

Section 6:

Monopoly, Oligopoly, and Game Theory

Final Exam
1994
Question 1

(2)

Answer all of the following five questions.

1. A monopolist faces an inverse demand curve given by $p = 10 - y$ where p is the price of output and y is its output. Input prices are fixed throughout this problem at $w \equiv 1$, and with these fixed input prices, this firm's cost function can be written simply as $c(w, y) = e^{ay}$ where a is a positive parameter.

Denote the firm's optimal output by y^* . You do not have to calculate the numerical value of y^* . (In fact, it is impossible to calculate the numerical value of y^* exactly.) Instead, find, as a function of y^* , how the firm's output changes when the parameter a changes by a small amount.

Final Exam
1994
Answer 1

Answers to Final Exam, Econ. 621, winter 1994

$$\textcircled{1} \quad \text{Total revenue} = \text{price} \cdot \text{quantity} = (10-y)y = 10y - y^2$$

$$\text{Total cost} = e^{ay}$$

$$\text{Profit} = 10y - y^2 - e^{ay} \quad (\text{call this } \pi) \quad \leftarrow \text{3 pts}$$

$$0 = \frac{d\pi}{dy} = 10 - 2y^* - ae^{ay^*} \quad \text{Take the total differential:} \quad \leftarrow \text{4 pts}$$

$$0 = \underbrace{[-2 - a^2 e^{ay^*}] dy}_{\frac{\partial}{\partial y} \frac{d\pi}{dy}} + \underbrace{[-e^{ay^*} - ay^* e^{ay^*}] da}_{\frac{\partial}{\partial a} \frac{d\pi}{dy}} \quad \leftarrow \text{12 pts}$$

$$\Rightarrow \frac{dy^*}{da} = - \frac{e^{ay^*} + ay^* e^{ay^*}}{2 + a^2 e^{ay^*}} = - \frac{1 + ay^*}{2e^{-ay^*} + a^2} \quad \leftarrow \text{1 pt}$$

Optional: $dy^*/da < 0$ because as $a \uparrow$, costs \uparrow , so output \downarrow . Also, using the F.O.C., $e^{ay^*} = (10 - 2y^*)/a$, so dy^*/da can be rewritten as

$$\left\{ \begin{array}{l} \frac{1 + ay^*}{(2a + a^2)} \\ - \frac{1 + ay^*}{(10 - 2y^* + a^2)} \end{array} \right.$$

Final Exam
1998

Question 1

(4)

Answer all of the following five questions.

1. Suppose two firms form a Stackelberg duopoly. Suppose the market demand curve is $p = 10 - Q$ where p is price and Q is market quantity. Suppose (for simplicity's sake) that each firm's cost of production is zero.
 - (a) Find the profit earned by each firm in the Stackelberg equilibrium.
 - (b) Find each firm's profit function.

Answers to Final Exam, Econ. 621, Winter 1998

① Stackelberg duopoly zero costs
 $p = 10 - Q$

a) "f" denotes follower, "l" denotes leader

$$\pi_f = p q_f = (10 - Q) q_f = (10 - q_e - q_f) q_f = 10q_f - q_e q_f - q_f^2$$

"f" takes q_e as given and maximizes π_f :

$$0 = \frac{\partial \pi_f}{\partial q_f} = 10 - q_e - 2q_f \Rightarrow 2q_f = 10 - q_e,$$

$$q_f^* = 5 - \frac{1}{2}q_e.$$

Then the leader, taking $q_f^* = 5 - \frac{1}{2}q_e$ as given, maximizes

$$\pi_e = p q_e = (10 - Q) q_e = (10 - q_e - q_f) q_e$$

$$= (10 - q_e - 5 + \frac{1}{2}q_e) q_e = (5 - \frac{1}{2}q_e) q_e = 5q_e - \frac{1}{2}q_e^2 \Rightarrow$$

$$0 = \frac{\partial \pi_e}{\partial q_e} = 5 - q_e \Rightarrow q_e^* = 5.$$

$$\text{Also, } q_f^* = 5 - \frac{1}{2}q_e = 5 - \frac{1}{2}(5) = \underline{2\frac{1}{2}}.$$

$$\text{Hence } p^* = 10 - Q = 10 - q_e - q_f = 10 - 5 - 2\frac{1}{2} = \underline{2\frac{1}{2}} \text{ and}$$

$$\pi_f^* = p^* q_f^* = (2\frac{1}{2})(2\frac{1}{2}) = \frac{5}{2} \cdot \frac{5}{2} = \frac{25}{4} = \underline{6\frac{1}{4}}$$

$$\pi_e^* = p^* q_e^* = (2\frac{1}{2})(5) = \frac{5}{2} \cdot 5 = \frac{25}{2} = \underline{12\frac{1}{2}}.$$

b) The "profit function" shows how a firm's profit is affected by changes in exogenous variables. For the follower, q_e is exogenous, so the follower's profit function is $\pi_f^*(q_e) = p q_f = (10 - q_e - q_f) q_f =$

Final Exam
 1998

Answer 1

Final Exam
1998
Answer 1 cont...

2

$$(10 - q_e - (5 - \frac{1}{2}q_e))(5 - \frac{1}{2}q_e) = (10 - q_e - 5 + \frac{1}{2}q_e)(5 - \frac{1}{2}q_e)$$

$\overbrace{\quad\quad\quad}^{q^*_f}$

$$= (5 - \frac{1}{2}q_e)(5 - \frac{1}{2}q_e)$$

$$= (5 - \frac{1}{2}q_e)^2$$

The leader faces no exogenous variables (since the demand curve has no parameters), so the leader's profit function is simply $\pi_L^* = 12\frac{1}{2}$.

↑ in this particular problem

Qualifying Exam
1997
Question 3

(4)

Question 3. Suppose two firms are Stackelberg duopolists, with the leader denoted by '1' and the follower by '2'. Let the output of firm i be q_i , where i equals 1 or 2. Suppose the cost function of each firm is given by $c \cdot q_i^2$ where c is a constant which is the same for both firms.

How will q_1 and q_2 respond when c rises?

(It may be a good idea not to start by trying to solve this problem in all its detail, but to get a general, or abstract, idea of what is going on first. These steps will earn you points even if you cannot solve the question in full detail.)

Qualifying Exam
1997

Answer 3

Answers to Qualifying Exam, Summer 1997
Econ. 621 Portion

Required Question

One way to begin is at a very general level, as the question suggests (but does not require) :

Leader

$$\pi_1(q_1)$$

First-Order Condition :

$$0 = \frac{d\pi_1(q_1)}{dq_1}$$

Comparative Statics :

$$0 = \frac{\partial^2 \pi_1}{\partial q_1^2} dq_1 + \frac{\partial^2 \pi_1}{\partial q_1 \partial c} dc$$

↓

$$\frac{dq_1}{dc} = - \frac{\frac{\partial^2 \pi_1}{\partial q_1 \partial c}}{\frac{\partial^2 \pi_1}{\partial q_1^2}}$$

negative for profit maximization

Follower

$$\pi_2(q_1, q_2)$$

↑
exogenous to follower

$$0 = \frac{\partial \pi_2}{\partial q_2}$$

The leader's profit only depends on q_1 , because q_2 is not independent but rather is itself a reaction to q_1 , and the leader in equilibrium knows $q_2(q_1)$'s form.

$$0 = \frac{\partial^2 \pi_2}{\partial q_2^2} dq_2 + \frac{\partial^2 \pi_2}{\partial q_1 \partial q_2} dq_1 + \frac{\partial^2 \pi_2}{\partial c \partial q_2} dc$$

↓

$$\frac{dq_2}{dc} = - \frac{\frac{\partial^2 \pi_2}{\partial q_1 \partial q_2}}{\frac{\partial^2 \pi_2}{\partial q_2^2}} \frac{dq_1}{dc}$$

$$- \frac{\frac{\partial^2 \pi_2}{\partial c \partial q_2}}{\frac{\partial^2 \pi_2}{\partial q_2^2}} ; \text{ since}$$

we know dq_1/dc from here, we can substitute in and collect terms to get

$$\frac{dq_2}{dc} = \left[\frac{\partial^2 \pi_2}{\partial q_2^2} \right]^{-1} \left[\frac{\partial^2 \pi_2}{\partial q_1 \partial q_2} \frac{\partial^2 \pi_1 / \partial q_1 \partial c}{\partial^2 \pi_1 / \partial q_1^2} - \frac{\partial^2 \pi_2}{\partial c \partial q_2} \right]$$

↑ negative for profit maximization

Another way is to start (or continue) using the given cost function. Let $p(Q)$ be the (inverse) demand curve, where $Q = q_1 + q_2$. Let $P' = dp/dQ$ and $q'_2 = dq_2/dq_1$.

Follower

$$\pi_2 = p(Q) q_2 - c q_2^2$$

$$F.O.C. \quad O = \frac{\partial \pi_2}{\partial q_2}$$

$$= \frac{dp}{dQ} \frac{d(q_1 + q_2)}{dq_2} q_2 + p(Q) - 2c q_2$$

$$= P' (O + 1) q_2 + p(Q) - 2c q_2$$

$$= P' q_2 + p(Q) - 2c q_2$$

Leader

$$\pi_1 = p(Q) q_1 - c q_1^2$$

$$F.O.C. \quad O = \frac{\partial \pi_1}{\partial q_1}$$

$$= \frac{dp}{dQ} \frac{d(q_1 + q_2)}{dq_1} q_1 + p(Q) - 2c q_1$$

$$= P' (1 + q'_2) q_1 + p(Q) - 2c q_1$$

$$= P' q_1 + P' q'_2 q_1 + p(Q) - 2c q_1$$

If you did the calculations on p.1 and the top of p.2, you now calculate those derivatives.

Otherwise, take total differentials (which entails the same calculations as finding the just-mentioned derivatives) :

$$O = \frac{\partial^2 \pi_2}{\partial q_2^2} dq_2$$

$$+ [P'' q_2 + P'] dq_1$$

$$- 2q_2 dc$$

$$O = \frac{\partial^2 \pi_1}{\partial q_1^2} dq_1 - 2q_1 dc$$

there's no dq_2
since q_2 can't
change independently
of q_1

we could calculate these but there's no
need (at least at this point) because they
have to be negative if an optimum is to be
guaranteed

Qualifying Exam

1997

Answer 3 cont...

Either by manipulating these equations or by substituting in for the quantities given in the $\frac{dq_1}{dc}$ formula on p. 1 and the $\frac{dq_2}{dc}$ formula at the top of p. 2, one obtains

$$\frac{dq_1}{dc} = \frac{2g_1}{\partial^2 \pi_1 / \partial g_1^2} < 0 \quad \text{so as } c \uparrow, g_1 \downarrow \quad (\text{which sounds reasonable: when costs go up, you produce less}).$$

$$\frac{dq_2}{dc} = \left[\frac{\partial^2 \pi_2}{\partial g_2^2} \right]^{-1} \left[\underbrace{\left(\frac{\partial^2 \pi_1}{\partial g_1^2} \right)^{-1}}_{\frac{d^2 P}{dQ^2} g_2 + \frac{df}{dQ}} \underbrace{\frac{\partial^2 \pi_2}{\partial g_1 \partial c}}_{-2g_1} - \underbrace{\frac{\partial^2 \pi_2}{\partial c \partial g_2}}_{-2g_2} \right]$$

$$= -2 \left[\frac{\partial^2 \pi_2}{\partial g_2^2} \right]^{-1} \left[\underbrace{\left(\frac{\partial^2 \pi_1}{\partial g_1^2} \right)^{-1}}_{\oplus} \underbrace{\left(\frac{d^2 P}{dQ^2} g_2 + \frac{df}{dQ} \right) g_1}_{\ominus} - \underbrace{g_2}_{\text{?}} \right].$$

\ominus since
demand curves
slope downwards
(usually)

Somewhat surprisingly, this cannot be signed.

Qualifying Exam
1997

Answer 3 cont...

Qualifying Exam

1996

Question 2

(4)

Question 2. Suppose an industry is a duopoly which is in a (quantity-setting) Stackelberg equilibrium.

Let the quantity of output produced by a Stackelberg leader be q_1 . Suppose the government imposes a tax on the Stackelberg leader; the amount of tax collected is τq_1 .

