

Section 5:

Uncertainty

(3)

621 Section (must answer one)

1. A consumer's utility function is $u(w) = w^\alpha$ where w is wealth. His initial wealth consists of \$4 plus a lottery ticket which pays off \$12 with probability $1/2$ and \$0 with probability $1/2$. This consumer's preferences over lotteries obey the expected utility hypothesis.
Suppose p denotes the lowest price at which this consumer will willingly part with his lottery ticket.

- (a) Show that

$$\frac{dp}{d\alpha} = \frac{-(4+p)^\alpha \ln(4+p) + \frac{1}{2}(\ln 16)16^\alpha + \frac{1}{2}(\ln 4)4^\alpha}{\alpha(4+p)^{\alpha-1}}.$$

Hint: although $\frac{d}{dx}x^y$ is yx^{y-1} , $\frac{d}{dx}y^x$ is $y^x \ln y$.

- (b) Does p increase or decrease as α increases slightly from $\alpha = 1$? (If you do not have a calculator or for any other reason you do not wish to numerically simplify your answer, that is fine; as long as your answer has only numbers and no variables, it is simplified enough. You should explain how you would answer the question if you had a calculator, though.)

Short Questions

$$\textcircled{1} \quad u(w) = w^\alpha$$

$$\text{initial wealth} = \$4 + (\frac{1}{2} \cdot \$12 \oplus \frac{1}{2} \cdot \$0)$$

If he doesn't sell the ticket: $w = (\frac{1}{2} \cdot \$16 \oplus \frac{1}{2} \cdot \$4)$ and

$$\begin{aligned}\text{utility} &= u(\frac{1}{2} \cdot \$16 \oplus \frac{1}{2} \cdot \$4) \\ &= \frac{1}{2} u(\$16) + \frac{1}{2} u(\$4) \text{ by Expected Utility} \\ &= \frac{1}{2} \cdot 16^\alpha + \frac{1}{2} \cdot 4^\alpha.\end{aligned}$$

If he does sell the ticket: $w = \$4 + p$ and $\text{utility} = u(4+p) = (4+p)^\alpha$.

In order to be indifferent between the two options,

$$\frac{1}{2} \cdot 16^\alpha + \frac{1}{2} \cdot 4^\alpha = (4+p)^\alpha. \quad (1)$$

Take the total derivative (total differential):

$$\begin{aligned}\frac{1}{2} (\ln 16) 16^\alpha d\alpha + \frac{1}{2} (\ln 4) 4^\alpha d\alpha &= \ln(4+p) \cdot (4+p)^\alpha d\alpha \\ &\quad + \alpha (4+p)^{\alpha-1} dp\end{aligned}$$

$$\text{so } [(4+p)^\alpha \ln(4+p) + \frac{1}{2} (\ln 16) 16^\alpha + \frac{1}{2} (\ln 4) 4^\alpha] d\alpha = \alpha (4+p)^{\alpha-1} dp$$

and the formula given in the exam follows by simple algebra.

b) Set $\alpha = 1$ in equation (1) to get $\frac{1}{2} \cdot 16 + \frac{1}{2} \cdot 4 = 4 + p$

$$10 = 4 + p \Rightarrow p = 6 \text{ when } \alpha = 1.$$

Now evaluate the answer to part (a) with $p = 6$ and $\alpha = 1$:

Qualifying Exam
1994
Answer 1 cont...

$$\begin{aligned}
 \frac{dp}{d\alpha} &= \frac{-10 \ln 10 + 8 \ln 16 + 2 \ln 4}{10^0} \\
 &= -10 \ln 10 + 8 \ln 2^4 + 2 \ln 2^2 \\
 &= -10 \ln 10 + 32 \ln 2 + 4 \ln 2 = -10 \ln 10 + 36 \ln 2 \\
 &\approx 1.927 > 0. \text{ Hence when } \\
 &\alpha \text{ increases slightly, } p \text{ will go up.}
 \end{aligned}$$

Interpretation (optional!): As $\alpha \uparrow$, the consumer becomes more risk-loving.
Hence as $\alpha \uparrow$, the consumer likes the lottery ticket more, and thus has to be paid more ($p \uparrow$) in order to willingly give the lottery ticket up.

Unsolved question: using (1), can it be shown that $dp/d\alpha > 0$ for all positive values of α ??

Final Exam
2000 ③

Question 1

Answer all of the following six questions.

1. In a lottery, suppose two outcomes are equally likely to occur, and only those two outcomes can occur. Suppose the lottery is a fair game.

Let a be the amount of money an agent would have to be paid in order to make him indifferent between participating in and not participating in the lottery.

Let b be the amount of money an agent would be willing to pay to have the right to avoid participating in the lottery.

Suppose the Expected Utility Hypothesis holds.

- If the agent is risk-neutral, find a and b .
- If the agent is risk-averse, is $a > b$, $a < b$, $a = b$, or can you not tell? Be sure to show all your work.

Answers to Econ. 6710 Final, Spring 2000

Final Exam

2000

Answer 1

① $p \cdot x \oplus (1-p) \cdot y$ is a lottery

outcomes equally likely $\Rightarrow p = \frac{1}{2}$ and the lottery is $\frac{1}{2} \cdot x \oplus \frac{1}{2} \cdot y$

fair game \Rightarrow the lottery's expected value is zero, so $\frac{1}{2}x + \frac{1}{2}y = 0$

$\Rightarrow y = -x$ and the lottery
is $\frac{1}{2} \cdot x \oplus \frac{1}{2} \cdot (-x)$.

Call the lottery " l ". Let initial wealth be " w ".

" a " is characterized by $u(w+l+a) = u(w)$

↑ this would be U instead of u in the notation of
Mas-Colell/Whinston/Greens

" b " is characterized by $u(w+l) = u(w-b)$ ← ' b ' is the risk premium

Applying the expected utility hypothesis, $u(w+l+a) = u(w+a + \frac{1}{2} \cdot x \oplus \frac{1}{2} \cdot (-x)) = \frac{1}{2}u(w+a+x) + \frac{1}{2}u(w+a-x)$ and $u(w+l) = u(w + \frac{1}{2} \cdot x \oplus \frac{1}{2} \cdot (-x)) = \frac{1}{2}u(w+x) + \frac{1}{2}u(w-x)$. Therefore,

$$\frac{1}{2}u(w+a+x) + \frac{1}{2}u(w+a-x) = u(w) \quad (1)$$

$$\frac{1}{2}u(w+x) + \frac{1}{2}u(w-x) = u(w-b). \quad (2)$$

a) If the agent is risk-neutral, $u(\cdot)$ is a linear function, so (1) becomes

$$\frac{1}{2}(w+a+x) + \frac{1}{2}(w+a-x) = w \Leftrightarrow \frac{1}{2}a + \frac{1}{2}a = 0 \Leftrightarrow a = 0$$

and (2) becomes

$$\frac{1}{2}(w+x) + \frac{1}{2}(w-x) = w-b \Leftrightarrow 0 = -b \Leftrightarrow b = 0.$$

The linear function I have used here is $u(y) = y$. If instead you use an affine transformation of $u(y) = g$, for example $u(y) = My + N$, you will of course get the same answer. (1) would be

$$\frac{1}{2} [M(w+a+x) + N] + \frac{1}{2} [M(w+a-x) + N] = Mw + N$$

$$Mw + Ma + N = Mw + N$$

$$a = 0$$

↑
Since behavior under uncertainty is unaffected by affine transformations

and (2) would be

$$\frac{1}{2} [M(w+x) + N] + \frac{1}{2} [M(w-x) + N] = M(w-b) + N$$

$$Mw + N = Mw - Mb + N$$

$$0 = b.$$

b) In this case, $u(w)$ becomes nonlinear (and concave). It is impossible to tell (1) or (2) for a or b . So it is not possible to tell whether a or b is larger.

Final Exam
2000
Answer 1 cont..

$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 \Rightarrow$$

$$\begin{aligned} u(w+\alpha) &\approx u(w) + u'(w)(w+\alpha-w) + \frac{1}{2}u''(w)(w+\alpha-w)^2 \\ &= u(w) + \alpha u'(w) + \frac{\alpha^2}{2}u''(w). \end{aligned}$$

Final Exam

2000

Answer 1 cont..

