p^2}(24w + \pi)}$$

which is b^D, the demand for bananas.

b) $b = L^2$

i) this is an increasing returns to scale production function:

$b(\alpha L) = (\alpha L)^2 = \alpha^2 L^2$ so if L increases by α (that is, $\alpha > 1$)

then b increases by $\alpha^2 > \alpha$, more than α.

ii) profit = total revenue − total cost

total revenue → pb → from part (a)

$$p \cdot \frac{w}{pw + p^2}(24w + \pi)$$

→ since w is the numéraire

$$\frac{p}{p+p^2}(24 + \pi)$$

$$= \frac{24+\pi}{1+p}$$

total cost → wL, but $b = L^2$ implies $\sqrt{b} = L$, so total cost is also equal to $w\sqrt{b}$, which is $\sqrt{b}$ taking w as the numéraire. From part (a), this is

$$\sqrt{\frac{24+\pi}{p+p^2}}.$$

So profit $= \dfrac{24+\pi}{1+p} - \sqrt{\dfrac{24+\pi}{p+p^2}} = \boxed{\dfrac{24+\pi}{1+p} - \sqrt{24+\pi}\,(p+p^2)^{-1/2}}.$

iii) Maximizing profit, set $d\,\text{profit}/dp = 0$. ← this firm can choose p Recall that the firm does not know that "π" means profit, and the firm takes "π" as a constant:

$$0 = \frac{d\,\text{profit}}{dp} = -\frac{24+\pi}{(1+p)^2} + \frac{1}{2}\sqrt{24+\pi}\,(p+p^2)^{-3/2}(1+2p)$$

Although the firm does not know that "π" is profit, we know that "π" is profit, so the first equation of the answer is

$$\pi = \frac{24+\pi}{1+p} - \sqrt{\frac{24+\pi}{p+p^2}}.$$

The second equation of the answer is the first-order condition,

$$0 = -\frac{24+\pi}{(1+p)^2} + \frac{1+2p}{2}\sqrt{24+\pi}\,(p+p^2)^{-3/2}.$$

These form a system of two equations in the two unknowns p and π.

Optional Material

Solving the second equation for π will make it much easier for a computer algebra system such as Mathematica to find p and π. We have:

$$\frac{24+\pi}{(1+p)^2} = \frac{1+2p}{2}\sqrt{24+\pi}\quad(p+p^2)^{-3/2}$$

$$\frac{\sqrt{24+\pi}}{(1+p)^2} = \frac{1+2p}{2}\quad(p+p^2)^{-3/2}$$

$$\begin{aligned}\sqrt{24+\pi} &= \tfrac{1}{2}(1+p)^2(1+2p)\,[p(1+p)]^{-3/2}\\ &= \tfrac{1}{2}(1+p)^2(1+2p)\,p^{-3/2}(1+p)^{-3/2}\\ &= \tfrac{1}{2}(1+p)^{+1/2}(1+2p)\,p^{-3/2}\end{aligned}$$

$$24+\pi = \frac{1}{4}(1+p)\quad(1+2p)^2 p^{-3} = \frac{(1+p)(1+2p)^2}{4p^3}$$

$$\boxed{\pi = \frac{(1+p)(1+2p)^2}{4p^3} - 24.}$$

Tell this to Mathematica:

```
In[1]:= pi = (1 + p) (1 + 2 p)^2 / (4 p^3) - 24
```

Out[1]= $-24 + \frac{(1+p)\,(1+2\,p)^2}{4\,p^3}$

Then ask Mathematica to solve the first equation on p. 3:

```
In[2]:= Solve[pi == (24 + pi) / (1 + p) - Sqrt[(24 + pi) / (p + p^2)], p] // N
```

Out[2]= {{p → 0. − 0.104257 ⅈ}, {p → 0. + 0.104257 ⅈ}, {p → −0.142261}, {p → 0.229217}}

```
In[3]:= p = p /. %[[4]]
```

← Set p equal to the fourth solution, since p should be a positive real number

Out[3]= 0.229217

← Optional Material →

```
In[4]:= pi
```

← the resulting level of profit

Out[4]= 30.275

```
In[5]:= b = (24 + pi) / (p + p^2)
```

← the demand for bananas

Out[5]= 192.63

```
In[6]:= L = Sqrt[b]
```

← hours worked

Out[6]= 13.8791

```
In[7]:= p b - 1 * L
```

← profit; should match with π above (it does)

Out[7]= 30.275

```
In[8]:= z = p^2 b
```

← finding z from the consumer's problem (could also use $z = 24 - L$)

Out[8]= 10.1209

```
In[9]:= utility = 2 Sqrt[b] + 2 Sqrt[z]
```

Out[9]= 34.1209

```
In[10]:= (* check FOC; this should be zero *)
         - (24 + pi) / (1 + p)^2 - Sqrt[24 + pi] (-1 / 2) (p + p^2)^(-3 / 2) (1 + 2 p)
```

Out[10]= 1.42109×10^{-14} ← close enough to zero

```
In[11]:= (* check profit; these should be equal *)
         {pi, (24 + pi) / (1 + p) - Sqrt[(24 + pi) / (p + p^2)]}
```

Out[11]= {30.275, 30.275}

If the monopoly had not existed and no firm existed then no bananas would be produced and utility would be $2\sqrt{b} + 2\sqrt{z} = 2\sqrt{0} + 2\sqrt{24} = 2\sqrt{6 \cdot 4} = 4\sqrt{6} \approx 9.8$, which is much less than the utility with the monopolist in charge of production.

2023 Qualifying Exam Sec. 1 Qu. 2 is new

2. **[19 points]** In this problem, you can get full credit even if you do not check any second-order conditions. (If this were a take-home exam, you would be expected to check second-order conditions.)

   Consider a Robinson Crusoe economy (that is, a one-person economy) in which Robinson Crusoe's utility function is

$$u(a, b, R) = 2\sqrt{a} + 2\sqrt{b} + 2\sqrt{R}$$

   where a is his consumption of apples, b is his consumption of bananas, and R is his consumption of "leisure" or "*r*est." Out of 24 hours in a day, Robinson spends some in rest, some in labor to produce apples, ℓ_a, and the remainder in labor to produce bananas, ℓ_b. In his role as a consumer, Robinson takes all prices as given.

   Let the price of apples be the numéraire, let the price of bananas be p_b, and let the price of labor be the wage rate, w.

   In Robinson's role as producer of apples, suppose he earns π_a in profit. In Robinson's role as producer of bananas, suppose he earns π_b in profit.

   (a) (4 points) Show that Robinson's demand for apples is

$$a^D = \frac{w p_b \left[\pi_a + \pi_b + 24w\right]}{p_b + w p_b + w},$$

   Robinson's demand for bananas is

$$b^D = \frac{w \left[\pi_a + \pi_b + 24w\right]}{p_b \left(p_b + w p_b + w\right)},$$

   and Robinson's supply of labor is

$$\ell_a + \ell_b = 24 - \frac{p_b \left[\pi_a + \pi_b + 24w\right]}{\left(w \left(p_b + w p_b + w\right)\right)}.$$

   (b) (1 point) Suppose that in Robinson's role as producer of apples, Robinson is a price taker (is "perfectly competitive"). Suppose the production function for apples is $a = \ell_a$. Argue that:

      i. The supply function is not "everywhere well-defined."

ii. The equilibrium wage is one.
iii. The equilibrium profit from producing apples is zero.

(c) (1 point) Suppose that in Robinson's role as producer of bananas, Robinson is a price taker (is "perfectly competitive"). Suppose the production function for bananas is $b = \ell_b$. Argue that:

i. The equilibrium price of bananas is one.
ii. The equilibrium profit from producing bananas is zero.

(d) (2 points) Show that in the general equilibrium of this economy, $a^* = 8$, $b^* = 8$, $\ell_a^* + \ell_b^* = 16$, and $R^* = 8$.

(e) Now suppose that in Robinson's role as producer of bananas, Robinson is *not* a price taker, but rather acts as a monopolist. This means that he knows what the demand curve for bananas is (it is given in part (a) above). He also completely understands how the demand for bananas depends on the profit from producing bananas. Nothing changes about Robinson's role as producer of apples; it is still as described in part (b) above. Find the equilibrium:

i. (1 point) wage rate and the profit from producing apples;
ii. (4 points) price of bananas (hint: it is the solution to $0 = 2p_b^2 - 4p_b - 1$, and you get full credit for showing that; you do not have to show the solution to that equation, which is $p_b = 1 + (\sqrt{6}/2) \approx 2.22$, but you can use this result in the other parts of this problem);
iii. (2 points) number of bananas produced (hint: it is

$$\frac{6\sqrt{6} + 12}{(1 + \frac{\sqrt{6}}{2})(3 + \sqrt{6})},$$

and you get full credit for showing that or an equivalent numerical expression; you do not have to show that that simplifies to $12 - 4\sqrt{6} \approx 2.20$, but you can use this result in the other parts of this problem);

iv. (2 points) number of apples produced (hint: it is

$$\frac{(1 + \frac{\sqrt{6}}{2})(6\sqrt{6} + 12)}{3 + \sqrt{6}},$$

and you get full credit for showing that or an equivalent numerical expression; you do not have to show that that simplifies to $2(\sqrt{6} + 3) \approx 10.9$, but you can use this result in the other parts of this problem);

v. (1 point) amount of rest R (hint: it is $6 + 2\sqrt{6} \approx 10.9$, which you should show).

(f) (1 point) Write a summary of the effect *on the market for apples* and *on the market for labor* when the banana market becomes monopolized.

Answer to Summer 2023 Qualifying Exam, Section 1 Question 2

(a) The constraint on labor hours is $\ell_a + \ell_b + R = 24$. Taking this into account by substituting it into the utility function, we have

$$u = 2\sqrt{a} + 2\sqrt{b} + 2\sqrt{24 - \ell_a - \ell_b}.$$

Income is $\pi_a + \pi_b + w(\ell_a + \ell_b)$: the profit of the apple firm, the profit of the banana firm, and the wage rate times the amount of labor supplied. In a Robinson Crusoe economy, Robinson Crusoe *as a consumer* considers π_a and π_b to be exogenous.