Will increases in τ increase or decrease the profit of the Stackelberg follower (or can you not tell)? (Notice I asked about the follower, not the leader.)

How would your answer change if the industry were a (quantity-setting) Cournot duopoly instead of a (quantity-setting) Stackelberg duopoly?

Optional Question #2:

Follower: $\max_{q_2} \underbrace{p(q_1 + q_2)q_2 - C(q_2)}_{\Pi_2}$

Envelope Theorem: $\frac{\frac{d\Pi_2}{dq_1}}{\frac{d}{dq_1}} = \left. \frac{\partial \Pi_2}{\partial q_1} \right|_*$

Qualifying Exam
1996

Answer 2

$= p' q_2 < 0$ since $p' = \frac{dp}{dQ}$ is
the slope of the demand curve (with
 $Q = q_1 + q_2$).

So the question is how the tax influences q_1 .

Leader: $\max_{q_1} \Pi_1 = \max_{q_1} p(q_1 + q_2)q_1 - C(q_1) - \tau q_1$

first-order condition: $0 = \frac{d\Pi_1}{dq_1}$

$$\begin{aligned} \text{take total differential: } 0 &= \frac{d^2\Pi_1}{dq_1^2} dq_1 + \left(\frac{\partial}{\partial \tau} \frac{d\Pi_1}{dq_1} \right) d\tau \\ &= \frac{d^2\Pi_1}{dq_1^2} dq_1 + \left(\frac{d}{dq_1} \frac{\partial \Pi_1}{\partial \tau} \right) d\tau \\ &= -'' + \left(\frac{d}{dq_1} (-q_1) \right) d\tau \\ &= -'' - d\tau \Rightarrow \end{aligned}$$

Qualifying Exam

1996

Answer 2 cont...

$$d\pi = \pi_1'' dq_1, \text{ where } \pi_1'' = \frac{d^2 \pi_1}{dq_1^2}$$

$\Rightarrow \frac{dq_1}{d\pi} = -\frac{1}{\pi_1''} < 0$ from second-order conditions. Therefore,

$$\uparrow \pi \Rightarrow \downarrow q_1 \Rightarrow \uparrow \pi_2^*$$

At this level of abstraction, nothing changes if the equilibrium is Cournot-Nash instead of Stackelberg. The leader would no longer take into account the effect of q_1 on q_2 , but I have left that effect implicit anyway.

Final Exam
2000 (4)

Question 2

2. Two firms in one market produce identical output. The market demand curve is $a - (a/2)Q$ where Q is quantity sold and "a" is a parameter. For what values of a will a firm have an incentive to deviate from a cartel (i.e., joint-profit-maximizing) output level? What is the cartel output level? To make this problem simpler, assume the firms' production costs are zero.

Final Exam

2000

Answer 2

$$(2) P = a - \frac{a}{2}Q$$

$$\text{Cartel } \Pi = PQ = (a - \frac{a}{2}Q)Q = aQ - \frac{a}{2}Q^2$$

$$\text{F.O.C. : } 0 = a - aQ \Rightarrow Q = 1. \text{ Then } P = a - \frac{a}{2} = \frac{a}{2}.$$

The symmetric answer is $q_1 = q_2 = \frac{1}{2}$.

$$\Pi_i = Pg_i = \frac{a}{2}q_i; \text{ in the symmetric equilibrium, } \Pi_i = \frac{a}{4}.$$

If one player plays $q = \frac{1}{2}$, the profit of the other player (firm) is

$$\left[a - \frac{a}{2}\left(\frac{1}{2} + q\right) \right]q = aq - \frac{a}{4}q - \frac{a}{2}q^2 = \frac{3}{4}aq - \frac{1}{2}aq^2.$$

$$\text{The F.O.C. is } 0 = \frac{3}{4}a - aq \Rightarrow q = \frac{3}{4}.$$

The S.O.C. for a maximum is $0 > -a \Leftrightarrow 0 < a$.

$$\text{The profit from deviating is } \frac{3}{4}a\left(\frac{3}{4}\right) - \frac{1}{2}a\left(\frac{3}{4}\right)^2 = \left(\frac{3}{4}\right)^2 \left[a - \frac{1}{2}a\right] = \left(\frac{3}{4}\right)^2 \frac{a}{2} = \frac{9}{32}a.$$

The previous profit was $\frac{a}{4} = \frac{8a}{32}$, so deviating is better, since $a > 0$.

Nonsymmetric Method I.

In a nonsymmetric equilibrium, for the cartel,

$$Q = 1 \Rightarrow q_{1c} + q_{2c} = 1$$

$$P = a/2$$

$$\pi_{1c} = \frac{a}{2} q_{1c}$$

$$\pi_{2c} = \frac{a}{2} q_{2c}$$

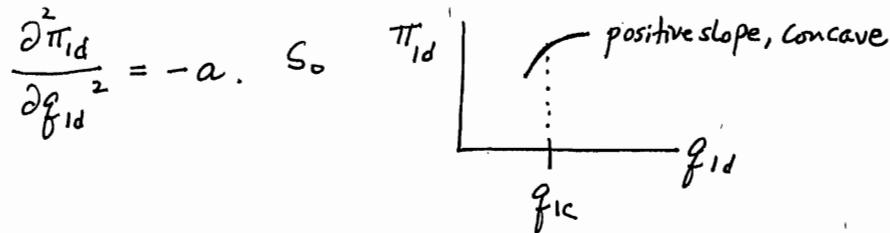
If firm 1 deviated,

$$\pi_{1d} = [a - \frac{a}{2}(q_{1d} + q_{2c})] q_{1d} = a q_{1d} - \frac{a}{2} q_{1d}^2 - \frac{a}{2} q_{2c} q_{1d}$$

$$\frac{\partial \pi_{1d}}{\partial q_{1d}} = a - q_{1d} - \frac{a}{2} q_{2c} \text{ . Evaluated at } q_{1c},$$

$$\begin{aligned} \frac{\partial \pi_{1d}(q_{1c})}{\partial q_{1d}} &= a - a q_{1c} - \frac{a}{2} q_{2c} \\ &= a - a q_{1c} - \frac{a}{2}(1 - q_{1c}) = \frac{a}{2} - \frac{a}{2} q_{1c}, \text{ which is positive} \end{aligned}$$

because q_{1c} is between zero and one. Furthermore,



and it follows that by increasing q_1 from q_{1c} , firm 1's profit will increase.

Non-symmetric, Method II.

In a non-symmetric equilibrium, for the cartel,

$$f_{2c} = 1 - f_{1c}$$

$$P = \frac{a}{2}$$

$$\pi_{1c} = \frac{a}{2} f_{1c}$$

$$\pi_{2c} = \frac{a}{2} (1 - f_{1c}).$$

$$\text{If firm 1 deviates, } \pi_{1d} = \left[a - \frac{a}{2} (f_{1d} + f_{2c}) \right] f_{1d} = a f_{1d} - \frac{a}{2} f_{1d}^2 - \frac{a}{2} f_{2c} f_{1d}$$

$$\text{F.O.C. } 0 = a - a f_{1d} - \frac{a}{2} f_{2c} \Rightarrow f_{1d} = 1 - \frac{1}{2} f_{2c}$$

$$\text{S.O.C. } 0 > -a \Leftrightarrow a > 0$$

$$\pi_{1d}^* = a (1 - \frac{1}{2} f_{2c}) - \frac{a}{2} (1 - \frac{1}{2} f_{2c})^2 - \frac{a}{2} f_{2c} (1 - \frac{1}{2} f_{2c}).$$

$$\text{Since } 1 - \frac{1}{2} f_{2c} = 1 - \frac{1}{2} (1 - f_{1c}) = 1 - \frac{1}{2} + \frac{1}{2} f_{1c} = \frac{1}{2} + \frac{1}{2} f_{1c},$$

$$\pi_{1d}^* = a (\frac{1}{2} + \frac{1}{2} f_{1c}) - \frac{a}{2} (\frac{1}{2} + \frac{1}{2} f_{1c})^2$$

$$- \frac{a}{2} (1 - f_{1c})(\frac{1}{2} + \frac{1}{2} f_{1c})$$

$$\begin{aligned} \pi_{1d}^* - \pi_{1c} &= \frac{a}{2} - \frac{a}{2} (\frac{1}{2} + \frac{1}{2} f_{1c})^2 - \frac{a}{2} (1 - f_{1c})(\frac{1}{2} + \frac{1}{2} f_{1c}) \\ &= \frac{a}{2} - \frac{a}{2} (\frac{1}{4} + \frac{1}{2} f_{1c} + \frac{1}{4} f_{1c}^2) - \frac{a}{2} (\underbrace{\frac{1}{2} + \frac{1}{2} f_{1c} - \frac{1}{2} f_{1c}}_{-\frac{1}{2} f_{1c}} - \frac{1}{2} f_{1c}) \\ &= \frac{a}{2} - \frac{a}{8} - \frac{a}{4} f_{1c} - \frac{a}{8} f_{1c}^2 - \frac{a}{4} + \frac{a}{4} f_{1c}^2 \end{aligned}$$

$$= \frac{4a - a - 2a}{8} - \frac{a}{4} f_{1c} + \frac{a}{8} f_{1c}^2 = \frac{a}{8} f_{1c}^2 - \frac{a}{4} f_{1c} + \frac{a}{8}$$

$$\pi_{1d}^* - \pi_{1c} > 0 \text{ iff } 0 < f_{1c}^2 - 2 f_{1c} + 1 \text{ (with } a > 0) = (f_{1c} - 1)^2, \text{ which is always true.}$$

So deviating is always better.

Final Exam
1994
Question 5

(4)

5. The demand curve for a product produced by a duopoly is $p(Y) = 10 - Y$ where p is price and Y is aggregate output. The output of firm 1 is denoted y_1 and its total costs are y_1^2 . The output of firm 2 is denoted y_2 and its total costs are y_2^2 . Assume the firms in this duopoly play a quantity-setting game, not a price-setting game.

- (a) Find each firm's reaction function.
- (b) Find the Cournot equilibrium output of each firm.
- (c) Find the Stackelberg equilibrium output of each firm.
- (d) Find the cartel output of each firm. Denote these outputs by y_1^j and y_2^j , where the "j" refers to "joint-profit-maximizing."
- (e) Suppose firm 1 produces y_1^j . If firm 2 wishes to cheat on the cartel agreement, how much will it produce?
- (f) Do not perform any mathematical calculations in this part of the question.

Denote the profits earned by the two firms in the Cournot equilibrium (part (b) above) by π_1^c and π_2^c .

Denote the profits earned by the two firms in the Stackelberg equilibrium (part (c) above) by π_1^s and π_2^s .

Denote the profits earned by the two firms in the cartel equilibrium (part (d) above) by π_1^j and π_2^j .

Suppose firm 1 produces y_1^j and firm 2 cheats on the cartel agreement (part (e) above). Denote this case by '1j2d' for '1 joint, 2 defects'. Denote the profits earned by the two firms in this case by π_1^{1j2d} and π_2^{1j2d} .

Suppose firm 2 produces y_2^j and firm 1 cheats on the cartel agreement. Denote this case by '1d2j' for '1 defects, 2 joint'. Denote the profits earned by the two firms in this case by π_1^{1d2j} and π_2^{1d2j} .

Using these π 's, fill in the following table describing the strategic-form game. Remember not to perform any mathematical calculations in this part of the question.

		Firm 2	
		cooperate	defect
Firm 1	cooperate		
	defect		

$$\textcircled{5} \quad p(y) = 10 - y$$

a) $\pi_1 = (10 - y_1 - \bar{y}_2)y_1 - y_1^2$ where \bar{y}_2 denotes the fixed output of firm 2

$$= 10y_1 - y_1^2 - \bar{y}_2 y_1 - y_1^2 = (10 - \bar{y}_2)y_1 - 2y_1^2$$

$$0 = \frac{d\pi_1}{dy_1} = 10 - \bar{y}_2 - 4y_1 \Rightarrow 4y_1 = 10 - \bar{y}_2 \Rightarrow y_1 = \frac{10 - \bar{y}_2}{4}$$
 and by symmetry,

$$y_2 = \frac{10 - \bar{y}_1}{4} \quad \text{(3 pts)}$$

b)

$$\begin{aligned} y_1 &= \frac{1}{4}(10 - \bar{y}_2) \\ y_2 &> \frac{1}{4}(10 - \bar{y}_1) \\ \bar{y}_1 &= y_1 \\ \bar{y}_2 &= y_2 \end{aligned} \quad \left. \begin{aligned} y_1 &= \frac{1}{4} \left[10 - \frac{10 - y_1}{4} \right] \\ 4y_1 &= 10 - \frac{10 - y_1}{4} \Rightarrow 16y_1 = 40 - (10 - y_1) \\ &= 30 + y_1 \Rightarrow 15y_1 = 30 \\ &\Rightarrow y_1 = 2 \end{aligned} \right.$$

symmetry $y_2 = 2$.

$$(or, y_2 = \frac{10 - 2}{4} = 2.) \quad \text{(3 pts)}$$

c) Since the firms are identical, we can without loss of generality let firm 1 be the leader and firm 2 be the follower.

