

Section 4:

Theory of the Firm

1. [10 points] Suppose a competitive firm has a production possibilities set denoted Y , an input requirement set denoted $V(y)$, and a production function denoted $f(\mathbf{x})$.
 - (a) Prove that if Y is a convex set then $V(y)$ is a convex set.
 - (b) Prove that the converse of the statement in part (a) is false; this is easily done by making a graph of a counterexample.
 - (c) Prove that $V(y)$ is a convex set if and only if $f(\mathbf{x})$ is a quasiconcave function.

Summer 2011 qualifying exam, Sec. 2 Qu. 1

Section 2 Question 1

production possibilities set Y

input requirement set $V(y)$

production function $f(x)$

a) Y being a convex set means that if

$$\begin{aligned} & (y, -\underline{x}) \in Y \\ \text{and } & (y, -\underline{x}') \in Y \end{aligned} \quad \left. \begin{array}{c} \\ \end{array} \right\} (1)$$

$$\text{then } t(y, -\underline{x}) + (1-t)(y, -\underline{x}') \in Y. \quad (2)$$

$$(2) \text{ implies } (ty + (1-t)y, -t\underline{x} - (1-t)\underline{x}') \in Y$$

$$(y, -t\underline{x} - (1-t)\underline{x}') \in Y$$

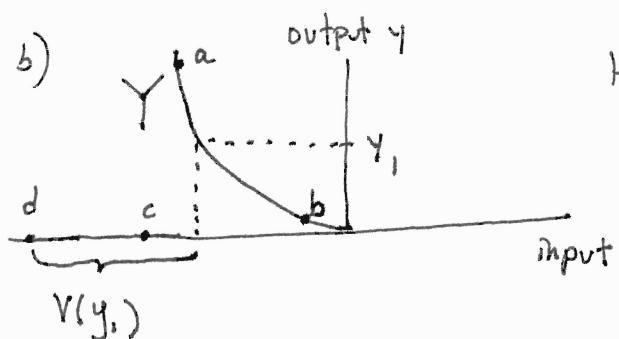
$$\Leftrightarrow t\underline{x} + (1-t)\underline{x}' \in V(y). \quad (3)$$

$$\begin{aligned} (1) \text{ implies } & \underline{x} \in V(y) \\ & \underline{x}' \in V(y). \end{aligned} \quad \left. \begin{array}{c} \\ \end{array} \right\} (4)$$

In particular,

$$\begin{array}{c} (1) \Rightarrow (2) \Rightarrow (3) \\ \Updownarrow \\ (4) \end{array}$$

so (4) \Rightarrow (3), meaning that $V(y)$ is a convex set.



Here Y is not a convex set (for example, $a \in Y$ and $b \in Y$ but points on a line between a and b are not in Y), but $V(y)$ for a typical y such as y_1 , is a convex

set (since points on a line between c and d are in $V(y_1)$).

c) $V(y)$ is $f(x)$'s upper contour set.

Quasiconcavity of f is defined to mean that f 's upper contour sets are convex sets.

So f is quasiconcave if and only if its upper contour sets, $V(y)$, are convex sets.

2. [11 points] The input requirement set given by

$$V(y) = \{(x_1, x_2) \in \mathbf{R}_+^2 : x_1^\alpha x_2^\beta \geq y\}$$

(for $y > 0$) is convex for what nonnegative values of α and β ?

(2)

V is the upper level set for $x_1^\alpha x_2^\beta$. For this to be a convex set, $x_1^\alpha x_2^\beta$ should be a quasiconcave function. This will be true if the bordered Hessian determinants of $x_1^\alpha x_2^\beta$, namely δ_r , alternate in sign beginning with (1) for $r = 2, \dots, n$. Here $n=2$, so all we need is $\delta_2 > 0$. Let $f = x_1^\alpha x_2^\beta$.

$$\begin{aligned} \delta_2 &= \begin{vmatrix} 0 & f'_1 & f'_2 \\ f'_1 & f''_{11} & f''_{12} \\ f'_2 & f''_{21} & f''_{22} \end{vmatrix} = \begin{vmatrix} 0 & \alpha \frac{x_1^\alpha x_2^\beta}{x_1} & \beta \frac{x_1^\alpha x_2^\beta}{x_2} \\ \alpha \frac{x_1^\alpha x_2^\beta}{x_1} & \alpha(\alpha-1) \frac{x_1^\alpha x_2^\beta}{x_1^2} & \beta \alpha \frac{x_1^\alpha x_2^\beta}{x_1 x_2} \\ \beta \frac{x_1^\alpha x_2^\beta}{x_2} & \alpha \beta \frac{x_1^\alpha x_2^\beta}{x_1 x_2} & \beta(\beta-1) \frac{x_1^\alpha x_2^\beta}{x_2^2} \end{vmatrix} \\ &= (x_1^\alpha x_2^\beta)^3 \begin{vmatrix} 0 & \frac{\alpha}{x_1} & \frac{\beta}{x_2} \\ \frac{\alpha}{x_1} & \frac{\alpha(\alpha-1)}{x_1^2} & \frac{\alpha\beta}{x_1 x_2} \\ \frac{\beta}{x_2} & \frac{\alpha\beta}{x_1 x_2} & \frac{\beta(\beta-1)}{x_2^2} \end{vmatrix} \begin{vmatrix} 0 & \frac{\alpha}{x_1} \\ \frac{\alpha}{x_1} & \frac{\alpha(\alpha-1)}{x_1^2} \\ \frac{\beta}{x_2} & \frac{\alpha\beta}{x_1 x_2} \end{vmatrix} \\ &\quad \text{positive (call it (1))} \qquad \text{to help calculate the determinant} \\ &= (1) \left\{ \frac{\alpha}{x_1} \frac{\alpha\beta}{x_1 x_2} \frac{\beta}{x_2} + \frac{\beta}{x_2} \frac{\alpha}{x_1} \frac{\alpha\beta}{x_1 x_2} - \left[\frac{\alpha}{x_1} \frac{\alpha}{x_1} \frac{\beta(\beta-1)}{x_2^2} + \frac{\beta}{x_2} \frac{\alpha(\alpha-1)}{x_1^2} \frac{\beta}{x_2} \right] \right\} \\ &= (1) \frac{\alpha\beta}{x_1^2 x_2^2} \left\{ 2\alpha\beta - \alpha(\beta-1) - (\alpha-1)\beta \right\} \\ &= (1) \frac{\alpha\beta}{x_1^2 x_2^2} \left\{ 2\alpha\beta - \alpha\beta + \alpha - \alpha\beta + \beta \right\} = (1) \frac{\alpha\beta}{x_1^2 x_2^2} \left\{ \alpha + \beta \right\} \end{aligned}$$

which is > 0 , as we desire, if $\alpha > 0$ and $\beta > 0$.

Final Exam

1997

Question 1

(2)

Answer all of the following five questions.

1. Suppose a firm's input requirement set is given by

$$V(y) = \{x_1, x_2 : ax_1 + \sqrt{x_1 x_2} + bx_2 \geq y\}.$$

Is $V(y)$ a convex set?

Answers to Final Exam, Econ. 621, Winter 1997

Final Exam

1997

Answer 1

- ① $V(y)$ will be a convex set if

$$f(x_1, x_2) = ax_1 + \sqrt{x_1 x_2} + bx_2$$

is a quasiconcave function, since V is f 's upper contour set, and since the definition of a quasiconcave function is a function whose upper contour set is convex.

One has

$$f'_1 = a + \frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}}$$

$$f'_2 = \frac{1}{2} x_1^{\frac{1}{2}} x_2^{-\frac{1}{2}} + b$$

$$f''_{11} = -\frac{1}{4} x_1^{-\frac{3}{2}} x_2^{\frac{1}{2}}$$

$$f''_{22} = -\frac{1}{4} x_1^{\frac{1}{2}} x_2^{-\frac{3}{2}}$$

$$f''_{12} = \frac{1}{4} x_1^{-\frac{1}{2}} x_2^{-\frac{1}{2}}$$

One approach is to directly test for $\overset{f}{\checkmark}$ quasiconcavity. We would form

$$\begin{bmatrix} 0 & f'_1 & f'_2 \\ f'_1 & f''_{11} & f''_{12} \\ f'_2 & f''_{21} & f''_{22} \end{bmatrix}$$

and the test would be whether the determinant of this matrix was positive. (Actually, this is the test for pseudconcavity.) Calculating this determinant

would be quite tedious.

Another approach is to test f for concavity. We would form $\begin{bmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{bmatrix}$ and the Hessian of f

Then check the principal minors as follows:

$$\Delta_1 = \{ f_{11}^{\prime\prime}, f_{22}^{\prime\prime} \} \stackrel{?}{\leq} 0$$

$$\Delta_2 = \{ f_{11}^{\prime\prime} f_{22}^{\prime\prime} - f_{12}^2 \} \stackrel{?}{\geq} 0.$$

Final Exam

1997

Answer 1 cont.

Recall that concave \Rightarrow quasiconcave.

A final approach is to check for strict concavity. Forming the Hessian of f as in the previous approach, we would require of the leading principal minors that:

$$D_1 = f_{11}^{\prime\prime} \stackrel{?}{<} 0$$

$$D_2 = f_{11}^{\prime\prime} f_{22}^{\prime\prime} - f_{12}^2 \stackrel{?}{>} 0.$$

Recall that strict concavity \Rightarrow concavity \Rightarrow quasiconcavity.

The approach with the fewest calculations is for strict concavity. The test is

$$D_1 = f_{11}^{\prime\prime} = \frac{-1}{4} x_1^{-3/2} x_2^{1/2} < 0 \text{ ok}$$

$$D_2 = \left(\frac{-1}{4} x_1^{-3/2} x_2^{1/2} \right) \left(\frac{-1}{4} x_1^{1/2} x_2^{-3/2} \right) - \left(\frac{1}{4} x_1^{-1/2} x_2^{-1/2} \right)^2$$

$$= \frac{1}{16} x_1^{-1} x_2^{-1} - \frac{1}{16} x_1^{-1} x_2^{-1} = 0 \text{ not ok (not } > 0).$$

So this test does not establish f as strictly concave.

The next test is for concavity. The test is $f_{11}^{\prime\prime} \leq 0$ (OK from above), $f_{22}^{\prime\prime} = \frac{-1}{4} x_1^{1/2} x_2^{-3/2} \leq 0$ (OK), and $f_{11}^{\prime\prime} f_{22}^{\prime\prime} - f_{12}^2 = 0 \geq 0$ (OK from above). So f is concave, so it is quasiconcave, so $V(y)$ is a convex set.

Exam 1

1999

Question 2

(2)

2. Is the input requirement set given by

$$V(y) = \{ (x_1, x_2) \in \mathbf{R}^2 : x_1 + x_1^{1/3} x_2^{2/3} + x_2 \geq y \}$$

convex, given that the output level y is strictly positive?

② Letting $f(\underline{x}) = x_1 + x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$, we have

$V(y) = \{ \underline{x} \in \mathbb{R}^2 : f(\underline{x}) \geq y \}$. Hence $V(y)$ is f 's upper contour set for $f(\underline{x}) = y$. The function f is quasiconcave if and only if its upper contour set ($V(y)$) is convex.

There are at least three ways of approaching this problem.

Method 1. Directly test if $f(\underline{x})$ is quasiconcave.

$$f'_1 = 1 + \frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}}$$

$$f'_2 = \frac{2}{3} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}} + 1$$

$$f''_{11} = -\frac{2}{9} x_1^{-\frac{5}{3}} x_2^{\frac{2}{3}}$$

$$f''_{22} = -\frac{2}{9} x_1^{\frac{1}{3}} x_2^{-\frac{4}{3}}$$

$$f''_{12} = \frac{2}{9} x_1^{-\frac{2}{3}} x_2^{-\frac{1}{3}}$$

Exam 1

1999

Answer 2

Form the bordered Hessian:

$$\begin{bmatrix} 0 & 1 + \frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}} & \frac{2}{3} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}} + 1 \\ 1 + \frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}} & -\frac{2}{9} x_1^{-\frac{5}{3}} x_2^{\frac{2}{3}} & \frac{2}{9} x_1^{-\frac{2}{3}} x_2^{-\frac{1}{3}} \\ \frac{2}{3} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}} + 1 & \frac{2}{9} x_1^{-\frac{2}{3}} x_2^{-\frac{1}{3}} & -\frac{2}{9} x_1^{\frac{1}{3}} x_2^{-\frac{4}{3}} \end{bmatrix}$$

Check whether $\delta_2 > 0$

δ_2 = determinant of the above 3×3 matrix

$$= (-1)^{1+2} \begin{vmatrix} 0 & 1 + \frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}} & \frac{2}{3} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}} + 1 \\ 1 + \frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}} & -\frac{2}{9} x_1^{-\frac{5}{3}} x_2^{\frac{2}{3}} & \frac{2}{9} x_1^{-\frac{2}{3}} x_2^{-\frac{1}{3}} \\ \frac{2}{3} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}} + 1 & \frac{2}{9} x_1^{-\frac{2}{3}} x_2^{-\frac{1}{3}} & -\frac{2}{9} x_1^{\frac{1}{3}} x_2^{-\frac{4}{3}} \end{vmatrix} \left(1 + \frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}} \right)$$

$$+ (-1)^{1+3} \begin{vmatrix} 0 & 1 + \frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}} & \frac{2}{3} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}} + 1 \\ 1 + \frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}} & -\frac{2}{9} x_1^{-\frac{5}{3}} x_2^{\frac{2}{3}} & \frac{2}{9} x_1^{-\frac{2}{3}} x_2^{-\frac{1}{3}} \\ \frac{2}{3} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}} + 1 & \frac{2}{9} x_1^{-\frac{2}{3}} x_2^{-\frac{1}{3}} & -\frac{2}{9} x_1^{\frac{1}{3}} x_2^{-\frac{4}{3}} \end{vmatrix} \left(\frac{2}{3} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}} + 1 \right)$$

$$= - \left[\frac{-2}{9} x_1^{1/3} x_2^{-4/3} - \frac{2}{27} x_1^{-1/3} x_2^{-2/3} - \frac{4}{27} x_1^{-1/3} x_2^{-2/3} - \frac{2}{9} x_1^{-2/3} x_2^{-1/3} \right] \left(1 + \frac{1}{3} x_1^{-2/3} x_2^{2/3} \right) \\ + \left[\frac{2}{9} x_1^{-2/3} x_2^{-1/3} + \frac{2}{27} x_1^{-4/3} x_2^{1/3} + \frac{4}{27} x_1^{-4/3} x_2^{1/3} + \frac{2}{9} x_1^{-5/3} x_2^{2/3} \right] \left(\frac{2}{3} x_1^{1/3} x_2^{-1/3} + 1 \right)$$

Then $\delta_2 > 0$ follows because
all the terms are positive by inspection.

Method 2: Test if $f(\mathbf{x})$ is strictly concave ; if it is, use : strict concavity \Rightarrow
Concavity \Rightarrow quasiconcavity.

If D_r of $\nabla^2 f$ alternate in sign beginning with < 0 for $r=1, 2$, then f is strictly
concave.

$$\nabla^2 f = \begin{bmatrix} \frac{-2}{9} x_1^{-5/3} x_2^{2/3} & \frac{2}{9} x_1^{-2/3} x_2^{-1/3} \\ \frac{2}{9} x_1^{-2/3} x_2^{-1/3} & \frac{-2}{9} x_1^{1/3} x_2^{-4/3} \end{bmatrix} \text{ from p.2.}$$

Exam 1
1999

Answer 2 cont...

$$D_1 = \frac{-2}{9} x_1^{-5/3} x_2^{2/3} < 0$$

$$D_2 = \left(\frac{-2}{9} x_1^{-5/3} x_2^{2/3} \right) \left(\frac{-2}{9} x_1^{1/3} x_2^{-4/3} \right) - \left(\frac{2}{9} x_1^{-2/3} x_2^{-1/3} \right) \left(\frac{2}{9} x_1^{-2/3} x_2^{-1/3} \right)$$

$$= \frac{4}{81} x_1^{-4/3} x_2^{-2/3} - \frac{4}{81} x_1^{-4/3} x_2^{-2/3}$$

$$= 0.$$

The sufficient condition for strict concavity fails.

Method 3: Test if $f(\underline{x})$ is concave; if it is, use: concave \Rightarrow quasiconcave.

All Δ_r of $\nabla^2 f$ alternate in sign beginning with ≤ 0 for $r=1, 2 \Leftrightarrow f$ concave.

$\nabla^2 f$ is given on p. 3.

$$\Delta_1 \text{ of } \nabla^2 f = \left\{ \frac{-2}{9} x_1^{-5/3} x_2^{2/3}, \frac{-2}{9} x_1^{2/3} x_2^{-5/3} \right\} = \{ 0, 0 \}$$

Exam 1
1999
Answer 2 cont...

$$\Delta_2 \text{ of } \nabla^2 f = |\nabla^2 f| = 0 \text{ from the bottom of p. 3.}$$

So f is concave, and hence quasiconcave.

Fall 2004
Final

3. [12 points] Consider the input requirement set

$$V(y) = \{\mathbf{x} \in \mathbf{R}_+^2 : x_1 \geq y, \frac{1}{2}x_2 \geq y\}.$$

- (a) Is this technology monotonic?
- (b) Is V convex? (Prove this formally.)
- (c) What is the production function for this technology?

(3)

$$V(y) = \{ \underline{x} \in \mathbb{R}^2 \mid x_1 \geq y, \frac{1}{2}x_2 \geq y \}$$

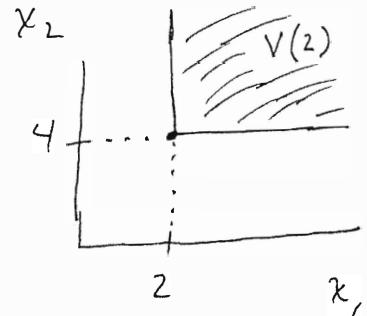
Fall 2004
Final

If $y=2$ then $x_1 \geq 2$ and $\frac{1}{2}x_2 \geq 2$

\Updownarrow

$x_2 \geq 4$ which looks like

The production function is
 $\min(x_1, \frac{1}{2}x_2)$.



$V(y)$ for $y \neq 2$ will look similar.

This is monotonic because if $\underline{x} \in V(y)$ then any $\underline{x}' \geq \underline{x}$ is also in $V(y)$. Graphically, if $\underline{x} \in V(y)$ then any point above or to the right of \underline{x} is also in $V(y)$. Analytically, if $(x_1, x_2) \in V(y)$

then $(x_1 + \delta, x_2 + \varepsilon) \in V(y)$ for $\delta \geq 0, \varepsilon \geq 0$ because

$$(x_1, x_2) \in V(y) \Leftrightarrow x_1 \geq y \text{ and } \frac{1}{2}x_2 \geq y$$

\Downarrow

\Downarrow

$$\underbrace{x_1 + \delta \geq y \text{ and } \frac{1}{2}(x_2 + \varepsilon) \geq y}_{\Downarrow} \dots \text{since } \delta \geq 0, \varepsilon \geq 0$$

\Downarrow

$$(x_1 + \delta, x_2 + \varepsilon) \in V(y).$$

$V(y)$ would be convex if $(a_1, a_2) = \underline{a} \in V(y)$ and

$(b_1, b_2) = \underline{b} \in V(y)$ implied that

$$\lambda \underline{a} + (1-\lambda) \underline{b} \in V(y), \text{ where } 0 < \lambda < 1.$$

Let's check:

$$\underline{a} \in V(y) \Leftrightarrow a_1 \geq y \text{ and } \frac{1}{2}a_2 \geq y \quad (1)$$

$$\underline{b} \in V(y) \Leftrightarrow b_1 \geq y \text{ and } \frac{1}{2}b_2 \geq y \quad (2)$$

$$\lambda \underline{a} + (1-\lambda) \underline{b} \in V(y) \Leftrightarrow \lambda a_1 + (1-\lambda) b_1 \geq y \text{ and } \frac{1}{2}(\lambda a_2 + (1-\lambda) b_2) \geq y \quad (3)$$

$$(1) \Rightarrow \lambda a_1 \geq \lambda y \text{ and } \frac{1}{2}\lambda a_2 \geq \lambda y \quad (4)$$

$$(2) \Rightarrow (1-\lambda) b_1 \geq (1-\lambda) y \text{ and } \frac{1}{2}(1-\lambda) b_2 \geq (1-\lambda) y \quad (5)$$

The first parts of (4) and (5) \Rightarrow (by addition) that

$$\lambda a_1 + (1-\lambda) b_1 \geq \lambda y + (1-\lambda) y = y$$

which is the first part of (3)'s conclusion.