Expenditure is $p_a a + p_b b = a + p_b b$ since the price of apples is the numéraire.

The Lagrangian is

$$\mathcal{L} = 2\sqrt{a}+2\sqrt{b}+2\sqrt{24 - \ell_a - \ell_b}+\lambda\left[\pi_a+\pi_b+w(\ell_a+\ell_b)-1\cdot a-p_b b\right].$$

The first-order conditions are

$$\begin{aligned}
0 = \mathcal{L}'_a &= \frac{1}{\sqrt{a}} - \lambda \\
0 = \mathcal{L}'_b &= \frac{1}{\sqrt{b}} - \lambda p_b \\
0 = \mathcal{L}'_{\ell_a} &= \frac{-1}{\sqrt{24 - \ell_a - \ell_b}} + w\lambda \\
0 = \mathcal{L}'_{\ell_b} &= \frac{-1}{\sqrt{24 - \ell_a - \ell_b}} + w\lambda.
\end{aligned}$$

Solving these equations for λ results in

$$\lambda = \frac{1}{\sqrt{a}} = \frac{1}{p_b\sqrt{b}} = \frac{1}{w\sqrt{24 - \ell_a - \ell_b}}.$$

The second equality leads to $\sqrt{a} = p_b\sqrt{b}$, so $b = a/p_b^2$. The second and third equalities lead to $\sqrt{a} = w\sqrt{24 - \ell_a - \ell_b}$, so $24 - \ell_a - \ell_b = a/w^2$ and $\ell_a + \ell_b = 24 - a/w^2$. Substituting these expressions for b

in terms of a and for $\ell_a + \ell_b$ in terms of a into the budget constraint yields

$$\begin{aligned}
0 &= \pi_a + \pi_b + w\,(24 - a/w^2) - a - p_b a/p_b^2 \\
&= \pi_a + \pi_b + 24w - \frac{a}{w} - a - \frac{a}{p_b} \\
&= \pi_a + \pi_b + 24w - a\left(\frac{1}{w} + 1 + \frac{1}{p_b}\right) \\
&= \pi_a + \pi_b + 24w - a \cdot \frac{p_b + wp_b + w}{wp_b}
\end{aligned}$$

and so the demand for apples is

$$a^D = (\pi_a + \pi_b + 24w)\,\frac{wp_b}{p_b + wp_b + w}.$$

Then using the expression for b in terms of a derived above, the demand for bananas is

$$b^D = \frac{a}{p_b^2} = (\pi_a + \pi_b + 24w)\,\frac{w}{p_b\,(p_b + wp_b + w)}.$$

Using the expression for $\ell_a + \ell_b$ in terms of a derived above, the supply of labor is

$$\ell_a^S + \ell_b^S = 24 - (\pi_a + \pi_b + 24w)\,\frac{p_b}{w\,(p_b + wp_b + w)}.$$

This proves all of the claims in this part of the problem.

(b) The profit from producing apples is $\pi_a = 1 \cdot a - w\ell_a$, but since the production function is $a = \ell_a$, this can be rewritten as $\pi_a = a - wa = a\,(1 - w)$. If $w < 1$, optimal apple supply is infinity, which cannot be an equilibrium. One says that the supply function "is not well-defined." If $w > 1$, optimal apple supply is zero, which could be an equilibrium but is unlikely to be an equilibrium, so we rule it out for now and will return to it if we cannot find an equilibrium with a strictly positive amount of apples. This leaves

$$w = 1.$$

That implies $\pi_a = a(1-1) = 0$, which is not surprising since the production function $a = \ell_a$ has constant returns to scale. The apple supply curve will be flat at a price of one (and zero for a price less than one).

(c) The profit from producing bananas is $\pi_b = p_b b - w\ell_b$, but since the production function is $b = \ell_b$, this can be rewritten as $\pi_b = p_b b - wb = b(p_b - w)$. If $p_b > w$, optimal banana supply is infinity, which cannot be an equilibrium. If $p_b < w$, optimal banana supply is zero, which could be an equilibrium but is unlikely to be an equilibrium, so we rule it out for now and will return to it if we cannot find an equilibrium with a strictly positive amount of bananas. This leaves

$$p_b = w.$$

That implies $\pi_b = b(w - w) = 0$, which is not surprising since the production function $b = \ell_b$ has constant returns to scale. The banana supply curve will be flat at a price of w, which from the answer to part (b) is equal to one. (The banana supply curve will be zero for a price less than one).

(d) We now have $1 \equiv p_a = w = p_b$ and $\pi_a = 0 = \pi_b$. Substituting these into the demand for apples gives

$$a^* = (0 + 0 + 24 \cdot 1)\,\frac{1 \cdot 1}{1 + 1 \cdot 1 + 1} = \frac{24}{3} = 8.$$

A similar substitution for bananas yields

$$b^* = (0 + 0 + 24 \cdot 1)\,\frac{1}{1 \cdot (1 + 1 \cdot 1 + 1)} = \frac{24}{3} = 8.$$

Using the production functions, $\ell_a^* = a^* = 8$ and $\ell_b^* = b^* = 8$, so $\ell_a^* + \ell_b^* = 16$ and $R^* = 24 - 16 = 8$.

(e) Banana profit is $\pi_b = p_b b - w\ell_b$. Using the production function, this is $\pi_b = p_b b - wb = (p_b - w)\,b$. Now, however, the banana firm knows the demand curve for bananas, which was given in part (a) above; substituting it in,

$$\pi_b = (p_b - w)\,(\pi_a + \pi_b + 24w)\,\frac{w}{p_b(p_b + wp_b + w)}.$$

Since there is still perfect competition and constant returns to scale in apple production, the reasoning in part (b)'s answer still applies, so it is still the case that $w = 1$ and $\pi_a = 0$ (**that answers part (i)**). Making these substitutions,

$$\begin{aligned}\pi_b &= (p_b - 1)(0 + \pi_b + 24 \cdot 1) \frac{1}{p_b (p_b + 1 \cdot p_b + 1)} \\ &= (p_b - 1)(\pi_b + 24) \frac{1}{2p_b^2 + p_b}\end{aligned}$$

so

$$\begin{aligned}2p_b^2 \pi_b + p_b \pi_b &= p_b \pi_b + 24 p_b - \pi_b - 24, \quad \text{thus} \\ (2p_b^2 + 1)\pi_b &= 24 p_b - 24 \quad \text{and} \\ \pi_b &= \frac{24 (p_b - 1)}{2p_b^2 + 1}.\end{aligned}$$

The monopolist chooses p_b in order to maximize π_b. The first-order condition is

$$\begin{aligned}0 &= \frac{24}{2p_b^2 + 1} - \frac{4p_b}{(2p_b^2 + 1)^2} 24 (p_b - 1) \\ &= \frac{2p_b^2 + 1 - 4p_b (p_b - 1)}{(2p_b^2 + 1)^2} \\ 0 &= 2p_b^2 + 1 - 4p_b^2 + 4p_b = -2p_b^2 + 4p_b + 1 \\ 0 &= 2p_b^2 - 4p_b - 1,\end{aligned}$$

which has roots of

$$\begin{aligned}p_b &= \frac{+4 \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{4 \pm \sqrt{16 + 8}}{4} = \frac{4 \pm \sqrt{24}}{4} = \frac{4 \pm 2\sqrt{6}}{4} \\ &= \frac{2 \pm \sqrt{6}}{2}.\end{aligned}$$

The negative sign will result in a numerator of $2 - \sqrt{6} < 2 - \sqrt{4} = 0$, which is incompatible with $p_b > 0$, so we take the positive root: $p_b = 1 + \sqrt{6}/2 \approx 2.22$. **That answers part (ii).**

The equilibrium profit in making bananas now is not zero but instead is

$$\begin{aligned}
\pi_b &= \frac{24\,(p_b - 1)}{2p_b^2 + 1} = \frac{24\,(1 + \frac{\sqrt{6}}{2} - 1)}{2\,(1 + \frac{\sqrt{6}}{2})^2 + 1} = \frac{12\sqrt{6}}{2\,(1 + \sqrt{6} + \frac{6}{4}) + 1} \\
&= \frac{12\sqrt{6}}{2 + 2\sqrt{6} + 3 + 1} = \frac{12\sqrt{6}}{6 + 2\sqrt{6}} = \frac{6\sqrt{6}}{3 + \sqrt{6}} \\
&= \frac{6\sqrt{6}}{3 + \sqrt{6}} \cdot \frac{3 - \sqrt{6}}{3 - \sqrt{6}} = \frac{18\sqrt{6} - 36}{9 - 6} = 6\sqrt{6} - 12 \approx 2.70.
\end{aligned}$$

Using Robinson's demand for bananas as given in part (a), we have

$$\begin{aligned}
b^* &= \frac{w\,[\pi_a + \pi_b + 24w]}{p_b\,(p_b + wp_b + w)} = \frac{1 \cdot [0 + (6\sqrt{6} - 12) + 24 \cdot 1]}{(1 + \frac{\sqrt{6}}{2})\,(1 + \frac{\sqrt{6}}{2} + 1 \cdot (1 + \frac{\sqrt{6}}{2}) + 1)} \\
&= \frac{6\sqrt{6} + 12}{(1 + \frac{\sqrt{6}}{2})\,(3 + \sqrt{6})} = \frac{6\sqrt{6} + 12}{3 + \sqrt{6} + \frac{3}{2}\sqrt{6} + 3} = \frac{6\sqrt{6} + 12}{6 + \frac{5}{2}\sqrt{6}} \\
&= 6\,\frac{\sqrt{6} + 2}{6 + \frac{5}{2}\sqrt{6}} = 6\,\frac{2\sqrt{6} + 4}{12 + 5\sqrt{6}} \cdot \frac{12 - 5\sqrt{6}}{12 - 5\sqrt{6}} \\
&= 6\,\frac{24\sqrt{6} - 60 + 48 - 20\sqrt{6}}{144 - 150} = 6\,\frac{4\sqrt{6} - 12}{-6} \\
&= 12 - 4\sqrt{6} \approx 2.20.
\end{aligned}$$

That answers part (iii). From the production function for bananas, this is also ℓ_b.