Then (1) \Rightarrow

$$\frac{1}{2} [u(w) + (a+x)u'(w) + \frac{1}{2}(a+x)^2u''(w)]$$

$$+ \frac{1}{2} [u(w) + (a-x)u'(w) + \frac{1}{2}(a-x)^2u''(w)] = u(w)$$

$$u(w) + a u'(w) + \frac{1}{4} [(a+x)^2 + (a-x)^2] u''(w) = u(w)$$

$$a u'(w) + \frac{1}{4} (a^2 + 2ax + x^2 + a^2 - 2ax + x^2) u''(w) = 0$$

$$a u'(w) + \frac{1}{4} (2a^2 + 2x^2) u''(w) = 0$$

$$a u'(w) + \frac{a^2 + x^2}{2} u''(w) = 0$$

$$\frac{u''(w)}{2} a^2 + u'(w) a + \frac{x^2}{2} u''(w) = 0$$

$$u'' a^2 + 2u' a + x^2 u'' = 0$$

$$a = \frac{-2u' \pm \sqrt{(2u')^2 - 4u'' x^2 u''}}{2u''}$$

$$= \frac{-u' \pm \sqrt{(u')^2 - (u'')^2 x^2}}{u''}$$

Final Exam
2000

Answer 1 cont.

and (2) \Rightarrow

$$\frac{1}{2} \left[u(\omega) + x u'(\omega) + \frac{x^2}{2} u''(\omega) \right]$$

$$+ \frac{1}{2} \left[u(\omega) - x u'(\omega) + \frac{x^2}{2} u''(\omega) \right] = u(\omega) - b u'(\omega) + \frac{b^2}{2} u''(\omega)$$

$$u(\omega) + \frac{1}{2} x^2 u''(\omega) = u(\omega) - b u'(\omega) + \frac{b^2}{2} u''(\omega)$$

$$0 = -b u'(\omega) + \frac{b^2 - x^2}{2} u''(\omega)$$

$$0 = \frac{u''}{2} b^2 - u' b - \frac{x^2}{2} u''$$

$$0 = u'' b^2 - 2 u' b - x^2 u''$$

$$b = \frac{2u' \pm \sqrt{(-2u')^2 + 4u''x^2u''}}{2u''}$$

$$= \frac{u' \pm \sqrt{(u')^2 + (u'')^2 x^2}}{u''}$$

a is not defined as $x \rightarrow \infty$, but b is always defined

a has two positive roots and so is not unique, while b has only one positive root and so is unique

Question 2.1 Suppose a consumer has monetary wealth W . There is some probability that he will lose an amount L —for example, there is some probability his house will burn down. The consumer can purchase insurance that will pay him q dollars in the event that he incurs this loss. The amount of money that he has to pay for q dollars of insurance coverage is πq ; here π is the premium per dollar of coverage.

The consumer can decrease the probability of the loss by performing a level of effort e . (For example, e might measure the number of times every month that the consumer checks the operation of the fire-detecting equipment in his home.) The consumer dislikes effort, so the only reason he might set $e \neq 0$ is because e decreases the probability of the loss. The probability of the loss can be written $p(e)$.

Assume that q is set by the consumer, not by the firm. Suppose there is only one firm and one consumer. Assume that the expected utility hypothesis holds.

- a) What is the consumer's objective function? Do not assume any specific functional forms beyond what the problem statement above allows you to assume.
- b) What is/are the consumer's first-order condition/conditions for a maximum?
- c) What is the firm's objective function? Do not assume any specific functional forms beyond what the problem statement above allows you to assume.
- d) What is/are the firm's first-order condition/conditions for a maximum?

Question 2.1

wealth W

amount of loss L

insurance coverage q : set by consumer

payment for insurance coverage πq

Consumer's effort at preventing losses e

probability of loss $p(e)$

a) If no loss occurs, wealth is $W - \pi q$.

If a loss occurs, wealth is $W - \pi q - L + q$.

So the lottery is:

$$p(e) = (W - \pi q - L + q) \oplus (1 - p(e)) = (W - \pi q).$$

The consumer's utility function is $u(\tilde{W}, e)$, where \tilde{W} is random wealth and e is effort, which he dislikes. Since expected utility holds, the consumer's objective function is

$$p(e) u(W - \pi q - L + q, e) + (1 - p(e)) u(W - \pi q, e).$$

b) The consumer chooses e and q . The firm will choose π .

Since there is only one firm and one consumer, one could assume that the consumer does not take π as given, but instead knows that π depends on the consumer's choice of e and q . However, the question did not mention anything explicit about this. It's easier to solve if the consumer takes π as given. You may work it either way. With the consumer taking π as given, the F.O.C. are

Answer 2.1

Answer 2.1 cont..

$$O = \frac{\partial \text{utility}}{\partial e} = p'(e) u(W - \pi q - L + q, e) + p(e) u'_1(W - \pi q - L + q, e) \\ - p'(e) u(W - \pi q, e) + (1-p(e)) u'_1(W - \pi q, e)$$

$$O = \frac{\partial \text{utility}}{\partial q} = p(e) u'_1(W - \pi q - L + q, e) (-\pi + 1) \\ + (1-p(e)) u'_1(W - \pi q, e) (-\pi).$$

If the consumer does not take π as given, then $\pi = \pi(e, q)$, and the obvious modifications to the above derivatives have to be made.

c) The firm's profit is $\pi q - q$ if a loss occurs and πq otherwise, so expected profit is

$$p(e)(\pi q - q) + (1-p(e)) \pi q \\ = p\pi q - pq + \pi q - p\pi q = \pi q - pq = (\pi - p)q. \quad \text{optional}$$

d) The firm chooses π . If the firm thinks p and q are given when it chooses π , and if $q > 0$, then $\underset{\text{expected}}{\text{profit}} \rightarrow \infty$ as $\pi \rightarrow \infty$, so no F.O.C. would make sense.

Optional: If the firm knows that the consumer's choice of e and q depend on π , then the firm's F.O.C. becomes

$$O = \frac{d}{d\pi} \text{profit} = \frac{d}{d\pi} [\pi - p(e)] q = \frac{d}{d\pi} [\pi q(\pi) - p(e(\pi)) q(\pi)] \\ = q + \pi \frac{dq}{d\pi} - \frac{dp}{de} \frac{\partial e}{\partial \pi} q - p \frac{dq}{d\pi}.$$

Final Exam
1995
Question 1

(3)

Answer all of the following six questions.

Each question is worth 10 points.

1. Assume a consumer has utility function $u(w) = -1/w$ where w is wealth, and assume the consumer obeys the Expected Utility Hypothesis.
 - (a) Denote the consumer's current wealth by w_0 . How much would he be willing to pay for a lottery ticket promising to pay \$1 with probability $1/2$ and \$0 with probability $1/2$? It is possible to solve this exactly but you only need to derive the appropriate formula which this price must obey.
 - (b) Suppose the consumer owns the lottery ticket described in part (a) and he also owns w_0 in sure wealth. How much money would he have to be paid in order to convince him to sell the lottery ticket? It is possible to solve this exactly but you only need to derive the appropriate formula which this price must obey.

$$\textcircled{1} \quad u(w) = -\frac{1}{w}$$

a) current wealth: w_0

$$\text{current utility: } \frac{-1}{w_0}$$

$$\begin{aligned} \text{wealth if he pays } " \hat{p} " \text{ to buy the lottery ticket: } & w_0 - \hat{p} + \left(\frac{1}{2} \circ *1 \oplus \frac{1}{2} \circ *0 \right) \\ &= \frac{1}{2} \circ (w_0 - \hat{p} + 1) \oplus \frac{1}{2} \circ (w_0 - \hat{p}) \end{aligned}$$

Utility if he pays " \hat{p} " to buy the lottery ticket:

$$u \left(\frac{1}{2} \circ (w_0 - \hat{p} + 1) \oplus \frac{1}{2} \circ (w_0 - \hat{p}) \right)$$

$$= \frac{1}{2} u(w_0 - \hat{p} + 1) + \frac{1}{2} u(w_0 - \hat{p}) \quad \therefore \text{Expected Utility Hypothesis}$$

$$= \frac{1}{2} \frac{-1}{w_0 - \hat{p} + 1} + \frac{1}{2} \frac{-1}{w_0 - \hat{p}}.$$