The second parts of (4) and (5) \Rightarrow (by addition) that

$$\frac{1}{2}\lambda a_2 + \frac{1}{2}(1-\lambda) b_2 \geq \lambda y + (1-\lambda) y \Leftrightarrow$$

$$\frac{1}{2}[\lambda a_2 + (1-\lambda) b_2] \geq y$$

which is the second part of (3)'s conclusion. So (3) holds and $V(y)$ is convex.

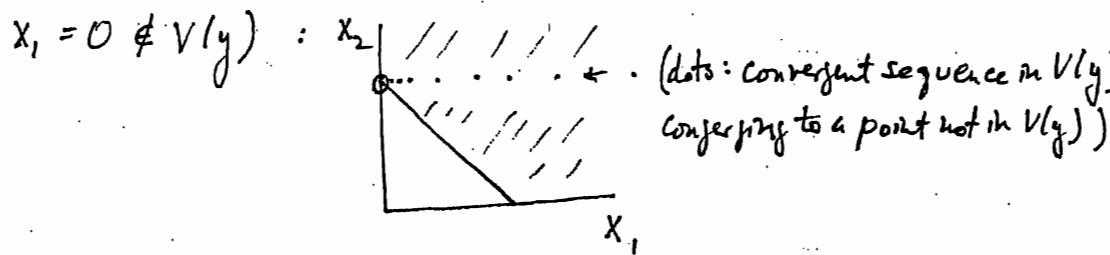
(2)

4. Suppose a firm produces output y from inputs x_1 and x_2 .
- (a) Give an example of an input requirement set for this technology which is not closed.
 - (b) Give an example of an input requirement set for this technology which is empty for at least some y .
 - (c) Give an example of an input requirement set for this technology which is not monotonic.
 - (d) Give an example of an input requirement set for this technology which is not convex.

4. Answers will vary here. From Varian's problem 1.11 :

a) $V(y) = \{x_1, x_2 : ax_1 + bx_2 \geq y, x_i > 0\}$ is not closed.

This is because for fixed a, b, x_2 , and y , one could construct a convergent sequence of x_1 's in $V(y)$ which would converge to



b) $V(y) = \{x_1, x_2 : x_1(1-y) \geq a, x_2(1-y) \geq b\}$ is empty for some y .

Reason: $x_1(1-y) \geq a \Rightarrow 1-y \geq \frac{a}{x_1} \Rightarrow 1 - \frac{a}{x_1} \geq y$, so there is no way to produce $y > 1$.

c) $V(y) = \{x_1, x_2 : ax_1 - \sqrt{x_1 x_2} + bx_2 \geq y\}$ is not always monotonic. Reason:

The production function is $f(x) = ax_1 - \sqrt{x_1 x_2} + bx_2$

$$\frac{\partial f}{\partial x_1} = a - \frac{1}{2} \sqrt{\frac{x_2}{x_1}}$$

which is not always positive

d) Calculate the Hessian of the production function in part (c) :

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1} \left[a - \frac{1}{2} x_1^{-1/2} x_2^{1/2} \right] = \frac{1}{4} x_1^{-3/2} x_2^{1/2}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = -\frac{1}{4} x_1^{-1/2} x_2^{-1/2}$$

$$\frac{\partial f}{\partial x_2} = -\frac{1}{2} x_1^{1/2} x_2^{-1/2} + b \Rightarrow \frac{\partial^2 f}{\partial x_2^2} = \frac{1}{4} x_1^{1/2} x_2^{-3/2} \quad \text{so}$$

Exam 1

1994

Answer 4 cont...

$$\nabla^2 f = \begin{bmatrix} \frac{1}{4} x_1^{-3/2} x_2^{1/2} & -\frac{1}{4} x_1^{-1/2} x_2^{-1/2} \\ -\frac{1}{4} x_1^{-1/2} x_2^{-1/2} & \frac{1}{4} x_1^{1/2} x_2^{-3/2} \end{bmatrix}$$

$$\tilde{D}_2 = |\nabla^2 f| = \frac{1}{16} x_1^{-1} x_2^{-1} - \frac{1}{16} x_1^{-1} x_2^{-1} = 0$$

$$\tilde{D}_1 = \frac{1}{4} x_1^{-3/2} x_2^{1/2} > 0 \text{ and } \frac{1}{4} x_1^{1/2} x_2^{-3/2} > 0$$

$\Rightarrow f$ is convex $\Rightarrow f$ is quasi-convex \Rightarrow lower contour

set is convex (with a boundary that is not a straight line) \Rightarrow upper contour set ($= V(y)$) is not convex

Section 2.
Answer one of the following two questions.

1. [8 points] Suppose a competitive firm uses water w to produce output q according to the production function $q = \sqrt{w}$. Suppose the price of water is 1.
 - (a) The firm's Board of Directors is considering hiring a new Chief Executive Officer ("CEO") who claims to be able to change the firm's production function to $q = 2\sqrt{w}$. How much should the firm be willing to pay this new CEO?
 - (b) The notion of a firm having *two* production functions—namely $q = \sqrt{w}$ under the old CEO and $q = 2\sqrt{w}$ under the new CEO—is completely inconsistent with the standard neoclassical idea of "a production function."
 - i. Describe this situation in a way that *is* compatible with standard neoclassical idea of "a production function."
 - ii. Once you have formulated the correct production function, graph it. This may require a three-dimensional graph, but it should be a rather easy one to draw.
 - iii. Is the technology monotonic? convex? regular? Why?
 - iv. Are all inputs to production being paid their marginal product? Why or why not? (If you think there are good arguments on both sides of this question, you need not decide which arguments are better: just give both sides' arguments.)

Note that in the US, the position of CEO is almost always a full-time position held by one person. Assume that is true in this problem.

Summer 2013 Qualifying Exam Section 2 Question 1

Section 2 Question 1.

Assume the price of water is 1.

a) Under the current CEO : $q = \sqrt{w}$. Also, $\pi = p_{\text{output}} \cdot \text{output} - p_{\text{input}} \cdot \text{input}$

$$= p \sqrt{w} - (1) w = p \sqrt{w} - w.$$

↑
price of
output

Maximizing π : $0 = \pi'_w = \frac{1}{2} p w^{-1/2} - 1 \Rightarrow$

$$1 = \frac{1}{2} p w^{-1/2}$$

$$\frac{2}{p} = w^{-1/2} \Rightarrow \frac{p}{2} = w^{1/2} \Rightarrow \frac{p^2}{4} = w^* \text{. Then}$$

$$q^* = \sqrt{w^*} = \sqrt{p^2/4} = p/2 \text{ and } \pi^* = p \sqrt{w^*} - w^* = p \sqrt{\frac{p^2}{4}} - \frac{p^2}{4} = p \cdot \frac{p}{2} - \frac{p^2}{4} = \frac{p^2}{2} - \frac{p^2}{4} = \frac{p^2}{4}.$$

Under the proposed new CEO : $\hat{q} = 2\sqrt{\hat{w}}$. (The $\hat{\cdot}$ just refers to contrast the variables under the new CEO with the variables under the current CEO.)

$$\hat{\pi} = p \cdot 2\sqrt{\hat{w}} - \hat{w}. \text{ (Note that } p \text{ is the same in both cases since the firm is competitive.)}$$

$$\text{Max } \hat{\pi} \Rightarrow 0 = \hat{\pi}'_w = 2p \cdot \frac{1}{2} \hat{w}^{-1/2} - 1 = p \hat{w}^{-1/2} - 1 \Rightarrow$$

$$1 = p \hat{w}^{-1/2} \Rightarrow 1 = p^2 \hat{w}^{-1} \Rightarrow p^{-2} = \hat{w}^{-1} \Rightarrow \hat{w}^* = p^2.$$

$$\text{Then } \hat{q}^* = 2\sqrt{\hat{w}^*} = 2\sqrt{p^2} = 2p \text{ and } \hat{\pi}^* = p \cdot 2\sqrt{\hat{w}^*} - \hat{w}^* \\ = p \cdot 2\sqrt{p^2} - p^2 = p \cdot 2p - p^2 = 2p^2 - p^2 = p^2.$$

$$\text{Therefore } \pi^* = p^2/4$$

$$\hat{\pi}^* = p^2.$$

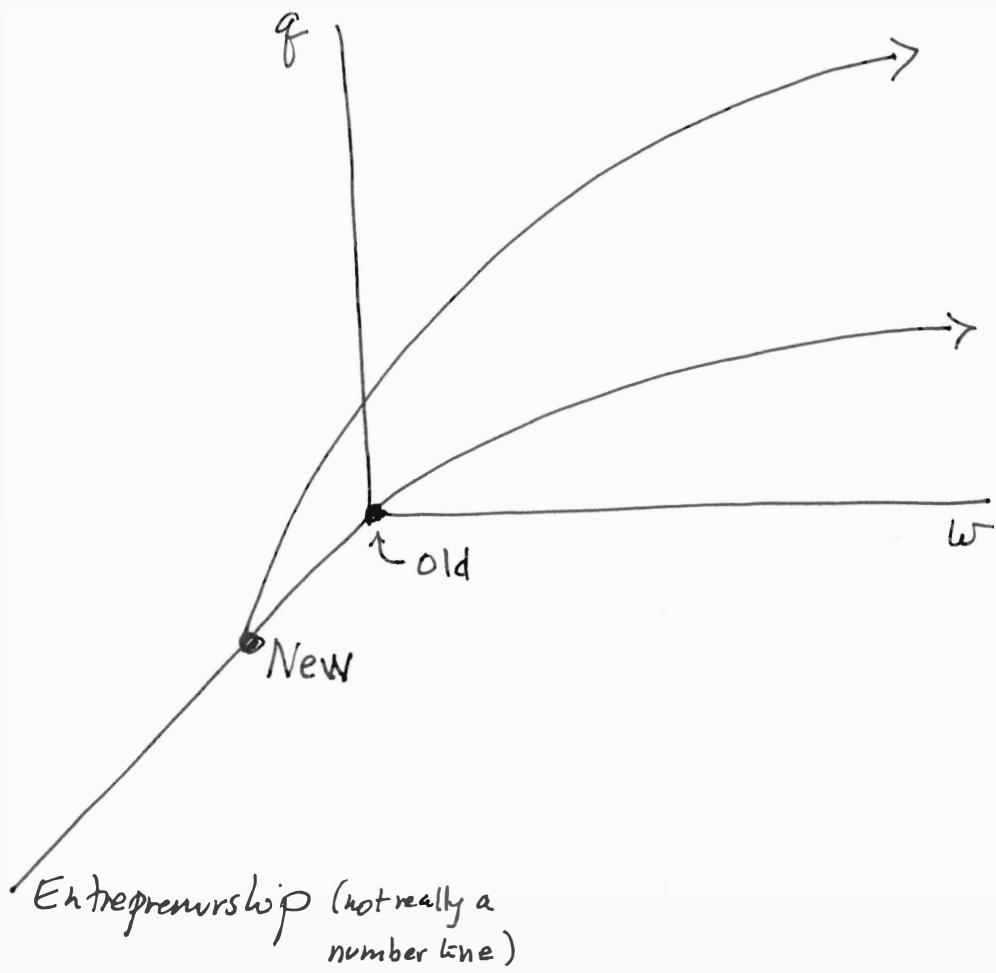
The new CEO would be responsible for profit rising from π^* to $\hat{\pi}^*$, so the firm would be willing to pay the new CEO at most

$$\hat{\pi}^* - \pi^* = p^2 - \frac{1}{4}p^2 = \underline{\frac{3}{4}p^2}.$$

b) (i) Firms cannot have more than one production function. To formulate the situation correctly, posit output as a function of two inputs, water and "CEO-ship" or "entrepreneurship" or "management" or some other term to that effect. Water is in \mathbb{R}_1^+ . "Entrepreneurship" is not in \mathbb{R}_1^+ , since it only takes two values, "current CEO" and "new CEO." So the production function is

$$q > f(w, \text{Entrepreneurship}) = \begin{cases} \sqrt{w} & \text{if Entrepreneurship = current CEO} \\ 2\sqrt{w} & \text{if Entrepreneurship = New CEO.} \end{cases}$$

(ii)



(iii) Monotonicity : It is monotonic in w because $\uparrow w \Rightarrow \uparrow g$.

Monotonicity in entrepreneurship seems hard to define, because the notion of "increasing entrepreneurship" requires entrepreneurship to have a numerical value, which it does not have here.

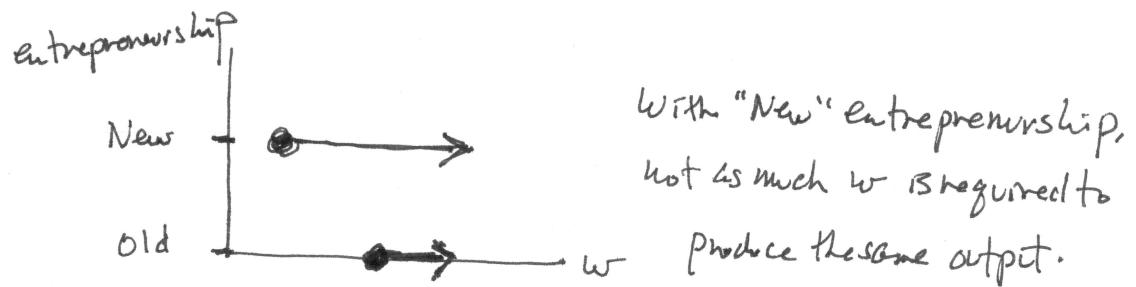
(Entrepreneurship here is a person, not a number.) Furthermore, monotonicity could not be defined beyond the {new, old} set.

Convex : Points strictly between "old CEO" and "new CEO" are undefined - one either has one CEO or the other. No mixtures are possible. So the set is not convex. (The question says to assume the CEO

Regular : This means the input requirement set

position is full-time and is held by only one person.)

is closed and nonempty. It is nonempty because some output can be produced. For closure : is the limit of any convergent sequence in $V(y)$ also in $V(y)$? $V(y)$ looks like :



No sequence which bounces between points on the "New" line and points on the "Old" line forever is going to converge. So the only convergent sequences eventually stay on one of the two lines. Each line is closed. So $V(y)$ is closed, and hence regular.

(iv) The marginal product of entrepreneurship cannot be defined because entrepreneurship is not differentiable. (A "small change" in entrepreneurship makes no sense.) So the CEO is not paid his or her "marginal product," despite the answer to part (a). (I.e., it's incorrect to interpret part (a) as calculating a "marginal product.")

Final Exam

1997

Question 2

(2)

2. Suppose a price-taking firm uses inputs x_1 and x_2 to produce output y according to the production function $y = \sqrt{x_1} + \sqrt{x_2}$. Find this firm's cost function.

You should state the second-order conditions for this problem, but you do not have to verify that these conditions hold. You may state the conditions symbolically in terms of the derivatives of the Lagrangian. (If this had been a qualifying exam, I may have asked you to verify that the second-order conditions held.)

(2) $\min_{x_1, x_2} w_1 x_1 + w_2 x_2 \text{ s.t. } y = \sqrt{x_1} + \sqrt{x_2} \quad (w_i \text{ is the price of input } i.)$

$$\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda (y - \sqrt{x_1} - \sqrt{x_2}) \quad \begin{array}{l} \text{Final Exam} \\ 1997 \end{array}$$

F.O.C.: $0 = \frac{\partial \mathcal{L}}{\partial \lambda} = y - \sqrt{x_1} - \sqrt{x_2} \quad \text{Answer 2}$

$$0 = \frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \frac{\lambda}{2} x_1^{-\frac{1}{2}} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \frac{w_1}{w_2} = \frac{\lambda/2}{\lambda/2} \frac{x_1^{-\frac{1}{2}}}{x_2^{-\frac{1}{2}}} \Rightarrow \frac{w_1}{w_2} = \frac{x_2^{\frac{1}{2}}}{x_1^{\frac{1}{2}}}$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_2} = w_2 - \frac{\lambda}{2} x_2^{-\frac{1}{2}} \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow x_2^{\frac{1}{2}} = \frac{w_1}{w_2} x_1^{\frac{1}{2}}$$

$$\Rightarrow x_2 = \left(\frac{w_1}{w_2} \right)^2 x_1.$$

$$\text{So } y = \sqrt{x_1} + \sqrt{x_2} = \sqrt{x_1} + \frac{w_1}{w_2} x_1^{\frac{1}{2}} = \left(1 + \frac{w_1}{w_2} \right) x_1^{\frac{1}{2}} = \frac{w_2 + w_1}{w_2} x_1^{\frac{1}{2}} \Rightarrow$$

$$x_1^{\frac{1}{2}} = \frac{w_2}{w_1 + w_2} y \quad \text{and} \quad x_1^* = \frac{w_2^2}{(w_1 + w_2)^2} y^2.$$

$$\text{Then } x_2^* = \frac{w_1^2}{w_2^2} \left(\frac{w_2^2}{(w_1 + w_2)^2} y^2 \right) = \frac{w_1^2}{(w_1 + w_2)^2} y^2 = \left(\frac{w_1}{w_1 + w_2} \right)^2 y^2.$$

The cost function then becomes

$$c(w, y) = w_1 x_1^* + w_2 x_2^* = \frac{w_1 w_2^2}{(w_1 + w_2)^2} y^2 + \frac{w_2 w_1^2}{(w_1 + w_2)^2} y^2$$

$$= (w_1 w_2^2 + w_2 w_1^2) \frac{y^2}{(w_1 + w_2)^2} = w_1 w_2 (w_2 + w_1) \frac{y^2}{(w_1 + w_2)^2} = \frac{w_1 w_2}{w_1 + w_2} y^2.$$

For second-order conditions, consider the Hessian of the Lagrangian:

Final Exam

1997

Answer 2 cont...

4

$$\nabla^2 \mathcal{L} = \begin{bmatrix} \mathcal{L}_{\lambda\lambda}'' & \mathcal{L}_{\lambda x_1}'' & \mathcal{L}_{\lambda x_2}'' \\ \mathcal{L}_{x_1\lambda}'' & \mathcal{L}_{x_1 x_1}'' & \mathcal{L}_{x_1 x_2}'' \\ \mathcal{L}_{x_2\lambda}'' & \mathcal{L}_{x_2 x_1}'' & \mathcal{L}_{x_2 x_2}'' \end{bmatrix}$$

number of constraints "m" = 1 ; number of variables "n" = 2
 $(-1)^m$ is negative

$$m+n = 1+2 = 3 : 2m+1 = 3$$

S.O.C.: D_3 of $\nabla^2 \mathcal{L}$ should be negative.

This is the same as $|\nabla^2 \mathcal{L}| < 0$.

The S.O.C. is: $D_{2m+1}, D_{2m+2}, \dots, D_{m+n}$ of $\nabla^2 \mathcal{L}$ all have the same sign as $(-1)^m$, which here is negative. This is the S.O.C. for minimization.

2015 Qualifying Exam Sec. 1 Qu. 1

1. **[14 points]** Suppose a price-taking firm uses inputs x_1 and x_2 to produce output y according to the production function $y = \sqrt{x_1} + \sqrt{x_2}$. Find this firm's cost function.

Explicitly verify that the second-order conditions hold in this particular problem. (It is not enough to state the conditions symbolically in terms of the derivatives of the Lagrangian.)

Answers to Prof. Lozada's Questions on the
Summer 2015 Qualifying Exam

Sector 1 Question 1.