Firm 2 will behave according to its reaction function, so $y_2 = \frac{10 - y_1}{4}$ from part(a). Firm 1 knows this, and therefore solves the problem

Answer 5 Cont.

$$\pi_1 = (10 - y_1 - y_2) y_1 - y_1^2$$

$$= \left[10 - y_1 - \frac{10 - y_1}{4} \right] y_1 - y_1^2 = \left[(10 - y_1) - \frac{(10 - y_1)}{4} \right] y_1 - y_1^2$$

$$= (10 - y_1) \left[1 - \frac{1}{4} \right] y_1 - y_1^2 = \frac{3}{4} (10 - y_1) y_1 - y_1^2 = \frac{30}{4} y_1 - \frac{3}{4} y_1^2 - y_1^2$$

$$= \frac{15}{2} y_1 - \frac{7}{4} y_1^2$$

$$0 = \frac{d\pi_1}{dy_1} = \frac{15}{2} - \frac{7}{2} y_1 \Rightarrow y_1 = \frac{15}{2} \cdot \frac{2}{7} = \frac{15}{7} \text{ leader}$$

$$y_2 = \frac{1}{4} (10 - \frac{15}{7}) = \frac{1}{4} \frac{70 - 15}{7} = \frac{55}{28} \text{ follower}$$

5 pts

d) $\pi_{\text{joint}} = (10 - y_1 - y_2)(y_1 + y_2) - y_1^2 - y_2^2$

$$0 = \frac{\partial \pi_{\text{joint}}}{\partial y_1} = -(y_1 + y_2) + (10 - y_1 - y_2) - 2y_1$$

$$0 = \frac{\partial \pi_{\text{joint}}}{\partial y_2} = -(y_1 + y_2) + (10 - y_1 - y_2) - 2y_2 \quad \left. \begin{array}{l} y_1 = y_2 \\ \text{Therefore} \end{array} \right\}$$

$$0 = -(y_1 + y_1) + (10 - y_1 - y_1) - 2y_1 = 10 - 6y_1 \Rightarrow y_1 = \frac{10}{6} = \frac{5}{3} \text{ and by symmetry, } y_2 = \frac{5}{3}$$

3 pts

e) $y_2 = \frac{10 - y_1}{4} = \frac{10 - \frac{5}{3}}{4} = \frac{1}{4} (10 - \frac{15}{3}) = \frac{1}{4} (\frac{25}{3}) = \frac{25}{12}$

3 pts

f)

		Firm 2	
		Cooperate	defect
Firm 1	cooperate	π_1^j, π_2^j	$\pi_1^{1j2d}, \pi_2^{1j2d}$
	defect	$\pi_1^{1d2j}, \pi_2^{1d2j}$	π_1^c, π_2^c

3 pts

over →

Final Exam
1994

Answer 5 cont...

- "cooperate, cooperate" means to form a cartel
- If one firm cooperates and the other defects, the situation is like in part(e).
- If both firms defect, the result is Cournot.
- Note that to "defect," given that the other firm will "cooperate", means to produce an output of $\frac{25}{12}$ from part(e), but to "defect," given that the other firm will "defect," means to produce an output of 2 (which is less than $\frac{25}{12}$) from part(b).

Hence while "cooperate" means "produce $\frac{5}{3}$,"

"defect" changes its meaning (in terms of $\overset{\text{to be}}{\text{output produced}}$) depending on the circumstances.

Final Exam
1995
Question 6

(4)

6. A Stackelberg duopoly faces a market demand curve of the form $p = A - y$ where y is industry output, p is market price, and A is a given constant. Both firms have zero costs of production.
- Does the profit of the firm which is the "leader" go up or down when A increases?
 - Does the profit of the firm which is the "follower" go up or down when A increases?

(6)

$$p = A - y$$

costs = 0 for both firms, so profit = total revenue

'f' : follower

'l' : leader

Final Exam

1995

Answer 6

$$\text{Follower: } \pi_f = p y_f = (A - y_f - y_e) y_f = A y_f - y_f^2 - y_e y_f.$$

$$\text{Maximize } \pi_f \text{ over } y_f \Rightarrow \text{F.O.C. } 0 = \frac{\partial \pi_f}{\partial y_f} = A - 2y_f - y_e \Rightarrow$$

$$2y_f = A - y_e \Rightarrow y_f^* = \frac{A - y_e}{2}.$$

$$\text{Leader: } \pi_e = p y_e = (A - y_f - y_e) y_e = (A - \frac{A - y_e}{2} - y_e) y_e \text{ from the follower's reaction function derived above; so } \pi_e = (\frac{1}{2}A - \frac{1}{2}y_e) y_e = \frac{1}{2}A y_e - \frac{1}{2}y_e^2. \text{ Maximize } \pi_e \text{ over } y_e \Rightarrow \text{F.O.C. } 0 = \frac{\partial \pi_e}{\partial y_e} = \frac{1}{2}A - y_e \Rightarrow y_e^* = A/2.$$

$$\text{Then } y_f^* = \frac{A - (A/2)}{2} = A/4.$$

$$\text{Using the formulas above, } \pi_f = (A - y_f - y_e) y_f = (A - \frac{1}{4}A - \frac{1}{2}A)(\frac{1}{4}A) = \frac{1}{4}A \cdot \frac{1}{4}A = A^2/16, \text{ so } \pi_f \uparrow \text{ if } A \uparrow. \text{ (By the way, } d\pi_f/dA = A/8 > 0 \text{ since } A > 0 \text{ is necessary for the problem to make sense.)}$$

$$\text{Also, } \pi_e = \frac{1}{2}A y_e - \frac{1}{2}y_e^2 = \frac{1}{2}A \cdot \frac{1}{2}A - \frac{1}{2}\left(\frac{A}{2}\right)^2 = \frac{1}{4}A^2 - \frac{1}{8}A^2 = A^2/8.$$

Again, if $A \uparrow$ then $\pi_e \uparrow$ (and $d\pi_e/dA = A/4 > 0$).

	y_f^*	y_e^*	π_f^*	π_e^*	A affects: π_f	π_e
points:	2	2	2	2	1	1

*Qualifying Exam
1999
Req. Question*

6710 Required Question

Suppose that the only way to produce commodity “ q ” is to use the single input “ x ” according to the production function “ f .” Suppose exactly two firms produce q . Suppose they are in a static Cournot equilibrium.

Assume that the price of x is “ w .” The firms in this problem cannot affect w . Assume that the inverse market demand curve for q is given by $p(q)$.

- (a) Suppose that w changes. What is the corresponding change in the amount of x which a firm buys? How is this related to

$$\frac{p''f'f'f + p'f''f + 2p'f'f' + pf'' - p''f'f'f - p'f'f'}{(p''f'f'f + p'f''f + 2p'f'f' + pf'')^2 - (p''f'f + p'f')^2(f')^2} ?$$

- (b) If w changes, how does the profit of each firm change?

You do not have to determine the sign of complicated expressions like the fraction given in part (a).

Hint 1. In answering part (a), I found it easier for most of the problem to express profit as something like π (though I did not use exactly π) instead of using something much more detailed.

Hint 2. Should dq be zero?

Answers to 6710 Portion of 1999 Micro Qualifying Exam

Required Ques.

Call the two firms "1" and "2". Let x_1 be the amount of x bought by the first firm and x_2 be the amount of x bought by the second firm. Let q_1 be the amount of q produced by the first firm and let q_2 be the amount of q produced by the second firm. Let π_1 be the profit of the first firm and let π_2 be the profit of the second firm.

The market price of q is $p(q_1 + q_2)$, where the first firm takes q_2 as exogenous and the second firm takes q_1 as exogenous. This is the Cournot assumption.

Therefore

$$\pi_1 = p(q_1 + q_2) q_1 - w x_1 \quad \text{and} \quad q_1 = f(x_1).$$

Firm 1's endogenous variables are x_1 and q_1 , and its exogenous variables are q_2 and w . The corresponding expressions for Firm 2 are obtained by replacing '1' subscripts with '2' and '2' subscripts with '1' in the previous two sentences, because of the problem's symmetry.

Qualifying Exam

1999

Req. Answer.

a) We wish to find dx_1/dw . Since

$$\pi_1 = p(f(x_1) + q_2) f(x_1) - w x_1,$$

Firm 1's profit-maximization problem can be expressed as having only one endogenous variable (x_1); the exogenous variables are q_2 and w , as stated above. The first-order condition for maximizing π_1 is

$$0 = \frac{d\pi_1}{dx_1} = \frac{d p(f(x_1) + q_2)}{dx_1} f(x_1) + p(f(x_1) + q_2) f'(x_1) - w$$

and if we let $Q = q_1 + q_2 = f(x_1) + q_2$ and then apply the Chain Rule,

$$0 = \frac{dp}{dQ} \left(\frac{d(f(x_1) + g_2)}{dx_1} f'(x_1) + p(f(x_1) + g_2) f''(x_1) - w \right)$$

where $p' = dp(Q)/dQ$ and $p = p(Q)$.

To find the comparative-statics derivative $\frac{dx_1}{dW}$, find the differential of the first-order condition $0 = \frac{d\pi_1}{dx_1}$. It is

$$O = \left(\frac{\partial}{\partial x_1} \frac{d\pi_1}{dx_1} \right) dx_1 + \left(\frac{\partial}{\partial g_2} \frac{d\pi_1}{dx_1} \right) dg_2 + \left(\frac{\partial}{\partial w} \frac{d\pi_1}{dx_1} \right) dw.$$

Notice however that one cannot set dg_2 equal to zero, because a change in w will cause firm 2 to change g_2 . Firm 2's behavior thus has to be taken into account. The differential of Firm 2's first-order condition $0 = d\pi_2/dx_2$ will, by symmetry, be

$$O = \left(\frac{\partial}{\partial x_2} \quad \frac{d\pi_2}{dx_2} \right) dx_2 + \left(\frac{\partial}{\partial q_1} \quad \frac{d\pi_2}{dq_1} \right) dq_1 + \left(\frac{\partial}{\partial w} \quad \frac{d\pi_2}{dw} \right) dw.$$

Since $g_1 = f(x_1)$ and $g_2 = f(x_2)$, we can replace the dg_1 and dg_2 terms which appear in these equations with $f'(x_1)dx_1$ and $f'(x_2)dx_2$, respectively. Making these substitutions yields

$$O = \frac{\partial^2 \pi_1}{\partial x_1^2} dx_1 + \frac{\partial^2 \pi_1}{\partial f_2 \partial x_1} f'(x_2) dx_2 + \frac{\partial^2 \pi_1}{\partial w \partial x_1} dw \quad \text{Qualifying Exam 1999}$$

$$D = \frac{\partial^2 \pi_2}{\partial x_2^2} dx_2 + \frac{\partial^2 \pi_2}{\partial f_1 \partial x_2} f'(x_1) dx_1 + \frac{\partial^2 \pi_2}{\partial w \partial x_2} dw \quad \text{Req. Answer cont..}$$

64

$$(\dot{x})_1 f \cdot (\dot{x})_2 f + (\dot{x})_2 f \cdot (\dot{x})_1 f = \frac{\dot{x}_2 e}{\dot{x}_1 e}$$

$$(\dot{x})_1 f \cdot (\dot{x})_2 f + (\dot{x})_2 f \cdot (\dot{x})_1 f = \frac{\dot{x}_2 e^2 e}{\dot{x}_1 e^2} - \frac{\dot{x}_1 e^2 e}{\dot{x}_2 e^2}$$

so

$$\text{we find before that } M - (\dot{x})_1 f \cdot (\dot{x})_2 f + (\dot{x})_2 f \cdot (\dot{x})_1 f = \frac{\dot{x}_2 p}{\dot{x}_1 p}$$

$$\frac{\frac{\partial^2 f}{\partial x_1^2} \frac{\partial^2 f}{\partial x_2^2} - \frac{\partial^2 f}{\partial x_1 \partial x_2} \frac{\partial^2 f}{\partial x_2 \partial x_1}}{\frac{\partial^2 f}{\partial x_1^2} \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_1 \partial x_2} \frac{\partial^2 f}{\partial x_2 \partial x_1}} = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix}^{-1} \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} = \frac{MP}{\dot{x}_1 p}$$

Req. Answer Cont.