Making the consumer indifferent between his current utility and the utility if he buys the lottery ticket implies

$$\frac{-1}{w_0} = \frac{-1}{2} \frac{1}{w_0 - \hat{p} + 1} - \frac{1}{2} \frac{1}{w_0 - \hat{p}} \quad \text{or}$$

$$\frac{\partial}{w_0} = \frac{1}{w_0 - \hat{p} + 1} + \frac{1}{w_0 - \hat{p}}.$$

Optional: After much algebra this simplifies to $2\hat{p}^2 - 2(w_0 + 1)\hat{p} + w_0 = 0$,

$$\text{whose solution is } \hat{p} = \frac{1}{2} \left[(w_0 + 1) \pm \sqrt{(w_0 + 1)^2 - 2w_0} \right]. \quad \text{If } w_0 = 10 \text{ then } \hat{p} = \frac{1}{2} [11 - \sqrt{101}] \approx 47.5\text{¢}.$$

1995

Answer 1. cont...

$$b) \text{ current wealth: } w_0 + \left(\frac{1}{2} \circ {}^s 1 \oplus \frac{1}{2} \circ {}^s 0 \right) = \frac{1}{2} \circ (w_0 + 1) \oplus \frac{1}{2} \circ w_0$$

current utility : $u\left(\frac{1}{2} \circ (w_0 + 1) \oplus \frac{1}{2} \circ w_0\right)$

$$= \frac{1}{2} u(w_0 + 1) + \frac{1}{2} u(w_0) \quad \text{by Expected Utility}$$

$$= \frac{1}{2} \cdot \frac{-1}{w_0 + 1} + \frac{1}{2} \cdot \frac{-1}{w_0}$$

wealth if he sells the lottery ticket for " \tilde{p} " dollars : $w_0 + \tilde{p}$

utility if he sells the lottery ticket for " \tilde{p} " dollars : $u(w_0 + \tilde{p}) = \frac{-1}{w_0 + \tilde{p}}$

Making the consumer indifferent between his current utility and the utility if he buys the lottery ticket implies

$$-\frac{1}{2} \frac{1}{w_0 + 1} - \frac{1}{2} \frac{1}{w_0} = \frac{-1}{w_0 + \tilde{p}} \quad \Rightarrow$$

$$\frac{2}{w_0 + \tilde{p}} = \frac{1}{w_0 + 1} + \frac{1}{w_0}$$

Optional: After algebra this simplifies to $\tilde{p} = \frac{w_0}{2w_0 + 1}$. If $w_0 = 10$ then

$$\tilde{P} = \frac{10}{21} \approx 47.6\text{¢.}$$

| pts | (a) | (b) |
|-----------------|-----|-----|
| current utility | 2 | 2 |
| new utility | 2 | 2 |
| equation | 1 | 1 |

Qualifying Exam
1996

Question 1

(3)

621 Section (must answer one)

Question 1. Suppose a consumer's utility function is $u(f) = \ln f$ where f is the number of pounds of good-tasting fish that the consumer eats.

This consumer goes to a store and sees two kinds of fish on sale, fish of type 1 and fish of type 2. The consumer knows that fish of type 1 is good tasting fish. The consumer believes that fish of type 2 is good-tasting fish with probability 1/2 and that fish of type 2 is bad-tasting fish with probability 1/2.

- a) Under what circumstances will the consumer buy some fish of type 2?
- b) Sketch the consumer's demand curve for fish of type 1.

Optional Question #1.

$$u(f) = \ln f$$

f_1 : pounds of fish of type 1
 f_2 : " " " "

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1996

Answer 1

The lottery is: $\frac{1}{2} \circ \ln(f_1 + f_2) + \frac{1}{2} \circ \ln f_1$. Explanation: with probability $\frac{1}{2}$, f_2 is good-tasting fish, just like f_1 ; with probability $\frac{1}{2}$, though, f_2 is bad-tasting fish which does not enter into the utility function.

Consumer's problem, assuming that the expected utility hypothesis holds:

$$\max \frac{1}{2} \ln(f_1 + f_2) + \frac{1}{2} \ln f_1 \text{ s.t. } p_1 f_1 + p_2 f_2 = m$$

\uparrow \uparrow \nwarrow
 price of each type of fish income

$$\mathcal{L} = \frac{1}{2} \ln(f_1 + f_2) + \frac{1}{2} \ln f_1 + \lambda [m - p_1 f_1 - p_2 f_2]$$

First-order conditions:

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 f_1 - p_2 f_2 \quad (1)$$

$$0 = \frac{\partial \mathcal{L}}{\partial f_1} = \frac{1}{2} \frac{1}{f_1 + f_2} + \frac{1}{2} \frac{1}{f_1} - \lambda p_1 \quad (2)$$

$$0 = \frac{\partial \mathcal{L}}{\partial f_2} = \frac{1}{2} \frac{1}{f_1 + f_2} - \lambda p_2 \quad (3)$$

$$(2) \& (3) \Rightarrow -\frac{1}{2} \frac{1}{f_1 + f_2} = \frac{1}{2f_1} - \lambda p_1 = -\lambda p_2 \quad \text{Answer 1 cont...}$$

$$\frac{1}{2f_1} = \lambda (p_1 - p_2) \Rightarrow \lambda = \frac{1}{2f_1 (p_1 - p_2)} \quad (4)$$

$$(4) \& (3) \Rightarrow$$

$$-\frac{1}{2} \frac{1}{f_1 + f_2} = \frac{-1}{2f_1 (p_1 - p_2)} p_2 \Rightarrow \cancel{2(f_1 + f_2)} = \cancel{2f_1 (p_1 - p_2)} / p_2 \Rightarrow$$

$$f_2 = \frac{f_1 (p_1 - p_2)}{p_2} - f_1 = f_1 \left[\frac{p_1 - p_2}{p_2} - 1 \right] = f_1 \frac{p_1 - p_2 - p_2}{p_2} = f_1 \frac{p_1 - 2p_2}{p_2} \quad (5)$$

This already means that if $p_1 = 2p_2$ then $f_2 = 0$. If $p_1 < 2p_2$ then (5) would give a negative number for f_2 . In this case the correct answer would be $f_2 = 0$ (for verification, use the Kuhn-Tucker theorem with the added constraints that $f_1 \geq 0$ and $f_2 \geq 0$). All this makes sense: if fish of type 1 is cheap enough, why take a chance by buying any fish of type 2?

That answers part (a). For part (b), substitute (5) into (1) :

$$m = p_1 f_1 + p_2 f_1 \left(\frac{p_1 - 2p_2}{p_2} \right) = p_1 f_1 + p_2 f_1 - 2p_2 f_1 = 2p_1 f_1 - 2p_2 f_1$$

$$= 2f_1 (p_1 - p_2) \Rightarrow f_1 = \frac{m}{2(p_1 - p_2)} \quad (6)$$

(6) is only true when (5), which (6) is based on, is true; that is, (6) is only true when $p_1 \geq 2p_2$. Otherwise, $f_2 = 0$ and all income

is spent on f_1 , yielding $f_1 = \frac{m}{p_1}$. So:

$$f_1 = \begin{cases} \frac{m}{p_1} & \text{if } p_1 < 2p_2 \\ \frac{m}{2(p_1 - p_2)} & \text{if } p_1 \geq 2p_2 \end{cases}$$