This is identical to the previous question, except for verifying the second-order conditions. One has

$$\nabla^2 \mathcal{L} = \begin{bmatrix} 0 & -\frac{1}{2}x_1^{-\frac{1}{2}} & -\frac{1}{2}x_2^{-\frac{1}{2}} \\ -\frac{1}{2}x_1^{-\frac{1}{2}} & \frac{1}{4}x_1^{-\frac{3}{2}} & 0 \\ -\frac{1}{2}x_2^{-\frac{1}{2}} & 0 & \frac{1}{4}x_2^{-\frac{3}{2}} \end{bmatrix}$$

$$|\nabla^2 \mathcal{L}| = \text{(expanding by the first row)}$$

$$\begin{aligned} & (-1)^{1+2} \left(-\frac{1}{2}x_1^{-\frac{1}{2}} \right) \left[-\frac{\lambda}{8}x_1^{-\frac{1}{2}}x_2^{-\frac{3}{2}} - 0 \right] + (-1)^{1+3} \left(-\frac{1}{2}x_2^{-\frac{1}{2}} \right) \left[0 + \frac{\lambda}{8}x_1^{-\frac{3}{2}}x_2^{-\frac{1}{2}} \right] \\ &= (-1) \frac{\lambda}{16} x_1^{-1} x_2^{-\frac{3}{2}} - \frac{\lambda}{16} x_1^{-\frac{3}{2}} x_2^{-1} \\ &= -\frac{\lambda}{16} \underbrace{x_1^{-\frac{3}{2}} x_2^{-\frac{3}{2}}}_{\textcircled{+}} \underbrace{\left[x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}} \right]}_{=y > 0} \end{aligned}$$

From the Envelope Theorem, $\frac{\partial(\text{minimized cost})}{\partial y} = \lambda$. The left-hand side is positive because it is Marginal Cost, so $|\nabla^2 \mathcal{L}| = -\frac{\lambda}{16} (\textcircled{+})(\textcircled{+}) < 0$ and the S.O.C. are satisfied. Optional: "Marginal cost is positive" can be shown either by:

•) calculating $\frac{\partial}{\partial y} c(w, y) = \frac{w_1 w_2}{w_1 + w_2} - 2y > 0$; or by

•) noting that if $MC < 0$ then there exist $y_1 < y_2$ such that

$$c(y_1) > c(y_2).$$

But y_1 can also be produced by producing y_2 (at a cost of $c(y_2)$)

and then throwing $y_2 - y_1$ away ; and since this only costs $c(y_2)$, it is cheaper than $c(y_1)$, contradicting the definition of $c(y_1)$ as being the least expensive way of producing y_1 . So $MC < 0$ leads to a contradiction.

2. [12 points]

Suppose a price-taking firm uses inputs x_1 and x_2 to produce output y according to the production function $y = \sqrt{x_1} + \sqrt{x_2}$.

- (a) Find this firm's cost function.
- (b) Do the second-order conditions for this problem hold?
- (c) How would the analysis in (a) and (b) change if the firm were not a price taker?

2008 Qualifier
Sec. 2

(Before like preceding
question.)

2008 Qualifier
Sec. C. 2 #2

\Rightarrow This question begins in the same way as

9

(2) $\min_{x_1, x_2} w_1 x_1 + w_2 x_2 \text{ s.t. } y = \sqrt{x_1} + \sqrt{x_2} \quad (w_i \text{ is the price of input } i.)$

$$\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda (y - \sqrt{x_1} - \sqrt{x_2})$$

F.O.C.: $0 = \frac{\partial \mathcal{L}}{\partial \lambda} = y - \sqrt{x_1} - \sqrt{x_2}$

Final Exam
1997

Answer 2

which precedes this in this packet.

$$\left. \begin{array}{l} 0 = \frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \frac{\lambda}{2} x_1^{-\frac{1}{2}} \\ 0 = \frac{\partial \mathcal{L}}{\partial x_2} = w_2 - \frac{\lambda}{2} x_2^{-\frac{1}{2}} \end{array} \right\} \quad \left. \begin{array}{l} \frac{w_1}{w_2} = \frac{\lambda/2}{\lambda/2} \frac{x_1^{-\frac{1}{2}}}{x_2^{-\frac{1}{2}}} \Rightarrow \frac{w_1}{w_2} = \frac{x_2^{\frac{1}{2}}}{x_1^{\frac{1}{2}}} \\ \Rightarrow x_2^{\frac{1}{2}} = \frac{w_1}{w_2} x_1^{\frac{1}{2}} \end{array} \right.$$

$$\Rightarrow x_2 = \left(\frac{w_1}{w_2} \right)^2 x_1.$$

$$\text{So } y = \sqrt{x_1} + \sqrt{x_2} = \sqrt{x_1} + \frac{w_1}{w_2} x_1^{\frac{1}{2}} = \left(1 + \frac{w_1}{w_2} \right) x_1^{\frac{1}{2}} = \frac{w_1 + w_2}{w_2} x_1^{\frac{1}{2}} \Rightarrow$$

$$x_1^{\frac{1}{2}} = \frac{w_2}{w_1 + w_2} y \quad \text{and} \quad x_1^* = \left(\frac{w_2}{w_1 + w_2} \right)^2 y^2.$$

$$\text{Then } x_2^* = \frac{w_1^2}{w_2^2} \left(\frac{w_2^2}{w_1 + w_2} \right)^2 y^2 = \frac{w_1^2}{(w_1 + w_2)^2} y^2 = \left(\frac{w_1}{w_1 + w_2} \right)^2 y^2.$$

The cost function then becomes

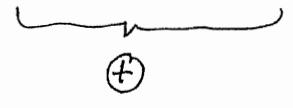
$$c(w, y) = w_1 x_1^* + w_2 x_2^* = \frac{w_1 w_2^2}{(w_1 + w_2)^2} y^2 + \frac{w_2 w_1^2}{(w_1 + w_2)^2} y^2$$

$$= (w_1 w_2^2 + w_2 w_1^2) \frac{y^2}{(w_1 + w_2)^2} = w_1 w_2 (w_2 + w_1) \frac{y^2}{(w_1 + w_2)^2} = \frac{w_1 w_2}{w_1 + w_2} y^2.$$

For second-order conditions, consider the Hessian of the Lagrangian:

$$\nabla^2 \mathcal{L} = \begin{bmatrix} 0 & -\frac{1}{2} x_1^{-1/2} & -\frac{1}{2} x_2^{-1/2} \\ -\frac{1}{2} x_1^{-1/2} & \frac{\lambda}{4} x_1^{-3/2} & 0 \\ -\frac{1}{2} x_2^{-1/2} & 0 & \frac{\lambda}{4} x_2^{-3/2} \end{bmatrix}$$

$$\begin{aligned}
 D_3 \text{ of } \nabla^2 \mathcal{L} &= (-1)^{2+1} \left(-\frac{1}{2} x_1^{-1/2} \right) \left[-\frac{1}{2} x_1^{-1/2}, \frac{\lambda}{4} x_2^{-3/2} \right] \\
 &\quad + (-1)^{2+2} \left(-\frac{1}{2} x_2^{-1/2} \right) \left[0 - \left(-\frac{1}{2} x_2^{-1/2} \right) \left(\frac{\lambda}{4} x_2^{-3/2} \right) \right] \\
 &= -\frac{\lambda}{16} x_1^{-1} x_2^{-3/2} - \frac{\lambda}{16} x_1^{-3/2} x_2^{-1} \\
 &= -\frac{\lambda}{16} x_1^{-3/2} x_2^{-3/2} \left[x_1^{1/2} + x_2^{1/2} \right] < 0 \text{ for positive } x_1 \text{ and } x_2,
 \end{aligned}$$

so the second-order conditions are satisfied. Note that $\lambda > 0$ since, for example, from the second F.O.C., $\lambda = 2w_1\sqrt{x_1} > 0$.

c) The problem of part (a),

$$\min_{x_1, x_2} w_1 x_1 + w_2 x_2 \text{ s.t. } y = \sqrt{x_1} + \sqrt{x_2} \text{ with } y \text{ fixed,}$$

becomes

$$\min_{x_1, x_2} w_1(x_1) x_1 + w_2(x_2) x_2 \text{ s.t. } y = \sqrt{x_1} + \sqrt{x_2} \text{ with } y \text{ fixed.}$$

The Lagrangian becomes

$$\mathcal{L} = w_1(x_1) x_1 + w_2(x_2) x_2 + \lambda (y - \sqrt{x_1} - \sqrt{x_2})$$

and the F.O.C. become

$$0 = \mathcal{L}'_{\lambda} = y - \sqrt{x_1} - \sqrt{x_2}$$

$$0 = \mathcal{L}'_{x_1} = w_1' x_1 + w_1 - \frac{\lambda}{2} x_1^{-1/2}$$

$$0 = \mathcal{L}'_{x_2} = w_2' x_2 + w_2 - \frac{\lambda}{2} x_2^{-1/2}$$

These can no longer be explicitly solved for x_1^* and x_2^* because of the presence of the unspecified functions $w_1'(x_1)$ and $w_2'(x_2)$ (and $w_1(x_1)$ and $w_2(x_2)$).
 Similarly, $\nabla^2 \mathcal{L}$ becomes

$$\begin{bmatrix} 0 & -\frac{1}{2} x_1^{-1/2} & -\frac{1}{2} x_2^{-1/2} \\ -\frac{1}{2} x_1^{-1/2} & w_1'' x_1 + w_1' + \frac{\lambda}{4} x_1^{-3/2} & 0 \\ -\frac{1}{2} x_2^{-1/2} & 0 & w_2'' x_2 + w_2' + \frac{\lambda}{4} x_2^{-3/2} \end{bmatrix}$$

which has unknown functions $w_1'(x_1)$, $w_1''(x_1)$, $w_2'(x_2)$, and $w_2''(x_2)$. Without more information, it is impossible to determine if the S.O.C. are satisfied or not.

4. [8 points] Suppose a price-taking firm has a production function

$$y = x_1 x_2$$

where y is output and x_1 and x_2 are inputs. These inputs have prices w_1 and w_2 , respectively.

Suppose that during one time period, the firm spent \$4 on inputs. How much output did it produce during this period? The only variables in your answer should be w_1 and w_2 .

Final Exam, Fall 2013

④ This problem is a variation of Varian's problem 4.8.
Method 1: Cost Minimization.

$$\min_{x_1, x_2} w_1 x_1 + w_2 x_2 \text{ s.t. } y = f(x_1, x_2) = x_1 x_2 .$$

$$\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda (y - x_1 x_2)$$

$$0 = \partial \mathcal{L} / \partial \lambda = y - x_1 x_2$$

$$0 = \partial \mathcal{L} / \partial x_1 = w_1 - \lambda x_2$$

$$0 = \partial \mathcal{L} / \partial x_2 = w_2 - \lambda x_1$$

Mistake here: should be

$$\lambda = w_1/x_2 = w_2/x_1 .$$

$$\lambda = \frac{x_2}{w_1} = \frac{x_1}{w_2} . \quad [\text{Note that you}]$$

can also arrive at this expression by recalling

the general optimality condition $\frac{MP_1}{w_1} = \frac{MP_2}{w_2}$ where "MP" is "marginal product." However, it's better to derive this condition, instead of merely recalling it.]

So $x_2 = \frac{w_1}{w_2} x_1$. (One could now continue to solve for x_1^* and x_2^* as functions of w_1, w_2 , and y , but it's not necessary in this problem.)

We know that $y = w_1 x_1 + w_2 x_2$. Using $x_2 = \frac{w_1}{w_2} x_1$, gives

$$y = w_1 x_1 + w_2 \frac{w_1}{w_2} x_1 = w_1 x_1 + w_1 x_1 = 2w_1 x_1 \Rightarrow$$

$$x_1 = \frac{2}{w_1}, \text{ so}$$

$$x_2 = \frac{w_1}{w_2} \left(\frac{2}{w_1} \right) = \frac{2}{w_2} \text{ and}$$

$$y = x_1 x_2 = \frac{2}{w_1} \frac{2}{w_2} = \boxed{\frac{4}{w_1 w_2}} .$$

Method 2. Profit Maximization.

$$\max_{x_1, x_2} \underbrace{pf(x_1, x_2) - w_1 x_1 - w_2 x_2}_{\text{profit}} \quad \text{s.t. } f(x_1, x_2) = x_1 x_2.$$

One could solve this by setting up the Lagrangian

$$\mathcal{L} = pf(x_1, x_2) - w_1 x_1 - w_2 x_2 + \lambda [f(x_1, x_2) - x_1 x_2]$$

but it's easier to use the constraint to eliminate $f(x_1, x_2)$:

$$\max_{x_1, x_2} p x_1 x_2 - w_1 x_1 - w_2 x_2.$$

$$0 = \frac{\partial \pi}{\partial x_1} = px_2 - w_1 \Rightarrow x_2 = w_1/p$$

$$0 = \frac{\partial \pi}{\partial x_2} = px_1 - w_2 \Rightarrow x_1 = w_2/p.$$

$$\text{We know that } 4 = w_1 x_1 + w_2 x_2$$

$$= w_1 \left(\frac{w_2}{p} \right) + w_2 \left(\frac{w_1}{p} \right) = \frac{2w_1 w_2}{p}$$

$$\Rightarrow 2 = \frac{w_1 w_2}{p} \Rightarrow p = \frac{w_1 w_2}{2}.$$

Then

$$y = x_1 x_2 = \frac{w_2}{p} \frac{w_1}{p} = \frac{w_1 w_2}{p^2} = \frac{w_1 w_2}{(w_1 w_2 / 2)^2} = \boxed{\frac{4}{w_1 w_2}}.$$

Wrong Method. $\max_{x_1, x_2} \pi = \max_{x_1, x_2} p x_1 x_2 - (w_1 x_1 + w_2 x_2) = \max_{x_1, x_2} p x_1 x_2 - 4.$

The problem is that, by substituting 4 in for $w_1 x_1 + w_2 x_2$ prematurely,

one has fixed total cost at 4 irrespective of the amounts of x_1 and x_2 used. This makes the problem $\max_{x_1, x_2} [\text{total revenue} - 4]$, and since 4 is a constant, that has the same optimal points as $\max_{x_1, x_2} [\text{total revenue}]$, whose answer — since total revenue is $p x_1 x_2$ — is $x_1^* = \infty$ and $x_2^* = \infty$. Such points do not maximize profit.

By the way, while the answer to " $\max_{x_1, x_2} p x_1 x_2$ " is clearly $x_1 = x_2 = \infty$, if one tries to solve it using first-order conditions, the result is

$$0 = \frac{\partial p x_1 x_2}{\partial x_1} = p x_2$$

$$0 = \frac{\partial p x_1 x_2}{\partial x_2} = p x_1 ,$$

which lead to $x_1^* = x_2^* = 0$. On \mathbb{R}_+^2 , this is a total revenue minimum, not maximum.

Method 3. Total Revenue Maximization with given costs. Consider

$$\max p \underbrace{x_1 x_2}_{f(x_1, x_2)} \text{ s.t. } \underbrace{w_1 x_1 + w_2 x_2}_{\text{cost}} = 4 .$$

With costs constrained to
be equal to 4, total revenue maximization
is equivalent to profit maximization.

($\pi = TR - TC$, so with TC fixed, TR will
 $\uparrow \pi$.)

Since p is a constant here, this problem will have the same x_1^* and

x_2^* , and hence the same y^* , as the output-maximization problem

$$\max x_1 x_2 \text{ s.t. } w_1 x_1 + w_2 x_2 = 4.$$

This can be solved by a Lagrangian, but another way is to solve the constraint for a variable, for example $x_1 = \frac{1}{w_1} (4 - w_2 x_2)$, and then solve

$$\max \frac{1}{w_1} (4 - w_2 x_2) x_2$$

$$= \max \frac{1}{w_1} (4x_2 - w_2 x_2^2) ; \text{F.O.C.} \Rightarrow$$

$$0 = \frac{1}{w_1} (4 - 2w_2 x_2)$$

$$\Rightarrow 4 = 2w_2 x_2 \Rightarrow x_2 = \frac{2}{w_2}, x_1 = \frac{1}{w_1} (4 - w_2 \cdot \frac{2}{w_2}) \\ = \frac{1}{w_1} (4 - 2) = \frac{2}{w_1},$$

$$\text{And then } y = x_1 x_2 = \frac{2}{w_1} \cdot \frac{2}{w_2} = \frac{4}{w_1 w_2}.$$

3. [11 points] A firm produces output y from inputs x_1 and x_2 whose prices are w_1 and w_2 , respectively. The firm takes these input prices as given. The production function is $y = x_1^{1/2}x_2^{1/2}$.
- (a) This firm has what kinds of returns to scale?
 - (b) Find this firm's cost function.
 - (c) Determine whether "the algebraic function you found in part (b)" is concave in \mathbf{w} .

Fall 2010 Ex. 1 Qu. 3

$$\textcircled{3} \quad a) \quad f(t\tilde{x}) = (tx_1)^{1/2} (tx_2)^{1/2} \\ = t^{1/2+1/2} x_1^{1/2} x_2^{1/2} = t f(\tilde{x})$$

where $f(\tilde{x})$ is the production function $x_1^{1/2} x_2^{1/2}$. Since f is homogeneous of degree one, this firm has constant returns to scale.

$$b) \quad \min_{x_1, x_2} w_1 x_1 + w_2 x_2 \quad \text{s.t.} \quad y = x_1^{1/2} x_2^{1/2} \quad (y \text{ fixed})$$

↑ ↑
 price of price of
 input 1 input 2

$$\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda [y - x_1^{1/2} x_2^{1/2}]$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = y - x_1^{1/2} x_2^{1/2}$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \frac{1}{2} \lambda x_1^{-1/2} x_2^{1/2} \Rightarrow 2w_1 = \lambda x_1^{-1/2} x_2^{1/2}$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_2} = w_2 - \frac{1}{2} \lambda x_1^{1/2} x_2^{-1/2} \quad \downarrow \quad \lambda = 2w_1 x_1^{1/2} x_2^{-1/2}$$

$$0 = w_2 - \frac{1}{2} (2w_1 x_1^{1/2} x_2^{-1/2}) x_1^{1/2} x_2^{-1/2}$$

$$= w_2 - w_1 x_1 x_2^{-1}$$

$$w_1 x_1 x_2^{-1} = w_2$$

$$x_1 = \frac{w_2}{w_1} x_2 ; \quad y = x_1^{1/2} x_2^{1/2} \Rightarrow y^2 = x_1 x_2, \text{ and}$$

$$\text{substituting in for } x_1 : \quad y^2 = \frac{w_2}{w_1} x_2 x_2 = \frac{w_2}{w_1} x_2^2 \Rightarrow x_2^* = y \sqrt{\frac{w_1}{w_2}} \text{ and}$$

$$x_1^* = \frac{w_2}{w_1} x_2 = \frac{w_2}{w_1} y \sqrt{\frac{w_1}{w_2}} = y \sqrt{\frac{w_2}{w_1}}.$$

Hence the cost function is

$$C^* = w_1 x_1^* + w_2 x_2^* = w_1 y \sqrt{\frac{w_2}{w_1}} + w_2 y \sqrt{\frac{w_1}{w_2}}$$

$$= y \sqrt{w_1 w_2} + y \sqrt{w_1 w_2} = \boxed{2y \sqrt{w_1 w_2}}.$$

c)

$$C'_{w_1} = 2y \cdot \frac{1}{2} w_1^{-\frac{1}{2}} w_2^{\frac{1}{2}} = y w_1^{-\frac{1}{2}} w_2^{\frac{1}{2}}$$

$$C'_{w_2} = 2y \cdot \frac{1}{2} w_1^{\frac{1}{2}} w_2^{-\frac{1}{2}} = y w_1^{\frac{1}{2}} w_2^{-\frac{1}{2}}$$

$$\nabla^2 C = \begin{bmatrix} C''_{11} & C''_{12} \\ C''_{21} & C''_{22} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} y w_1^{-\frac{3}{2}} w_2^{\frac{1}{2}} & \frac{1}{2} y w_1^{-\frac{1}{2}} w_2^{-\frac{1}{2}} \\ \frac{1}{2} y w_1^{-\frac{1}{2}} w_2^{-\frac{1}{2}} & -\frac{1}{2} y w_1^{\frac{1}{2}} w_2^{-\frac{3}{2}} \end{bmatrix}$$

Sufficient Condition for strict concavity

$$0 > D_1 = \frac{1}{2} y w_1^{-\frac{3}{2}} w_2^{\frac{1}{2}} \text{ OK since } y > 0, w_1 > 0, \text{ and } w_2 > 0$$

$$0 < D_2 = \left(\frac{1}{2} y w_1^{-\frac{3}{2}} w_2^{\frac{1}{2}} \right) \left(\frac{1}{2} y w_1^{\frac{1}{2}} w_2^{-\frac{3}{2}} \right) - \left(\frac{1}{2} y w_1^{-\frac{1}{2}} w_2^{-\frac{1}{2}} \right) \left(\frac{1}{2} y w_1^{\frac{1}{2}} w_2^{-\frac{1}{2}} \right)$$

$$= \frac{1}{4} y^2 w_1^{-\frac{1}{2}} w_2^{-\frac{1}{2}} - \frac{1}{4} y^2 w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} = 0 \text{ not OK.}$$

Sufficient conditions for concavity:

$$0 \geq D_1 \text{ OK from above}$$

$$0 \geq \text{the (2,2) element of } \nabla^2 C = -\frac{1}{2} y w_1^{\frac{1}{2}} w_2^{-\frac{3}{2}} \text{ OK } \} \text{ the } \Delta_1 \text{ of } \nabla^2 C$$

$$0 \leq D_2 = 0 \text{ from above } \} \text{ the } \Delta_2 \text{ of } \nabla^2 C$$

So C is concave in \underline{w} . (Optional: Note that along this line $w_2 = w_1$, C is $2y \sqrt{w_1 w_1} = 2y w_1$, which is linear in w_1 . So C is not strictly concave in \underline{w} .)