Using Robinson's demand for apples as given in part (a), we have

$$\begin{aligned}
a^* &= \frac{wp_b\,[\pi_a + \pi_b + 24w]}{p_b + wp_b + w} = \frac{1 \cdot (1 + \frac{\sqrt{6}}{2})\,[0 + 6\sqrt{6} - 12 + 24 \cdot 1]}{1 + \frac{\sqrt{6}}{2} + 1 \cdot (1 + \frac{\sqrt{6}}{2}) + 1} \\
&= \frac{(1 + \frac{\sqrt{6}}{2})\,(6\sqrt{6} + 12)}{3 + \sqrt{6}} = \frac{(2 + \sqrt{6})\,(3\sqrt{6} + 6)}{3 + \sqrt{6}} \\
&= \frac{(2 + \sqrt{6})\,\sqrt{6}\,(3 + \sqrt{6})}{3 + \sqrt{6}} = 2\sqrt{6} + 6 \approx 10.9.
\end{aligned}$$

That answers part (iv). From the production function for apples, this is also ℓ_a.

The equilibrium labor supplied is $\ell_a^* + \ell_b^* = (2\sqrt{6} + 6) + (12 - 4\sqrt{6}) = 18 - 2\sqrt{6} \approx 13.1$, and so the equilibrium amount of rest is $24 - (\ell_a + \ell_b) = 6 + 2\sqrt{6} \approx 10.9$. **That answers part (v).**

(f) The purely competitive equilibrium was $\ell_a^* = a^* = 8$, $\ell_b^* = b^* = 8$, $\ell_a^* + \ell_b^* = 16$ and $R^* = 8$, with $p_a \equiv 1 = w = p_b$ and $\pi_a = \pi_b = 0$.

When the banana production was monopolized, the equilibrium was $\ell_a^* = a^* \approx 10.9$, $\ell_b^* = b^* \approx 2.2$, $\ell_a^* + \ell_b^* = 13.1$ and $R^* = 10.9$, with $p_a \equiv 1 = w$ and $p_b \approx 2.22$, and $\pi_a = 0$ while $\pi_b \approx 2.7$.

The fact that after monopolization of banana production, banana quantity fell a great deal, and banana prices and profits went up, is not a surprise, and the question did not ask about this. What may be more surprising was that the apple market was affected, with apple production going up (although apple price was unchanged), and the labor market was affected also, with the total amount of labor going down (though the wage rate was unchanged). The demand for labor falls a great deal in the banana sector, because the monopoly wants to decrease output of bananas and charge a higher price for bananas. Some of this labor is shifted to the apple sector, increasing apple production; the rest is shifted to leisure.

Optional: Robinson Crusoe's utility in the purely competitive equilibrium was $2\sqrt{8} + 2\sqrt{8} + 2\sqrt{8} \approx 16.97$, while in the "banana monopoly" equilibrium, it was $2\sqrt{10.9} + 2\sqrt{2.2} + 2\sqrt{10.9} \approx 16.17$.

Completely Optional Remarks

Chapter 10 of Mas-Colell, Whinston and Green's book is about how to make normative judgments (assess welfare changes) in a partial equilibrium framework. (So is Varian's Chapter 10.) Mas-Colell, Whinston and Green write (pages 311–312):

> Starting in Section 10.C, we narrow our focus to the partial equilibrium context. The partial equilibrium approach, which originated in Marshall (1920), envisions the market for a single good (or group of goods) for which each consumer's expenditure constitutes only a small portion of his overall budget. When

> this is so, *it is reasonable to assume that changes in the market for this good will leave the prices of all other commodities approximately unaffected* [emphasis added] and that there will be, in addition, negligible wealth effects in the market under study. We capture these features in the simplest possible way by considering a two-good model in which the expenditure on all commodities other than that under consideration is treated as a single composite commodity (called the *numeraire* commodity), and in which consumers' utility functions take a quasilinear form with respect to this numeraire. Our study of the competitive equilibria of this simple model lends itself to extensive demand-and-supply graphical analysis. . . .
>
> In Section 10.D, we analyze the properties of Pareto optimal allocations in the partial equilibrium model. Most significantly, we establish for this special context the validity of the *fundamental theorems of welfare economics:* Competitive equilibrium allocations are necessarily Pareto optimal, and any Pareto optimal allocation can be achieved as a competitive equilibrium if appropriate lump-sum transfers are made.
>
> [. . .] In Section 10.E, we consider the measurement of welfare changes in the partial equilibrium context. We show that these can be represented by areas between properly defined demand and supply curves.

In other words, all the normative tools economics has outside of general equilibrium—for example, all the normative tools of undergraduate economics, such as consumer surplus, which are often used to teach that "competitive equilibrium is socially optimal"—pertain only to relatively unimportant commodities. This problem shows that if there are only three commodities (labor, apples, and bananas), then monopolization of one of them leads to large effects on the other two. This violates the assumptions underlying the standard partial equilibrium normative analysis. (So does assuming the utility function given in this problem, which is not of the highly restrictive "quasilinear" type.)

Fall 2004

Final

4. [17 points] Consider a Robinson Crusoe economy where the consumer is endowed with one unit of time which he can divide between labor time and non-labor time. There is only one technology available in this economy. It can be employed to make butter "b" from labor according to the production function

$$b = 2 * \text{labor}^2 .$$

Suppose the consumer's utility function is

$$U = b * \text{"non-labor time"} .$$

Finally, take the price of labor as the numéraire.

(a) Find the competitive market-clearing prices and quantities for butter and labor, *ignoring* any strange signs that you come across.

(b) Considering now the strange sign or signs you should have encountered in part (a), and considering the type of production function, describe why the competitive equilibrium you found in part (a) is actually not a competitive equilibrium, and why in fact this economy does not have a competitive equilibrium.

(4)

Fall 2004 Final

One could denote working time by a for labor and denote non-working time by e for leisure, but since $a + e = 1$ in this problem, I will just set $e = 1 - a$ and only use "a" from now on. Take $p_a = 1$

The firm's problem is to maximize

the price of butter

$$\pi = TR - TC \text{ where } TR = p_b \cdot 2a^2 \text{ and } TC = p_a a = a$$

$$\Rightarrow \quad \pi = 2p_b a^2 - a.$$

$$F.O.C.: \; 0 = \frac{d\pi}{da} = 4p_b a - 1 \Rightarrow a = \frac{1}{4p_b}, \text{ labor demand.}$$

Butter supply then is $2(a^*)^2 = 2\left(\frac{1}{4p_b}\right)^2 = 2\left(\frac{1}{16p_b^2}\right) = \frac{1}{8p_b^2}$.

These results are already strange because one would expect that as $p_b \uparrow$, the butter firm's labor demand and butter supply would $\uparrow$ not $\downarrow$.

The firm does not have constant returns to scale, so its $\pi^* \neq 0$. One calculates

$$\pi^* = 2p_b(a^*)^2 - a^* = 2p_b\left(\frac{1}{4p_b}\right)^2 - \frac{1}{4p_b} = 2p_b \cdot \frac{1}{16p_b^2} - \frac{1}{4p_b}$$

$$= \frac{1}{8p_b} - \frac{1}{4p_b} = \frac{-1}{8p_b}$$ which is strange because it is negative.

Indeed it is wrong because the firm could always do better by just shutting down and getting $\pi^* = 0$. But the question says to ignore this for now.

The consumer's expenditure is $p_b b$ and his income is $p_a a + \pi^* = a + \pi^*$. So his problem is to max $b\underbrace{(1-a)}_{=e}$ s.t. $a + \pi^* = p_b b$. (The consumer is "competitive"; he therefore knows nothing about the production function.)

The Lagrangian is

$$\mathcal{L} = b(1-a) + \lambda\,(a + \pi^* - p_b b). \quad \text{F.O.C.'s:}$$

$$0 = \partial\mathcal{L}/\partial\lambda = a + \pi^* - p_b b \;\Rightarrow\; p_b b = a + \pi^*$$

$$0 = \partial\mathcal{L}/\partial a = -b + \lambda \;\Rightarrow\; b = \lambda$$

$$0 = \partial\mathcal{L}/\partial b = 1 - a - \lambda p_b$$

$$\Rightarrow 0 = 1 - a - b p_b$$

$$0 = 1 - a - (a + \pi^*) = 1 - a - a - \pi^*$$

$$2a = 1 - \pi^*$$

$$a = \frac{1-\pi^*}{2} = \frac{1}{2}\left(1 - \frac{-1}{8p_b}\right) \text{ from above}$$

$$= \frac{1}{2}\,\frac{8p_b+1}{8p_b} = \frac{8p_b+1}{16p_b}, \text{ labor supply.}$$