Qualifying Exam
1996

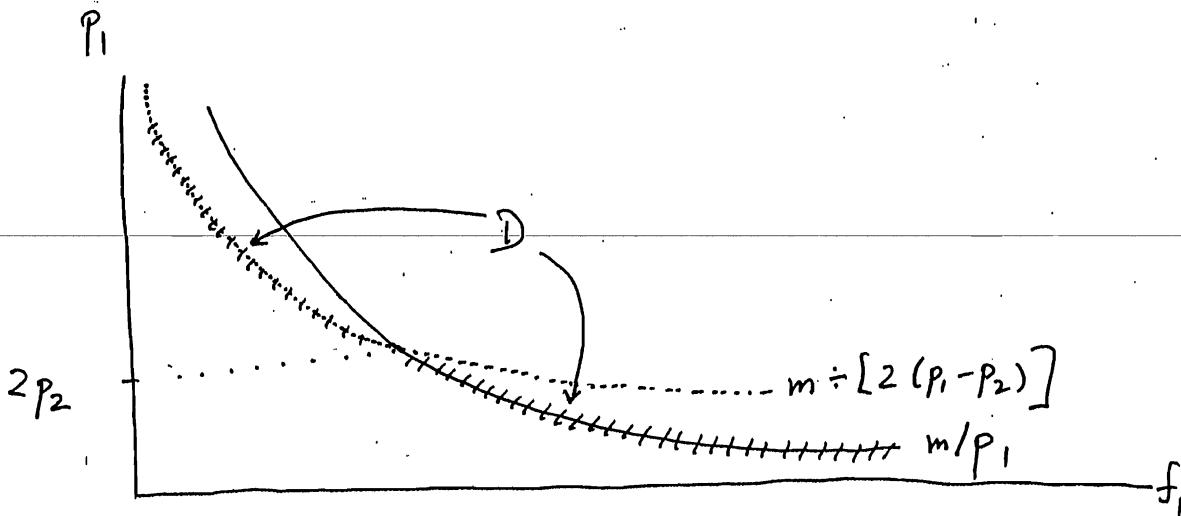
Answer 1 cont...

Note that $\frac{m}{p_1} = \frac{m}{2(p_1 - p_2)}$ whenever $p_1 = 2(p_1 - p_2) \Leftrightarrow 2p_2 = p_1$,

so at $p_1 = 2p_2$ the two demand functions are equal. If $p_1 > 2p_2$ then

$$\frac{m}{p_1} - \frac{m}{2(p_1 - p_2)} = m \left[\frac{2(p_1 - p_2) - p_1}{2p_1(p_1 - p_2)} \right] = m \frac{p_1 - 2p_2}{2p_1(p_1 - p_2)}$$

which is positive. So the graph looks like this:



Notice that f_1 is always positive. This is because if the consumer bought no f_1 , and if f_2 turned out to be bad, his utility would be $\ln 0 = -\infty$.

Qualifying Exam

1995

Question 1

(3)

621 Section (must answer one)

1. There are two types of lottery tickets available on the market. One type of lottery ticket pays the owner 1 apple with probability $1/2$ and 2 apples with probability $1/2$. This type of lottery ticket costs \$1 each. The other type of lottery ticket pays the owner 0 oranges with probability $1/2$ and 5 oranges with probability $1/2$. This type of lottery ticket costs \$2 each.

If the utility an agent enjoys from consuming x apples and y oranges is given by $u(x, y) = xy$, and this agent's income is \$4, how many of each type of lottery ticket will the consumer buy? Assume the consumer obeys the Expected Utility Hypothesis.

Be very clear and very careful with the notation you use in your explanation.

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1995

Answer 1

Short Questions.

① $\frac{1}{2} \circ 1 \text{ apple} \oplus \frac{1}{2} \circ 2 \text{ apples}$ ↪
 $\frac{1}{2} \circ 0 \text{ oranges} \oplus \frac{1}{2} \circ 5 \text{ oranges}$ ↪ the two lotteries

If the consumer buys ' α ' of the first lottery tickets, he has $\frac{1}{2} \circ \alpha$ apples \oplus $\frac{1}{2} \circ 2\alpha$ apples = $(\frac{1}{2} \circ \alpha \oplus \frac{1}{2} \circ 2\alpha)$ apples. If he buys ' β ' of the second lottery tickets, he similarly has $(\frac{1}{2} \circ 0 \oplus \frac{1}{2} \circ 5\beta)$ oranges.

At least this is one interpretation : there are only four states of the world (you get both apples and oranges ; you get apples but not oranges ;
 $\underbrace{\text{lucky with}}$ $\underbrace{\text{lucky with}}$)

Answer 1 cont.

You get lucky with oranges but not apples; and you don't get lucky at all).

There is another interpretation I will discuss below.

Continuing on the first interpretation, the consumer's utility is as follows, presuming he obeys the Expected Utility Hypothesis:

$$u(\text{apples, oranges}) = u\left(\frac{1}{2}\alpha + \frac{1}{2}\alpha 2\beta, \frac{1}{2}\alpha 0 + \frac{1}{2}\alpha 5\beta\right)$$

$$= \frac{1}{2}u(\alpha, \frac{1}{2}\alpha 0 + \frac{1}{2}\alpha 5\beta) + \frac{1}{2}u(2\alpha, \frac{1}{2}\alpha 0 + \frac{1}{2}\alpha 5\beta)$$

$$= \underbrace{\frac{1}{2} \cdot \frac{1}{2}u(\alpha, 0)}_{\text{upper term}} + \underbrace{\frac{1}{2} \cdot \frac{1}{2}u(\alpha, 5\beta)}_{\text{lower term}} + \underbrace{\frac{1}{2} \cdot \frac{1}{2}u(2\alpha, 0)}_{\text{upper term}} + \underbrace{\frac{1}{2} \cdot \frac{1}{2}u(2\alpha, 5\beta)}_{\text{lower term}}$$

$$= \frac{1}{4}u(\alpha, 0) + \frac{1}{4}u(\alpha, 5\beta) + \frac{1}{4}u(2\alpha, 0) + \frac{1}{4}u(2\alpha, 5\beta)$$

$u(x, y) = xy$, this utility equals $\frac{1}{4}(0) + \frac{1}{4}5\alpha\beta + \frac{1}{4}(0) + \frac{1}{4}10\alpha\beta =$

$$\frac{5}{4}\alpha\beta + \frac{10}{4}\alpha\beta = \frac{15}{4}\alpha\beta$$

The consumer wants to maximize this subject to his budget constraint,

which is $\alpha + 2\beta = 4$. Thus:

$$\mathcal{L} = \frac{15}{4}\alpha\beta + \lambda(4 - \alpha - 2\beta)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha} = \frac{15}{4}\beta - \lambda \quad \text{and} \quad 0 = \frac{\partial \mathcal{L}}{\partial \beta} = \frac{15}{4}\alpha - 2\lambda \quad \text{so} \quad \lambda = \frac{15}{4}\beta \text{ and}$$

Answer 1 cont...

$$0 = \frac{15}{4}\alpha - 2\left(\frac{15}{4}\beta\right) \Rightarrow 0 = \alpha - 2\beta \Rightarrow \alpha = 2\beta \Rightarrow (\text{budget})$$

$$\text{constraint}) \quad 2\beta + 2\beta = 4 \Rightarrow 4\beta = 4 \Rightarrow \underline{\beta^* = 1} \text{ and } \underline{2\beta = \alpha^* = 2}$$

Optional:

A strange thing about this setup is that while changing the 'apple' lottery from ' $\frac{1}{2} \circ 1 \text{ apple} \oplus \frac{1}{2} \circ 2 \text{ apples}$ ' to ' $\frac{1}{2} \circ 1 \text{ apple} \oplus \frac{1}{2} \circ 1,000,000 \text{ apples}$ ' increases utility, utility remains in the form constant: $\alpha\beta$, and hence one still has $\alpha^* = 2$ and $\beta^* = 1$. This bizarre result presumably comes from the fact that using the Expected Utility Hypothesis implies that utility is cardinaly measurable; the rest of the micro theory we studied only required ordinal measurability.

This ends my discussion of the "first interpretation" of the question, which in some respects resembles the problem of how much money an investor should invest in two stocks. The "second interpretation" of the question is that each (say) apple lottery ticket's outcome is independent of any other apple lottery ticket's outcome. If you bought three apple lottery tickets, the possible states of the world would be $\{3 \text{ apples}, 4 \text{ apples}, 5 \text{ apples}, 6 \text{ apples}\}$ (whereas in the "first interpretation" there would only be two states of the world, $\{3 \text{ apples}, 6 \text{ apples}\}$ - for "good luck" and "bad luck" respectively).

The "second interpretation" leads to a very difficult problem. Here is a sketch

Answer 1 cont...

of the answer.