Final Exam
1994
Question 4

(2)

4. A competitive firm buys goods x_1 and x_2 and combines them in order to produce good y according to the production function

$$y = \sqrt{x_1} + 2\sqrt{x_2}.$$

The price of x_1 is w_1 , the price of x_2 is w_2 , and the price of y is p .

- (a) Find the firm's cost function. Hint: the answer is $\frac{w_1 w_2 y^2}{4w_1+w_2}$.
- (b) Find the firm's demand for x_1 as a function of prices and output.
Hint: the answer is $\frac{y^2 w_2^2}{(4w_1+w_2)^2}$.
- (c) Find the firm's profit function. Hint 1: this is easier if you use the answer to part (a). Hint 2: the answer is $\frac{4w_1+w_2}{4w_1 w_1} p^2$.
- (d) Find the firm's demand for x_1 as a function of prices only. Hint: the answer is $p^2/(4w_1^2)$.
- (e) In two sentences or less, describe *but do not carry out* the procedure you would use to verify that the answers to parts (b) and (d) are compatible with each other.

Final Exam
1994
Answer 4

(4)

a) minimize $\underset{x}{w_1 x_1 + w_2 x_2}$ subject to $y = \sqrt{x_1} + 2\sqrt{x_2}$.

$$\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda \left[x_1^{\frac{1}{2}} + 2x_2^{\frac{1}{2}} - y \right]$$

Setup & FOC's:
4 pts

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial x_1} = w_1 + \frac{1}{2} \lambda x_1^{-\frac{1}{2}} \\ 0 &= \frac{\partial \mathcal{L}}{\partial x_2} = w_2 + \lambda x_2^{-\frac{1}{2}} \end{aligned} \quad \left\{ \begin{array}{l} \frac{w_1}{w_2} = \frac{x_2^{\frac{1}{2}}}{2x_1^{\frac{1}{2}}} \Rightarrow x_2^{\frac{1}{2}} = 2x_1^{\frac{1}{2}} \left(\frac{w_1}{w_2} \right) \text{ and} \\ \end{array} \right.$$

$$\text{Then } y = x_1^{\frac{1}{2}} + 2 \left[2x_1^{\frac{1}{2}} \left(\frac{w_1}{w_2} \right) \right] \quad \leftarrow x_2^{\frac{1}{2}} = 4x_1^{\frac{1}{2}} \left(\frac{w_1}{w_2} \right)^2.$$

$$= x_1^{\frac{1}{2}} + 4x_1^{\frac{1}{2}} \left(\frac{w_1}{w_2} \right) = x_1^{\frac{1}{2}} \left[1 + 4 \frac{w_1}{w_2} \right] = x_1^{\frac{1}{2}} \frac{w_2 + 4w_1}{w_2} \Rightarrow$$

$$x_1^{\frac{1}{2}} = \frac{w_2}{w_2 + 4w_1}, \text{ and } x_1 = \frac{w_2^2 y^2}{(4w_1 + w_2)^2}, \quad \leftarrow 2 \text{ pts} \quad \leftarrow 2 \text{ pts}$$

$$x_2 = 4 \frac{w_2^2 y^2}{(4w_1 + w_2)^2} \left(\frac{w_1}{w_2} \right)^2 = \frac{4w_1^2 y^2}{(4w_1 + w_2)^2}.$$

$$\text{Therefore } c(w, y) = w_1 x_1^* + w_2 x_2^* = w_1 \frac{w_2^2 y^2}{(4w_1 + w_2)^2} + w_2 \frac{4w_1^2 y^2}{(4w_1 + w_2)^2}$$

$$= \frac{w_1 w_2 y^2}{(4w_1 + w_2)^2} \left[w_2 + 4w_1 \right] = \frac{w_1 w_2 y^2}{4w_1 + w_2}. \quad \leftarrow 2 \text{ pts}$$

b) $x_1 = \frac{w_2^2 y^2}{(4w_1 + w_2)^2}$ as given above. (Equivalently, $x_1 = \frac{\partial c(w, y)}{\partial w_1}.$)

Answer 4 cont..

c) profit equation $\pi^*(w, p, y) = py - \text{total cost}$

$$= py - \frac{w_1 w_2 y^2}{4w_1 + w_2}.$$

Maximize π^* w.r.t. y : $0 = \frac{d\pi^*}{dy} = p - \frac{2w_1 w_2}{4w_1 + w_2} y \Rightarrow y = \frac{4w_1 + w_2}{2w_1 w_2} p$. This

is the supply curve. Hence the profit function

$$\pi(w, p) = py^* - \frac{w_1 w_2}{4w_1 + w_2} (y^*)^2 = \frac{4w_1 + w_2}{2w_1 w_2} p^2 - \frac{w_1 w_2}{4w_1 + w_2} \frac{(4w_1 + w_2)^2}{4w_1^2 w_2^2} p^2$$

$$= \frac{4w_1 + w_2}{2w_1 w_2} p^2 - \frac{4w_1 + w_2}{4w_1 w_2} p^2 = \frac{4w_1 + w_2}{w_1 w_2} p^2 \left(\frac{1}{2} - \frac{1}{4} \right)$$

$$= \frac{4w_1 + w_2}{4w_1 w_2} p^2. \quad (5 \text{ pts})$$

Note: Another way to solve (d) is on the next page.

Get $\pi(w, p)$ from part (c)

d) $x_1 = \frac{\partial \pi(w, p)}{\partial w_1} = p^2 \left[\frac{4}{4w_1 w_2} - \frac{4w_1 + w_2}{4w_1^2 w_2} \right] = p^2 \frac{4w_1 - 4w_1 - w_2}{4w_1^2 w_2} = -\frac{p^2 w_2}{4w_1^2 w_2}$

$$= -p^2 / (4w_1^2) \quad \text{where the negative sign means good 1 is an input.} \quad (3 \text{ pts})$$

e) $y = \frac{\partial \pi(w, p)}{\partial p}$, which would give y as a function of (w, p) only; substitute this into the answer to part (b). (2 pts)

or see the third line in part (c)

Optional: $y = \frac{\partial \pi}{\partial p} = \frac{4w_1 + w_2}{4w_1 w_2} (2p) = \frac{4w_1 + w_2}{2w_1 w_2} p$ (this was also derived in part (c)).

Substituting into part (b),

Final Exam
1994

Answer 4 cont...

$$x_1 = \frac{w_2^2}{(4w_1+w_2)^2} \cdot \frac{(4w_1+w_2)^2}{4w_1^2 w_2^2} p^2 = \frac{p^2}{4w_1^2}, \text{ just as in part (d).}$$

Another way to solve (d) :

$$\text{From (b), } x_1 = \frac{w_2^2}{(4w_1+w_2)^2} y^2.$$

From the third line of the answer to (c), $y = \frac{4w_1+w_2}{2w_1 w_2} p$. Substituting

this into the formula for x_1 yields

$$x_1 = \frac{w_2^2}{(4w_1+w_2)^2} \cdot \frac{(4w_1+w_2)^2}{4w_1^2 w_2^2} p^2 = \frac{p^2}{4w_1^2},$$

as in the answer to (d).

2015 Final Exam Qu. 4

4. [13 points] A firm has two plants with cost functions $c_1(y_1) = y_1^2$ and $c_2(y_2) = y_2$. What is the cost function of the firm?

(which was assigned as homework)

④ This is a trivial variation on Varian's problem 4.3. Following its reasoning,

$$\left. \begin{array}{l} MC_1 = \frac{\partial c_1(y_1)}{\partial y_1} = 2y_1 \\ MC_2 = \frac{\partial c_2(y_2)}{\partial y_2} = 1 \end{array} \right\} \begin{array}{l} MC_1 = MC_2 \Rightarrow 2y_1 = 1, \\ y_1 = \frac{1}{2}. \end{array} *$$

If $y > \frac{1}{2}$ then $y_1 + y_2 = y$

$y_2 = y - y_1$ and since $y_1 = \frac{1}{2}$,

$y_2 = y - \frac{1}{2}$. Then

$$\begin{aligned} c(y) &= c_1(y_1) + c_2(y_2) = c_1\left(\frac{1}{2}\right) + c_2\left(y - \frac{1}{2}\right) \\ &= \left(\frac{1}{2}\right)^2 + y - \frac{1}{2} = \frac{1}{4} + y - \frac{1}{2} = y - \frac{1}{4}. \end{aligned}$$

But if $y < \frac{1}{2}$, it's better to produce everything in plant 1. (Since $MC_2 = 1$

and $MC_1 = 2y_1 \leq 2y < 2 \cdot \frac{1}{2} = 1 = MC_2$, that is, $MC_1 < MC_2$.)

So

$$c(y) = \begin{cases} y^2 & \text{if } y \leq \frac{1}{2} \\ y - \frac{1}{4} & \text{if } y \geq \frac{1}{2} \end{cases}$$

This also follows from $\min_{y_1, y_2} y_1^2 + y_2^2$ s.t. $y_1 + y_2 = y$, the cost-minimization problem, if you don't want to use $MC_1 = MC_2$. Proof: $\min_{y_1} y_1^2 + (y - y_1)^2 \Rightarrow 0 = 2y_1 - 1 \Rightarrow y_1^ = \frac{1}{2}$.

2017 Qualifying Exam Sec. 3 Qu. 1

1. [10 points]

- (a) A firm has two plants with cost functions $c_1(y_1) = y_1^2/2$ and $c_2(y_2) = y_2$.
- Sketch the marginal cost curves of the two plants on one graph.
 - What is the cost function of the firm?
 - According to your cost function, how much would it cost this firm to produce an output of $1/4$? (If you got part (ii) right, this question is so easy that you may wonder why I ask it. The reason I ask it is that if you got part (ii) wrong, this question may reveal your error to you.)
- (b) A firm has two plants. One plant produces output according to the production function $x_1^a x_2^{1-a}$. The other plant has a production function $x_1^b x_2^{1-b}$.
- What is the cost function for this technology?
 - If this firm wanted to produce a total output of 2, how much would be produced in each plant?

Section 3 Qu. 1

a) See Varian homework problem 4.3.

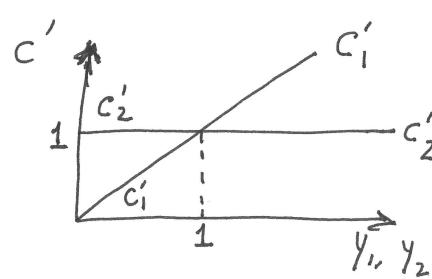
$$c_1(y_1) = y_1^2/2$$

$$c_2(y_2) = y_2$$

$$c'_1(y_1) = y_1$$

$$c'_2(y_2) = 1$$

Marginal Cost



i) Typically, for cost minimization $MC_1 = MC_2$. (Else keep total

output the same - and hence total revenue the same - while shifting production from the plant with high MC to the plant with low MC , thus \downarrow total cost.)

$$MC_1 = MC_2 \Leftrightarrow y_1 = 1 \Rightarrow c_1(y_1) = c_1(1) = 1^2/2 = 1/2$$

$$\text{Since } y_1 + y_2 = y, y_2 = y - y_1 = y - 1 \Rightarrow$$

$$c_2(y_2) = y_2 = y - 1.$$

$$\text{Total cost } c(y) = c_1(y_1) + c_2(y_2) = \frac{1}{2} + y - 1 = \boxed{y - \frac{1}{2}} \leftarrow \text{for } y > 1.$$

However, if $y < 1$, the equation $c_2(y) = y - 1$ would be negative, which makes no sense. If $y < 1$, graph (i) shows that it's better for y_2 to be zero, because c'_2 cannot be made equal to c'_1 when $y < 1$ (implying $y_1 < 1$ and $y_2 < 1$): c'_2 will always be larger than c'_1 .

$$\text{So when } y < 1, \text{ total cost } c(y) = c_1(y) + 0 = \boxed{y^2/2} \text{ for } y < 1.$$

$$\text{iii)} \quad y = \frac{1}{4} < 1 \text{ so } c(y) = y^2/2 = (\frac{1}{4})^2/2 = \frac{1}{32}.$$

Using the wrong formula $c(y) = y - \frac{1}{2}$ would lead here to $c(\frac{1}{4}) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$, an impossibility.

$$b) i) \quad y_1 = x_1^a x_2^{1-a}$$

↑ ↑
prices w_1, w_2

$$y_2 = x_1^b x_2^{1-b}$$

See Varian's homework
problem 4.4.

Cost minimization:

$$\min w_1 x_1 + w_2 x_2 \text{ s.t. } y_1 = x_1^a x_2^{1-a}$$

$$\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda (y_1 - x_1^a x_2^{1-a})$$

$$\text{F.O.C.'s} \quad 0 = \mathcal{L}'_1 = w_1 - a \lambda x_1^{a-1} x_2^{1-a} \Rightarrow \lambda = \frac{w_1}{a} x_1^{1-a} x_2^{a-1}$$

$$0 = \mathcal{L}'_2 = w_2 - (1-a) \lambda x_1^a x_2^{-a} \Rightarrow$$

$$0 = \mathcal{L}'_2 = y_1 - x_1^a x_2^{1-a} \quad w_2 = (1-a) \left[\frac{w_1}{a} x_1^{1-a} x_2^{a-1} \right] x_1^a x_2^{-a}$$

$$\frac{w_2}{w_1} = \frac{1-a}{a} x_1 x_2^{-1}$$

$$\frac{w_2}{w_1} x_2 = \frac{1-a}{a} x_1$$

$$y_1 = x_1^a \left(\frac{w_1}{w_2} \frac{1-a}{a} x_1 \right)^{1-a} \quad x_2 = \frac{w_1}{w_2} \frac{1-a}{a} x_1$$

$$= \left(\frac{w_1}{w_2} \frac{1-a}{a} \right)^{1-a} x_1 \Rightarrow x_1 = \underline{\left(\frac{w_2}{w_1} \frac{a}{1-a} \right)^{1-a} y_1},$$

$$x_2 = \frac{w_1}{w_2} \frac{1-a}{a} x_1 = \frac{w_1}{w_2} \frac{1-a}{a} \left(\frac{w_2}{w_1} \frac{a}{1-a} \right)^{1-a} y_1$$

$$= \underline{\left(\frac{w_2}{w_1} \frac{a}{1-a} \right)^{-1} \left(\frac{w_2}{w_1} \frac{a}{1-a} \right)^{1-a} y_1}$$

$$= \underline{\left(\frac{w_2}{w_1} \frac{a}{1-a} \right)^{-a} y_1},$$

$$\text{Cost function} = w_1 x_1 + w_2 x_2 = w_1 \left(\frac{w_2}{w_1} \frac{a}{1-a} \right)^{-a} y_1 + w_2 \left(\frac{w_2}{w_1} \frac{a}{1-a} \right)^{-a} y_1$$

$$= \left[w_1 w_2^{1-a} \left(\frac{a}{1-a} \right)^{1-a} + w_1^a w_2^{1-a} \left(\frac{a}{1-a} \right)^{-a} \right] y_1$$

$$= w_1^a w_2^{1-a} \left(\frac{a}{1-a} \right)^{-a} \left[\frac{a}{1-a} + 1 \right] y_1 = w_1^a w_2^{1-a} \left(\frac{a}{1-a} \right)^{-a} \frac{a+1-a}{1-a} y_1$$

$$= w_1^a w_2^{1-a} \left(\frac{a}{1-a} \right)^{-a} \frac{1}{1-a} y = \underbrace{w_1^a w_2^{1-a} \frac{1}{a} (1-a)^{a-1}}_{\text{call this "A"}} y$$

$$\text{so } c_1(w_1, w_2, y) = A y.$$

$$\underbrace{w_1^b w_2^{1-b} b^{-b} (1-b)^{b-1}}_{\downarrow}$$

The second plant by symmetry will have $c_2(w_1, w_2, y) = B y$ for some B .

Hence $MC_1 = A$ and $MC_2 = B$.

Case 1. $A = B$. Then $MC_1 \equiv MC_2$ regardless of how much is produced in each plant. In this case,

$$\text{total cost} = A y_1 + B y_2 = A y_1 + A y_2 = A(y_1 + y_2) = A y = B y.$$

Case 2. $A \neq B$. Then $MC_1 \neq MC_2$.

2a. $A < B \Rightarrow MC_1 < MC_2 \Rightarrow$ output in the second plant is zero,
 $\text{total cost} = A y$.

2b. $A > B \Rightarrow MC_1 > MC_2 \Rightarrow$ output in the first plant is zero,
 $\text{total cost} = B y$.

Summary. total cost = $\min \{A, B\} y$.

ii) If $A < B$, everything is produced in the first plant.

If $A > B$, " second".

If $A = B$, it does not matter how much is produced where.

2. [10 points] Suppose a competitive firm has a profit function denoted $\pi(\mathbf{p})$ and a production possibilities set denoted by Y whose generic element is denoted by \mathbf{y} .

In solving the problems below, if you use Hotelling's Lemma, you should prove it (using the Envelope Theorem).

- (a) Show that $\pi(\mathbf{p})$ is increasing in output prices and decreasing in input prices.
- (b) Show that $\pi(\mathbf{p})$ is homogeneous of degree one in \mathbf{p} .
- (c) Show that $\mathbf{y}^*(\mathbf{p})$ is homogeneous of degree zero in \mathbf{p} .
- (d) Varian (p. 41) writes that properties such as these (emphasis added by me):

... follow from the definition of the profit function alone
and *do not rely on any properties of the technology*.

Why do such properties actually depend on technology, in the sense that there are some technologies for which $\pi(\mathbf{p})$ is not even defined for a competitive firm?

Summer 2011 qualifying exam, Sec. 2 Qu. 2

Section 2 Question 2.

a) $\pi(p) = \max_{\tilde{y} \in Y} \tilde{p} \cdot \tilde{y}$ by definition

$$\frac{\partial \pi}{\partial p_i} = \frac{\partial \mathcal{L}^*}{\partial p_i} \text{ by the Envelope Theorem}$$

$$= \frac{\partial}{\partial p_i} \left(\tilde{p} \cdot \tilde{y} \right)^*$$

$= y_i^*$, which is Hotelling's Lemma;

$$= y_i^* \begin{cases} < 0 & \text{if } i \text{ is an input} \\ > 0 & \text{if } i \text{ is an output} \end{cases}$$

b) $\pi(\lambda p) = \max_{\tilde{y} \in Y} \lambda \tilde{p} \cdot \tilde{y} = \lambda \max_{\tilde{y} \in Y} \tilde{p} \cdot \tilde{y} = \lambda \pi(p).$

c) $\nabla_p \pi(p) = y$ as shown in part (a).