1999

Qualifying Exam

so using Gramm's Rule,

$$\begin{bmatrix} MP / \dot{x}_1 p \\ MP / \dot{x}_2 p \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} \\ \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_2^2} & - \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_2^2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} \end{bmatrix}$$

$$\frac{MP}{\dot{x}_2 p} = \frac{\frac{\partial^2 f}{\partial x_2^2}}{\frac{\partial^2 f}{\partial x_1^2}} + \frac{MP}{\dot{x}_1 p} \frac{\frac{\partial^2 f}{\partial x_1^2}}{\frac{\partial^2 f}{\partial x_2^2}}$$

$$\frac{MP}{\dot{x}_2 p} + \frac{\frac{\partial^2 f}{\partial x_1^2}}{\frac{\partial^2 f}{\partial x_1^2}} = \frac{\frac{\partial^2 f}{\partial x_1^2}}{\frac{\partial^2 f}{\partial x_2^2}}$$

$$\frac{\partial^2 \pi_2}{\partial x_2^2} = p'' f'(x_2) f'(x_2) f(x_2) + p' f''(x_2) f(x_2) + p' f'(x_2) f'(x_2) \\ + p' f'(x_2) f'(x_2) + p f''(x_2)$$

$$\frac{\partial^2 \pi_2}{\partial w \partial x_2} = -1, \quad \frac{\partial^2 \pi_2}{\partial g_1 \partial x_2} = p'' f'(x_2) f(x_2) + p' f'(x_2).$$

From second-order conditions for a maximum, $\frac{\partial^2 \pi_1}{\partial x_1^2} \leq 0$ and $\frac{\partial^2 \pi_2}{\partial x_2^2} \leq 0$.

By symmetry, $x_1^* = x_2^*$. So $f^{(n)}(x_1^*) = f^{(n)}(x_2^*)$ and we can drop subscripts:

$$\frac{dx_1}{dw} = \frac{-(-1) \frac{\partial^2 \pi_2}{\partial x_2^2} + (-1)(p'' f' f + p' f') f'}{\frac{\partial^2 \pi_1}{\partial x_1^2} \frac{\partial^2 \pi_2}{\partial x_2^2} - (p'' f' f + p' f')(p'' f' f + p' f') f' f'} \quad \begin{matrix} \text{Qualifying Exam} \\ 1999 \end{matrix}$$

Reg. Answer Cont.

$$= \frac{\frac{\partial^2 \pi_2}{\partial x_2^2} - (p'' f' f + p' f') f'}{\frac{\partial^2 \pi_1}{\partial x_1^2} \frac{\partial^2 \pi_2}{\partial x_2^2} - (p'' f' f + p' f')^2 (f')^2} \quad \begin{matrix} \textcircled{\text{S}} - \textcircled{\text{I}} \textcircled{\text{G}} \\ \textcircled{\text{G}} - \textcircled{\text{I}} \end{matrix}$$

$$= \frac{p'' f' f' f + p' f'' f + 2 p' f' f' + p f'' - p'' f' f' f - p' f' f'}{(p'' f' f' f + p' f'' f + 2 p' f' f' + p f'')^2 - (p'' f' f + p' f')^2 (f')^2}$$

which does not show much promise in terms of finding a definite sign. Since these firms are cost minimizers, one can differentiate Shephard's Lemma to get $\frac{\partial}{\partial w} \frac{\partial C_i}{\partial w} = \frac{\partial}{\partial w} x_i$; this is ≤ 0 by concavity of the cost function.

b) From before, $\pi_1 = p(f(x_1) + g_2) f(x_1) - w x_1$, so calculating differentials,

$$d\pi_1 = \underbrace{\frac{\partial \pi_1}{\partial x_1} dx_1}_{} + \underbrace{\frac{\partial \pi_1}{\partial g_2} dg_2}_{} + \underbrace{\frac{\partial \pi_1}{\partial w} dw}_{} \\ = 0 \text{ at the optimum}$$

therefore

$$\begin{aligned}\frac{d\pi_1}{dw} &= \underbrace{\frac{\partial\pi_1}{\partial f_2}}_{p'f} \frac{df}{dx_2} \frac{dx_2}{dw} + \underbrace{\frac{\partial\pi_1}{\partial w}}_{-x_1} \\ &= \underbrace{\frac{\partial\pi_1}{\partial f_2}}_{p'f} \underbrace{\frac{df}{dx_2}}_{f'} \underbrace{\frac{dx_2}{dw}}_{?} + \underbrace{\frac{\partial\pi_1}{\partial w}}_{-x_1}\end{aligned}$$

$$= p' f' f \frac{dx_2}{dw} - x_1$$

$p' < 0$ because demand curves slope downwards

$f' > 0$ for positive marginal product

$x > 0$ sign convention for production functions

$\frac{dx_2}{dw} \leq 0$ from last sentence of part (a)

This sign is indeterminate.

Alternative method for (b): if

$\pi_1(x_1) = p(f(x_1) + g_2) f(x_1) - w x_1$, then let the profit function be

$\pi_1^*(g_2, w) = \max_{x_1} \pi_1(x_1)$. For this maximization problem,

$$\mathcal{L} = p [f(x_1) + g_2] f(x_1) - w x_1$$

so by the Envelope Theorem,

$$\begin{aligned}\frac{\partial \pi_1^*}{\partial w} &= \frac{\partial \mathcal{L}^*}{\partial w}, \text{ which equals } p' \frac{\partial g_2}{\partial w} f - x_1 \\ &= p' \frac{df}{dx_2} \frac{dx_2}{dw} f - x_1 \\ &= p' f' f \frac{dx_2}{dw} - x_1\end{aligned}$$

as before.

Qualifying Exam

1999

Req. Answer Cont. .

Final Exam
1996
Question 3

(4)

3. The demand curve for a product produced by a symmetric duopoly is $p(Y) = 14 - Y$ where p is price and Y is aggregate output. The output of firm 1 is denoted y_1 and its total costs are $2y_1$. The output of firm 2 is denoted y_2 and its total costs are $2y_2$. Assume the firms in this duopoly play a quantity-setting game, not a price-setting game. Assume the game is played only once and that it is a simultaneous-move game.

Fill in the following table describing the game. It is up to you to figure out what "cooperate" and "defect" means in this situation, and to calculate what the appropriate payoffs are. Hint: half of the eight numbers in the table should be integers.

		Firm 2	
		cooperate	defect
Firm 1	cooperate		
	defect		

Final Exam

1996

Answer 3

③. "Defect, Defect" means the Cournot equilibrium. This is the following :

$$\begin{aligned}\pi_1 &= p(y) y_1 - 2y_1 = p(y_1 + y_2) y_1 - 2y_1 = (14 - y_1 - y_2) y_1 - 2y_1 \\ &= 14y_1 - y_1^2 - y_2 y_1 - 2y_1 = 12y_1 - y_1^2 - y_2 y_1.\end{aligned}$$

Maximize this

taking y_2 as given. F.O.C. : $0 = \frac{d\pi_1}{dy_1} = 12 - 2y_1 - y_2 \Rightarrow$
 $y_1 = 6 - \frac{1}{2}y_2$.

By symmetry, $y_2 = 6 - \frac{1}{2}y_1$. These are two equations in the two unknowns y_1, y_2 . One solves as follows:

$$y_1 = 6 - \frac{1}{2}(6 - \frac{1}{2}y_1) = 6 - 3 + \frac{1}{4}y_1 = 3 + \frac{1}{4}y_1$$

$$\frac{3}{4}y_1 = 3 \Rightarrow y_1^* = 4 \text{ and } y_2^* = 6 - \frac{1}{2}(4) = 4.$$

Hence $\pi_1^* = 12(4) - 4^2 - 4 \cdot 4 = 48 - 16 - 16 = \underline{\underline{16}}$. By symmetry, $\pi_2^* = \underline{\underline{16}}$ too.

- "Cooperate, Cooperate" means the "collusive," or "cartel," equilibrium.

$$\pi = p(y) y - 2y_1 - 2y_2 = (14 - y_1 - y_2)(y_1 + y_2) - 2y_1 - 2y_2$$

$0 = \frac{\partial \pi}{\partial y_1}$ and $0 = \frac{\partial \pi}{\partial y_2}$ so both firms 1 and 2 try to maximize total profits π . These lead to

$$0 = \frac{\partial \pi}{\partial y_1} = -(y_1 + y_2) + (14 - y_1 - y_2) - 2$$

$$0 = \frac{\partial \pi}{\partial y_2} = -(y_1 + y_2) + (14 - y_1 - y_2) - 2$$

linearly dependent so there is 1 degree of freedom. To simplify, substitute y for $y_1 + y_2$:

Final Exam

1996 $0 = -y + (14 - y) - 2 = 14 - 2y - 2 = 12 - 2y \Rightarrow$

Answer 3 cont... $0 = 6 - y \Rightarrow y^* = 6$. Since the duopoly is symmetric, we can assume $y_1^* = y_2^* = 3$ (although there are other possibilities). Then

$$P(y) = 14 - y = 14 - 6 = 8$$

$$\pi_1^* = P(y) y_1 - 2y_1 = 8 \cdot 3 - 2 \cdot 3 = \underline{18} = \pi_2^* \text{ by symmetry.}$$

• "Cooperate, Defect" could be taken to mean that firm 1 plays $y_1 = 3$

as in the "cooperate, cooperate" equilibrium, and that firm 2 plays

$y_2 = 4$ as in the "defect, defect" equilibrium. It is probably better, though,

to assume that y_1 stays at 3 and then calculate the optimal level of y_2

when firm 2 decides to defect.

From the "defect, defect" calculation, we have $y_2 = 6 - \frac{1}{2}y_1$, when firm 2 takes firm 1's output as fixed. By assumption, $y_1 = 3$. Then $y_2 = 6 - \frac{1}{2}(3) = 6 - \frac{3}{2} = 4\frac{1}{2}$. Next, $y = y_1 + y_2 = 3 + 4\frac{1}{2} = 7\frac{1}{2}$ and $P = 14 - y = 6\frac{1}{2}$. So

$$\pi_1^* = 6\frac{1}{2} \cdot 3 - 2 \cdot 3 = (6\frac{1}{2} - 2) 3 = 4\frac{1}{2} \cdot 3 = \underline{13\frac{1}{2}} \text{ and}$$

$$\pi_2^* = 6\frac{1}{2} \cdot 4\frac{1}{2} - 2 \cdot 4\frac{1}{2} = (6\frac{1}{2} - 2) 4\frac{1}{2} = 4\frac{1}{2} \cdot 4\frac{1}{2} = \frac{9}{2} \cdot \frac{9}{2} = \frac{81}{4} = \underline{20\frac{1}{4}}$$

To conclude:

		cooperate	defect
cooperate	cooperate	18, 18	13 $\frac{1}{2}$, 20 $\frac{1}{4}$
	defect	20 $\frac{1}{4}$, 13 $\frac{1}{2}$	16, 16

Final Exam

1996

Answer 3 cont...

1 point
each

Final Exam
1998

Question 3

(4)

3. Consider the following one-round (not repeated) game.

		Column	
		left	right
Row	top	-3, -3	6, 1
	bottom	1, 6	4, 4

- (a) Find the mixed strategy Nash Equilibrium.
(b) Find the expected payoffs.

(3)

	L	R
T	-3, -3	6, 1
B	1, 6	4, 4

(This is the game of "Chicken.")