Then butter demand is

$$b = \frac{a+\pi^*}{p_b} = \frac{1}{p_b}\left[\frac{8p_b+1}{16p_b} + \frac{-1}{8p_b}\right] = \frac{1}{p_b}\;\frac{8p_b+1-2}{16p_b}$$

$$= \frac{8p_b-1}{16p_b^2}.$$

Equate labor demand and labor supply: (or one could equate butter demand and supply)

$$\frac{1}{4p_b} = \frac{8p_b+1}{16p_b} \iff \frac{16p_b}{4p_b} = 8p_b+1 \iff 4 = 8p_b+1 \iff$$

$$8p_b = 3 \Rightarrow p_b = 3/8$$

Optional → Check that this makes an equilibrium in both markets:

$$\text{labor supply} = \frac{8p_b+1}{16p_b} = \frac{8\left(\frac{3}{8}\right)+1}{16\left(\frac{3}{8}\right)} = \frac{3+1}{2\cdot 3} = \frac{4}{6} = \frac{2}{3} \quad \text{and}$$

$$\text{labor demand} = \frac{1}{4p_b} = \frac{1}{4(\frac{3}{8})} = \frac{1}{3/2} = \frac{2}{3} \quad \text{OK} \ ;$$

$$\text{butter demand} = \frac{8p_b - 1}{16p_b^2} = \frac{8(\frac{3}{8})-1}{16(\frac{3}{8})^2} = \frac{3-1}{4 \cdot 4 \cdot \frac{9}{4 \cdot 2 \cdot 4 \cdot 2}} = \frac{2}{\frac{9}{4}}$$

$$= \frac{8}{9}$$

$$\text{butter supply} = \frac{1}{8p_b^2} = \frac{1}{8(\frac{3}{8})^2} = \frac{1}{\frac{9}{8}} = \frac{8}{9}, \quad \text{OK too.}$$

b) Check the firm's S.O.C.: $\frac{d^2\pi}{da^2} = 4p_b > 0$, which means we found a (local) π minimum, not maximum! The graph of π versus a is a parabola pointing up, so maximum profit occurs as $a \to \infty$, making $\pi \to \infty$. This always happens when the technology is increasing returns to scale and one has assumed a competitive firm. Since the supply of a will be ∞, no equilibrium can exist in the "a" market, so there is no (competitive) equilibrium in this economy.

Summer 2006 Qualifier

Section 1.
Answer all of the following three questions.

1. Suppose there are N price-taking (i.e., competitive) consumers, all of whom earn the same income \$$m$, all of whom consume two commodities x and y, and all of whom have the identical utility function $U = x^{\alpha}y^{\beta}$ where α and β are positive.

   Suppose there are F price-taking (i.e., competitive) producers of good x, each having the same cost function $C(x)$ for producing x.

   (a) If $N = 1$ and $F = 1$ but the agents still act competitively, and if $C(x) = x^2$, how will changes in β affect the equilibrium price of x?

   (b) If N and F are arbitrary natural numbers and if the form of $C(x)$ is unspecified (but the firms' second-order conditions are met), how will changes in β affect the equilibrium price of x?

   (c) If $N = 1$ and $F = 1$ and $C(x) = x^2$ but the agents still act competitively, how does the consumer think changes in β will affect the equilibrium price of x?

Answers to 7005 portion of Micro Qualifying Exam,
Summer 2006

Section 1.

Qu. 1

N consumers ; income \$m ; $U = x^\alpha y^\beta$

F firms producing x ; cost function $C(x)$

a) $N=1,\ F=1,\ C(x)=x^2$

Demand for x : max U s.t. $p_x x + p_y y = m$

$$\mathcal{L} = x^\alpha y^\beta + \lambda(m - p_x x - p_y y)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - p_x x - p_y y$$

$$\left.\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial x} = \alpha \frac{x^\alpha y^\beta}{x} - \lambda p_x \\ 0 &= \frac{\partial \mathcal{L}}{\partial y} = \beta \frac{x^\alpha y^\beta}{y} - \lambda p_y \end{aligned}\right\} \quad \frac{p_x}{p_y} = \frac{\alpha}{\beta}\frac{y}{x}$$

$$\Rightarrow y = \frac{\beta}{\alpha}\frac{p_x}{p_y} x$$

and so $m = p_x x + p_y \left[\frac{\beta}{\alpha}\frac{p_x}{p_y} x\right]$

$$= p_x x + \frac{\beta}{\alpha} p_x x = \left(1 + \frac{\beta}{\alpha}\right) p_x x = \frac{\alpha+\beta}{\alpha} p_x x \Rightarrow$$

$$x = \frac{\alpha}{\alpha+\beta}\frac{m}{p_x} \quad \text{and}$$

$$y = \frac{\beta}{\alpha}\frac{p_x}{p_y} x = \frac{\beta}{\alpha+\beta}\frac{m}{p_y}.$$

demand curve for x

Supply for x : $\max \pi = \max\ p_x x - C(x) \Rightarrow 0 = p_x - C'(x) \Rightarrow$

$$p_x = C'(x) = 2x \Rightarrow x = p_x/2.$$

Demand = supply (partial equilibrium) $\Rightarrow$

$$\frac{\alpha}{\alpha+\beta}\frac{m}{p_x} = \frac{p_x}{2}$$

$$p_x = \sqrt{\frac{2\alpha}{\alpha+\beta} m} = \sqrt{2\alpha m}\,(\alpha+\beta)^{-1/2}$$

$$\frac{\partial p_x}{\partial \beta} = -\frac{1}{2}\sqrt{2\alpha m}\,(\alpha+\beta)^{-3/2}.$$ Optional: This is $-\frac{1}{2}(\alpha+\beta)^{-1}\sqrt{2\alpha m}(\alpha+\beta)^{-1/2}$

$$= \frac{-p_x}{2(\alpha+\beta)} < 0.$$

b) Demand for x: $x = N\cdot\frac{\alpha}{\alpha+\beta}\frac{m}{p_x}$

Supply for x: by one firm, $p_x = C'(x) \Rightarrow x = (C')^{-1}(p_x)$.

$\underbrace{\text{whole industry supply of } x}_{\text{call this } x^s}$ is $F(C')^{-1}(p_x)$.

$$x^s = F\,(C')^{-1}(p_x)$$

$$x^s/F = (C')^{-1}(p_x)$$

$$C'(x^s/F) = p_x.$$

So

$$0 = x - N\frac{\alpha}{\alpha+\beta}\frac{m}{p_x}$$

$$0 = p_x - C'(x/F).$$

Method 1: Use the first equation to solve for x and substitute

$$0 = p_x - C'\left(\frac{Nm}{Fp_x}\frac{\alpha}{\alpha+\beta}\right)$$

$$0 = \left[1 - C''\cdot\frac{(-Nm)}{Fp_x^2}\frac{\alpha}{\alpha+\beta}\right]dp_x + \left[-C''\cdot\frac{Nm}{Fp_x}\frac{(-\alpha)}{(\alpha+\beta)^2}\right]d\beta$$

$$\Rightarrow \frac{dp_x}{d\beta} = \frac{-C''\cdot\frac{Nm}{Fp_x}\frac{\alpha}{(\alpha+\beta)^2}}{1 + C''\cdot\frac{Nm}{Fp_x^2}\frac{\alpha}{\alpha+\beta}}.$$

Method 2: Take differentials of the two equations.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & N \frac{\alpha}{\alpha+\beta} \frac{m}{p_x^2} \\ \frac{-1}{F} C'' & 1 \end{bmatrix} \begin{bmatrix} dx \\ dp_x \end{bmatrix} + \begin{bmatrix} \frac{Nm}{p_x} \frac{\alpha}{(\alpha+\beta)^2} \\ 0 \end{bmatrix} d\beta$$

$$\begin{bmatrix} -\frac{Nm}{p_x} \frac{\alpha}{(\alpha+\beta)^2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & N \frac{\alpha}{\alpha+\beta} \frac{m}{p_x^2} \\ \frac{-1}{F} C'' & 1 \end{bmatrix} \begin{bmatrix} \partial x / \partial \beta \\ \partial p_x / \partial \beta \end{bmatrix}$$

$$\frac{\partial p_x}{\partial \beta} = \frac{\begin{vmatrix} 1 & -\frac{Nm}{p_x} \frac{\alpha}{(\alpha+\beta)^2} \\ \frac{-1}{F} C'' & 0 \end{vmatrix}}{\begin{vmatrix} 1 & \frac{Nm}{p_x^2} \frac{\alpha}{\alpha+\beta} \\ \frac{-1}{F} C'' & 1 \end{vmatrix}} = \frac{-C'' \frac{Nm}{Fp_x} \frac{\alpha}{(\alpha+\beta)^2}}{1 + C'' \frac{Nm}{Fp_x^2} \frac{\alpha}{\alpha+\beta}}$$

Both methods give the same answer.

c) The consumer thinks $\partial p_x / \partial \beta = 0$ because he thinks his actions (and his preferences) do not affect the price; he is a price-taker.

2016 Qualifying Exam Sec. 1 Qu. 2

2. **[14 points]** Suppose an economy consists of two agents, "a" and "b," and two goods, "1" and "2," and the agents have the following utility functions and endowments:

$$u_a = \ln(x_{1a}) + x_{2a} \qquad \omega_a = (1, 1),$$
$$u_b = \ln(x_{1b}) + x_{2b} \qquad \omega_b = (9, 9).$$

(a) Find the competitive equilibrium prices (or price ratio) and allocation $(x^*_{1a}, x^*_{2a}, x^*_{1b}, x^*_{2b})$ for this economy. You may work either with the price of Good 1, "p_1," and the price of Good 2, "p_2," or with their ratio (for example, $\rho = p_2/p_1$ or $\gamma = p_1/p_2$), or you may choose a numéraire.

(b) Suppose that while the behavior of Person "b" is identical to that in part (a), Person "a" now behaves in the following noncompetitive way regarding Good 2:

- Person "a" knows that $x_{2a} + x_{2b} = 10$;
- Person "a" knows the demand curve for Good 2 by Person "b" (which you worked out in part (a)); and
- Person "a" can choose p_2 (or, equivalently, Person "a" can choose ρ or γ).

i. Find the demand by Person "a" for Good 2 as a function of price(s). You should be able to do this without solving an optimization problem by taking into account the first and second "bullet points" above.

ii. Find the resulting non-competitive equilibrium prices (or price ratio) and allocation $(\hat{x}_{1a}, \hat{x}_{2a}, \hat{x}_{1b}, \hat{x}_{2b})$ for this economy. Hint: in working this out, at one point I got to

$$0 = \frac{p_2^2}{9p_1} - 10p_1 + p_2 \,.$$

If you get to the same point, you should be able to use algebra and the quadratic formula to simplify this to

$$0 = \rho^2 + 9\rho - 90 = (\rho + 15)(\rho - 6)$$

using $\rho = p_2/p_1$ as mentioned in part (a).