If you buy ' α ' apple lottery tickets, the probability that i of them will be good (i.e., pay off 2 apples) is given by the binomial distribution

$$\binom{\alpha}{i} \left(\frac{1}{2}\right)^{\alpha-i} \left(\frac{1}{2}\right)^i = \binom{\alpha}{i} 2^{-\alpha} \text{ for } 0 \leq i \leq \alpha. \quad \text{The payoff if } i \text{ of}$$

$\downarrow \text{prob. of } 1 \text{ apple.}$ $\downarrow \text{prob. of } 2 \text{ apples}$ $\begin{array}{c} \text{apples from good tickets} \\ \text{apples from bad tickets} \end{array}$

the tickets are good is $\overbrace{2i + (\alpha-i)1}^{\alpha} = i + \alpha.$ Similarly, if you buy ' β '

orange lottery tickets, the probability that j of them will be good is $\binom{\beta}{j} 2^{-\beta}$ for $0 \leq j \leq \beta$, and the payoff if j of them are good is $5j + 0(\beta-j) = 5j$ oranges.

Somewhat abusing notation, the consumer's utility is therefore

$$u\left(\binom{\alpha}{0} 2^{-\alpha} \circ (0+\alpha) \oplus \binom{\alpha}{1} 2^{-\alpha} \circ (1+\alpha) \oplus \dots \oplus \binom{\alpha}{\alpha} 2^{-\alpha} (\alpha+\alpha), \right. \\ \left. \binom{\beta}{0} 2^{-\beta} \circ (5 \cdot 0) \oplus \binom{\beta}{1} 2^{-\beta} \circ (5 \cdot 1) \oplus \dots \oplus \binom{\beta}{\beta} 2^{-\beta} (5 \cdot \beta) \right).$$

Repeatedly applying the Expected Utility Hypothesis leads to an expected utility of

$$\sum_{j=0}^{\beta} \sum_{i=0}^{\alpha} \binom{\alpha}{i} 2^{-\alpha} \binom{\beta}{j} 2^{-\beta} u(i+\alpha, 5j). \quad \text{Since } u(x, y) = xy, \text{ this equals}$$

$$\sum_{j=0}^{\beta} \sum_{i=0}^{\alpha} \binom{\alpha}{i} \binom{\beta}{j} 2^{-\alpha-\beta} \cdot 5j (i+\alpha). \quad \text{One wishes to maximize this over non-}$$

negative integers α and β such that the budget constraint $1\alpha + 2\beta = 4$ holds.

The general solution to this type of problem is well beyond the level of this course. However, given this particular budget constraint, there are only three feasible (α, β) combinations : $(0, 2)$, $(2, 1)$, and $(4, 0)$. The first choice makes the $(i+\alpha)$ term in expected utility identically equal to zero. The third

Qualifying Exam
1995

Answer 1 cont...

8

choice makes the S_j term in expected utility identically equal to zero. It is not therefore hard to guess that the optimal choice must be $(2, 1)$, but here I verify that $(2, 1)$ gives an expected utility greater than zero:

$$\sum_{j=0}^1 \sum_{i=0}^2 \binom{2}{i} \binom{1}{j} 2^{-2-1} \cdot S_j(i+2) = \binom{2}{0} \binom{1}{0} \frac{1}{8} \cdot 5 \cdot 0(0+2) + \\ \binom{2}{1} \binom{1}{0} \frac{1}{8} \cdot 5 \cdot 0(1+2) + \binom{2}{2} \binom{1}{0} \frac{1}{8} \cdot 5 \cdot 0(2+2) + \binom{2}{0} \binom{1}{1} \frac{1}{8} \cdot 5 \cdot 1(0+2) + \\ \binom{2}{1} \binom{1}{1} \frac{1}{8} \cdot 5 \cdot 1(1+2) + \binom{2}{2} \binom{1}{1} \frac{1}{8} \cdot 5 \cdot 1(2+2) = 0 + 0 + 0 + \\ \frac{1}{8}(10) + \frac{1}{4}(15) + \frac{1}{8}(20) = \frac{15}{2} > 0.$$

Orange ticket bad; 2 good apple tickets
 { " " ; 1 " "
 { " " ; apple tickets bad

Note: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Also: I think it is coincidence that the "first interpretation" and the "second interpretation" give the same answer.

Final Exam

1998

Question 2

(3)

2. A consumer with initial wealth \$200 and utility function $\ln w$ (where w is "wealth") faces an uncertain prospect in which with probability $1/2$ he will win \$100 and with probability $1/2$ he will lose \$100. Assume the consumer's preferences satisfy the Expected Utility Hypothesis.
- The consumer has an opportunity to pay $\$p$ to an insurance company. In return, the insurance company will pay the consumer nothing if the consumer wins \$100, and the insurance company will pay the consumer $\$3p/2$ if the consumer loses \$100. What is the consumer's optimal value of p ?
 - The consumer has an opportunity to pay $\$p$ to an insurance company. Unlike the case in part (a), in return, the insurance company will pay the consumer nothing if the consumer wins \$100, and the insurance company will pay the consumer $\$p/2$ if the consumer loses \$100. What is the consumer's optimal value of p ? [Hint: the answer is not \$250, although that answer is worth some partial credit.]

$$② w_0 = \$200 \quad u(w) = \ln w \quad \text{lottery: } \frac{1}{2} \circ \$100 \oplus \frac{1}{2} \circ (-\$100).$$

a) wealth now becomes

$$\frac{1}{2} \circ (200 + 100 - p) \oplus \frac{1}{2} \circ (200 - 100 - p + \frac{3p}{2}).$$

Expected utility is $\frac{1}{2} \ln(200 + 100 - p) + \frac{1}{2} \ln(200 - 100 - p + \frac{3p}{2})$

since $u(w) = \ln w$ and since the expected utility hypothesis is satisfied.

Simplifying, expected utility $Eu = \frac{1}{2} \ln(300 - p) + \frac{1}{2} \ln(100 + \frac{p}{2})$.

Maximizing this with respect to p gives

$$0 = \frac{d Eu}{dp} = \frac{1}{2} \frac{-1}{300-p} + \frac{1}{2} \frac{1}{100+\frac{p}{2}} (\frac{1}{2}) = \frac{1}{2} \frac{-1}{300-p} + \frac{1}{2} \frac{1}{200+p}$$

$$\Rightarrow 0 = \frac{-1}{300-p} + \frac{1}{200+p} \Rightarrow \frac{1}{300-p} = \frac{1}{200+p} \Rightarrow 200+p = 300-p \Rightarrow$$

$$2p = 100 \Rightarrow p^* = 50.$$

b) wealth is $\frac{1}{2} \circ (200 + 100 - p) \oplus \frac{1}{2} \circ (200 - 100 - p + \frac{p}{2})$. Final Exam

$$Eu = \frac{1}{2} u(200 + 100 - p) + \frac{1}{2} u(200 - 100 - p + \frac{p}{2}) \quad 1998$$

$$= \frac{1}{2} \ln(200 + 100 - p) + \frac{1}{2} \ln(200 - 100 - p + \frac{p}{2}) \quad \text{Answer 2}$$

$$= \frac{1}{2} \ln(300 - p) + \frac{1}{2} \ln(100 - \frac{p}{2}).$$

By inspection, $\uparrow p$ will always \downarrow expected utility. Hence $p^* = 0$.

$$0 = \frac{d E_u}{d p} = \frac{1}{2} \frac{-1}{300-p} + \frac{1}{2} \frac{1}{100-\frac{p}{2}} \left(-\frac{1}{2}\right) \Rightarrow$$

$$0 = \frac{1}{300-p} + \frac{1}{2} \frac{1}{100-\frac{p}{2}} = \frac{1}{300-p} + \frac{1}{200-p} \Rightarrow$$

$$\frac{-1}{200-p} = \frac{1}{300-p}$$

$$p-300 = 200-p$$

$$2p = 500$$

$p = 250$ is the wrong answer. Indeed, since

$$E_u = \frac{1}{2} \ln(300-p) + \frac{1}{2} \ln(100-\frac{p}{2}),$$

any $p \geq 500$ will make this term \uparrow , and thus E_u , equal to $-\infty$ or undefined quantity.