- a). Row's problem is the following (where p_T and p_B are the probabilities of Row playing T and B, and π_L and π_R are Row's subjective guesses of the probabilities that Column will play L and R) :

$$\max_{p_T, p_B} \quad p_T (-3\pi_L + 6\pi_R) + p_B (1\pi_L + 4\pi_R) \quad \text{s.t. } p_T + p_B = 1$$

$$\Leftrightarrow \max_{p_T} \quad p_T (-3\pi_L + 6\pi_R) + (1-p_T) (\pi_L + 4\pi_R)$$

$$\Rightarrow O = \frac{\partial \text{Row's payoff}}{\partial p_T} = -3\pi_L + 6\pi_R - \pi_L - 4\pi_R = -4\pi_L + 2\pi_R$$

$$\Rightarrow 4\pi_L = 2\pi_R \Leftrightarrow 2\pi_L = \pi_R \text{ and since } \pi_L + \pi_R = 1,$$

Final Exam
1998

$$2\pi_L = 1 - \pi_L$$

Answer 3

$$3\pi_L = 1 \Rightarrow \pi_L = \frac{1}{3} \text{ and } \pi_R = 1 - \frac{1}{3} = \frac{2}{3}.$$

Since the payoff matrix is symmetric, Column's problem will be analogous.

Hence the mixed strategy Nash Equilibrium will be $p_L = \frac{1}{3}$, $p_R = \frac{2}{3}$,

$$p_T = \frac{1}{3}, \quad p_B = \frac{2}{3}.$$

b) Row's expected payoff is

$$\begin{aligned}
 & p_T (-3\pi_L + 6\pi_R) + p_B (\pi_L + 4\pi_R) \\
 &= p_T \left(-3 \cdot \frac{1}{3} + 6 \cdot \frac{2}{3} \right) + p_B \left(\frac{1}{3} + 4 \cdot \frac{2}{3} \right) \\
 &= p_T (-1 + 4) + p_B \left(\frac{1}{3} + \frac{8}{3} \right) \\
 &= p_T (3) + p_B (3) \\
 &= 3(p_T + p_B) \quad \leftarrow \text{As usual, in mixed strategy equilibrium, any choice of } p_T \text{ and } p_B \text{ yields the same expected utility to Row: the "indeterminacy of mixed strategy equilibrium."}
 \end{aligned}$$

Final Exam

1998

Answer 3 cont...

Column's expected payoff will be 3 also, by symmetry.

Qualifying Exam

1998

Question 2

(4)

Question 2. Consider the following game, called "Chicken."

		Column	
		stay	swerve
Row	stay	-3, -3	6, 1
	swerve	1, 6	4, 4

Suppose this game is repeated an infinite number of times.

The "Tit-for-tat" strategy is defined as follows: swerve at $t = 1$; then, play at date t whatever your opponent played at date $t - 1$.

- What is the outcome of the Tit-for-Tat strategy?
- Is this outcome a Nash Equilibrium?
- If the outcome is a Nash Equilibrium, is the Nash Equilibrium subgame perfect?

Qualifying Exam
1998

Optional Question 2.

Answer 2

"Tit-for-Tat"

Strategy: "Tit-for-tat:" Swerve at $t = 1$; then, play at date t whatever your opponent played at date $t - 1$.

Outcome: (sw, sw)

Is this a Nash equilibrium?

- If Row doesn't deviate, Row gets: 4, 4, 4, 4, ...
- If Row deviates, Row gets:
 1. 6 (st, sw) (Row deviates, Column plays "tit-for-tat")
 2. 1 (sw, st) (Row plays "tit-for-tat," Column plays "tit-for-tat")
 3. 6 (st, sw) (Row plays "tit-for-tat," Column plays "tit-for-tat")
 4. 1 (sw, st) (Row plays "tit-for-tat," Column plays "tit-for-tat")

Etc.

- Row will not deviate when:

$$4 + \frac{4}{1+r} + \frac{4}{(1+r)^2} + \frac{4}{(1+r)^3} + \dots > 6 + \frac{1}{1+r} + \frac{6}{(1+r)^2} + \frac{1}{(1+r)^3} + \dots$$

By symmetry, it is only necessary to compare the first two terms. So Row will not deviate when:

$$4 + \frac{4}{1+r} > 6 + \frac{1}{1+r}$$

\iff

$$r < 1/2 = 0.5.$$

- Column: Since the payoff matrix is symmetric and since both players play the same strategy, the calculations for Column's deviating would be the same as the above calculations for Row's deviating.
- Conclusion: This is a Nash Equilibrium strategy whenever the players' discount rates satisfy $r < 0.5$.

Is the Nash equilibrium subgame perfect?

Suppose that at $t = 1$, Column deviated. Should Row deviate at $t = 2$?

<i>t</i>	R	C	R gets	R	C	R gets
1	C deviated	sw	st			
2	tit-for-tat	st	sw	6	R deviates	sw sw 4
3	tit-for-tat	sw	st	1	tit-for-tat	sw sw 4
4	tit-for-tat	st	sw	6	tit-for-tat	sw sw 4
			:			:

Qualifying Exam
1998

Answer 2 Cont...

- Row will not deviate when:

$$6 + \frac{1}{1+r} + \frac{6}{(1+r)^2} + \frac{1}{(1+r)^3} + \dots > 4 + \frac{4}{1+r} + \frac{4}{(1+r)^2} + \frac{4}{(1+r)^3} + \dots$$

As we just saw, this occurs if and only if

$$r > 1/2 = 0.5.$$

- Column: Since the payoff matrix is symmetric and since both players play the same strategy, the calculations for Column would be the same as the above calculations for Row.
- Conclusion: This condition for subgame perfection, $r > 0.5$, contradicts the condition needed for Nash Equilibrium, $r < 0.5$. Therefore, this equilibrium is not subgame perfect.

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Question 3

(4)

Question 3. Consider the following one-shot (that is, not repeated) game.

		Column	
		heads	tails
Row	heads	1, 0	0, β
	tails	0, 1	1, 0

Sketch a graph of Row's expected utility (in equilibrium) versus the value of β . Consider both positive and negative values of β .

Sketch a graph of Column's expected utility (in equilibrium) versus the value of β . Consider both positive and negative values of β .

Hint: mixed-strategy equilibria.

Optional Question 3.

		Column	
		H	T
Row	H	1, 0	0, β
	T	0, 1	1, 0

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1997

Answer 3

The arrows show: if Column plays H, Row prefers H to T as his play.
 " " " " T, " " T to H " " "

" Row " T, Column " H to T " " "

What will Column do if Row plays H? In this case, Column will get 0 by playing H and β by playing T. If $\beta < 0$, playing H is better for column, the arrows look like

		H	T
		1, 0	0, 0
H	H	1, 0	0, 0
	T	0, 1	1, 0

is the unique Nash Equilibrium. If $\beta = 0$, the arrows look like

		H	T
		1, 0 ↔ 0, 0	0, 0
H	H	1, 0 ↔ 0, 0	0, 0
	T	0, 1	1, 0

so again, (H, H) is a Nash Equilibrium. So if $\beta \leq 0$, Row's payoff is 1 and column's payoff is 0.

If $\beta > 0$, however, the arrows look like

		H	T
		1, 0 → 0, 0	0, 0
H	H	1, 0 → 0, 0	0, 0
	T	0, 1	1, 0

and there

is no Nash Equilibrium in pure strategies. So we look for a mixed-strategy Nash equilibrium. Suppose Row plays H with probability p and Column plays H

with probability q .

		Column	
		H prob. q	T prob. $1-q$
Row	H prob. p	1, 0	0, β
	T prob. $1-p$	0, 1	1, 0

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1997

Answer 3 cont..

One way of solving would be to set up the players' optimization problems:

Column wants to $\max_q pq(0) + p(1-q)(\beta) + (1-p)q(1) + (1-p)(1-q)(0)$

and Row wants to $\max_p pq(1) + p(1-q)(0) + (1-p)q(0) + (1-p)(1-q)(1)$

However, it's easier just to use the fact that if a player is supposed to mix between H and T, then he must get the same utility (in the expected sense) by playing H or T. (Varian discusses this.) Therefore:

Row's expected payoff from playing H = Row's expected payoff from playing T

$$q(1) + (1-q)(0) = q(0) + (1-q)(1)$$

$$q = 1 - q$$

$$2q = 1 \Rightarrow q = \frac{1}{2}$$

Similarly,

Column's expected payoff from playing H = Column's expected payoff from playing T

$$p(0) + (1-p)(1) = p(\beta) + (1-p)(0)$$

$$1-p = p\beta$$

Qualifying Exam
1997

$$-p\beta - p = -1$$

$$\text{Answer 3 cont... } p\beta + p = 1 \Rightarrow p(\beta+1) = 1 \Rightarrow p = \frac{1}{\beta+1}$$

$$\text{and } 1-p = \frac{\beta}{\beta+1}.$$

$$\text{So } g = \frac{1}{2}, 1-g = \frac{1}{2}, p = \frac{1}{\beta+1}, \text{ and } 1-p = \frac{\beta}{\beta+1}.$$

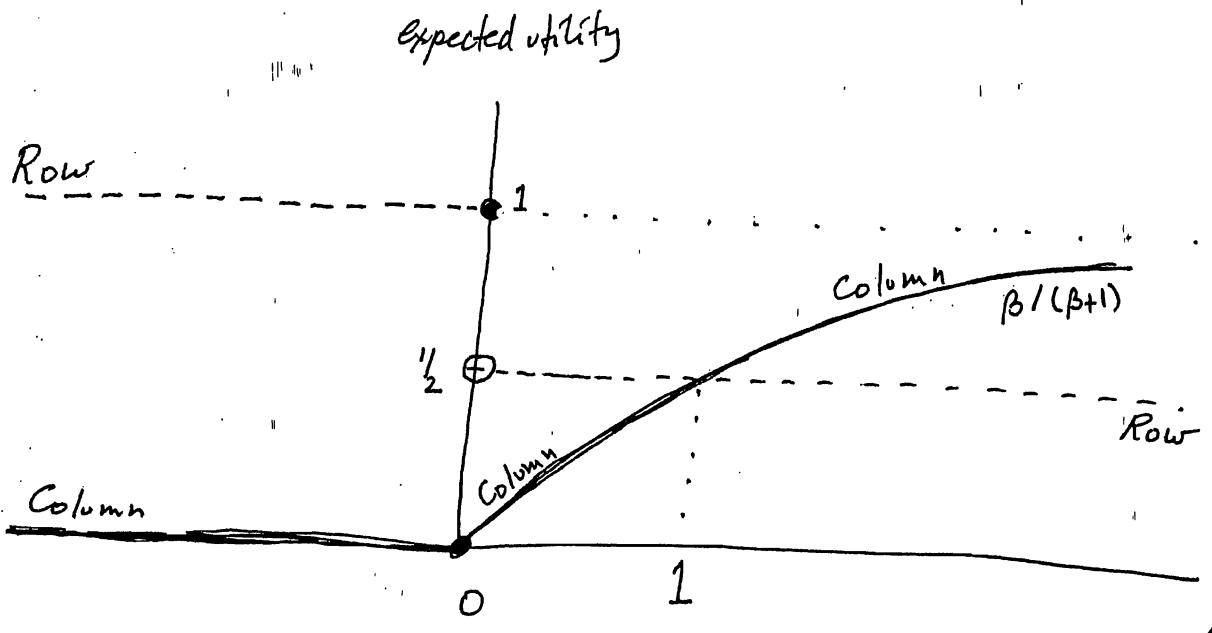
Row's expected utility is therefore

$$\begin{aligned} & \frac{1}{\beta+1} \cdot \frac{1}{2} (1) + \frac{1}{\beta+1} \cdot \frac{1}{2} (0) + \frac{\beta}{\beta+1} \cdot \frac{1}{2} (0) + \frac{\beta}{\beta+1} \cdot \frac{1}{2} (1) \\ &= \frac{1}{\beta+1} \cdot \frac{1}{2} + \frac{\beta}{\beta+1} \cdot \frac{1}{2} = \frac{1}{2} \left(\frac{1}{\beta+1} + \frac{\beta}{\beta+1} \right) = \frac{1}{2}. \end{aligned}$$

Column's expected utility is

$$\begin{aligned} & \frac{1}{\beta+1} \cdot \frac{1}{2} (0) + \frac{1}{\beta+1} \cdot \frac{1}{2} (\beta) + \frac{\beta}{\beta+1} \cdot \frac{1}{2} (1) + \frac{\beta}{\beta+1} \cdot \frac{1}{2} (0) \\ &= \frac{\beta}{\beta+1} \cdot \frac{1}{2} + \frac{\beta}{\beta+1} \cdot \frac{1}{2} = \frac{\beta}{\beta+1} = \frac{1}{1+\frac{1}{\beta}}. \text{ At } \beta=0, \end{aligned}$$

thus is 0; as $\beta \rightarrow \infty$, it approaches 1, but it is always < 1 .
 ↓ (use the $\frac{\beta}{\beta+1}$ form)
 over →



Qualifying Exam

1997

Answer 3 cont...

optional : when $\beta = 1$
 the game is symmetric so
 the payoffs have to be the same
 (or: $\frac{\beta}{\beta+1} = \frac{1}{2}$ if $\beta = 1$)

For $\beta \leq 0$, answers on p. 8.