(c) It turns out that the equilibrium allocation for part (a) occurs at the point labeled “C” (for “competitive”) in the Edgeworth Box illustrated in Figure 1, and that the equilibrium allocation for part (b) occurs at the point labeled “NC” (for “non-competitive”) in Figure 1. Figure 1 is not drawn to scale, and it omits portions of the Edgeworth Box in order to better show its lower-lefthand corner. The numbers next to the figure’s indifference curves (the numbers 1, 1.81, 1.89, 11.2, 11.3, and 11.4) represent the approximate corresponding value of u_a or u_b.

The straight line through NC represents the non-competitive equilibrium price vector of part (b). At NC, is the indifference curve of Person “b” tangent to this price vector? Why? At NC, is the indifference curve of Person “a” tangent to this price vector? Why?

(d) Suppose that each year, the economy starts at (ω_a, ω_b) and, because Person “a” has market power, each year the economy ends up at point NC. Furthermore, suppose the US Department of Justice has filed a lawsuit against Person “a” in which a court is asked to prohibit Person “a” from engaging in non-competitive behavior. If the Department of Justice is successful, in future years the economy will be at C. If the Department of Justice is unsuccessful, in future years the economy will continue ending up at NC.

i. If you were an economic consultant for the Department of Justice, what argument or arguments might you make to the court?

ii. If you were an economic consultant for Person “a,” what argument or arguments might you make to the court?

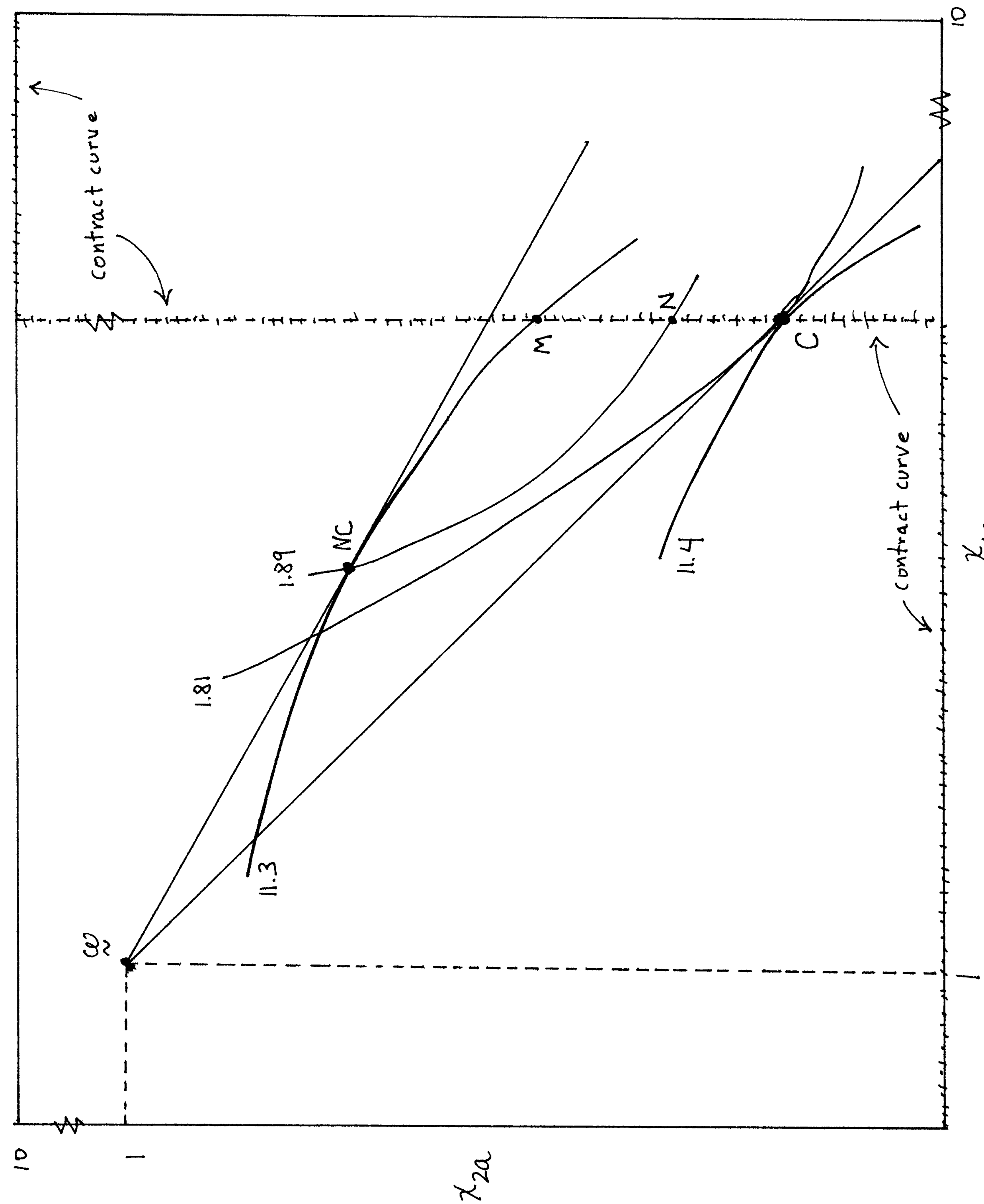

contract curve
contract curve
M
N
C
NC
1.89
1.81
11.3
11.4
10
1
10
1
χ_{2a}
χ_{1a}
$\underset{\sim}{\omega}$

Figure 1

② a) Person a's budget constraint

$$p_1 x_{1a} + p_2 x_{2a} = p_1(1) + p_2(2)$$
$$= p_1 + p_2.$$

So $p_1(x_{1a}-1) + p_2(x_{2a}-1) = 0.$

$$\mathcal{L} = \underbrace{\ln x_{1a} + x_{2a}}_{u_a} - \lambda\left[p_1(x_{1a}-1) + p_2(x_{2a}-1)\right]$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_{1a}} = \frac{1}{x_{1a}} - \lambda p_1 \Rightarrow \lambda = \frac{1}{p_1 x_{1a}}$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_{2a}} = 1 - \lambda p_2 \Rightarrow \lambda = \frac{1}{p_2}$$

$$\lambda = \frac{1}{p_1 x_{1a}} = \frac{1}{p_2}$$

$$\Rightarrow \boxed{x_{1a} = \frac{p_2}{p_1}.}$$

Substituting this into the budget constraint:

$$p_1\left(\frac{p_2}{p_1} - 1\right) + p_2(x_{2a} - 1) = 0$$

$$p_2 - p_1 + p_2(x_{2a}-1) = 0$$

$$p_2(x_{2a}-1) = p_1 - p_2$$

$$x_{2a} - 1 = \frac{p_1}{p_2} - 1$$

$$\boxed{x_{2a} = \frac{p_1}{p_2}.}$$

Person b's budget constraint

$$p_1 x_{1b} + p_2 x_{2b} = p_1(9) + p_2(9)$$

so

$$p_1(x_{1b} - 9) + p_2(x_{2b} - 9) = 0.$$

$$\mathcal{L} = \underbrace{\ln x_{1b} + x_{2b}}_{u_b} - \lambda\left[p_1(x_{1b}-9) + p_2(x_{2b}-9)\right]$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_{1b}} = \frac{1}{x_{1b}} - \lambda p_1 \Rightarrow \lambda = \frac{1}{p_1 x_{1b}}$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_{2b}} = 1 - \lambda p_2 \Rightarrow \lambda = \frac{1}{p_2}$$

$$\lambda = \frac{1}{p_1 x_{1b}} = \frac{1}{p_2}$$

$$\Rightarrow \boxed{x_{1b} = \frac{p_2}{p_1}.}$$

Substituting this into the budget constraint:

$$p_1\left(\frac{p_2}{p_1} - 9\right) + p_2(x_{2b} - 9) = 0$$

$$p_2 - 9p_1 + p_2 x_{2b} - 9p_2 = 0$$

$$p_2 x_{2b} = 9p_1 + 8p_2$$

$$\boxed{x_{2b} = 9\frac{p_1}{p_2} + 8.}$$

Good 1 has an aggregate supply of 10 and an aggregate demand of $x_{1a} + x_{1b} = \frac{p_2}{p_1} + \frac{p_2}{p_1}$ $= \frac{2p_2}{p_1}$. Setting demand equal to supply, $10 = \frac{2p_2}{p_1}$ so $\boxed{p_2/p_1 = 5.}$