If f is homogeneous of degree k , its derivative is homogeneous of degree $k-1$.

$\pi(p)$ is homogeneous of degree 1 from part (b).

Hence its derivative, $\nabla \pi = y$, is homogeneous of degree zero.

d) If Y reflects increasing returns to scale, then the optimal \tilde{y} for any fixed p is $\tilde{y}^* = \infty$, so $\pi^* = \infty$ for all p and the properties do not follow.

(competitive firms think p is fixed)

Exam 1 (2)
2000

Question 4

(2)

4. [6 points] Prove to me all you can about the dependence of the profit function on prices. If you use an Envelope Theorem result, you do not have to prove the Envelope Theorem, but you should state it and explain how it is relevant.

① \underline{p} : price vector \underline{y} : input-output vector π : profit

a) Proof of Hotelling's Lemma:

Envelope Theorem

$$M(a) =$$

$$\max_{\underline{x}} g(\underline{x}, a)$$

$$\Rightarrow \frac{dM(a)}{da} = \frac{\partial g^*}{\partial a}$$

(unconstrained version of
the Envelope Theorem)

So for a price-taking firm:

$$\pi(\underline{p}) =$$

$$\max_{\underline{y}} \underline{p} \cdot \underline{y}$$

$$\Rightarrow \nabla_{\underline{p}} \pi(\underline{p}) = \nabla_{\underline{p}} \max_{\underline{y}} \underline{p} \cdot \underline{y}^* = \underline{y}^*$$

$$\text{Or: } \boxed{\frac{\partial \pi}{\partial p_i} = y_i^*}.$$

b)

$$\frac{\partial \pi}{\partial p_i} = y_i^* \begin{cases} < 0 \text{ for inputs} \\ > 0 \text{ for outputs} \end{cases}$$

from part (a)

So when an input price \uparrow , $\pi \downarrow$, and when
"output" \uparrow , $\pi \uparrow$.

c)

$$\pi(t\underline{p}) = t \pi(\underline{p}) : \boxed{\text{profit is homogeneous of degree one in } \underline{p}}$$

$$\text{Proof: } \pi(t\underline{p}) = \max_{\underline{y} \in Y} t\underline{p} \cdot \underline{y} = t \max_{\underline{y} \in Y} \underline{p} \cdot \underline{y} = t \pi(\underline{p}).$$

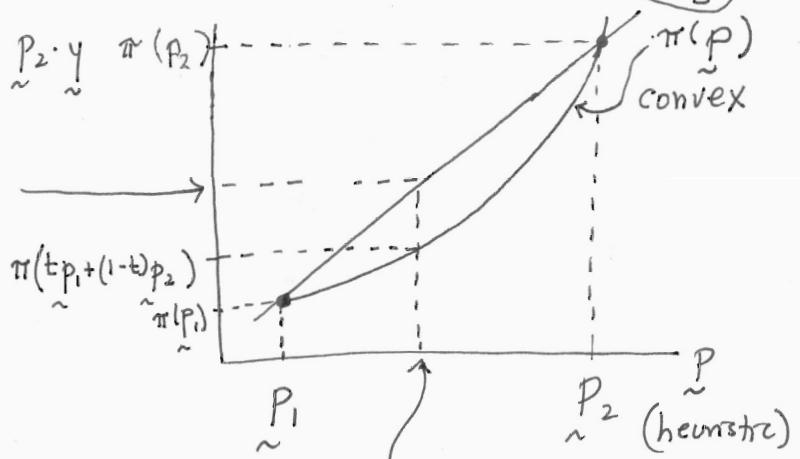
d) $\pi(\underline{p})$ is convex. Proof: $\pi(t\underline{p}_1 + (1-t)\underline{p}_2) =$

$$\max_{\underline{y} \in Y} [t\underline{p}_1 + (1-t)\underline{p}_2] \cdot \underline{y} = \max_{\underline{y} \in Y} [t\underline{p}_1 \cdot \underline{y} + (1-t)\underline{p}_2 \cdot \underline{y}]$$

$$\leq \max_{\substack{y \in Y \\ \sim}} t \underset{\sim}{p_1} \cdot \underset{\sim}{y} + \max_{\substack{y \in Y \\ \sim}} (1-t) \underset{\sim}{p_2} \cdot \underset{\sim}{y} \quad \text{because } \max(f+g) \leq \underbrace{\max f + \max g}_{\sim}$$

$$= t \max_{\substack{y \in Y \\ \sim}} \underset{\sim}{p_1} \cdot \underset{\sim}{y} + (1-t) \max_{\substack{y \in Y \\ \sim}} \underset{\sim}{p_2} \cdot \underset{\sim}{y} \quad \pi(\underset{\sim}{p}_2)$$

$$= t \pi(\underset{\sim}{p}_1) + (1-t) \pi(\underset{\sim}{p}_2).$$



$$\underset{\sim}{t p_1 + (1-t) p_2}$$

e) $\pi(\underset{\sim}{p})$ is continuous. Proof:

$$\pi(\underset{\sim}{p}) = \max_{\substack{y \in Y \\ \sim}} \underset{\sim}{p} \cdot \underset{\sim}{y}$$

this is continuous in $\underset{\sim}{p}$

γ is trivially continuous in $\underset{\sim}{p}$ since it does not depend on $\underset{\sim}{p}$;

so the result follows from the Theorem of the Maximum.

Final Exam

2000

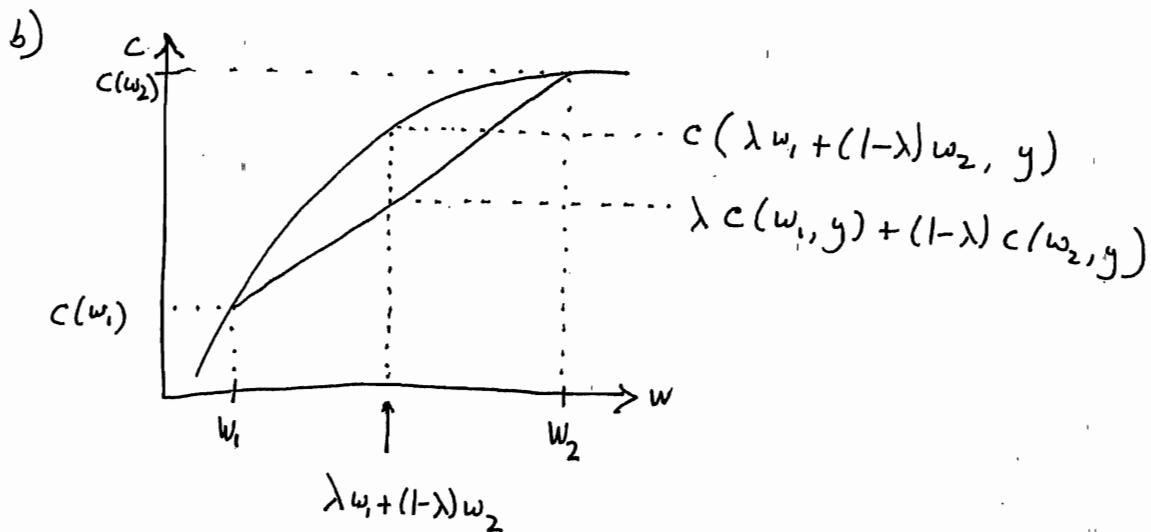
Question 3 (2)

3. (a) Prove that the cost function is homogeneous of degree one in input prices.
- (b) Prove that the cost function is concave in input prices.

$$\textcircled{3} \quad a) \quad c(t\omega, y) = \min_{\underline{x}} t\omega \cdot \underline{x} \text{ s.t. } f(\underline{x}) \geq y$$

$$= t \min_{\underline{x}} \omega \cdot \underline{x} \text{ s.t. } f(\underline{x}) \geq y$$

$$= t c(\omega, y).$$



$$c(\lambda \underline{\omega}_1 + (1-\lambda) \underline{\omega}_2, y) = \min_{\underline{x}} (\lambda \underline{\omega}_1 + (1-\lambda) \underline{\omega}_2) \cdot \underline{x} \text{ s.t. } f(\underline{x}) \geq y$$

$$= \min_{\underline{x}} [\lambda \underline{\omega}_1 \cdot \underline{x} + (1-\lambda) \underline{\omega}_2 \cdot \underline{x}] \text{ s.t. } f(\underline{x}) \geq y$$

$$\geq \left(\min_{\underline{x}} \lambda \underline{\omega}_1 \cdot \underline{x} \text{ s.t. } f(\underline{x}) \geq y \right) + \left(\min_{\underline{x}} (1-\lambda) \underline{\omega}_2 \cdot \underline{x} \text{ s.t. } f(\underline{x}) \geq y \right)$$

Since separate minimizations can achieve smaller values;

$$= \left(\lambda \min_{\underline{x}} \underline{\omega}_1 \cdot \underline{x} \text{ s.t. } f(\underline{x}) \geq y \right) + \left((1-\lambda) \min_{\underline{x}} \underline{\omega}_2 \cdot \underline{x} \text{ s.t. } f(\underline{x}) \geq y \right)$$

$$= \lambda c(\underline{\omega}_1, y) + (1-\lambda) c(\underline{\omega}_2, y).$$

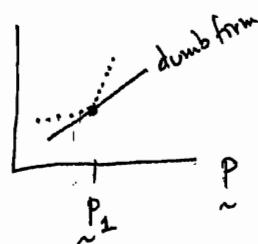
Final Exam
1995
Question 4

(2)

4. A profit-maximizing firm's profit function $\pi(p)$ is convex. What is the intuition for this? (Instead of giving the intuition you may give the proof of convexity, but you do not have to give the proof of convexity.) What are the implications of convexity of the firm's profit function for the firm's demand functions for inputs and supply function(s) for output(s)? (This has something to do with the Envelope Theorem, though I am *not* asking you for a formal proof.)

④ $\pi(\tilde{p}) = \tilde{p} \cdot \tilde{y}^*(\tilde{p})$. Suppose that when prices are \tilde{p}_1 , the optimal input-output vector is \tilde{y}_1 . Then $\pi(\tilde{p}_1) = \tilde{p}_1 \cdot \tilde{y}_1$. If price now deviates from \tilde{p}_1 , a "dumb" firm which is not profit-maximizing and which stubbornly persists in having input-output vector \tilde{y}_1 has profit of $\tilde{p} \cdot \tilde{y}_1$, which is linear in \tilde{p} . (For example, with one output and one input, $\tilde{p} \cdot \tilde{y}_1 = P_1 y_{11} + P_2 y_{12}$ with y_{11} and y_{12} fixed.)

pts
4 motivation
3 $y = \nabla \pi$, $\nabla y = \nabla^2 \pi$
3 implications



A smarter, profit-maximizing firm will do better (or at least as well) as the "dumb" firm. So its profit function looks like the dotted line, which is convex.

By Hotelling's Lemma — which is an "Envelope Theorem Result" — one has

$$y = \nabla_p \pi(p). \text{ Differentiating both sides with respect to } p \text{ gives } \nabla_p y = \nabla^2_p \pi(p).$$

Since $\pi(p)$ is convex, $\nabla^2_p \pi(p)$ is positive semidefinite. $\nabla^2 \pi$ is also symmetric,

as all Hessians are. Therefore $\nabla_p y$ is positive semidefinite symmetric, which

means that $\frac{\partial y_i}{\partial p_j} = \frac{\partial y_j}{\partial p_i}$ (by symmetry) and that the diagonal

elements, $\frac{\partial y_i}{\partial p_i}$, are ≥ 0 . So if $p_i \uparrow$ then $y_i \uparrow$, meaning that output

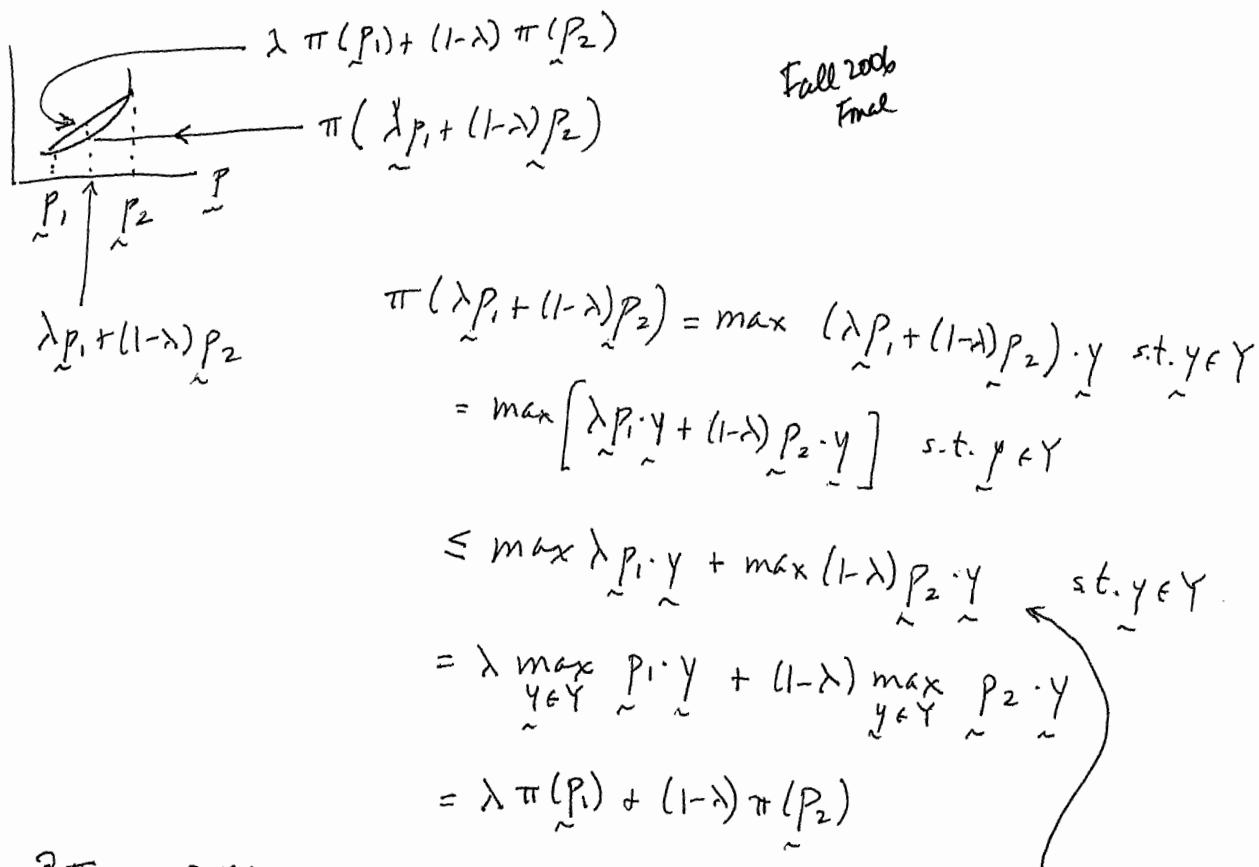
increases if $y_i > 0$ or that input use decreases (say, from -10 to -1, a

mathematical increase) if $y_i < 0$: output supply functions are upward sloping and input demand functions are downward sloping.

3. [12 points]

- (a) Prove that the profit function of a competitive firm is convex in p .
- (b) Suppose $y(p)$ is the net supply function of a competitive firm.
 - i. Prove that $\partial y_i / \partial p_i \geq 0$.
 - ii. Explain what $\partial y_i / \partial p_i \geq 0$ means.

③ a)



b) (i) $\frac{\partial \pi}{\partial p_i} = \frac{\partial \mathcal{L}^*}{\partial p_i}$ Envelope Theorem

$$= \frac{\partial}{\partial p_i} \underset{\gamma}{\sim} P \cdot \gamma = \underset{\gamma}{\sim} y_i^* \quad (\text{Hotelling's Lemma's proof})$$

so $\frac{\partial y_i^*}{\partial p_i} = \frac{\partial}{\partial p_i} \frac{\partial \pi}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_i^2}$. But since $\pi(P)$ is convex,

Two separate maximizations allow customized choices for γ in each of the two terms, while one maximization forces γ for both terms to be the same, resulting in the \leq sign

its Hessian $\underset{P}{\sim} \frac{\partial^2 \pi}{\partial p_i^2}$ is positive semidefinite, and so the main diagonal terms of $\underset{P}{\sim} \frac{\partial^2 \pi}{\partial p_i^2}$ are all ≥ 0 . These terms are $\partial^2 \pi / \partial p_i^2$.

(ii) if y_i is an input then: $p_i \uparrow \Rightarrow y_i \uparrow \Leftrightarrow y_i$ becomes less negative,

gets closer to zero \Rightarrow less of input i.e.
(input demand curves slope downwards)

if y_i is an output then: $p_i \uparrow \Rightarrow y_i \uparrow$: output supply curves slope upwards

*Final Exam
Fall 2009*

2. [15 points] A competitive profit-maximizing firm's profit function $\pi(p)$ is convex.
 - (a) Give the intuition for this result.
 - (b) Give a formal proof of this result.

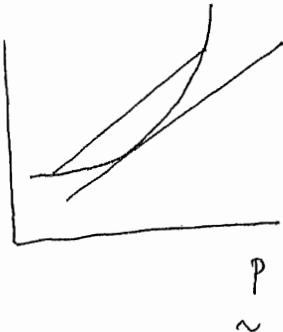
② a) See the first paragraph of two questions ago.

b) See part (a) of one question ago.

The point of asking this question was to implicitly compare and contrast



with each other, on the contrary : π



Qualifying Exam
1995

Question 2.1

(2)

- 2.1 In class, we were able to show that Hicksian demand curves are downward-sloping by differentiating both sides of the "Envelope Theorem" result $\mathbf{h}(\mathbf{p}, u) = \nabla_{\mathbf{p}} e(\mathbf{p}, u)$ with respect to \mathbf{p} , then invoking the concavity of $e(\mathbf{p}, u)$ in \mathbf{p} .

Derive the analogous testable implication for the profit-maximizing competitive firm by applying the same type of reasoning. In other words, apply the Envelope Theorem ("Hotelling's Lemma"); invoke the convexity of the profit function, and describe mathematically and non-mathematically the result you end up with.

You do not have to prove the convexity of the profit function, but you should give the intuitive explanation of why it is convex. You should prove Hotelling's Lemma using the Envelope Theorem.

Qualifying Exam
1995

Answer 2.1

②

Envelope Theorem (unconstrained problem)

$$M(a) = \max_x f(x, a)$$

$$\Rightarrow \frac{d M(a)}{d a} = \left. \frac{\partial f(x, a)}{\partial a} \right|_{x^*}$$

Profit Maximization for a Competitive Firm

$$\pi(p) = \max_y p \cdot y$$

$$\Rightarrow \nabla_p \pi(p) = \left. \frac{\partial p \cdot y}{\partial p} \right|_{y^*}$$

$$= \underline{y}^*$$

1995

Answer 2.1 cont...

analogies:

 \underline{x} , input-output vector \underline{p} , given prices of inputs and outputs M , $\pi(\underline{p})$, profit function.

The result $\underline{y}^* = \nabla_{\underline{p}} \pi(\underline{p})$ is Hotelling's Lemma.

Next, differentiate both sides of Hotelling's Lemma with respect to \underline{p} :

$$\nabla_{\underline{p}} \underline{y}^* = \nabla_{\underline{p}}^2 \pi(\underline{p}).$$

But $\pi(\underline{p})$ is convex (see below), so $\nabla^2 \pi$ is positive semi-definite symmetric. Therefore $\nabla_{\underline{p}} \underline{y}^*$ is positive semi-definite symmetric, which implies, for example, that its diagonal elements $\partial y_i / \partial p_i$ are ≥ 0 and that $\partial y_i / \partial p_j = \partial y_j / \partial p_i$. If i is an input, then $y_i < 0$ and $\partial y_i / \partial p_i \geq 0$ means that as $p_i \uparrow$, y_i gets closer to zero (smaller in absolute value): input demand curves are downward-sloping. If i is an output, then $y_i > 0$ and $\partial y_i / \partial p_i \geq 0$ means that as $p_i \uparrow$, y_i gets farther from zero (larger in absolute value): output supply curves are upward-sloping.