Another explanation: you'd never pay \$250 for sure in order to avoid a possible loss of only \$100.

Final Exam

1998

Answer 2 Cont...

Final Exam

1997

Question 3

(3)

3. Suppose a consumer has a utility function $u(w)$ where w is wealth, and suppose this consumer's preferences obey the Expected Utility Hypothesis. Suppose this consumer is offered the opportunity to bet on the flip of a coin that has a probability π of coming up heads. If he bets \$ x , he will have $w + x$ if heads comes up and $w - x$ if tails comes up.

Find how the amount of money " x " the consumer wishes to bet changes as π increases. You may ignore the possibility that $x = 0$ for the purposes of this exam.

③ The consumer's utility is

$$u(\pi \circ (w+x) + (1-\pi) \circ (w-x))$$

$$= \pi u(w+x) + (1-\pi) u(w-x) \text{ by the Expected Utility Hypothesis.}$$

The consumer wishes to maximize this over "x", the amount he bets. So

the F.O.C. is

$$\begin{aligned} 0 &= \frac{d \text{ utility}}{dx} = \pi u'(w+x^*) + (1-\pi) u'(w-x^*) \frac{d(w-x^*)}{dx} && \text{Final Exam} \\ &= \pi u'(w+x^*) - (1-\pi) u'(w-x^*) && 1997 \end{aligned}$$

Answer 3

The total differential of this is

$$\begin{aligned} 0 &= [u'(w+x^*) + u'(w-x^*)] d\pi + [\pi u''(w+x^*) - (1-\pi) u''(w-x^*) (-1)] dx \\ &\quad + [\dots] dw \text{ but here } dw = 0; \text{ so} \end{aligned}$$

$$dx [-\pi u''(w+x^*) - (1-\pi) u''(w-x^*)] = [u'(w+x^*) + u'(w-x^*)] d\pi \text{ and}$$

$$\frac{dx}{d\pi} = \frac{u'(w+x^*) + u'(w-x^*)}{-\pi u''(w+x^*) - (1-\pi) u''(w-x^*)}$$

Usually $u' > 0$, and $u'' < 0$ will ensure the second-order conditions (which you did not have to find) hold. With those signs, $dx/d\pi > 0$, which makes sense: as the probability of the good outcome increases, you increase the size of your bet.

Exam I
1999

Question 1

(3)

Answer all of the following three questions.

1. Suppose a consumer's utility function under certainty is $u(w)$ where w is his wealth. This consumer is forced to play a lottery in which there are only two outcomes, $-\$1$ and $+\$1$. The probability of each outcome is one-half. However, this consumer can change the probabilities by paying a bribe to the person running the lottery. If the consumer pays a bribe of x , the probability of the bad outcome, namely $-\$1$, will be

$$\frac{1}{2} \frac{1}{1+x}$$

Assume the expected utility hypothesis holds.

- Does this form of the probability of the bad outcome make sense?
- What is the consumer's first-order condition?

Answers to Exam 1, Econ 6710, Spring 1999

- ① a) If no bribe is paid, then $x=0$ and the lottery is $\frac{1}{2} \circ (\$1) \oplus \frac{1}{2} \circ (-\$1)$, as described earlier in the problem.

The higher the bribe x , the lower the probability of the bad outcome. This makes sense. If $x \rightarrow \infty$, the probability of the bad outcome $\rightarrow 0$.

(Note that no agent would give a bribe of more than $\$1$, because a bad outcome only involves the loss of $\$1$.)

- b) The lottery alone is $(1 - \frac{1}{x+1}) \circ \$1 \oplus (\frac{1}{x+1}) \circ (-\$1)$, but if w stands for the consumer's initial wealth, the amount of money he is left with is $w-x \neq 1$. Therefore, the consumer's total wealth is

$$(1 - \frac{1}{x+1}) \circ (w-x+1) \oplus (\frac{1}{x+1}) \circ (w-x-1).$$

If the expected utility hypothesis holds, then utility is equal to expected utility, which is

$$(1 - \frac{1}{x+1}) u(w-x+1) + \frac{1}{x+1} u(w-x-1) \quad \text{Exam 1}$$

1999

Answer 1

The first-order condition is obtained by setting the derivative of this w.r.t. x equal to zero:

$$0 = \frac{1}{2} \frac{1}{(x+1)^2} u(w-x+1) + (1 - \frac{1}{x+1}) u'(w-x+1)(-1)$$

$$- \frac{1}{2} \frac{1}{(x+1)^2} u(w-x-1) + \frac{1}{2} \frac{1}{x+1} u'(w-x-1)(-1).$$

Solving the problem would involve solving this equation for x and then checking second-order conditions.

621 Qualifying Exam Questions
Summer 1998

Question 2.1 Suppose the consumption bundle of a slave is fixed and unchanging. Suppose the utility of this slave depends on the number of hours he works each day, h , and the number of times he is beaten each day, b . Suppose the maximum working day for a slave is 18 hours.

Let A be a number between zero and one. This slave's owner has decided to set b equal to a fixed number " B " > 0 with probability $A(1 - \frac{h}{18})$ and this slave's owner has decided to set b equal to zero with probability $1 - A(1 - \frac{h}{18})$. Therefore, b is never equal to anything other than B or zero.

- a) What happens if $A = 0$? if $A = 1$? Why does the above specification of the probabilities for $b = B$ and $b = 0$ make sense? Why do these specifications allow one to mathematically model h as a "free" choice of the slave?
- b) What is the slave's first order condition? Be as specific as you can.
- c) Suppose slave labor is the only input to the production function $f(h)$. The slave owner sells output at an exogenous price of p and incurs monitoring costs of A . Ignore all other production costs (assume they are fixed in this problem).

If p increases, does A go up or down? You will probably not be able to come up with a definitive answer, but tell me as much as you can mathematically about the answer.

According to your intuition, what should the answer be (up or down)? Why?

Qualifying Exam
1998

Question 2.1

(3)

Answers to Qualifying Exam Questions from Econ. 621.

Summer 1998

Required Question 2.1

a)

Utility of slave is $u(h, b)$

Probability of $b = B > 0$ is $A(1 - \frac{h}{18})$; graphically,
 $0 \leq A \leq 1$.

Probability of $b = 0$ is $1 - (\text{the probability of } b = B)$.

Interpretation: As $h \uparrow$, the probability of the slave being beaten \downarrow .

If $h = 18$, _____ " _____ is zero.

If $h = 0$, _____ " _____ is A.

"A" represents how vigilant the slave owner's monitoring is. If $A = 0$, the slave is never beaten. If $A = 1$, then whenever the slave sets $h = 0$ the slave will be beaten with probability one.

The slave's utility is

$$u(h, A(1 - \frac{h}{18})) \circ B \oplus (1 - A(1 - \frac{h}{18})) \circ 0).$$

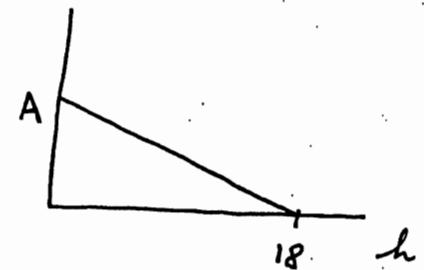
Applying the Expected Utility Hypothesis, this is equal to

$$A(1 - \frac{h}{18}) u(h, B) + (1 - A(1 - \frac{h}{18})) u(h, 0).$$

The slave wishes to maximize this subject to $0 \leq h \leq 18$. The slave is "free" to choose h in this — admittedly narrow-sense. When a neoclassical

Qualifying Exam
1998

Answer 2.1



model posits free choice, that does not mean the underlying social relationship lacks elements of coercion.

Qualifying Exam
1998

b) Assuming an interior maximum, the F.O.C. would be

Answer 2.1 Cont...

$$\begin{aligned} 0 = & \frac{-A}{18} u(h^*, B) + A \left(1 - \frac{h^*}{18}\right) u'_1(h^*, B) \\ & + \frac{A}{18} u(h^*, 0) + \left[1 - A \left(1 - \frac{h^*}{18}\right)\right] u'_1(h^*, 0). \end{aligned}$$

This implicitly defines h^* as a function of A and B .

c) Profit is $p f(h) - A$. However, it is the slave who chooses h , according to parts (a) & (b). So the slave owner is a Stackelberg leader (choosing A and B), and the slave is a Stackelberg follower (taking A and B as given and choosing h on the basis of the given A and B).