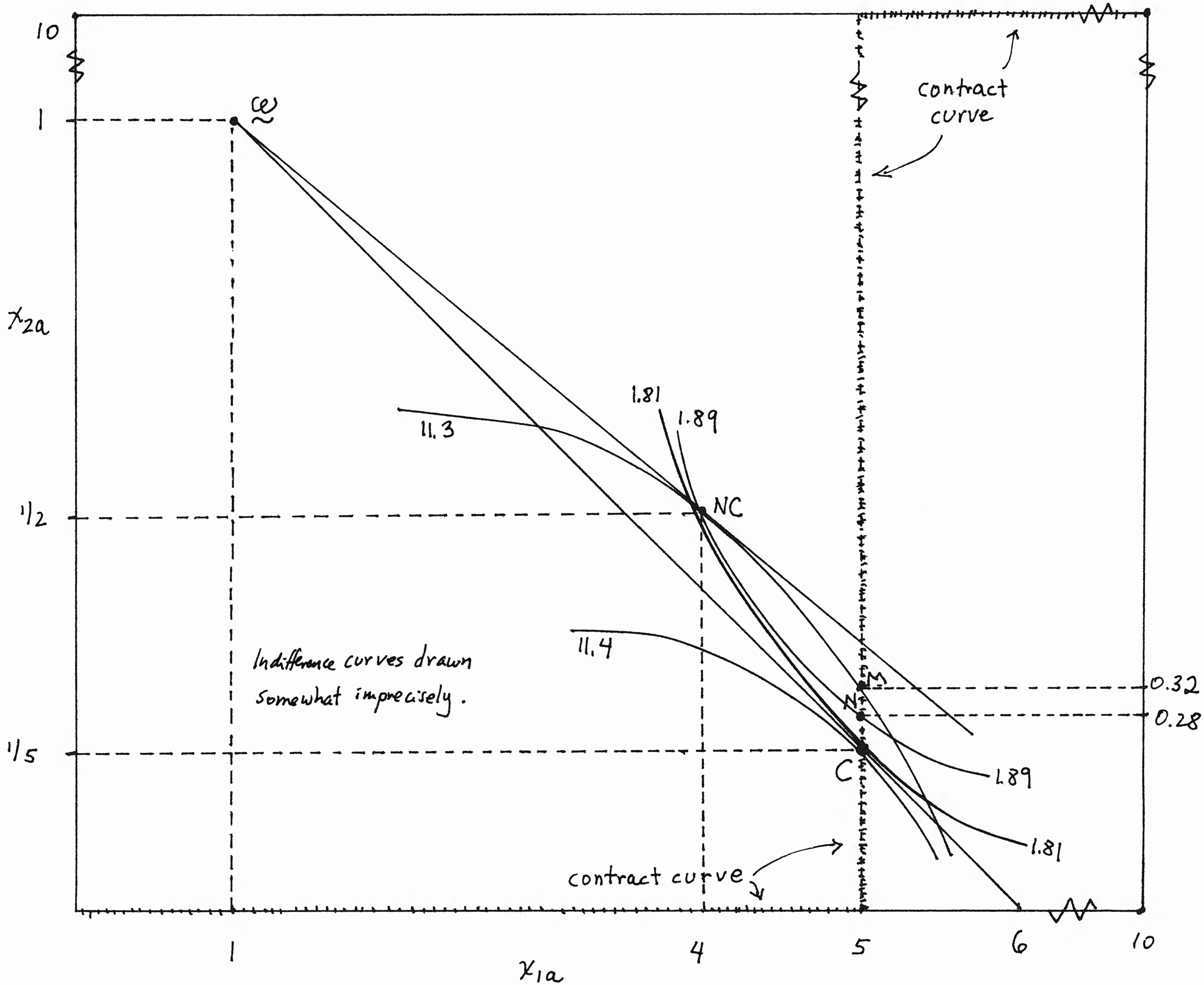
Optional: This should clear the Good 2 market. Does it? The supply of Good 2 is 10 and the aggregate demand for Good 2 is $x_{2a} + x_{2b} = \frac{p_1}{p_2} + 9\frac{p_1}{p_2} + 8 = \frac{10p_1}{p_2} + 8$. At the equilibrium price ratio established three lines ago (namely $p_2/p_1 = 5$), this aggregate demand is $\frac{10}{5} + 8 = 2 + 8 = 10$, so the Good 2 market does clear with this price ratio.

The equilibrium allocation is

$$x_{1a}^* = \frac{p_2}{p_1} = 5 \qquad x_{1b}^* = \frac{p_2}{p_1} = 5$$

$$x_{2a}^* = \frac{p_1}{p_2} = \frac{1}{5} \qquad x_{2b}^* = 9\frac{p_1}{p_2} + 8 = \frac{9}{5} + 8 = 1\tfrac{4}{5} + 8 = 9\tfrac{4}{5}.$$

This is at Point C of the diagram below. That diagram resembles the one provided on the exam, but the exam's diagram was not drawn to scale and it is easier to read, whereas the diagram below is drawn to scale, although it also omits parts of the Edgeworth Box.



10
1
x_{2a}
1/2
1/5
ω
contract curve
contract curve
1.81
1.89
11.3
NC
11.4
Indifference curves drawn somewhat imprecisely.
M
N
C
0.32
0.28
1.89
1.81
1
4
5
6
10
x_{1a}

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One could find the contract curve as follows:

$$\max \; \alpha u_a + (1-\alpha) u_b \quad \text{s.t.} \quad x_{1a} + x_{1b} = 10$$
$$x_{2a} + x_{2b} = 10.$$

Varying α from 0 to 1 will yield the contract curve.

$$\max \; \alpha \left[\ln x_{1a} + x_{2a} \right] + (1-\alpha) \left[\ln x_{1b} + x_{2b} \right] \quad \text{s.t. (above conditions hold)}$$

$\Longleftrightarrow$

$$\max \; \alpha \left[\ln x_{1a} + x_{2a} \right] + (1-\alpha) \left[\ln (10 - x_{1a}) + (10 - x_{2a}) \right].$$

First-order conditions:

$$0 = \frac{\partial \,(\text{objective function})}{\partial x_{1a}} = \frac{\alpha}{x_{1a}} - \frac{1-\alpha}{10 - x_{1a}}$$

$$10\alpha - \alpha x_{1a} = x_{1a} - \alpha x_{1a}$$

$$\boxed{10\alpha = x_{1a}}$$

$$0 = \frac{\partial \,(\text{objective function})}{\partial x_{2a}} = \alpha - (1-\alpha) = 2\alpha - 1$$

? We need to be able to vary α between 0 and 1. We cannot have α always equal to ½. Something is wrong. What's wrong is that the objective function is linear in x_{2a},

$$\text{Objective Function} = \alpha \ln x_{1a} + (1-\alpha) \ln (10 - x_{1a}) + \underbrace{\alpha x_{2a} + (1-\alpha)(10 - x_{2a})}$$

$$= \alpha x_{2a} + 10 - x_{2a} - 10\alpha + \alpha x_{2a}$$

$$= 2\alpha x_{2a} - x_{2a} + 10 - 10\alpha$$

$$= (2\alpha - 1) x_{2a} + 10 - 10\alpha$$

so you maximize the objective function with respect to x_{2a} by setting x_{2a} equal to its extreme values (a corner solution), $x_{2a} = 0$ or 10, depending on the sign of $2\alpha - 1$:

if $2\alpha - 1 < 0$ then $x_{2a}^* = 0$

$\Leftrightarrow 2\alpha < 1 \Leftrightarrow \alpha < 1/2$

if $2\alpha - 1 > 0$ then $x_{2a}^* = 10$

$\Leftrightarrow 2\alpha > 1 \Leftrightarrow \alpha > 1/2$

if $2\alpha - 1 = 0$ then any value of x_{2a} which is feasible (that is, is between 0 and 10 inclusive) is optimal.

$\alpha = 1/2$

Since $x_{1a} = 10\alpha$ from the previous page, the contract curve is given by $(x_{1a}, x_{2a}) =$

$$= \begin{cases} (10\alpha, 0) & \text{if } 0 \le \alpha < 1/2 \\ (10\alpha, [0, 10]) = (5, [0,10]) & \text{if } \alpha = 1/2 \\ (10\alpha, 10) & \text{if } \frac{1}{2} < \alpha \le 1. \end{cases}$$

This is the contract curve position illustrated in the diagram.

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b) (i)

Person a controls p_2.

Person a knows that $x_{2a} + x_{2b} = 10$, that is, that $x_{2a} = 10 - x_{2b}$.

Person a knows that $x_{2b}^D = 9\frac{p_1}{p_2} + 8$ from part (a).

Therefore Person a knows that $x_{2a} = 10 - \left(9\frac{p_1}{p_2} + 8\right)$

$$\boxed{x_{2a} = 2 - 9\frac{p_1}{p_2}.} \qquad (1)$$

(ii) In Part (a), Person a's Lagrangian was

$$\ln x_{1a} + x_{2a} - \lambda\left[p_1(x_{1a} - 1) + p_2(x_{2a} - 1)\right]$$

and he maximized over x_{1a} and x_{2a}. Here, the Lagrangian will be the same except we should use (1) to replace x_{2a} with $2 - 9\frac{p_1}{p_2}$, and the maximization should be over x_{1a} and p_2.

$$\mathcal{L} = \ln x_{1a} + 2 - 9\frac{p_1}{p_2} - \lambda\left[p_1(x_{1a} - 1) + p_2\left(2 - 9\frac{p_1}{p_2} - 1\right)\right]$$

$$= \text{——}\text{"}\text{——} - \lambda\left[p_1(x_{1a} - 1) + 2p_2 - 9p_1 - p_2\right]$$

$$= \text{——}\text{"}\text{——} - \lambda\left[p_1(x_{1a} - 1) + p_2 - 9p_1\right].$$

F.O.C.'s:

$$\left.\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial x_{1a}} = \frac{1}{x_{1a}} - \lambda p_1 \Rightarrow \lambda = \frac{1}{p_1 x_{1a}} \\ 0 &= \frac{\partial \mathcal{L}}{\partial p_2} = 9\frac{p_1}{p_2^2} - \lambda \Rightarrow \lambda = 9\frac{p_1}{p_2^2} \end{aligned}\right\} \Rightarrow \frac{1}{p_1 x_{1a}} = \frac{9p_1}{p_2^2} \Rightarrow \boxed{x_{1a} = \frac{p_2^2}{9p_1^2}}.$$

Substitute this into the budget constraint:

↑ note optimization with respect to p_2

$$0 = p_1\left(\frac{p_2^2}{9p_1^2} - 1\right) + p_2\left(1 - 9\frac{p_1}{p_2}\right)$$

$$= \frac{p_2^2}{9p_1} - p_1 + p_2 - 9p_1 = \frac{p_2^2}{9p_1} - 10p_1 + p_2.$$ Divide by p_2:

$$0 = \frac{p_2}{9p_1} - 10\frac{p_1}{p_2} + 1.$$ Let $\boxed{\rho = p_2/p_1}$:

$$0 = \frac{\rho}{9} - 10/\rho + 1.$$ Multiply by ρ:

$$0 = \frac{1}{9}\rho^2 - 10 + \rho = \frac{1}{9}\rho^2 + \rho - 10.$$ Multiply by 9:

$$0 = \rho^2 + 9\rho - 90 = (\rho + 15)(\rho - 6).$$ I got these roots from the quadratic formula

$$\frac{-9 \pm \sqrt{9^2 - 4(1)(-90)}}{2(1)} = \frac{-9 \pm \sqrt{81 + 360}}{2} = \frac{-9 \pm \sqrt{441}}{2}$$

$$= \frac{-9 \pm 21}{2} = \left\{\frac{-30}{2}, \frac{12}{2}\right\} = \{-15, 6\}.$$

The equilibrium price vector is therefore $\boxed{\frac{p_2}{p_1} = \rho = 6}$ since $\rho = -15$ violates the condition that prices be positive.