To show intuitively that $\pi(\underline{p}) = \underline{p} \cdot \underline{y}^*$ is convex, suppose that \underline{y}_a is the firm's optimal response to \underline{p}_a . Now suppose prices deviate from \underline{p}_a . A "dumb" (not profit-maximizing) firm might continue to use the input-output vector \underline{y}_a ,

Qualifying Exam

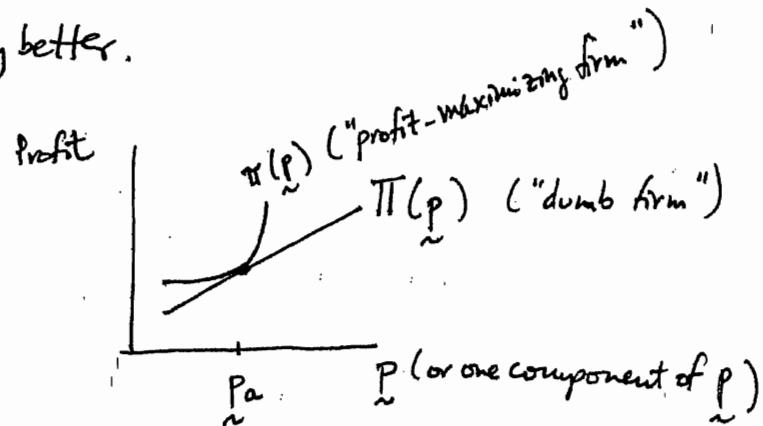
1995

Answer 2.1 cont...

10

yielding profit of $\Pi(p) = p \cdot \gamma_a$, which is linear in p . (See graph below.)

A profit-maximizing firm will do at least as well as this dumb firm, and probably better.



So $\pi(p)$ is convex.

Final Exam

1999

Question 2

(2)

2. [15 points] Suppose a cost-minimizing firm which is competitive in its input markets has a production function $f(\mathbf{x})$. Show that this firm's input demand curves are downward-sloping. Prove all your assertions except the Envelope Theorem.

(2)

$$c(\underline{w}) = \min_{\underline{x}} \underline{w} \cdot \underline{x} \text{ s.t. } f(\underline{x}) = y$$

$$\mathcal{L} = \underline{w} \cdot \underline{x} + \lambda (y - f(\underline{x}))$$

$$\text{By the Envelope Theorem, } \frac{\partial c}{\partial w_i} = \frac{\partial \mathcal{L}^*}{\partial w_i}$$

$= x_i^*$ Shephard's Lemma.

$$\nabla_{\underline{w}} c = \underline{x} \quad \text{" " " vector form.}$$

Differentiate w.r.t. \underline{w} :

$$\nabla_{\underline{w}}^2 c = \nabla_{\underline{x}} \underline{x}$$

If I can prove that c is concave, then $\nabla_{\underline{w}}^2 c$ is negative semidefinite, so its diagonal terms are ≤ 0 , so $\partial x_i / \partial w_i \leq 0$ for all i , as was to be shown.
So I must prove that c is concave.

$$c[\lambda \underline{w}_1 + (1-\lambda) \underline{w}_2] = \min_{\underline{x}} [\lambda \underline{w}_1 + (1-\lambda) \underline{w}_2] \cdot \underline{x} \text{ s.t. } f(\underline{x}) = y$$

$$= \min_{\underline{x}} [\lambda \underline{w}_1 \cdot \underline{x} + (1-\lambda) \underline{w}_2 \cdot \underline{x}] \text{ s.t. } f(\underline{x}) = y$$

$$\geq \min_{\underline{x}} \lambda \underline{w}_1 \cdot \underline{x} + \min_{\underline{x}} (1-\lambda) \underline{w}_2 \cdot \underline{x} \text{ s.t. } f(\underline{x}) = y \text{ for both minimizations,}$$

because two separate minimizations might be better and could never be worse than one combined minimization, in which the minimizing \underline{x} would have to be the same for both $\lambda \underline{w}_1 \cdot \underline{x}$ and $(1-\lambda) \underline{w}_2 \cdot \underline{x}$.



Alternate method: $\underline{x} = \nabla_{\underline{w}} c \Rightarrow$

$$d\underline{x} = \nabla_{\underline{w}}^2 c \cdot d\underline{w} \text{ and } d\underline{w} \cdot d\underline{x} =$$

$$d\underline{w} \cdot \nabla_{\underline{w}}^2 c \cdot d\underline{w} \leq 0 \text{ since } \nabla_{\underline{w}}^2 c$$

is n.s.d. since c is concave.

Second alternate method: WACM.

Final Exam

1999

Answer 2

$$\begin{aligned} &= \left[\lambda \min_{\underline{x}} \underline{w}_1 \cdot \underline{x} \text{ s.t. } f(\underline{x}) = y \right] + \left[(1-\lambda) \min_{\underline{x}} \underline{w}_2 \cdot \underline{x} \text{ s.t. } f(\underline{x}) = y \right] \\ &= \lambda c(\underline{w}_1) + (1-\lambda) c(\underline{w}_2) \end{aligned}$$

Exam 1

2004

Question 3

(2)

3. [11 points] A cost-minimizing firm buys two inputs x_1 and x_2 at given prices w_1 and w_2 , respectively, in order to produce output y according to the production function $y = f(\mathbf{x})$. What conditions must the production function f satisfy in order to ensure that when output increases, the firm's purchases of x_1 also increase? (You may assume that the firm's second-order conditions are satisfied.)

Exam 1
2004
Answer 3

③ $\min w_1 x_1 + w_2 x_2 \text{ s.t. } y = f(x)$

$$\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda (y - f(x))$$

FOC: Differential of FOC's:

$$0 = y - f(x)$$

$$0 = w_1 - \lambda f'_1$$

$$0 = w_2 - \lambda f'_2$$

$$0 = 0 d\lambda - f'_1 dx_1 - f'_2 dx_2 + 0 dw_1 + 0 dw_2 + dy$$

$$0 = -f'_1 d\lambda - \lambda f''_{11} dx_1 - \lambda f''_{12} dx_2 + dw_1 + 0 dw_2 + 0 dy$$

$$0 = -f'_2 d\lambda - \lambda f''_{21} dx_1 - \lambda f''_{22} dx_2 + 0 dw_1 + dw_2 + 0 dy$$

$$\underbrace{d\lambda}_{\text{endogenous}} \quad \underbrace{dx_1}_{\text{endogenous}} \quad \underbrace{dx_2}_{\text{endogenous}} \quad \underbrace{dw_1}_{\text{exogenous}} \quad \underbrace{dw_2}_{\text{exogenous}} \quad dy$$

$$dw_1 = dw_2 = 0$$

$$\begin{bmatrix} 0 & f'_1 & f'_2 \\ f'_1 & \lambda f''_{11} & \lambda f''_{12} \\ f'_2 & \lambda f''_{21} & \lambda f''_{22} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} dy$$

$$\Rightarrow \underline{\underline{\begin{bmatrix} d\lambda/dy \\ dx_1/dy \\ dx_2/dy \end{bmatrix}}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Cramer's Rule \Rightarrow

$$\frac{dx_1}{dy} = \frac{\begin{vmatrix} 0 & 1 & f'_2 \\ f'_1 & 0 & \lambda f''_{12} \\ f'_2 & 0 & \lambda f''_{22} \end{vmatrix}}{\begin{vmatrix} 0 & f'_1 & f'_2 \\ f'_1 & \lambda f''_{11} & \lambda f''_{12} \\ f'_2 & \lambda f''_{21} & \lambda f''_{22} \end{vmatrix}}$$

→ numerator is $-[\lambda f'_1 f''_{22} - \lambda f'_2 f''_{12}]$

Exam 1
2004
Answer 3 cont...

↳ denominator is
 $\det(-1 * \nabla^2 L)$

For a min, D_{2m+1}, \dots, D_{n+m} of $\nabla^2 L$

should all have sign of $(-1)^m$. Here,

$m=1, n=2$, so D_3 of $\nabla^2 L$'s sign

should be same as -1 :

$$D_3 \text{ of } \nabla^2 L < 0.$$

$$\text{So } |\nabla^2 L| < 0.$$

Since $\nabla^2 L$ is 3×3 ,

$$\det(-1 * \nabla^2 L) = (-1)^3 \det \nabla^2 L$$

$$= - \underbrace{\det \nabla^2 L}_{\ominus}$$

so the denominator should be \oplus .

$$\text{Therefore } \frac{dx_1}{dy} > 0 \text{ iff } -(\lambda f'_1 f''_{22} - \lambda f'_2 f''_{12}) > 0$$

$$\lambda (f'_1 f''_{22} - f'_2 f''_{12}) < 0.$$

3. [10 points] Suppose a firm produces one output. The output is produced using one purchased input called " x ," but production is adversely affected by air pollution made by other firms, the amount of which is called " d " (for "dirty air"). Let the price of x be " w ."
- (a) How is this firm's purchases of x changed by a change in the amount of air pollution? Find a symbolic answer to this question, then speculate about its sign.
 - (b) How is this firm's profits affected by a change in the amount of air pollution?

Econ 6710, Spring 2000 Exam 1

$$(3) \pi = p f(x, d) - w x$$

↑
production function
output price

a) F.O.C. : $0 = \frac{\partial \pi}{\partial x} = p \frac{\partial f}{\partial x} - w.$

Find the differential of both sides :

$$0 = (p f_{xx}'') dx + (p f_{xd}'') dd. \quad (dw=0 \text{ and } dp=0)$$

Therefore
$$\boxed{\frac{dx}{dd} = -\frac{f_{xd}''}{f_{xx}''}}.$$

S.O.C. : $0 > \frac{\partial^2 \pi}{\partial x^2} = p f_{xx}''$, so it is reasonable to take $f_{xx}'' < 0$.

f_{xd}'' is $\frac{\partial}{\partial d} f_x' = \frac{\partial}{\partial d}$ (marginal product of x). It is reasonable to assume that as $d \uparrow$, the marginal product of $x \downarrow$. If this were the case, then f_{xd}'' would be negative, and, together with $f_{xx}'' < 0$, we would have $\frac{dx}{dd} < 0$.

b) $\frac{d\pi}{dd} = \frac{\partial L^*}{\partial d}$ (envelope theorem)
 $= \frac{\partial}{\partial d} \left[p f(x, d) - w x \right] = p \frac{\partial f}{\partial d} < 0.$

Or, without the envelope theorem,

$$d\pi = \underbrace{\left(p \frac{\partial f}{\partial x} - w \right) dx}_{=0 \text{ by F.O.C.}} + \left(p \frac{\partial f}{\partial d} \right) dd \Rightarrow \frac{d\pi}{dd} = p \frac{\partial f}{\partial d} < 0.$$

Air pollution hurts profits of this firm.

Final Exam
1996
Question 4

- ②
4. Refer to Figure 1 and consider the following argument:

Suppose the price of input 1 falls. A competitive firm may respond to this by increasing output. If it does, the amount of input 1 purchased by this firm may actually fall, as illustrated in Figure 1.

Explain why this argument is false.

If part of your reasoning involves claiming that some function is concave or convex, you should *prove* that it really is concave or convex. To help you, here is the proof that a consumer's expenditure function is concave in prices:

$$\begin{aligned} e(t\mathbf{p} + (1-t)\mathbf{p}', u_0) &= \min_{\mathbf{x} \text{ s.t. } u(\mathbf{x})=u_0} (t\mathbf{p} + (1-t)\mathbf{p}') \cdot \mathbf{x} \\ &\geq \min_{\mathbf{x} \text{ s.t. } u(\mathbf{x})=u_0} (t\mathbf{p} \cdot \mathbf{x}) + \min_{\mathbf{x} \text{ s.t. } u(\mathbf{x})=u_0} ((1-t)\mathbf{p}' \cdot \mathbf{x}) \\ &= t e(\mathbf{p}, u_0) + (1-t) e(\mathbf{p}', u_0). \end{aligned}$$

Final Exam

1996

Question 4 cont..

Input 2

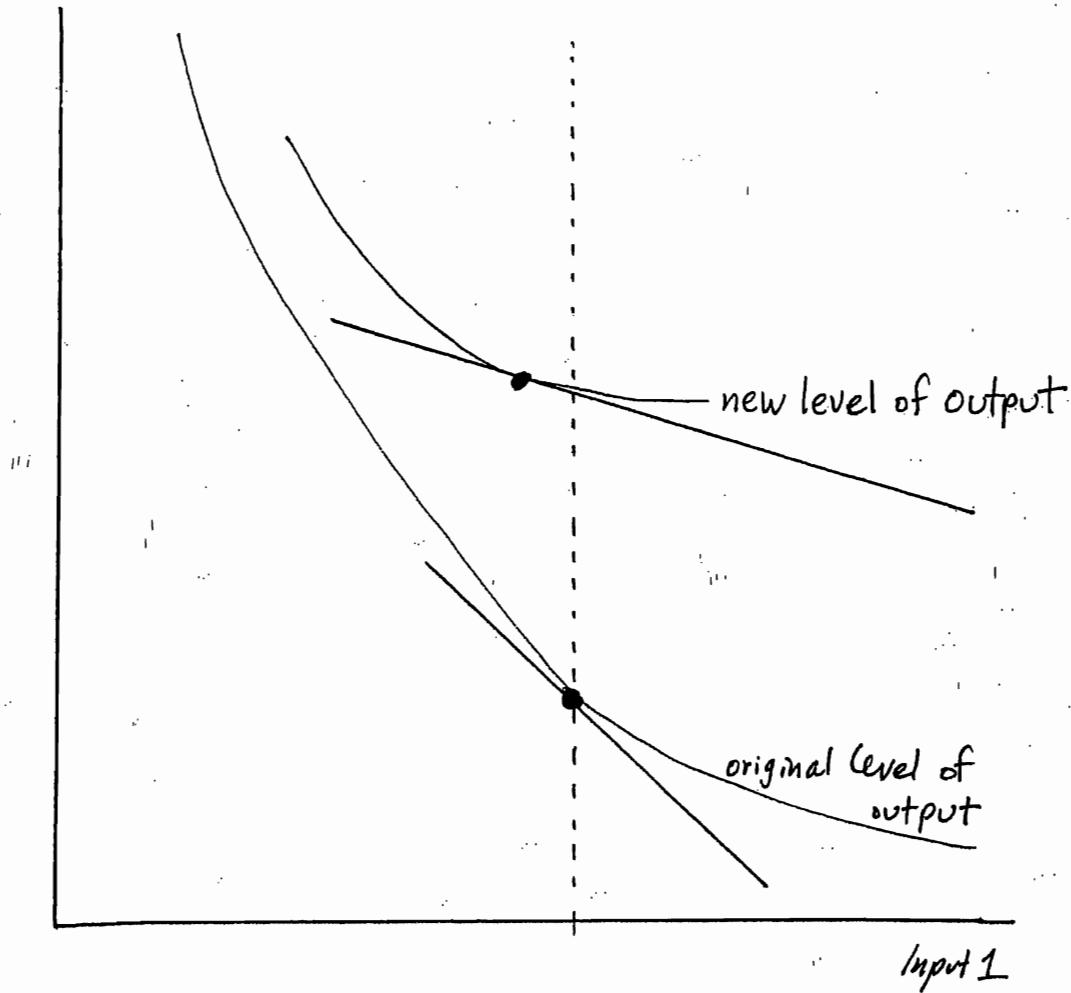
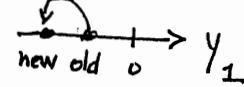


Figure 1

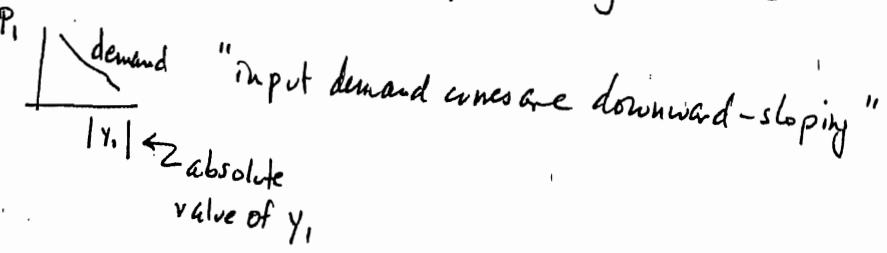
④ The profit function $\pi(p)$ is convex (proof follows). This is the proper place to start because the cost function $c(w, y)$ depends on output y , which is changing in this problem.

$y = \nabla_p \pi(p)$ Hotelling's Lemma (which I didn't ask you to prove, though you should know how to do it.)
 (Input-output vector)

$\nabla_p y = \nabla^2 \pi(p)$. This is a positive semi-definite matrix since $\pi(p)$ is convex. Hence the diagonal entries are ≥ 0 . This proves that $\partial y_i / \partial p_i \geq 0$. So if p_1 falls then

y_1 falls, and if y_1 is an input (as assumed) then: 

Since inputs are negative numbers in this framework, y_1 falling means the firm buys more y_1 :



This shows that the argument is false and that what seems possible from Figure 1 actually can't happen to a competitive firm.

Now for the proof that $\pi(p)$ is convex =

Final Exam

1996

8

Answer 4 cont..

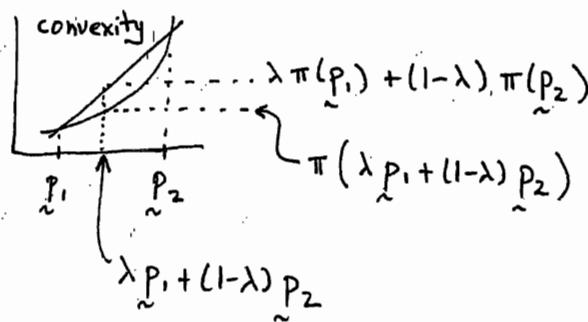
$$\pi(\lambda \tilde{p}_1 + (1-\lambda) \tilde{p}_2) = \max_{y \in Y} [\lambda \tilde{p}_1 + (1-\lambda) \tilde{p}_2] \cdot y \quad \text{where } Y \text{ is the production possibilities set}$$

$$\leq \max_{y \in Y} \lambda \tilde{p}_1 \cdot y + \max_{y \in Y} (1-\lambda) \tilde{p}_2 \cdot y \quad \text{because the latter makes possible}$$

different choices of y for the first and second terms, instead of forcing y to be the same for both.

$$= \lambda \max_{y \in Y} \tilde{p}_1 \cdot y + (1-\lambda) \max_{y \in Y} \tilde{p}_2 \cdot y \quad \text{moving constants}$$

$$= \lambda \pi(\tilde{p}_1) + (1-\lambda) \pi(\tilde{p}_2) \quad \text{by definition.}$$



basic idea: 4 pts.

convexity: 2 pts.

rest of math: 2 pts.

2. [12 points]

- (a) Prove Hotelling's Lemma $\mathbf{y} = \nabla_{\mathbf{p}} \pi(\mathbf{p})$.
- (b) Prove that the profit function $\pi(\mathbf{p})$ is convex.
- (c) Refer to Figure 1 and consider the following argument:

Suppose the price of Input 1 (energy) rises. A competitive firm may respond to this by decreasing output. If it does, the amount of Input 1 (energy) purchased by this firm may actually rise, as illustrated in Figure 1.

Explain why this argument is false.