Profit can thus be written $p f(h^*(A, B)) - A$.

The F.O.C. with respect to A is

$$0 = p \frac{df(h^*(A, B))}{dh} \frac{\partial h^*(A, B)}{\partial A} - 1$$

The total differential is

$$0 = \left[\frac{df}{dh} \frac{\partial h^*}{\partial A} \right] d_p + \left[p \frac{d^2 f}{dh^2} \frac{\partial h^*}{\partial A} \frac{\partial h^*}{\partial A} + p \frac{df}{dh} \frac{\partial^2 h^*}{\partial A^2} \right] d_A$$

and therefore

1998

Answer 2.1 cont...

$$\frac{dA}{dp} = \frac{-\frac{df}{dh} \frac{\partial h^*}{\partial A}}{p \frac{d^2f}{dh^2} \left(\frac{\partial h^*}{\partial A} \right)^2 + p \frac{df}{dh} \frac{\partial^2 h^*}{\partial A^2}}$$

+ -? + + + + ?

(1)

Notice that the second-order condition with respect to A (assuming B is not a choice variable) would be

$$0 > \frac{d^2(\text{profit})}{dA^2} = \text{denominator of equation (1) above.}$$

(If B were a choice variable then everything gets more tedious — along the lines of
 $\begin{bmatrix} dA/dP \\ dB/dP \end{bmatrix} = - \begin{bmatrix} pf''(h_A')^2 + pf'h_{AA}'' & pf''L'Ah_B + pf'h_{AB}'' \\ pf''h_A'h_B + pf'L''_{AB} & pf''(h_B')^2 + pf'h_{BB}'' \end{bmatrix}^{-1} \begin{bmatrix} f'h_A' \\ f'h_B' \end{bmatrix}$ — which is why the problem says B is fixed.)

To actually determine $\partial h^*/\partial A$ would require doing comparative statics on the slave's problem (totally differentiate the F.O.C. in part (b)).

Foregoing that, we could guess that $\partial h^*/\partial A > 0$ (as the owner's monitoring becomes more vigilant, the slave works more). Assuming this,

$$\frac{dA}{dp} = \frac{-\oplus \oplus}{\text{negative}} > 0$$

when the slave's output becomes more valuable, the owner's monitoring increases.

Optional: $\partial h^*/\partial A$ turns out to be $- \left\{ \frac{u(h, 0) - u(h, B)}{18} + \left(1 - \frac{L}{18}\right) [u'_h(h, B) - u'_h(h, 0)] \right\} \div \approx u''_{hB}(h, \cdot) \cdot B$

(a term that's negative according to S.O.C. for the slave). As long as $|u''_{hB}|$ is small, or $u''_{hB} > 0$, $\partial h^*/\partial A$ will be positive as expected.

Final Exam

1996

Question 1

(2)

Answer all of the following three questions.

1. Suppose a firm has a fixed amount of money, T , which the firm wants to spend to give two of its employees extra training. Let t_1 be the amount of money it spends to give training to employee number 1 and let t_2 be the amount of money it spends to give training to employee number 2.

If both employees continue working for this firm, suppose the benefit which the firm will receive from their training is $f(t_1) + f(t_2)$. The firm is completely sure that employee 1 will continue working for the firm, but the firm thinks there is a probability of p that employee 2 will quit his job after he gets the training. If that happens then the benefit which the firm will receive from training the two workers is only $f(t_1)$.

- (a) [2 points] Guess what the sign of dt_2/dp is and explain your guess.
- (b) [6 points] Mathematically determine the sign of dt_2/dp , assuming that the firm maximizes the expected benefit of training the workers. Concerning second-order conditions: make whatever assumptions you think are appropriate, then explain.

Answers to Econ. 621 Final, Winter 1996

Final Exam
1996

Answer 1

- ① a) If $p \uparrow$, there's a greater chance the firm's investment in training worker 2 will be wasted. So the firm will be less likely to train that worker: $t_2 \downarrow$. So $dt_2/dp < 0$. (This assumes that $\frac{df}{dt} > 0$.) 2pts
- b) With probability p , employee 2 quits and the firm's benefit is $f(t_1)$
 " " "(1-p), " " " doesn't quit " " " " " $f(t_1) + f(t_2)$.

So the firm's expected benefit is $p f(t_1) + (1-p) [f(t_1) + f(t_2)]$
 $= p f(t_1) + (1-p) f(t_1) + (1-p) f(t_2)$
 $= p f(t_1) + f(t_1) - p f(t_1) + (1-p) f(t_2)$
 $= f(t_1) + (1-p) f(t_2).$

The firm wants to maximize this subject to $t_1 + t_2 = T$.

$$\mathcal{L} = f(t_1) + (1-p) f(t_2) + \lambda [T - t_1 - t_2]$$

F.O.C.:

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = T - t_1 - t_2$$

$$0 = \frac{\partial \mathcal{L}}{\partial t_1} = f'(t_1) - \lambda$$

$$0 = \frac{\partial \mathcal{L}}{\partial t_2} = (1-p) f'(t_2) - \lambda$$

S.O.C.: Form $\nabla^2 \mathcal{L}$:

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda^2} = 0 \quad \frac{\partial^2 \mathcal{L}}{\partial t_1 \partial \lambda} = -1 \quad \frac{\partial^2 \mathcal{L}}{\partial t_2 \partial \lambda} = -1$$

$$\frac{\partial^2 \mathcal{L}}{\partial t_1^2} = f''(t_1) \quad \frac{\partial^2 \mathcal{L}}{\partial t_2 \partial t_1} = 0$$

$$\frac{\partial^2 \mathcal{L}}{\partial t_2^2} = (1-p)f''(t_2)$$

FOC, SOC: 3pts

Therefore:

$$\nabla^2 \mathcal{L} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & f''(t_1) & 0 \\ -1 & 0 & (1-p)f''(t_2) \end{bmatrix}$$

D_{2m+1} should have $(-1)^{m+1}$'s sign.

$m = 1$ here. So

D_3 of $\nabla^2 \mathcal{L}$ should have $(-1)^2 = +1$'s sign.

In more detail: D_3 of $\nabla^2 \mathcal{L} = |\nabla^2 \mathcal{L}| =$ (expanding along the first row)

$$\begin{aligned} & (-1)^3 (-1) (-1) (1-p) f''(t_2) + (-1)^4 (-1) [-(-1) f''(t_1)] \\ &= -(1-p) f''(t_2) - f''(t_1) > 0 \Rightarrow (1-p) f''(t_2) + f''(t_1) < 0. \end{aligned}$$

A sufficient condition for this is $f''(t) < 0$ (diminishing returns to training).

Next, find the total differential of the F.O.C.'s. endogenous variables: λ, t_1, t_2
exogenous " : p, T .

Since T is constant in this problem, $dT = 0$.

$$0 = 0 d\lambda - 1 dt_1 - 1 dt_2 + 0 dp \quad \text{Final Exam 1996}$$

$$0 = -1 d\lambda + f''(t_1) dt_1 + 0 dt_2 + 0 dp \quad \text{Answer 1 cont..}$$

$$0 = -1 d\lambda + 0 dt_1 + (1-p)f''(t_2) dt_2 - f'(t_2) dp \Rightarrow$$

$$Q = \begin{bmatrix} 0 & -1 & -1 \\ -1 & f''(t_1) & 0 \\ -1 & 0 & (1-p)f''(t_2) \end{bmatrix} \begin{bmatrix} d\lambda \\ dt_1 \\ dt_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ f'(t_2) \end{bmatrix} dp \Rightarrow$$

Final Exam

1996

Answer 1 cont...