The equilibrium is therefore

$$\hat{x}_{1a} = \frac{p_2^2}{9p_1^2} = \frac{e^2}{9} = \frac{6^2}{9} = \frac{36}{9} = 4$$

$$\hat{x}_{2a} = 2 - 9\frac{p_1}{p_2} = 2 - \frac{9}{e} = 2 - \frac{9}{6} = 2 - 1\tfrac{1}{2} = \tfrac{1}{2}.$$

One can find Person b's allocation either by using his demand curves, which were derived in Part (a),

$$\hat{x}_{1b} = \frac{p_2}{p_1} = e = 6$$

$$\hat{x}_{2b} = 9\frac{p_1}{p_2} + 8 = \frac{9}{e} + 8 = \frac{9}{6} + 8 = 1\tfrac{1}{2} + 8 = 9\tfrac{1}{2},$$

or by using the feasibility conditions that $10 = \hat{x}_{1a} + \hat{x}_{1b} = 4 + \hat{x}_{1b} \Rightarrow \hat{x}_{1b} = 6$

$$10 = \hat{x}_{2a} + \hat{x}_{2b} = \tfrac{1}{2} + \hat{x}_{2b} \Rightarrow \hat{x}_{2b} = 9\tfrac{1}{2}.$$

Answer to Question 2(c):

At NC, the indifference curve of Person "b" is tangent to his budget constraint, which is the straight line going through ω whose slope is the equilibrium price vector $-p_1/p_2 = -6$. This is because Person "b" is a price-taker in part (b), just as he was in part (a), so the usual condition that "at the optimum, the indifference curve is tangent to the budget constraint" applies to him. However, at NC the indifference curve of Person "a" is not tangent to that line. That is because in part (b), Person "a" is not a price-taker; he controls the price.

Answer to Question 2(d)(i):

Point NC is clearly inefficient, since it is not on the contract curve and (equivalently) the indifference curves of the two people are not tangent there. Both people could be made better off by moving into the lens-shaped area enclosed by the lines passing through M, NC, and N. In particular, points between M and N are efficient, and points strictly between M and N would make both people strictly better off than they are at NC. (In other words: NC is not Pareto Efficient; M and N are Pareto Efficient; and a move from NC to any point between M and N is a Pareto-improving move.) Hence NC is a socially bad point. On the other hand, point C is efficient (that is, it is Pareto Optimal). So it's socially better than NC.

Answer to Question 2(d)(ii):

Person "a" has the same preferences as Person "b," but Person "a" is much poorer, since he has $\omega_a = (1, 1)$ in contrast to $\omega_b = (9, 9)$. By giving Person "a" some market power, part (b) of the question improved his utility at least a little bit compared to the competitive equilibrium (a utility level of 1.89 instead of 1.81, if cardinal utility levels matter). At NC, the rich Person "b" is not damaged much compared to point C; this can be seen either by observing that NC is not far away from ω, which is quite lopsided in favor of Person "b," or (if cardinal utility levels matter) by observing that the utility level of Person "b" only falls from 11.4 to 11.3 if the economy was at NC instead of at C. It is true that NC is inefficient. A point between M and N would be better than NC. However, the question before the court is not a choice between, on the one hand, points between M and N, and, on the other hand, NC or C. Instead the question before the court is a choice between NC and C. NC is more fair than C (because it has a higher utility for the poorer person). Society may certainly care about fairness, and it may, in some or even in all situations, value fairness over efficiency. So it is certainly possible for society to prefer NC to C. Society should indeed have this preference, because Person "a" is so disadvantaged.

Optional details related to part (d) follow.

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Utility levels. Part 1a). $u_a = \ln(x_{1a}^*) + x_{2a}^* = \ln 5 + \frac{1}{5} \approx 1.81$

$u_b = \ln(x_{1b}^*) + x_{2b} = \ln 5 + 9\frac{4}{5} \approx 11.4$

Part (b). $u_a = \ln(\hat{x}_{1a}) + \hat{x}_{2a} = \ln 4 + \frac{1}{2} \approx 1.89$

$u_b = \ln(\hat{x}_{1b}) + \hat{x}_{2b} = \ln 6 + 9\frac{1}{2} \approx 11.3$

At $\underset{\sim}{\omega}$: $u_a = \ln(1) + 1 = 1$

$u_b = \ln(9) + 9 \approx 11.2$

x_{2a} at Point M: Person b's utility is $\ln 6 + 9\frac{1}{2} \approx 11.3$ and $x_{1a} = 5$ so $x_{1b} = 10-5$
$= 5$ and

$\ln 6 + 9\frac{1}{2} = u_b = \ln x_{1b} + x_{2b}$

$= \ln 5 + x_{2b}$; but $x_{1b} + x_{2b} = 10$ so $x_{2b} = 10 - x_{1b}$ and

$\ln 6 + 9\frac{1}{2} = \ln 5 + 10 - x_{2b} \Rightarrow$

$x_{2b} = 10 - 9\frac{1}{2} + \ln 5 - \ln 6 = \frac{1}{2} + \ln\frac{5}{6} \approx 0.32$

x_{2a} at Point N: Person a's utility is $\ln 4 + \frac{1}{2} \approx 1.89$ and $x_{1a} = 5$ so

$\ln 4 + \frac{1}{2} = u_a = \ln 5 + x_{2a} \Rightarrow x_{2a} = \ln 4 + \frac{1}{2} - \ln 5$

$= \ln\frac{4}{5} + \frac{1}{2} \approx 0.28$

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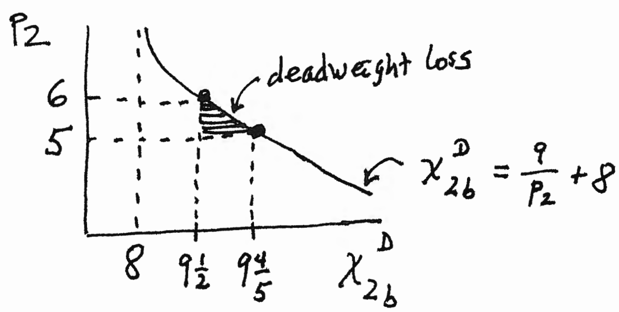
The deadweight loss to Person b from having to face Person a's market power can be found as follows. In both parts (a) and (b), Person b is a net buyer of Good 2 and Person a is a net seller of Good 2 (since, in the diagram, Points NC and C both lie below $\underset{\sim}{\omega}$). Let $\boxed{p_1 = 1}$ be the numéraire. Then in part (a), $\frac{p_2}{p_1} = 5 \Rightarrow p_2 = 5$; and in part (b), $\frac{p_2}{p_1} = 6 \rightarrow p_2 = 6$. Person a's market power over Good 2 in part (b) causes the price of Good 2 (of which Person a is a net seller) to rise from 5 to 6. Person b's demand for Good 2 is, from part (a), $9\frac{p_1}{p_2} + 8 = \frac{9}{p_2} + 8$. So:

In part (a) [competition], $x_{2b}^D = \frac{9}{5} + 8 = 1\frac{4}{5} + 8 = 9\frac{4}{5}$;

in part (b) [non-competitive], $x_{2b}^D = \frac{9}{6} + 8 = 1\frac{1}{2} + 8 = 9\frac{1}{2}$.



Consumer surplus loss $= \int_5^6 \left(\frac{9}{p_2} + 8\right) dp_2 = 9 \ln p_2 + 8p_2 \Big|_5^6 = 9\ln 6 + 48 - 9\ln 5 - 40$

$= 9 \ln \frac{6}{5} + 8$

deadweight loss $= 9 \ln \frac{6}{5} + 8 - (6-5) \cdot 9\frac{1}{2}$

$= 9 \ln \frac{6}{5} - 1\frac{1}{2} \approx \boxed{0.14.}$

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Suppose the court were to rule in favor of Person a. Could the winner (Person a) compensate the loser (Person b) and still come out ahead of where the winner would be if he had lost?