2009
Qualifier
Sec. 2

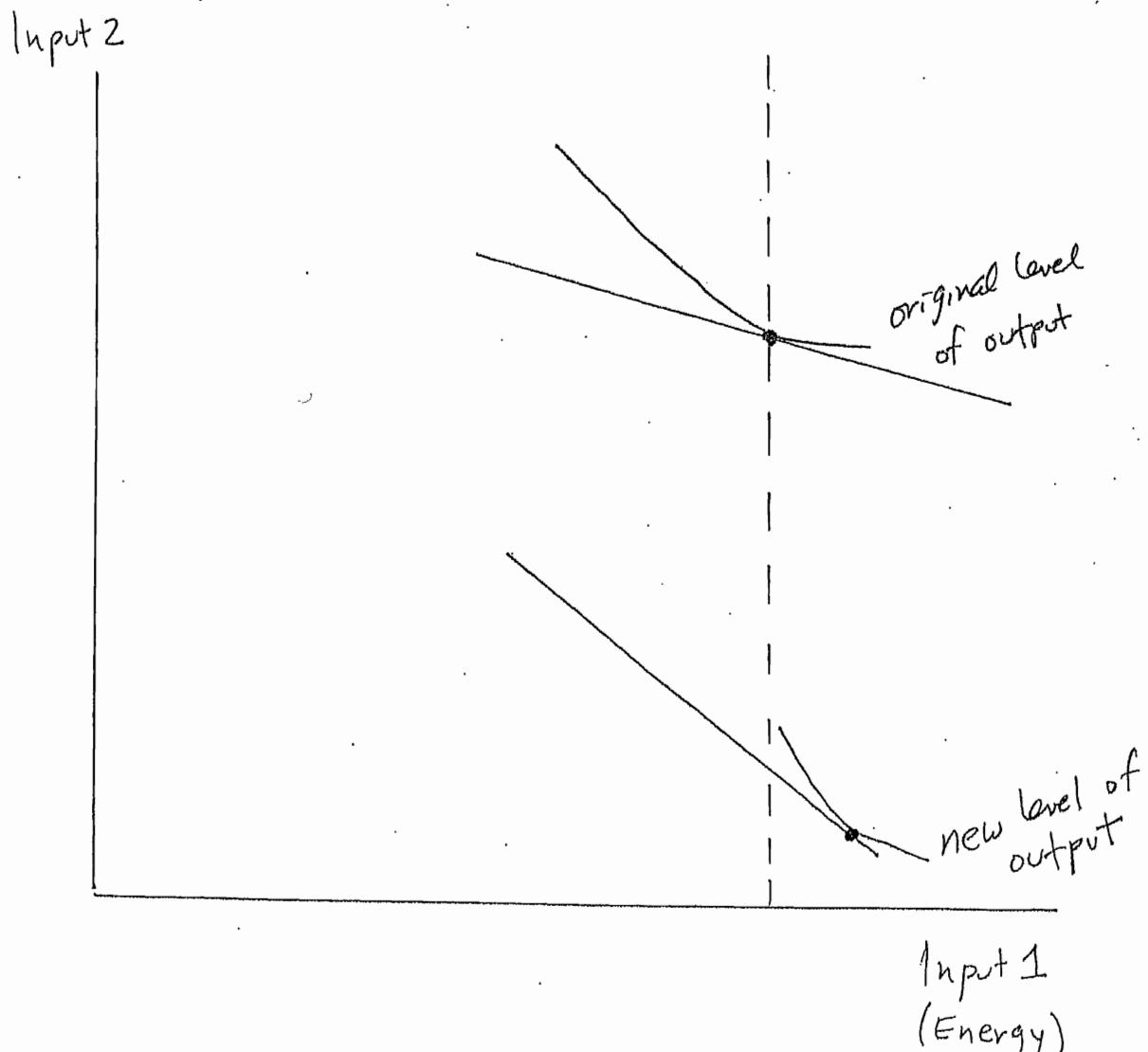


Fig. 1

(2)

a) Proof of Hotelling's Lemma:

Envelope Theorem (unconstrained
problem)

$$M(a) = \max_a f(x, a)$$

$$\Rightarrow \frac{dM}{da} = \left. \frac{\partial f(x, a)}{\partial a} \right|_{x^*}$$

Profit Maximization for a
Competitive Firm

$$\pi(p) = \max_y p \cdot y$$

$$\Rightarrow \left. \nabla_p \pi(p) = \left. \frac{\partial p \cdot y}{\partial p} \right|_{y^*} \right.$$

$$= y^*$$

b)

See the second (the last) page of the
answer to "Final Exam 1996 Answer 4"
which immediately precedes this question.

c) See the first page of this! \uparrow

2015 Qualifying Exam Sec. 1 Qu. 2

2. [14 points]

- (a) Answer the following questions assuming a competitive cost-minimizing firm.

- i. Prove that input demand curves $\mathbf{x}(\mathbf{w}, y)$ are homogeneous of degree zero in \mathbf{w} .
- ii. As some of you may already know, Euler proved the following: if $f(\mathbf{x})$ is differentiable and is homogeneous of degree k , then

$$\nabla_{\mathbf{x}} f(\mathbf{x}) \cdot \mathbf{x} = k f(\mathbf{x}).$$

(Do not forget that the left-hand side has a “ $\cdot \mathbf{x}$ ” in it.) What property of input demand curves can you derive from this result, given what you already know from part (i)?

- iii. Rewrite your answer to part (ii) for the special case when the total number of inputs is exactly three.
- iv. For any two inputs j and k , here are two definitions:

$$\begin{aligned}\partial x_j(\mathbf{w}, y)/\partial w_k \geq 0 &\iff j \text{ and } k \text{ are “substitutes”} \\ \partial x_j(\mathbf{w}, y)/\partial w_k < 0 &\iff j \text{ and } k \text{ are “complements.”}\end{aligned}$$

Use the previous parts of this question, and other information, to prove that if the total number of inputs is three, then every input has at least one substitute.

- v. Prove that every input has at least one substitute (regardless of what the total number of inputs may be).
- (b) Answer the following questions assuming a competitive profit-maximizing firm.

- i. Prove that net output curves $\mathbf{y}(\mathbf{p})$ are homogeneous of degree zero in \mathbf{p} .
- ii. What property of net output curves can you derive from the result that if $f(\mathbf{x})$ is differentiable and is homogeneous of degree k , then $\nabla_{\mathbf{x}} f(\mathbf{x}) \cdot \mathbf{x} = k f(\mathbf{x})$, given what you already know from the part (i)?
- iii. Rewrite your answer to the part (ii) for the special case when $\mathbf{y} \in \mathbf{R}^3$.

iv. For the special case when $\mathbf{y} \in \mathbf{R}^3$, tell me everything you know about the signs of

$$\frac{\partial y_1}{\partial p_1}, \quad \frac{\partial y_1}{\partial p_2}, \quad \text{and} \quad \frac{\partial y_1}{\partial p_3}$$

and thoroughly describe what these results mean intuitively.

Section 1 Question 2.

a) i) Cost-minimizing input demand functions solve

$$\min_{\underline{x}} \underline{w} \cdot \underline{x} \text{ s.t. } f(\underline{x}) = y. \quad (\text{P1})$$

If \underline{w} changes to $\lambda \underline{w}$, the problem becomes

$$\min_{\underline{x}} \lambda \underline{w} \cdot \underline{x} \text{ s.t. } f(\underline{x}) = y \quad (\text{P2})$$

$$\Leftrightarrow \lambda \min_{\underline{x}} \underline{w} \cdot \underline{x} \text{ s.t. } f(\underline{x}) = y. \quad (\text{P3})$$

(P1) and (P3) have the same optimal point $\underline{x}^*(\underline{w}, y)$. This is the same as the optimal point for (P2), which is $\underline{x}^*(\lambda \underline{w}, y)$. Hence writing

$$\underline{x}^*(\lambda \underline{w}, y) = \lambda^k \underline{x}^*(\underline{w}, y),$$

one requires $k=0$, proving that \underline{x}^* is homogeneous of degree zero in \underline{w} . (Alternatively: show that $c(\underline{w}, y)$ is homogeneous of degree 1 in \underline{w} , so $\underline{x}^* = \nabla_{\underline{w}} c$ is homogeneous of degree 0 in \underline{w} .)

ii)

$$\nabla_{\underline{x}} f(\underline{x}) \cdot \underline{x} = k f(\underline{x}) \text{ so}$$

$$\nabla_{\underline{w}} \underline{x}^*(\underline{w}, y) \cdot \underline{w} = k \underline{x}^*(\underline{w}, y) \text{ with } k=0. \text{ For input } i,$$

$$\nabla_{\underline{w}} x_i(\underline{w}, y) \cdot \underline{w} = k x_i(\underline{w}, y) \text{ with } k=0. \text{ So}$$

$$\sum_j \frac{\partial x_i}{\partial w_j} w_j = 0 \text{ for all } i.$$

iii)

$$\frac{\partial x_i}{\partial w_1} w_1 + \frac{\partial x_i}{\partial w_2} w_2 + \frac{\partial x_i}{\partial w_3} w_3 = 0 \text{ for } i=1, 2, 3.$$

(iv) Take $i=1$ for example. Then

$$\frac{\partial x_1}{\partial w_1} w_1 + \frac{\partial x_1}{\partial w_2} w_2 + \frac{\partial x_1}{\partial w_3} w_3 = 0.$$

this is negative
because "own" input
demand curves are
always downward-

scoring (proof):

$$\nabla_w^2 c = \nabla_w \nabla_w c = \nabla_w \left[\frac{\partial \text{min}}{\partial w} w \cdot x \right]$$

$\nabla_w x$ is negative semi-definite since
 $c(w, y)$ is concave in w , so $\nabla_w^2 c$ has
negative diagonal terms)

Since the first term is negative but the
sum is zero, the second or third term
(or both of them) must be positive.

Hence $\frac{\partial x_1}{\partial w_2} > 0$ or $\frac{\partial x_1}{\partial w_3} > 0$.

1 and 2 are
substitutes

1 and 3 are
substitutes.

The cases of $i=2$ and $i=3$ are analogous.

(v) $\sum_j \frac{\partial x_i}{\partial w_j} w_j = 0$ but $\frac{\partial x_i}{\partial w_i} < 0$ and all the w_j 's are positive,

so the i^{th} term of the sum, $\frac{\partial x_i}{\partial w_i} w_i$, is negative, so some other term
must be positive: $\frac{\partial x_i}{\partial w_j} w_j > 0$ for some j . Since $w_j > 0$, this

means $\frac{\partial x_i}{\partial w_j} > 0$ for some j , so i and this j are substitutes.

b) i) Net output curves $\underline{y}(\underline{p})$ solve

$$\max_{\underline{y}} \underline{p} \cdot \underline{y} \text{ s.t. } \underline{y} \in Y. \quad (P1')$$

If \underline{p} changes to $\lambda \underline{p}$, the problem becomes

$$\max_{\underline{y}} \lambda \underline{p} \cdot \underline{y} \text{ s.t. } \underline{y} \in Y \quad (P2')$$

$$\Leftrightarrow \lambda \max_{\underline{y}} \underline{p} \cdot \underline{y} \text{ s.t. } \underline{y} \in Y. \quad (P3')$$

$(P1')$ and $(P3')$ have the same optimal point $\underline{y}^*(\underline{p})$. This is the same as the optimal point for $(P2')$, which is $\underline{y}^*(\lambda \underline{p})$. Hence writing

$$\underline{y}^*(\lambda \underline{p}) = \lambda^k \underline{y}^*(\underline{p}),$$

one requires $k=0$, proving that \underline{y}^* is homogeneous of degree zero in \underline{p} .

ii) $\nabla_{\underline{x}} f(\underline{x}) \cdot \underline{x} = k f(\underline{x})$ so

$$\nabla_{\underline{p}} \underline{y}(\underline{p}) \cdot \underline{p} = k \underline{y}(\underline{p}) \text{ with } k=0. \text{ For commodity } i,$$

$$\nabla_{\underline{p}} y_i(\underline{p}) \cdot \underline{p} = k y_i(\underline{p}) \text{ with } k=0. \text{ So}$$

$$\sum_j \frac{\partial y_i}{\partial p_j} p_j = 0 \text{ for all } i.$$

iii) Take $i=1$ for example. Then

$$\frac{\partial y_1}{\partial p_1} p_1 + \frac{\partial y_1}{\partial p_2} p_2 + \frac{\partial y_1}{\partial p_3} p_3 = 0.$$

↳ (Alternatively: show that $\pi(\underline{p})$ is homogeneous of degree 1 in \underline{p} , so $\underline{y} = \nabla_{\underline{p}} \pi$ is homogeneous of degree 0 in \underline{p} .)

(iv)

$\frac{\partial y_i}{\partial p_i} > 0$. Optional proof: $\underbrace{\pi(p)}$ convex \Rightarrow this is positive semi-definite

and thus has positive diagonal terms:

$$\underbrace{\nabla_p^2 \pi}_{\sim} = \underbrace{\nabla_p}_{\sim} \nabla_p \underbrace{\pi(p)}_{\sim} = \underbrace{\nabla_p}_{\sim} \nabla_p \max_{\underbrace{y \in \mathbb{R}^n}_{\sim}} p \cdot y = \nabla_p y.$$

Use the Envelope Theorem

Since $p_1 > 0$, $p_2 > 0$, and $p_3 > 0$, this and (iii) implies that either

$$\frac{\partial y_1}{\partial p_2} < 0 \text{ or } \frac{\partial y_1}{\partial p_3} < 0 \text{ (or both).}$$

For interpretation, suppose that $\partial y_1 / \partial p_2 < 0$ (even though $\partial y_1 / \partial p_3 < 0$ is just as likely), and suppose that

$$\underbrace{y_1 > 0}_{\swarrow} \text{ or } \underbrace{y_1 < 0}_{\searrow}.$$

"1" is an output.

"1" is an input.

When $p_2 \uparrow$, $y_1 \downarrow$: when the price of input 2 \uparrow , output falls.

When $p_2 \uparrow$, $y_1 \downarrow$ (y_1 gets further away from zero; y_1 gets more negative): when the price of input 2 \uparrow , the amount of input 1 goes "up" treating all numbers as positive.

Optional: "1" is an output and "2" is an output. Then:

When the price of output 2 \uparrow , the amount of output 1 \downarrow .

Optional: "1" is an input and "2" is an output. Then: when

the price of output 2 \uparrow , the amount of input 1 goes "up" treating all numbers as positive.

2. Consider a cost-minimizing firm which produces an output y using two inputs x_1 and x_2 according to the production function $y = f(x_1, x_2)$. Make the usual assumptions that the firm takes the prices of x_1 and x_2 to be fixed, that both $\partial f / \partial x_1$ and $\partial f / \partial x_2$ are strictly positive, and that there are diminishing returns to each input.

Call an input "inferior" if when output increases, the firm chooses to use less of this input.

- (a) Is it possible for both x_1 and x_2 to be inferior (simultaneously)? You should be able to answer this without solving any optimization problem; indeed, undergraduate students who do not know calculus should be able to answer this.
- (b) Under what conditions is x_1 inferior? (This is not a question undergraduates could solve.)
- (c) Under what conditions is x_2 inferior?
- (d) Use the answers to (a), (b), and (c) to argue that it is impossible for f''_{12} to be very negative.
- (e) Could x_1 or x_2 ever actually be inferior? When?

Qualifying Exam
2004

Question 2

(2)

Section 2 #2

cost minimization

Qualifying Exam

2004

Answer 2

a) if $y \uparrow \Rightarrow x_1 \downarrow$ and $x_2 \downarrow$ then $f(x_1, x_2)$ would have to fall, so $y = f(x_1, x_2)$ would have to fall, contradicting $y \uparrow$

b) $\min w_1 x_1 + w_2 x_2$ s.t. $y = f(x_1, x_2)$

$$\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda [y - f(x_1, x_2)]$$

$$\begin{aligned} \lambda: 0 &= y - f(x_1, x_2) \\ x_1: 0 &= w_1 - \lambda f'_1 \\ x_2: 0 &= w_2 - \lambda f'_2 \end{aligned}$$

λ x_1 x_2

exogenous
↓ ↓
 y $d w_1 = d w_2 = 0$

endogenous

$0 d\lambda - f'_1 dx_1 - f'_2 dx_2 + dy = 0$

$-f'_1 d\lambda - \lambda f''_{11} dx_1 - \lambda f''_{12} dx_2 + 0 = 0$

$-f'_2 d\lambda - \lambda f''_{21} dx_1 - \lambda f''_{22} dx_2 + 0 = 0$

Find the differentials

$$\begin{bmatrix} dy \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & f'_1 & f'_2 \\ f'_1 & \lambda f''_{11} & \lambda f''_{12} \\ f'_2 & \lambda f''_{21} & \lambda f''_{22} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -dy \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -f'_1 & -f'_2 \\ -f'_1 & -\lambda f''_{11} & -\lambda f''_{12} \\ -f'_2 & -\lambda f''_{21} & -\lambda f''_{22} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & f'_1 & f'_2 \\ f'_1 & \lambda f''_{11} & \lambda f''_{12} \\ f'_2 & \lambda f''_{21} & \lambda f''_{22} \end{bmatrix} \begin{bmatrix} d\lambda / dy \\ dx_1 / dy \\ dx_2 / dy \end{bmatrix}$$

Cramer's Rule:

$$\frac{dx_1}{dy} = \frac{\begin{vmatrix} 0 & 1 & f'_1 \\ f'_1 & 0 & \lambda f''_{12} \\ f'_2 & 0 & \lambda f''_{22} \end{vmatrix}}{\begin{vmatrix} 0 & f'_1 & f'_2 \\ f'_1 & \lambda f''_{11} & \lambda f''_{12} \\ f'_2 & \lambda f''_{21} & \lambda f''_{22} \end{vmatrix}} \quad \text{or} \quad \frac{\begin{vmatrix} 0 & -1 & -f'_2 \\ -f'_1 & 0 & -\lambda f''_{12} \\ -f'_2 & 0 & -\lambda f''_{22} \end{vmatrix}}{\begin{vmatrix} 0 & -f'_1 & -f'_2 \\ -f'_1 & -\lambda f''_{11} & -\lambda f''_{12} \\ -f'_2 & -\lambda f''_{21} & -\lambda f''_{22} \end{vmatrix}}$$

The S.O.C. for a min. are that $\underbrace{|\nabla^2 \mathcal{L}_{2m+1}|}_3 \dots \underbrace{|\nabla^2 \mathcal{L}_{m+n}|}_{42=3}$ have the sign of $(-1)^m$.

So $|\nabla^2 \mathcal{L}|$ should have the sign of $(-1)^{142} < 0$.

$$|\nabla^2 \mathcal{L}| \text{ is } \begin{vmatrix} 0 & -f'_1 & -f'_2 \\ -f'_1 & -\lambda f''_{11} & -\lambda f''_{12} \\ -f'_2 & -\lambda f''_{12} & -\lambda f''_{22} \end{vmatrix} \text{ which equals } (-1)^{142} \begin{vmatrix} 0 & f'_1 & f'_2 \\ f'_1 & \lambda f''_{11} & \lambda f''_{12} \\ f'_2 & \lambda f''_{12} & \lambda f''_{22} \end{vmatrix}.$$

$$\text{So } \operatorname{sign}\left(\frac{dx_1}{dy}\right) = \frac{(-1)^{142}(1)(f'_1 \lambda f''_{22} - f'_2 \lambda f''_{12})}{\oplus} \text{ or } \frac{(-1)^{142}(-1)(f'_1 \lambda f''_{22} - f'_2 \lambda f''_{12})}{\ominus}$$

$$= \lambda(f'_2 f''_{12} - f'_1 f''_{22}) \quad \text{or} \quad \lambda(f'_2 f''_{12} - f'_1 f''_{22}) \text{ which are the same!}$$

From the envelope theorem, note that

$$\frac{\partial \text{minimized cost}}{\partial y} = \frac{\partial \mathcal{L}^*}{\partial y} = \lambda$$

Qualifying Exam
2004

so $\lambda > 0$ because as $y \uparrow$, cost must \uparrow .