For $\beta > 0$, answers on p. 10.

(It's interesting that Row's expected utility is discontinuous at $\beta = 0$.)

Final Exam
1996

Question 2

(4)

2. Suppose two players are playing the following game:

		Column	
		Left	Right
Row	Top	5, 1	4, 0
	Middle	6, 0	3, 1
	Bottom	6, 4	4, 4

- (a) Identify all the pure-strategy Nash equilibria in the game. Extensive explanations are not necessary; if you are confident about your answer, it is enough to put arrows leading out of each table entry which is not a Nash equilibrium, indicating which player will deviate and how. If you are not very confident about your answer, it might be a good idea to explain some more.
- (b) Show that in this game, the equilibria which remain after "iterative deletion of weakly dominated strategies" depend on the order of deletion. Hint: eliminate only one strategy per player in each iteration.

Final Exam

1996

Answer 2

(2) a)

4pts
 BL, BR: 1 pt
 others: 1 pt
 explanations: 2 pts
 Row
 ("R")

		Column ("C")	
		L	R
		5, 1	4, 0
T		↓	↑
M		6, 0	3, 1
B		↓	↑
		(6, 4)	(4, 4)

- Given that R plays T, C should play L, not R since $1 > 0$
- Given that R plays M, C should play R, not L, since $0 < 1$
- Given that R plays B, C gets the same payoff from L or R since $4 = 4$.

Given that C plays L, R should play M or B but not T since $5 < 6 = 6$.
 " " " " R, Row " " T " B " " M " " $4 > 3 < 4$.

Final Exam

1996

The only Nash equilibria therefore are (B, L) and (B, R) . Answer 2 cont.

b) For Row, T (namely 5 or 4) is weakly dominated by B (namely 6 or 4).

4pts

" " M (" 6 " 3) " " " " " " " "

Neither of Column's strategies $\left(\frac{1}{4} \text{ or } \frac{6}{4}\right)$ weakly dominates the other.

First let's eliminate Top. Then the game form is

	L	R
M	6, 0	3, 1
B	6, 4	4, 4

Right weakly dominates Left. Eliminating Left, B strictly dominates M for Row since $3 < 4$. So all that remains is (B, R) .

On the other hand, let's start again, this time by eliminating M. Then the game

form is

	L	R
T	5, 1	4, 0
B	6, 4	4, 4

so Left weakly dominates Right. Elim-

mating Right, B strictly dominates T for Row since $1 < 4$. So all that remains is (B, L) .

Therefore the order in which the weakly dominated strategies were removed mattered: one way gave (B, R) , the other way gave (B, L) .

2. The payoff matrix for a game called "Chicken" looks like this.

		Column	
		stay	swerve
Row	stay	-3, -3	2, 0
	swerve	0, 2	1, 1

(a) Find the mixed strategy equilibrium.

(b) Given that Row plays his equilibrium mixed strategy, find the payoff to Column if Column plays:

- i. stay;
 - ii. swerve;
 - iii. Column's equilibrium mixed strategy.
- (c) Consider the following game:

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1994

Question 2

(4)

		Column	
		left	right
Row	up	x, x	2, 0
	down	0, 2	1, 1

If $x < 0$ then this game is "Chicken" and not "Prisoner's Dilemma"; why? For what values of x would this be a Prisoner's Dilemma game? Hint: the following game is a Prisoner's Dilemma game; does it have dominant strategies?

		Column	
		cooperate	defect
Row	cooperate	3, 3	0, 4
	defect	4, 0	1, 1

Qualifying Exam 1994

Answer 2

		Column	
		p	1-p
		-3, -3	2, 0
Row	q	-3, -3	2, 0
	1-q	0, 2	1, 1

Column:

$$\max_p -3(pq) + 0((1-p)q) \\ + 2(p(1-q)) + 1((1-p)(1-q))$$

$$\Rightarrow F.O.C. D = -3q + 0 \\ + 2(1-q) - (1-q) =$$

first order condition

$$0 = -3q + 2 - 2q - 1 + q = 1 - 4q \Rightarrow q = \frac{1}{4} = 25\%$$

Row: $\max_q -3(pq) + 2(1-p)q$

Qualifying Exam 1994 $+ 0(p(1-q)) + 1((1-p)(1-q)) \Rightarrow \text{F.O.C. } 0 = -3p + 2(1-p)$

Answer 2 cont..

$$+ 0 - (1-p)$$

$$\Rightarrow p = \frac{1}{4} \text{ after some algebra.}$$

In the mixed-strategy equilibrium, each player plays 'stay' with probability $\frac{1}{4}$.

(This problem is identical to a homework problem in Varian's textbook.)

b) See end of question, on p. 7.

c)

		Column
		L R
Row	Up	2, x 2, 0
	Down	0, 2 1, 1

One way to answer is to observe that 'Chicken' has two Nash Equilibria (the off-diagonal elements) while 'Prisoner's Dilemma' only has one Nash Equilibrium.

Here's another way to answer: In 'Chicken,' suppose you are Row.

If Column plays 'stay,' you should swerve
but if Column plays 'swerve,' you should stay.

So in Chicken, Row has no dominant

strategy; by symmetry, Column has no dominant strategy in 'Chicken' either.

In 'Prisoner's Dilemma,' regardless of what the other player does, it's better to 'defect.' So each player has a dominant strategy.

If $x < 0$, Row should play down if Column plays left, but

" " " " up " " " right. So Row has

no dominant strategy and the game isn't Prisoner's Dilemma.

If $x > 0$, Row should play 'up' regardless of what Column does.

Qualifying Exam

1994

Answer 2 cont...

7

Also, Column should play 'left' regardless of what Row does. So each has a dominant strategy and the game isn't 'Chicken.'

Finally, if $x > 0$ then the outcome is ' x, x '. If $x \geq 1$ then this is Pareto Optimal so there's no 'dilemma.' So Prisoner's Dilemma occurs

when $0 < x < 1$.

$$\text{PART B: i) } -3(q) + 2(1-q) = -3\left(\frac{1}{4}\right) + 2\left(\frac{3}{4}\right) = -\frac{3}{4} + \frac{6}{4} = \frac{3}{4}$$

$$\text{ii) } 0(q) + 1(1-q) = 0 + 1\left(\frac{3}{4}\right) = \frac{3}{4}$$

$$\text{iii) } -3(pq) + 2(p(1-q)) + 0((1-p)q) + 1((1-p)(1-q))$$

$$= -3 \cdot \frac{1}{4} \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} \cdot \frac{3}{4} + 0 + \frac{3}{4} \cdot \frac{3}{4}$$

$$= \frac{-3}{16} + \frac{6}{16} + \frac{9}{16} = \frac{12}{16} = \frac{3}{4}$$

That the answers to (i), (ii), and (iii) are identical is not a coincidence, as Varian discusses.

(4)

2. In each of the three parts of this question you are given one symmetric game in strategic form. Suppose each game is played sequentially instead of simultaneously. Find the extensive form of each game and determine whether it is better to be the leader or the follower in each game.

(a)

		Column	
		Left	Right
Row	Top	2, 1	0, 0
	Bottom	0, 0	1, 2

(b)

		Column	
		Left	Right
Row	Top	1, -1	-1, 1
	Bottom	-1, 1	1, -1

(c)

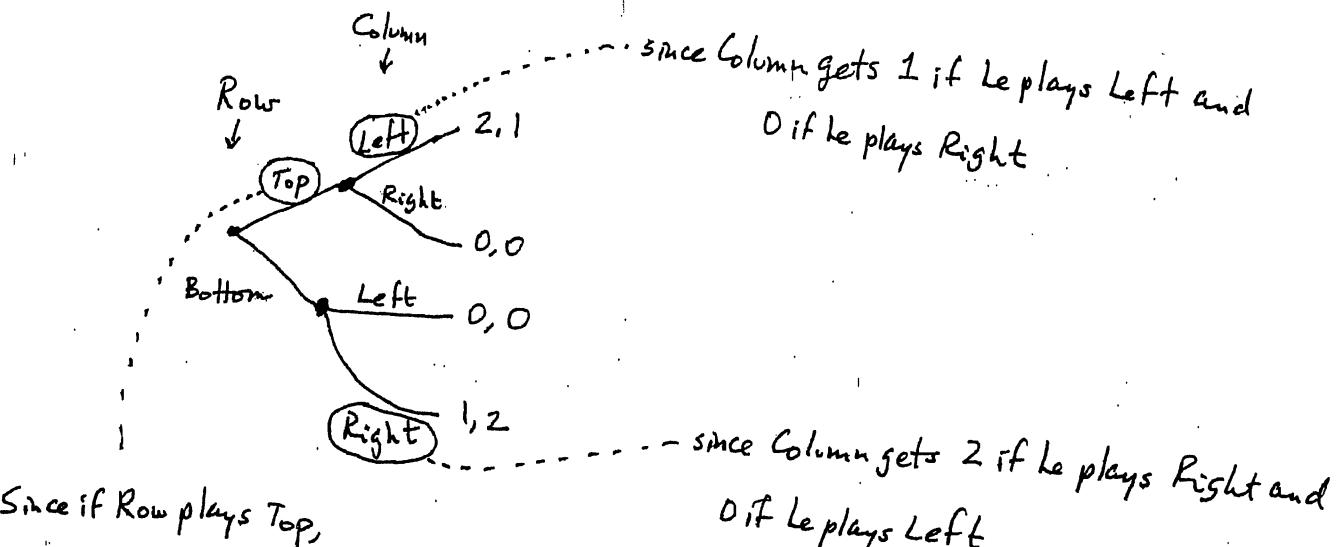
		Column	
		Left	Right
Row	Top	3, 3	0, 4
	Bottom	4, 0	1, 1

Final Exam

1995

(2) Game 1 (Battle of the Sexes)

Answer 2

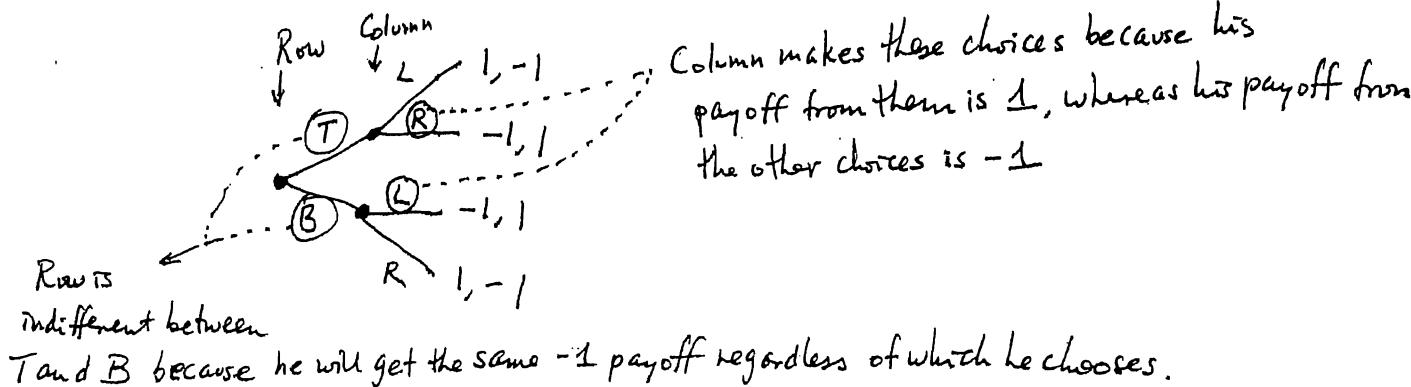


Since if Row plays Top, Column will play Left and Row will get 2, whereas if Row plays Bottom, Column will play Right and Row will only get 1, not 2.