Person b's willingness to accept ("WTA") point NC $\overbrace{\ }^{\text{instead of point C}}$ is (measuring in terms of the always-competitive Good 1)

$$\underbrace{u_b(\hat{x}_{1b}+WTA,\ \hat{x}_{2b})}_{\text{at NC}} = \underbrace{u_b(x^*_{1b},\ x^*_{2b})}_{\text{at C}}$$

$$u_b(6+WTA,\ 9\tfrac{1}{2}) = u_b(5,\ 9\tfrac{4}{5})$$

$$\ln(6+WTA)+9\tfrac{1}{2} = \ln 5 + 9\tfrac{4}{5}$$

$$\ln\frac{6+WTA}{5} = \frac{3}{10}$$

$$\frac{6+WTA}{5} = e^{3/10} \Rightarrow WTA_b = 5e^{3/10}-6$$

Person a's willingness to pay ("WTP") for point NC instead of point C is

$$u_a(\hat{x}_{1a}-WTP,\ \hat{x}_{2a}) = u_a(x^*_{1a},\ x^*_{2a})$$

$$u_a(4-WTP,\ \tfrac{1}{2}) = u_a(5,\ \tfrac{1}{5})$$

$$\ln(4-WTP)+\tfrac{1}{2} = \ln 5+\tfrac{1}{5}$$

$$\frac{5}{10}-\frac{2}{10} = \ln\frac{5}{4-WTP}$$

$$e^{3/10} = \frac{5}{4-WTP}$$

$$4-WTP = 5e^{-3/10} \Rightarrow WTP_a = 4-5e^{-3/10}$$

Is the winner's WTP $\geq$ the loser's WTA? (over →)

$$4-5e^{-3/10} \overset{?}{\geq} 5e^{3/10}-6$$

$$4+6 \overset{?}{\geq} 5\left(e^{3/10}+e^{-3/10}\right)$$

Math trivia note:

$$\underbrace{e^{3/10}+e^{-3/10}=2\cosh\tfrac{3}{10}}$$

hyperbolic cosine

$10 \overset{?}{\geq} 10.453$, <u>No</u>: The winner (Person a) cannot fully compensate the loser (Person b) and still be glad he won. This is another reflection of the fact that NC is inefficient.

Note: $WTA_b = 5e^{3/10}-6 \approx 0.75$, $WTP_a = 4-5e^{-3/10} \approx 0.30$

Suppose the court were to rule in favor of Person b. Could the winner (Person b) compensate the loser (Person a) and still come out ahead of where the winner would be if he had lost?

← Optional Page →

The loser's WTA compensation for being at C instead of at NC is (in terms of Good 1)

$$\underbrace{u_a(x_{1a}^* + WTA,\ x_{2a}^*)}_{\text{at } C} = \underbrace{u_a(\hat{x}_{1a},\ \hat{x}_{2a})}_{\text{at } NC}$$

$$u_a(5+WTA,\ \tfrac{1}{5}) = u_a(4,\ \tfrac{1}{2})$$

$$\ln(5+WTA) + \tfrac{1}{5} = \ln 4 + \tfrac{1}{2}$$

$$\ln\frac{5+WTA}{4} = \frac{1}{2}-\frac{1}{5} = \frac{5-2}{10} = \frac{3}{10}$$

$$\frac{5+WTA}{4} = e^{3/10} \Rightarrow WTA_a = 4e^{3/10}-5$$

$$\approx 0.40$$

The winner's WTP for being at C instead of at NC is

$$\underbrace{u_b(x_{1b}^* - WTP,\ x_{2b}^*)}_{\text{at } C} = \underbrace{u_b(\hat{x}_{1b},\ \hat{x}_{2b})}_{\text{at } NC}$$

$$u_b(5-WTP,\ 9\tfrac{4}{5}) = u_b(6,\ 9\tfrac{1}{2})$$

→ over →

$$\ln(5-WTP) + 9\tfrac{4}{5} = \ln 6 + 9\tfrac{1}{2}$$

$$9\tfrac{8}{10} - 9\tfrac{5}{10} = \ln \frac{6}{5-WTP}$$

$$e^{3/10} = \frac{6}{5-WTP}$$

$$5 - WTP = 6e^{-3/10}$$

$$WTP_b = 5 - 6e^{-3/10} \approx 0.56$$

Is the winner's WTP $\geq$ the loser's WTA?

$$5 - 6e^{-3/10} \overset{?}{\geq} 4e^{3/10} - 5$$

$$10 \overset{?}{\geq} 4e^{3/10} + 6e^{-3/10} \approx 9.84$$ Yes. The

← Optional Page →

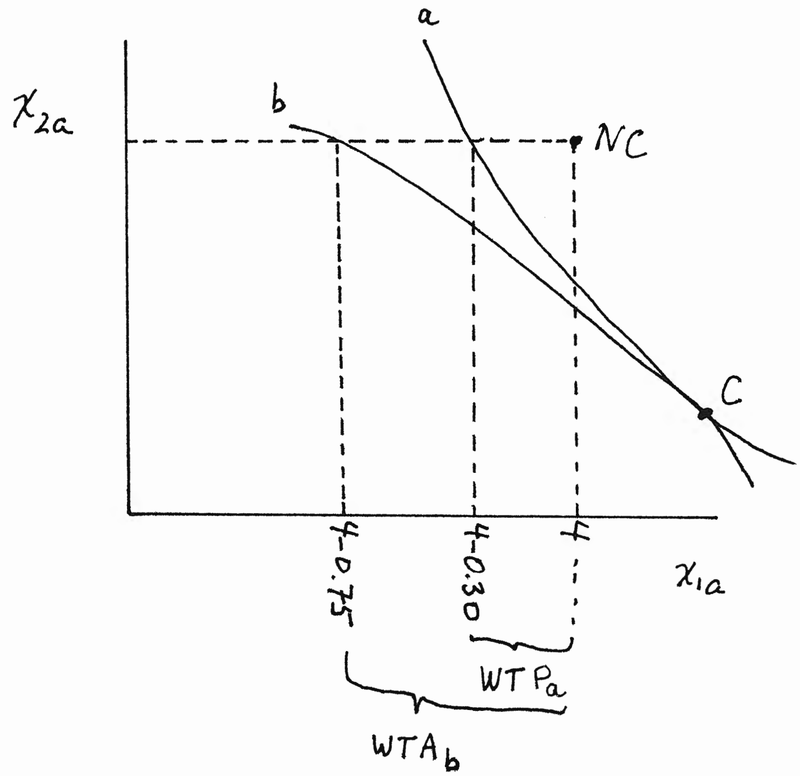
winner (Person b) can fully compensate the loser (Person a) and still be glad he won. This reflects the facts that NC is inefficient and C is efficient. (However, this still does not mean that C is better than NC unambiguously. All this means is that there exists some point to the right of C (more x_{1a} than C) which is Pareto-superior to NC.)

These two results are illustrated on the next page. The first graph shows the first result (that at NC the winner cannot compensate the loser). Because the indifference curves are tangent at C, this conclusion could have been proven just using the graph, without calculating any numbers. The second graph, showing the second result (that at C the winner can compensate the loser), needs to have numbers calculated in order to ensure the graph is correct.

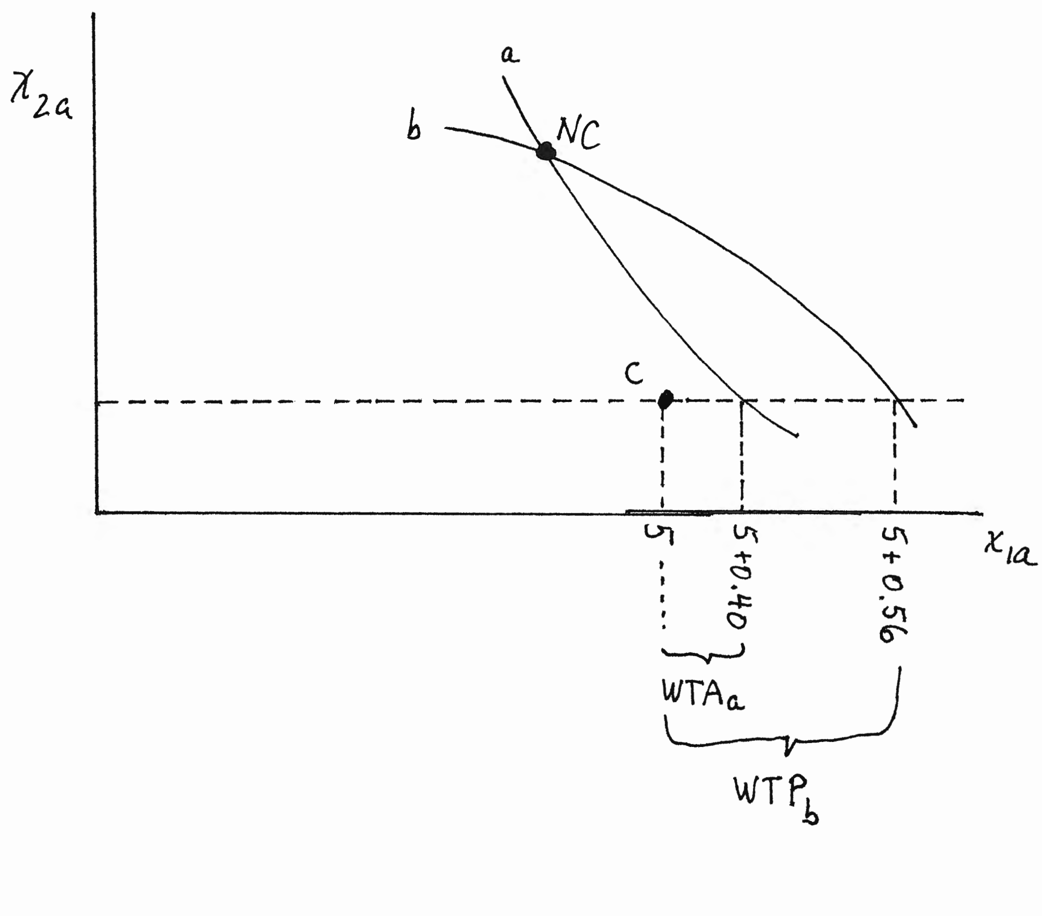
Finally, note that:

	WTA		WTP
a	0.40	>	0.30
b	0.75	>	0.56

That is, an individual's WTA > their WTP. Also, a's WTA and WTP is less than b's, possibly because a is poorer than b.


a
b
x_{2a}
NC
C
$4-0.75$
$4-0.30$
4
x_{1a}
WTP_a
WTA_b

← Optional →


a
b
x_{2a}
NC
C
5
$5+0.40$
$5+0.56$
x_{1a}
WTA_a
WTP_b