3

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & f''(t_2) & 0 \\ -1 & 0 & (1-p)f''(t_2) \end{bmatrix} \begin{bmatrix} dt_1 \\ dt_2 \\ dt_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f'(t_2) dp \end{bmatrix} \Rightarrow \text{Using Cramer's Rule}$$

(2pts)

$$\frac{dt_2}{dp} = \frac{\begin{vmatrix} 0 & -1 & 0 \\ -1 & f''(t_2) & 0 \\ -1 & 0 & f'(t_2) \end{vmatrix}}{\begin{vmatrix} 0 & -1 & -1 \\ -1 & f''(t_2) & 0 \\ -1 & 0 & (1-p)f''(t_2) \end{vmatrix}} \quad \left. \begin{array}{l} \rightarrow = f'(t_2) (-1) \text{ expanding along third column} \\ > 0 \text{ by S.O.C.} \\ = (-1)[(1-p)f''(t_2) + f''(t_1)] \end{array} \right\} \text{from p. 2.}$$

$$\frac{dt_2}{dp} = \frac{f'(t_2)}{(1-p)f''(t_2) + f''(t_1)} \quad \left. \begin{array}{l} \} > 0 \text{ if } \uparrow t_2 \text{ yields } \uparrow \text{ benefits to the firm} \\ \} < 0 \quad \text{when the worker remains employed, a natural assumption.} \end{array} \right.$$

(1pt)

So $dt_2/dp < 0$ as expected.

4. [15 points] Suppose there are 8000 used cars in an economy. Their true values are uniformly distributed from \$2000 to \$6000 (which means there are 2 cars which are worth \$2000, 2 cars which are worth

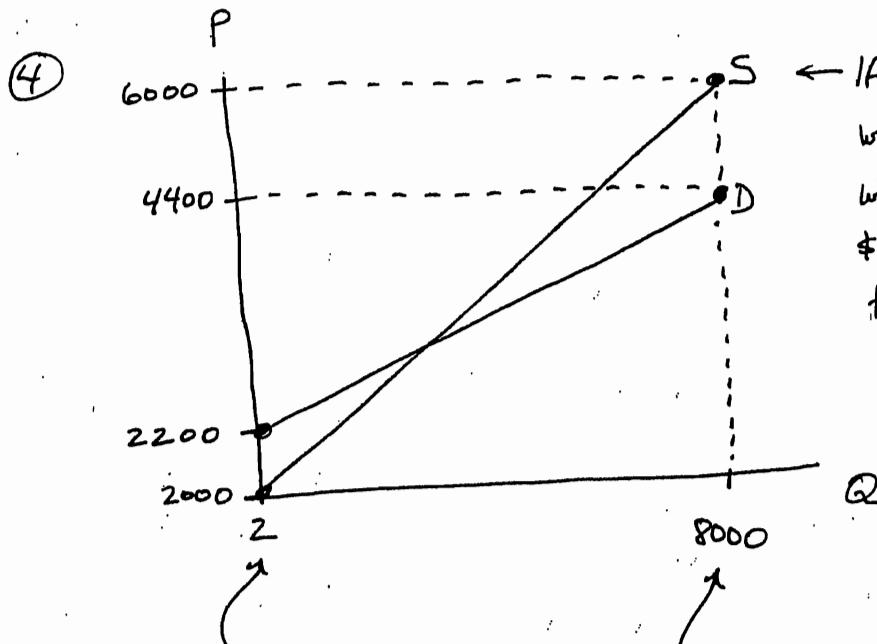
Final Exam
1999
Question 4

(3)

\$2001, 2 cars which are worth \$2002, etc.). Car sellers know the true value of each car, but car buyers do not.

Suppose car buyers are willing to pay 1.1θ for a car of value θ . Suppose car sellers are willing to sell a car of value θ for any price equal to or greater than θ .

Graphing price p on the vertical axis and the quantity of cars sold, Q , on the horizontal axis, draw the demand and supply curves and find the equilibrium p and Q . (Note these graphs are different from those drawn in class, where the horizontal axis was θ .)



← If the price is \$6000, all sellers will want to sell, so 8000 cars will be supplied. If the price is \$2000, only the owners of the two worst cars will want to sell.

Final Exam
1999

Answer 4

If 2 cars are sold,
 $\bar{\theta} = 2000$, so
consumers are willing
to pay \$2200.

If 8000 cars are sold,
the average quality $\hat{\theta}$ is
4000, so consumers are
willing to pay \$4400.

Supply curve: $P = \frac{4000}{7998}Q + 2000$

Demand curve: $P = \frac{2200}{7998}Q + 2200$

$$\frac{4000}{7998}Q + 2000 = \frac{2200}{7998}Q + 2200$$

$$\frac{1800}{7998}Q = 200$$

$$Q = \frac{200 \cdot 7998}{1800} = \frac{7998}{9} = \frac{2664}{3} = 888\frac{2}{3}$$

$$P = \frac{4000}{7998} \cdot \frac{7998}{9} + 2000$$

$$= \frac{4000}{9} + 2000 = \frac{4000}{9} + \frac{18,000}{9}$$

$$= \frac{22,000}{9} = 2444.\overline{4} = 2444\frac{4}{9}$$

Final Exam
1994
Question 3

(3)

3. A consumer's utility function is given by $u(w) = \sqrt{w}$ where w is wealth. The consumer is an expected utility maximizer, and his initial wealth is \$16. Denote by v the amount of money the consumer would be willing to pay in order to obtain the following lottery ticket.

"With probability 1/4, you are the Row player in the following game, and with probability 3/4, you are the Column player in the same game. The game is as follows, with payoffs in dollars.

| | | Column | | |
|-----|---|--------|------|------|
| | | l | m | r |
| Row | t | 2, 3 | 0, 3 | 0, 0 |
| | m | 3, 0 | 2, 2 | 0, 3 |
| | b | 0, 0 | 2, 0 | 1, 2 |

(Do not consider mixed strategies.)

- (a) Find the equation which v must satisfy. (You do not have to calculate the numerical value of v . In fact, it is impossible to calculate the numerical value of v exactly.)
- (b) What is the expected value of the lottery?

Final Exam
1994
Answer 3

(3)

First find the Nash Equilibria of the game. Method 1:

If Row plays top then Column would play left or middle.

If Column played left, Row would not play top.

" " " middle, " " " "

So Row playing top is not a N.E. strategy.

If Row plays middle then Column would play right. If Column plays right, Row wou

not play middle, so Row playing middle is not a N.E. strategy. Answer 3 cont..

If Row plays bottom then Column will play right. If Column plays right, Row will play bottom. So (b, r) is a N.E.

Method 2:

If Column plays left then Row will play middle. If Row plays middle, Column will not play left. So Column playing left is not a N.E. strategy.

If Column plays middle then Row will play middle or bottom.

If Row plays middle, Column will not play middle.

If Row plays bottom, Column will not play middle.

So Column playing middle is not a N.E. strategy.

If Column plays right then Row will play bottom. If Row plays bottom, Column will play right. So (b, r) is a N.E.

(10pts)

Using either Method 1 or Method 2, (b, r) is the unique Nash Equilibrium in pure strategies.

So the payoff to the Row player is \$1 and the payoff to the Column player is \$2.

If r is the price of the lottery ticket, then :

- if the consumer does not buy the ticket, his wealth is $\$16$ and his utility is $\sqrt{16} = 4$

- if the consumer does buy the ticket, his wealth is $\$16 - r + (\frac{1}{4} \cdot \$1 + \frac{3}{4} \cdot \$2)$

and his utility is $u(\$16 - r + (\frac{1}{4} \cdot \$1 + \frac{3}{4} \cdot \$2))$

$$= u\left(\frac{1}{4}u(\$16 - r + \$1) + \frac{3}{4}u(\$16 - r + \$2)\right)$$

$$= \frac{1}{4}u(\$17 - r) + \frac{3}{4}u(\$18 - r) \quad (\text{because of expected utility hypothesis})$$

$$= \frac{1}{4}u(\$17 - r) + \frac{3}{4}u(\$18 - r)$$

(4pts)

Final Exam
1994
Answer 3 cont.

a) To be indifferent between buying and not buying, v must satisfy

$$4 = \frac{1}{4} \sqrt{17-v} + \frac{3}{4} \sqrt{18-v}. \quad (2 \text{ pts})$$

b) The expected value of the lottery is $\frac{1}{4}(\$1) + \frac{3}{4}(\$2) = \frac{\$7}{4} = \$1.75. \quad (4 \text{ pts})$