So the outcome is (Top, Left) and Row (the leader) gets 2 while Column (the follower) gets 1: it's better to be the leader.

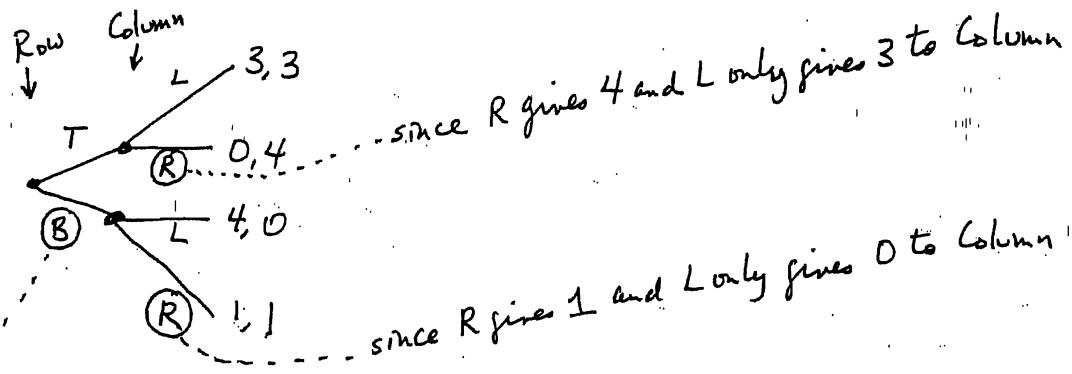
Since all three games in this question are symmetric, assuming that Row is the leader can be done without loss of generality. You can check that in Game 1, if Column were the leader it would still be better to be the leader.

Game 2 (Matching Pennies)



So the leader (Row) gets -1 and the follower (Column) gets +1 : it's better to be the follower.

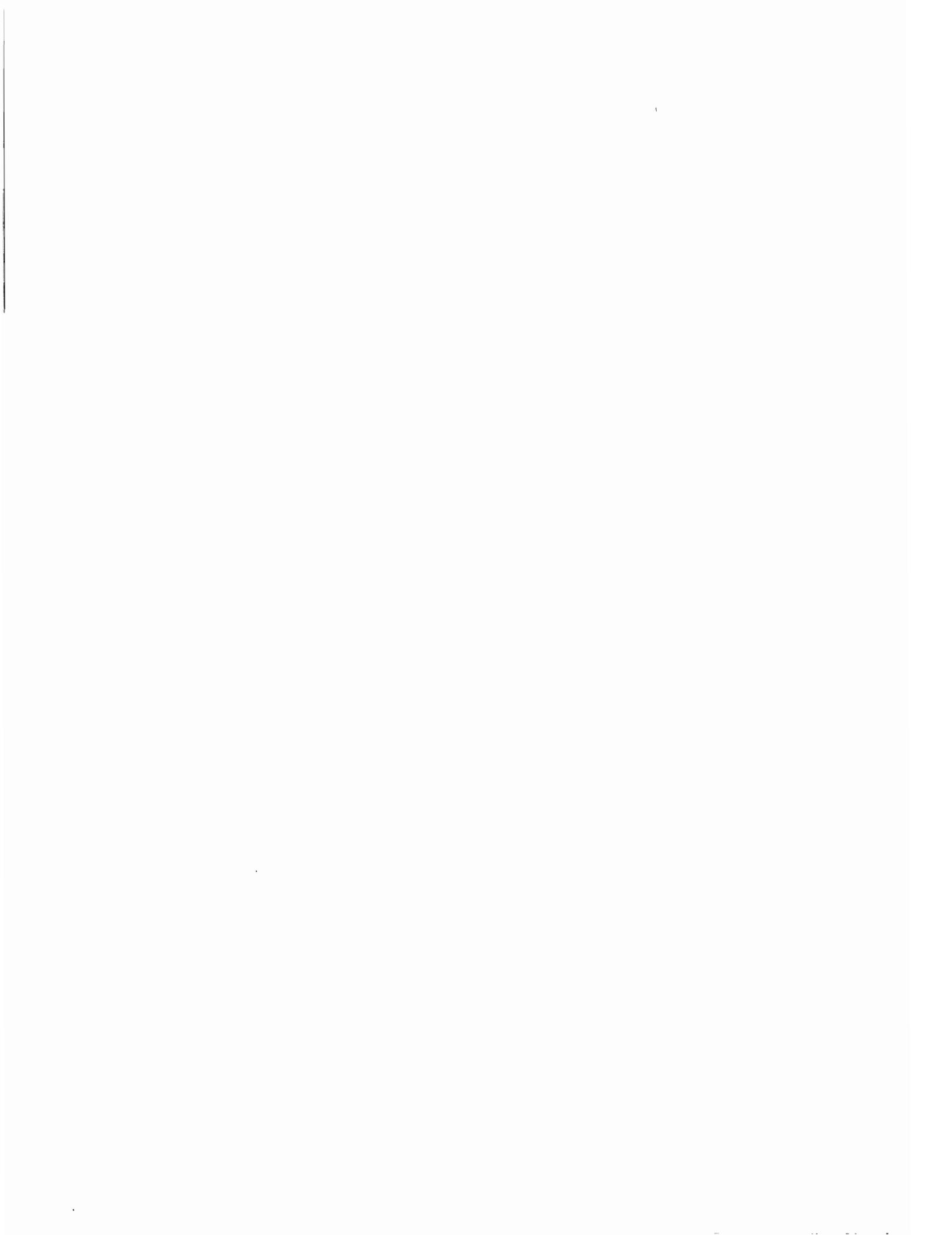
Game 3 (Prisoner's Dilemma)



If Row plays B, Column will play R and Row will end up with 1; whereas
 " " " T, " " " R " " " " " 0.

In this game, both players get 1, so there is neither an advantage nor a disadvantage to being the leader.

Player	Part
3	(a)
4	(b)
3	(c)



6. Consider the following game:

		Column	
		left	right
Row	up	8, 8	1, 2
	down	2, 1	0, 0

- (a) What is the Nash Equilibrium of the one-round game?
- (b) Consider the following strategy for the infinitely-repeated game:
if the number of past rounds when (up, left) occurred is even,
play (up, right), else play (down, left). "0" counts as an even
number.
What is the outcome? Is it a Nash Equilibrium? Is it a subgame
perfect Nash Equilibrium?

(6)

a)

	L	R
U	8, 8	1, 2
D	2, 1	0, 0

(U, L) is the NE (in fact, it is a dominant strategy equilibrium)

- b) At $t=1$, (U, L) has occurred zero times, and since zero counts as even, (U, R) are the actions. At $t=2$, (U, R) are again the actions, for the same reason. So (U, R) are the actions $\forall t \geq 1$.

- Is this Nash?

If, at $t=1$, Row deviated, then : (D, R)	<u>Row gets</u> 0	<u>If no deviation, Row gets</u> 1
$t=2 : (U, R)$	1	1
$t=3 : (U, R)$	1	1
	:	:

So Row would not want to deviate.

<u>t</u>	<u>If Column deviates at $t=1$</u>	<u>If no deviation</u>
1	(U, L) Column gets 8	2
2	(D, L)	1
3	(D, L)	2

Column does not want to deviate iff

$$8 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots < 2 + \frac{2}{1+r} + \frac{2}{(1+r)^2} + \dots$$

$$6 < \frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots$$

Final Exam

2000

Answer 6 Cont..

$$S = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots$$

$$\frac{1}{1+r} S = \frac{1}{(1+r)^2} + \dots$$

$$S - \frac{1}{1+r} S = \frac{1}{1+r}$$

$$\left(\frac{1}{1+r} - \frac{1}{1+r}\right) S = \frac{1}{1+r}$$

$$\frac{r}{1+r} S = \frac{1}{1+r} \Rightarrow S = \frac{1}{r} \text{ so Column does not want to deviate if}$$

$$6 < \frac{1}{r} \Leftrightarrow r < \frac{1}{6}.$$

Since Row never wants to deviate, this is the condition for Nash Equilibrium.

• Subgame Perfection.

If Column deviated at $t=1$, would Row want to deviate at $t=2$?

t		
1	(U, L)	
2	(D, L)	R gets 2
		<u>If R deviates at $t=2$</u>
2	(U, L)	R gets 8
3	(D, L)	R gets 2
		(U, R) R gets 1
	⋮	⋮

Row does not want to deviate iff

$$2 + \frac{2}{1+r} + \frac{2}{(1+r)^2} + \dots > 8 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots$$

$$\Leftrightarrow r < \frac{1}{6} \text{ as shown above.}$$

If Row deviated at $t=1$, would Column want to deviate at $t=2$?

<u>t</u>		<u>If C deviates at $t=2$</u>	
1	(D, R)		
2	(U, R)	C gets 2	(U, L) C gets 8
3	(U, R)	2	(D, L)
		:	:

So Column does not want to deviate if $r < \frac{1}{6}$, as shown above.

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1999

Question 2

(4)

2. Suppose two players play the following game infinitely many times.

		Column	
		cooperate	defect
Row	cooperate	3, 3	0, 4
	defect	4, 0	1, 1

Consider the following strategy: "Play Cooperate on the current move unless:

(a) the other player defected on the last move, or

(b) you defected on the last move and the other player defected on the next-to-last move,

in which case play defect."

(a) What outcome will occur?

(b) Is this outcome a Nash equilibrium?

(c) Is this outcome a subgame-perfect Nash equilibrium?

Optional Question 2.

		COLUMN	
		coop.	defect
Row	coop.	3, 3	0, 4
	defect	4, 0	1, 1

optional: this game is a Prisoners' Dilemma with what Varian calls the "punishment strategy".

a) (Cooperate, Cooperate)

Qualifying Exam
1999

b) If Row does not deviate, Row gets 3, 3, 3, 3, ...

Answer 2

If Row deviates, Row gets:

- 4 (Row deviates, Column does not deviate, so Row plays defect and Column plays cooperate)

- 0 (Neither player deviates, Row plays cooperate, Column plays defect)

- 1 (Neither player deviates, both play defect)

- 1
etc.

Remark: Because Varian does not apply the one-stage deviation principle, on p. 271 of his Third Edition he gets 4, 1, 1, 1, ... instead of 4, 0, 1, 1, ...

Row will not deviate when

$$3 + \frac{3}{1+r} + \sum_{i=2}^{\infty} \frac{3}{(1+r)^i} > 4 + \frac{0}{1+r} + \sum_{i=2}^{\infty} \frac{1}{(1+r)^i}$$

$$\sum_{i=2}^{\infty} \frac{2}{(1+r)^i} > 1 + \frac{-3}{1+r}. \quad (1)$$

$$\text{If } S = \sum_{i=2}^{\infty} \frac{2}{(1+r)^i} \text{ then } S = \frac{2}{(1+r)^2} + \frac{2}{(1+r)^3} + \dots$$

$$\frac{1}{1+r} S = \frac{2}{(1+r)^3} + \dots$$

$$S - \frac{1}{1+r} S = \frac{2}{(1+r)^2} \Rightarrow$$

$$\frac{1+r}{1+r} S - \frac{1}{1+r} S = \frac{2}{(1+r)^2}$$

$$\frac{r}{1+r} S = \frac{2}{(1+r)^2}$$

$$S = \frac{2}{(1+r)^2} \cdot \frac{1+r}{r} = \frac{2}{r(1+r)}$$

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1999

Answer 2 cont...

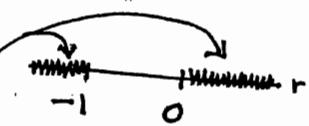
So (1) \Leftrightarrow

$$\frac{2}{r(1+r)} > 1 - \frac{3}{1+r} \quad \text{and if } r(1+r) > 0 \text{ then}$$

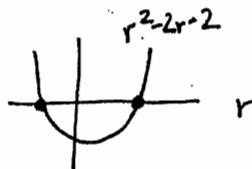
$$2 > r(1+r) - 3r$$

$$2 > r + r^2 - 3r$$

$$0 > r^2 - 2r - 2.$$

this is true on  but usually in economics we ignore $r < 0$

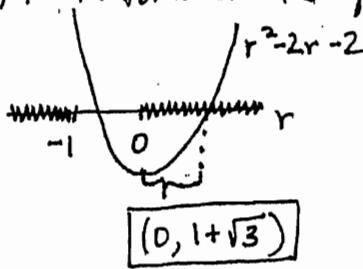
$$\text{If } 0 = r^2 - 2r - 2 \text{ then } r = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$



I'll accept an answer of $0 \leq r \leq 1 \pm \sqrt{3}$ for the equilibrium to be Nash, just ignoring negative interest rates. (The game & strategy are symmetric, so r would be the same to prevent column from deviating.) (Note that Varian gets $r < 2$, which is not right.)

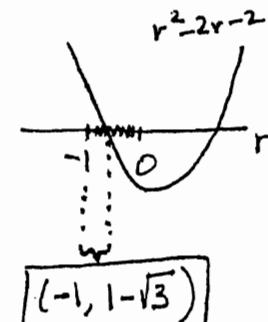
OPTIONAL: Technically, the r 's for which the equilibrium is Nash are

$$0 > r^2 - 2r - 2 \text{ on }$$



$$(0, 1 + \sqrt{3})$$

$$\text{and } 0 < r^2 - 2r - 2 \text{ on }$$



$$(-1, 1 - \sqrt{3})$$

Note that $1 - \sqrt{3} \approx -0.73$.

Final Exam

1997

Question 5

(4)

5. Suppose two players are involved in infinite repetitions of the following game, called "Chicken":

		Column	
		stay	swerve
Row	stay	-3, -3	6, 1
	swerve	1, 6	4, 4

Suppose both players play the "Unrelenting" strategy, which is: for $t = 1$, swerve; for $t > 1$, play stay if the other player has ever played stay at any time in the past; otherwise, play swerve.

- (a) What is the outcome of the game?
- (b) Is this an equilibrium outcome?
- (c) Is this a subgame-perfect equilibrium outcome?

If these answers depend on the discount rate, then describe this dependence.

You may wish to know that $(1 + \sqrt{57})/4 \approx 2.14$.

(5)

		Column	
		stay	swerve
Row	stay	-3, -3	6, 1
	swerve	1, 6	4, 4

Final Exam
1997
Answer 5

Abbreviate "stay" by "st" and "swerve" by "sw". Abbreviate "unrelenting" by "UR".

a) (sw, sw) forever. Explanation: at $t=1$, the UR strategy specifies "sw." For $t>1$, each player's opponent has played "sw", so each player plays "sw".

b) Would Row want to deviate, assuming Column plays the UR strategy? (We don't ask here whether Column actually would play the UR strategy; that's part (c).) We check one-period deviations only; that's all that's needed.

t	Row	Column	Row's payoff			Row	Column	Row's payoff
				st	sw			
1	UR	sw	sw	4	Row deviates	st	sw	6
2	UR	sw	sw	4	UR	sw	st	1
3	UR	sw	sw	4	UR	st	st	-3
4	UR	sw	sw	4	UR	st	st	-3
				:				:

So we have to compare $4 + \frac{4}{1+r} + \sum_{i=2}^{\infty} \frac{4}{(1+r)^i}$ to $6 + \frac{1}{1+r} + \sum_{i=2}^{\infty} \frac{-3}{(1+r)^i}$.

If we let

$$S = \sum_{i=2}^{\infty} \frac{x}{(1+r)^i} = \frac{x}{(1+r)^2} + \frac{x}{(1+r)^3} + \frac{x}{(1+r)^4} + \dots \text{ then}$$

$$\frac{1}{1+r} S = \frac{x}{(1+r)^3} + \frac{x}{(1+r)^4} + \dots \text{ and}$$

$$S - \frac{1}{1+r} S = \frac{x}{(1+r)^2}$$

$$(1 - \frac{1}{1+r}) S = \frac{x}{(1+r)^2}$$

$$\frac{1+r-1}{1+r} S = \frac{x}{(1+r)^2} \Rightarrow \frac{r}{1+r} S = \frac{x}{(1+r)^2} \Rightarrow S = \frac{1}{r(1+r)} x$$

So for deviating not to pay off, it must be true that

$$4 + \frac{4}{1+r} + \frac{4}{r(1+r)} > 6 + \frac{1}{1+r} - \frac{3}{r(1+r)}$$

$$0 > 2 - \frac{3}{1+r} - \frac{7}{r(1+r)}$$

$$0 > 2r(1+r) - 3r - 7 = 2r + 2r^2 - 3r - 7$$

$$0 > 2r^2 - r - 7$$

If this is equal to zero then $r = \frac{1 \pm \sqrt{1+4(14)}}{4} = \frac{1 \pm \sqrt{57}}{4}$; the positive

root (there is only one) is $\frac{1}{4} + \frac{1}{4}\sqrt{57} \approx 2.14$. We want $2r^2 - r - 7$ to be less than $2r^2 - r - 7$

zero.

So the outcome is an equilibrium if $r < \frac{1}{4} + \frac{1}{4}\sqrt{57}$.

c) If Row deviated, would Column actually play the UR strategy?

Again, we check one-period deviations.

$t=1$: Row stayed, Column swerved

t	Row	Column	Column's payoff		Row	Column	Column's payoff	
			st	6			sw	sw
2	UR	sw	st	6	Column deviates	UR	sw	4
3	UR	st	st	-3	UR	sw	st	6
4	UR	st	st	-3	UR	st	st	-3
				:				:

In order for "UR" to be better for Column, we need

$$6 - \frac{3}{1+r} - \sum_{i=2}^{\infty} \frac{3}{(1+r)^i} > 4 + \frac{6}{1+r} - \sum_{i=2}^{\infty} \frac{3}{(1+r)^i}$$

$$6 - \frac{3}{1+r} > 4 + \frac{6}{1+r}$$

$$2 - \frac{9}{1+r} > 0$$

$$2 > \frac{9}{1+r} \Rightarrow 1+r > \frac{9}{2} \Rightarrow r > \frac{7}{2}$$

So subgame perfection requires $r > 3\frac{1}{2}$ while from part (b) equilibrium requires $r < 2.14$. These contradict, so the equilibrium is not subgame-perfect.

Final Exam

1997

Answer 5 cont...

Qualifying Exam

1995

Question 3

(4)

3. Consider an industry with two firms, each having zero costs. The (inverse) demand curve facing this industry is

$$P(Y) = 100 - Y,$$

where $Y = y_1 + y_2$ is total output.

- (a) Find the payoff to each firm if the two firms are in a symmetric duopoly and they produce the level of output that maximizes their joint profits. You may wish to call this payoff π_j .

Qualifying Exam
1995

Question 3 cont...

- (b) Find the payoff to each firm if the two firms are in a symmetric duopoly and they produce the Cournot level of output. You may wish to call this payoff π_c .
- (c) Find the maximum payoff that one firm can get if the other chooses the joint-profit maximizing output. You may wish to call this payoff π_d .
- (d) Let the discount rate be r . Suppose the firms are in an infinitely repeated, symmetric duopoly. Suppose the firms adopt the punishment strategy of reverting to the Cournot game if either player defects from the joint-profit maximizing level of output. How large can r be? (If you do not have a calculator, you need not arithmetically simplify the answer, but your answer should only contain numbers, not variables.)

Hint: if the discount rate is r then the discounted present value of a constant stream of earnings of $\$x$ per period for the infinite number of periods $1, 2, 3, \dots$ is $\sum_{t=1}^{\infty} (x/(1+r)^t) = x/r$.

③ (This resembles a combination of homework problems 15.7 and 16.10 from Varian's book.)

$$P = 100 - Y$$

zero costs

a) $(\text{cartel profit } \pi) = P(Y) \cdot Y = (100 - Y)Y = 100Y - Y^2$

maximize $\pi \Rightarrow D = \frac{d\pi}{dY} = 100 - 2Y \Rightarrow Y^* = 50 \Rightarrow Y_1^* = Y_2^* = 25$ due

to symmetry. Then $P(Y) = 100 - Y = 100 - 50 = 50$ and

$$\pi_j = P \cdot Y_1^* = 50 \cdot 25 = \$1250 \text{ for each firm.}$$

b) Suppose firm 1 is producing y_1 . Then firm 2 wishes to maximize its profits, which

$$\text{are } P(Y_1 + Y_2) \cdot Y_2 = (100 - Y_1 - Y_2) \cdot Y_2 = 100Y_2 - Y_1Y_2 - Y_2^2.$$

The first-order condition is $D = \frac{\partial \pi_2}{\partial Y_2} = 100 - Y_1 - 2Y_2 \Rightarrow 2Y_2 = 100 - Y_1 \Rightarrow Y_2 = 50 - \frac{1}{2}Y_1$.

By symmetry, though, $Y_1 = 50 - \frac{1}{2}Y_2$, which equals $50 - \frac{1}{2}(50 - \frac{1}{2}Y_1)$ \Rightarrow

Qualifying Exam
1995

Answer 3

$$y_1 = 50 - 25 + \frac{1}{4} y_1 \Rightarrow \frac{3}{4} y_1 = 25 \Rightarrow y_1 = \frac{100}{3} \text{ and, again by}$$

$$\text{symmetry, } y_2 = \frac{100}{3}. \text{ Then } P(Y) = 100 - (y_1 + y_2) = 100 - \frac{100}{3} - \frac{100}{3} =$$

$$\frac{300}{3} - \frac{200}{3} = \frac{100}{3} \text{ and } \pi_1 = \pi_2 = P(Y) \cdot y_1 = \frac{100}{3} \cdot \frac{100}{3} = \frac{10,000}{9}. \text{ This}$$

$$\text{equals } \$1111\frac{1}{9} = \pi_c.$$

c) Suppose firm 1 chooses the joint-profit maximizing output, which is $y_1^* = 25$

from part (a). Firm 2's optimal response, from part (b), is $y_2 = 50 - \frac{1}{2} y_1 =$

$$50 - \frac{25}{2} = \frac{100}{2} - \frac{25}{2} = \frac{75}{2}. \text{ Then } P(Y) = 100 - 25 - \frac{75}{2} = 75 - \frac{75}{2} =$$

$$\frac{75}{2} \text{ and } \pi_d = P(Y) \cdot y_2 = \frac{75}{2} \cdot \frac{75}{2} = \frac{5625}{4} = \$1406.25.$$

Optional: in this case firm 1 gets $P(Y) \cdot y_1 = \frac{75}{2} \cdot 25 = \frac{1875}{2} = \$937.5.$

d) If you cooperate with the other firm forever, your payoffs are $(\pi_j, \pi_j, \pi_j, \dots)$.

If you defect, your opponent plays Cournot forever thereafter, so your payoffs are $(\pi_d, \pi_c, \pi_c, \pi_c, \dots)$. The present value of cooperation is $\pi_j + \frac{\pi_j}{r}$

and the present value of defecting is $\pi_d + \frac{\pi_c}{r}$ (see the problem's hint; for example, 'cooperate forever' gives a present discounted value of $\pi_j + \frac{\pi_j}{1+r} +$

$$\frac{\pi_j}{(1+r)^2} + \frac{\pi_j}{(1+r)^3} + \dots = \pi_j + \sum_{t=1}^{\infty} \frac{\pi_j}{(1+r)^t} = \pi_j + \frac{\pi_j}{r}).$$

'Cooperation' therefore

is better if $\pi_j + \frac{\pi_j}{r} > \pi_d + \frac{\pi_c}{r} \Rightarrow \pi_j - \pi_d > \frac{\pi_c - \pi_j}{r} \Rightarrow$

$$r < \frac{\pi_c - \pi_j}{\pi_j - \pi_d} = \frac{(10,000/9) - 1250}{1250 - (5625/4)} = \frac{8}{9} = 0.\bar{8} \text{ (i.e., } 0.8888\dots).$$

\downarrow since $\pi_j < \pi_d$, $\pi_j - \pi_d < 0$, and dividing by a negative number changes the direction of an inequality

Qualifying Exam
1995

Answer 3 Cont...