Answer 2 cont...

$$\text{Thus } \operatorname{sign}\left(\frac{dx_1}{dy}\right) = \operatorname{sign}(f'_2 f''_{12} - f'_1 f''_{22}).$$

x_1 is inferior if $f'_2 f''_{12} - f'_1 f''_{22} < 0$.

$$\text{c)} \quad \frac{dx_2}{dy} = \frac{\begin{vmatrix} 0 & -f'_1 & -1 \\ -f'_1 & -\lambda f''_{11} & 0 \\ -f'_2 & -\lambda f''_{12} & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -f'_1 & -f'_2 \\ -f'_1 & -\lambda f''_{11} & -\lambda f''_{12} \\ -f'_2 & -\lambda f''_{12} & -\lambda f''_{22} \end{vmatrix}} \leftarrow \ominus$$

So $\text{sign}(dx_2/dy) = f'_1 f''_{21} - f'_2 f''_{11}$ (recall from above that $\lambda > 0$).

d) $x_2 \text{ inferior} \Leftrightarrow f'_1 f''_{21} - f'_2 f''_{11} < 0$

$$x_1 \text{ inferior} \Leftrightarrow f'_2 f''_{12} - f'_1 f''_{22} < 0$$

$$\begin{array}{c} + \\ \diagdown \\ \end{array} \quad \begin{array}{c} + \quad - \\ \downarrow \\ \text{diminishing returns} \end{array}$$

f''_{12} very negative $\Rightarrow x_1 \text{ inferior}$

$$x_2 \text{ inferior} \Leftrightarrow f'_1 f''_{21} - f'_2 f''_{11} < 0$$

$$\begin{array}{c} + \\ \diagdown \\ \end{array} \quad \begin{array}{c} + \quad - \\ \downarrow \\ \end{array}$$

$f''_{21} = f''_{12}$ very negative $\Rightarrow x_2 \text{ inferior}$

So if f''_{12} were very negative, both x_1 and x_2 would be inferior, which can't happen.

e) $x_1 \text{ inferior iff } f''_{12} < \underbrace{f'_1 f''_{22}}_{\ominus} / f'_2 \quad (1)$

$$x_2 \text{ inferior iff } f''_{12} = f''_{21} < \underbrace{f'_2 f''_{11}}_{\ominus} / f'_1 \quad (2)$$

So x_2 would be inferior if f''_{12} satisfied (2) but not (1).

In other words, f''_{12} would have to be in between $f'_1 f''_{22} / f'_2$ and $f'_2 f''_{11} / f'_1$, and $f'_1 f''_{22} / f'_2$ would have to be less than $f'_2 f''_{11} / f'_1$ (otherwise whenever (2) was true, (1) would also be true).

Section 2.

Answer one of the following two questions.

1. In this problem, if you would like to use the Envelope Theorem, you need not prove it. On the other hand, if you assert convexity or concavity of a function, you should prove that.

Summer
2006
Qualifier

- (a) Derive the basic comparative-statics results for a competitive profit-maximizing firm when all relevant prices change.
- (b) Derive the basic comparative-statics results for a competitive cost-minimizing firm when all relevant prices change.
- (c) Compare and contrast your answers to (a) and (b). Are they contradictory or consistent with each other?

Section 2.

Qn1. a) $\pi(\underline{p}) = \max_{\underline{y}} \underline{p} \cdot \underline{y}$ s.t. $\underline{y} \in Y$

$\pi(\underline{p})$ is convex

$\Rightarrow \nabla_{\underline{p}}^2 \pi(\underline{p})$ is positive semidefinite

$\Rightarrow \nabla_{\underline{p}} \nabla_{\underline{p}} \pi(\underline{p})$ is " "

$\Rightarrow \boxed{\nabla_{\underline{p}} \underline{y}^* \text{ is } " "}$

because $\nabla_{\underline{p}} \pi(\underline{p}) = \underline{y}^*$

(Hotelling's Lemma)

$$\nabla_{\underline{p}} \pi(\underline{p}) = \nabla_{\underline{p}} (\underline{p} \cdot \underline{y}^*) = \underline{y}^* \text{ by}$$

the Envelope Theorem

b) $c(\underline{w}, \underline{y}) = \min_{\underline{x}} \underline{w} \cdot \underline{x}$ s.t. $f(\underline{x}) = \underline{y}$

$c(\underline{w}, \underline{y})$ is concave in \underline{w}

$\nabla_{\underline{w}}^2 c$ is negative semi-definite

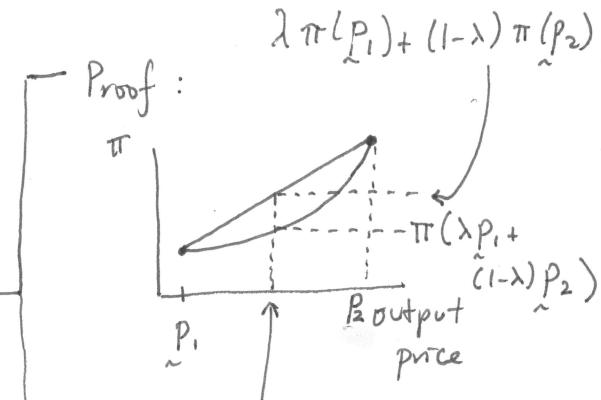
$\nabla_{\underline{w}} \nabla_{\underline{w}} c$ is " "

$\boxed{\nabla_{\underline{w}} \underline{x}^* \text{ is } " "}$ because

$\nabla_{\underline{w}} c = \underline{x}^*$ (Shephard's Lemma):

$\nabla_{\underline{w}} E = \nabla_{\underline{w}} (\underline{w} \cdot \underline{x}^*) = \underline{x}^*$ by the

Envelope Theorem



$$\lambda \underline{p}_1 + (1-\lambda) \underline{p}_2$$

$$\pi(\lambda \underline{p}_1 + (1-\lambda) \underline{p}_2)$$

$$= \max_{\underline{y}} (\lambda \underline{p}_1 + (1-\lambda) \underline{p}_2) \cdot \underline{y}$$

$$\leq \max_{\underline{y}} \lambda \underline{p}_1 \cdot \underline{y} + \max_{\underline{y}} (1-\lambda) \underline{p}_2 \cdot \underline{y}$$

$$= \lambda \pi(\underline{p}_1) + (1-\lambda) \pi(\underline{p}_2) \quad Q.E.D.$$

The third step come from

$$\max_x [f(x) + g(x)]$$

$$\leq \max_x f(x) + \max_x g(x).$$

Proof: similar to above

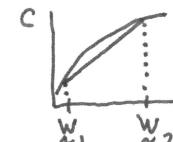
$$c(\lambda \underline{w}_1 + (1-\lambda) \underline{w}_2, \underline{y})$$

$$= \min_{\underline{x}} (\lambda \underline{w}_1 + (1-\lambda) \underline{w}_2) \cdot \underline{x} \text{ s.t. } f(\underline{x}) = \underline{y}$$

$$= \min_{\underline{x}} [\lambda \underline{w}_1 \cdot \underline{x} + (1-\lambda) \underline{w}_2 \cdot \underline{x}]$$

$$\geq \min_{\underline{x}} \lambda \underline{w}_1 \cdot \underline{x} + \min_{\underline{x}} (1-\lambda) \underline{w}_2 \cdot \underline{x}$$

$$= \lambda c(\underline{w}_1, \underline{y}) + (1-\lambda) c(\underline{w}_2, \underline{y})$$



c) $P_w \tilde{x}^*$ neg. semi-def. \Rightarrow diagonal terms negative (or zero)

$$\Rightarrow \partial x_i / \partial w_i \leq 0$$

\Leftrightarrow if the price of the i^{th} input \uparrow ,

the firm's demand for that input \downarrow
(or is unchanged)

$V_p \tilde{y}^*$ pos. semi-def. \Rightarrow diagonal terms positive (or zero)

$$\Rightarrow \partial y_i / \partial p_i \geq 0$$

\Rightarrow if the price of the i^{th} input \uparrow , then

the firm's demand for that input, which is
 $y_i \leq 0$, becomes algebraically more positive,
meaning that y_i moves closer to zero, so

$|y_i|$ falls; if you redefine input demands
to be positive, input demand would fall.

So these two results are the same.

In (a), changes in output price can be studied (\Rightarrow upward-sloping
Supply curves), but in (b) output price does not appear.

Question 3. Consider a profit-maximizing firm that produces a good which is sold in a competitive market. It is observed that when the price of the output good rises, the firm hires more skilled workers but fewer unskilled workers. Now the unskilled workers unionize and succeed in getting their wages increased. Assume that all other prices remain constant.

- a) What will happen to the firm's demand for unskilled workers?
- b) What will happen to the firm's supply of output?

Qualifying Exam

1998

Question 3

(2)

Optional Question 3. Varian 5.13.

Use the profit function $\pi(\tilde{p})$ here, not the cost function $c(w, y)$, because output is changing. Notation:

Qualifying Exam
1998

Answer 3

price commodity quantity

p_1 output $y_1 > 0$

p_2 skilled labor $y_2 < 0$

p_3 unskilled labor $y_3 < 0$

} remember the sign convention
for input-output
vectors!

- When $p_1 \uparrow$, more skilled workers are hired $\Rightarrow y_2 \downarrow$ (becomes more negative - more input)
- When $p_1 \uparrow$, less unskilled workers are hired $\Rightarrow y_3 \uparrow$ (gets closer to zero - less input)

$$\therefore \frac{\partial y_2}{\partial p_1} < 0 \text{ and } \frac{\partial y_3}{\partial p_1} > 0.$$

Now suppose $p_3 \uparrow$.

a) Hotelling's Lemma: $\tilde{y} = \nabla_{\tilde{p}} \pi(\tilde{p})$. Differentiate both sides w.r.t. \tilde{p} :

$\nabla_{\tilde{p}} \tilde{y} = \nabla_{\tilde{p}}^2 \pi(\tilde{p})$ which is positive semi-definite symmetric
since $\pi(\tilde{p})$ is convex.

Hence $\frac{\partial y_3}{\partial p_3} > 0$: as $p_3 \uparrow$, $y_3 \uparrow$ (becomes less negative) : fewer unskilled workers are hired.

b) $\frac{\partial y_1}{\partial p_3} = \frac{\partial y_3}{\partial p_1} > 0$. So as $p_3 \uparrow$, $y_1 \uparrow$: more output
by symmetry from above is produced (!).

Qualifying Exam
1998

Answer 3 cont...

new: 2020 Final Exam, Qu. 1.

1. [17 points]

- (a) Given that Hotelling's Lemma can be written as $\mathbf{y} = \nabla_{\mathbf{p}} \pi(\mathbf{p})$, what does this imply for $\nabla_{\mathbf{p}} \mathbf{y}$?
- (b) Argue that the profit of a competitive firm using two inputs y_2 and y_3 to produce output y_1 with a concave production function $f(y_2, y_3) = y_1$ could be modeled as having profit

$$\pi = f(y_2, y_3) p_1 + y_2 p_2 + y_3 p_3 .$$

Be sure to state the sign of each of the variables in this equation.

- (c) Given profit as in part (b), find the comparative statics derivative $\partial y_2 / \partial p_3$.
- (d) What would it mean intuitively for $\partial y_2 / \partial p_3$ to be positive? Do not forget the sign convention for y_2 . Throughout this problem, you can assume that the second-order conditions for a profit maximum are satisfied.
- (e) Given profit as in the preceding parts of this question, find the comparative statics derivative $\partial y_2 / \partial p_1$.
- (f) Argue that if $\partial y_2 / \partial p_3 > 0$ then $\partial y_2 / \partial p_1 > 0$.
- (g) Argue that if $\partial y_2 / \partial p_1 > 0$ then $\partial y_1 / \partial p_2 > 0$. Do not use a comparative statics argument here; instead, reason directly from part (a) of this problem.
- (h) Parts (f) and (g) together mean that $\partial y_2 / \partial p_3 > 0$ implies $\partial y_1 / \partial p_2 > 0$. What does this last condition mean intuitively? Do not forget the sign convention for y_1 .

Answer to Question 1, Final Exam, Econ. 7005, Fall 2020

a) $\underline{y} = \nabla_p \underline{\pi}(p)$. Differentiate both sides with respect to \underline{p} :

$\nabla_{\underline{p}} \underline{y} = \nabla_{\underline{p}}^2 \underline{\pi}(p)$. Since $\underline{\pi}(p)$ is convex, the right-hand side is positive semidefinite. Since it is a Hessian, it is symmetric. Hence the left-hand side, which is a Jacobian, is positive semidefinite and symmetric.

b) $p_1 > 0, p_2 > 0, p_3 > 0$

$y_1 > 0, y_2 < 0, y_3 < 0$ for inputs y_2 and y_3 and output y_1 . (This is the sign convention for the multiple-output framework, which is OK to use even though there's only one output here.)

$$\pi = f(y_2, y_3) p_1 + y_2 p_2 + y_3 p_3$$

↑ ↑ ⊖ ⊖
 output output
 ↓ ↓
 price input costs
 total revenue

These signs are correct:

output contributes positively to profit and "absolute value of inputs" contribute negatively to profit, ceteris paribus.

c) F.O.C. for π -maximization:

$$0 = \partial \pi / \partial y_2 = f'_2 p_1 + p_2$$

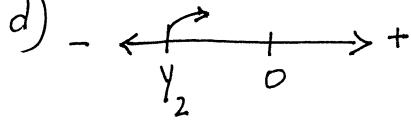
$0 = \partial \pi / \partial y_3 = f'_3 p_1 + p_3$. Find the differentials with $dp_2 = dp_1 = D$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f''_{22} p_1 & f''_{23} p_1 \\ f''_{32} p_1 & f''_{33} p_1 \end{bmatrix} \begin{bmatrix} dy_2 \\ dy_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dp_3 \Rightarrow$$

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} f''_{22} p_1 & f''_{23} p_1 \\ f''_{32} p_1 & f''_{33} p_1 \end{bmatrix} \begin{bmatrix} \partial y_2 / \partial p_3 \\ \partial y_3 / \partial p_3 \end{bmatrix}. \text{ Using Cramer's Rule,}$$

$$\frac{\partial y_2}{\partial p_3} = \frac{\begin{vmatrix} 0 & f''_{23} p_1 \\ -1 & f''_{33} p_1 \end{vmatrix}}{\begin{vmatrix} f''_{22} p_1 & f''_{23} p_1 \\ f''_{32} p_1 & f''_{33} p_1 \end{vmatrix}} = \frac{f''_{23} p_1}{p_1^2 \begin{vmatrix} f''_{22} & f''_{23} \\ f''_{32} & f''_{33} \end{vmatrix}} = \frac{f''_{23}}{p_1 \begin{vmatrix} f''_{22} & f''_{23} \\ f''_{32} & f''_{33} \end{vmatrix}}$$

↑ ↑
⊕ ⊕ since
f is concave

d)  $\frac{\partial y_2}{\partial p_3} > 0$ means that as $p_3 \uparrow$, y_2 goes to the right. Intuitively, $p_3 \uparrow \Rightarrow$ using "less" y_2 (y_2 gets closer to zero).

e) From part (c) find the differentials of the F.O.C.s with $dp_2 = dp_3 = 0$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f''_{22} p_1 & f''_{23} p_1 \\ f''_{32} p_1 & f''_{33} p_1 \end{bmatrix} \begin{bmatrix} dy_2 \\ dy_3 \end{bmatrix} + \begin{bmatrix} f'_2 \\ f'_3 \end{bmatrix} dp_1 \Rightarrow$$

$$\begin{bmatrix} -f'_2 \\ -f'_3 \end{bmatrix} = \begin{bmatrix} f''_{22} p_1 & f''_{23} p_1 \\ f''_{32} p_1 & f''_{33} p_1 \end{bmatrix} \begin{bmatrix} \partial y_2 / \partial p_1 \\ \partial y_3 / \partial p_1 \end{bmatrix} \Rightarrow$$

$$\frac{\partial y_2}{\partial p_1} = \frac{\begin{vmatrix} -f'_2 & f''_{23} p_1 \\ -f'_3 & f''_{33} p_1 \end{vmatrix}}{\begin{vmatrix} f''_{22} p_1 & f''_{23} p_1 \\ f''_{32} p_1 & f''_{33} p_1 \end{vmatrix}} = \frac{-f'_2 f''_{33} p_1 + f'_3 f''_{23} p_1}{p_1^2 \begin{vmatrix} f''_{22} & f''_{23} \\ f''_{32} & f''_{33} \end{vmatrix}} = \frac{-f'_2 f''_{33} + f'_3 f''_{23}}{p_1 \begin{vmatrix} f''_{22} & f''_{23} \\ f''_{32} & f''_{33} \end{vmatrix}}$$

⊕ marginal products of inputs

⊖ since f is concave

↑
⊕ since f is concave

f) From (c), $\frac{\partial y_2}{\partial p_3} = \frac{f''_{23}}{\oplus}$. So if $\frac{\partial y_2}{\partial p_3} > 0$ then $f''_{23} > 0$.

$$\text{From (e), } \frac{\partial y_2}{\partial p_1} = \frac{-\ominus\ominus + \oplus f''_{23}}{\oplus} = \frac{\oplus + \ominus f''_{23}}{\oplus}.$$

So if $f''_{23} > 0$ then $\frac{\partial y_2}{\partial p_1} > 0$.

g) $\frac{\partial y_2}{\partial p_1} = \frac{\partial y_1}{\partial p_2}$ follows from the equality of the (1,2) and (2,1)

elements of the $\nabla_p y$
matrix, which is symmetric
from part (a).

$$\nabla_p y = \begin{bmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} & \frac{\partial y_1}{\partial p_3} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} & \frac{\partial y_2}{\partial p_3} \\ \frac{\partial y_3}{\partial p_1} & \frac{\partial y_3}{\partial p_2} & \frac{\partial y_3}{\partial p_3} \end{bmatrix}$$

h) y_1 is the output; part (b) says $y_1 > 0$. So $\frac{\partial y_1}{\partial p_2} > 0$ means that as the price of the second input goes up, output goes up.

Optimal: Intuitively (not referring to inputs as negative numbers), the finding is: if $\uparrow p_3 \Rightarrow \uparrow y_2$ then $\uparrow p_2 \Rightarrow \uparrow \text{output}$. This is a rather surprising result. It is the abstract version of Varian's problem 5.13.

Varian 5.13

output

skilled workers

unskilled workers

this problem

y_1

y_3

y_2

However, Varian assumes $\partial y_1 / \partial p_1 > 0$, while this problem shows that a sufficient condition for that is $\partial y_2 / \partial p_3 > 0$.

Final Exam

1999

Question 3

(2)

3. [15 points] Suppose a profit-maximizing competitive firm uses two inputs, x_1 and x_2 , to produce output according to the production function $y = f(x_1, x_2)$. Under what circumstances will increases in w_1 , which is the price of the first input, cause output to fall?

$$(3) \max_{\underline{x}} p f(x_1, x_2) - w_1 x_1 - w_2 x_2. \text{ Profit } \pi = p f(x_1, x_2) - w_1 x_1 - w_2 x_2$$

F.O.C.

$$0 = \frac{\partial \pi}{\partial x_1} = p f'_1 - w_1$$

$$0 = \frac{\partial \pi}{\partial x_2} = p f'_2 - w_2$$

S.O.C. $\mathcal{L} = \pi$

$$\nabla^2 \mathcal{L} = \begin{bmatrix} \pi''_{11} & \pi''_{12} \\ \pi''_{21} & \pi''_{22} \end{bmatrix} = \begin{bmatrix} p f''_{11} & p f''_{12} \\ p f''_{21} & p f''_{22} \end{bmatrix}$$

$$m=0, (-1)^{m+1} = -1, D_{2m+1} = D,$$

Conditions: $p f''_{11} < 0$

$$\begin{vmatrix} p f''_{11} & p f''_{12} \\ p f''_{21} & p f''_{22} \end{vmatrix} > 0$$

Call this " Δ "; it is equal
to $p^2 f''_{11} f''_{22} - p^2 (f''_{12})^2$

Find the total differential of the
F.O.C.

endogenous: x_1, x_2

exogenous: w_1, w_2 , but we are
not interested in changes in w_2
in this problem

$$p f''_{11} dx_1 + p f''_{12} dx_2 - dw_1 = 0$$

$$p f''_{21} dx_1 + p f''_{22} dx_2 - 0 = 0$$

$\downarrow dw_2 = 0$ so it can be
dropped

Final Exam

1999

Answer 3

$$\begin{bmatrix} p f''_{11} & p f''_{12} \\ p f''_{21} & p f''_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dw_1 \Rightarrow \begin{bmatrix} p f''_{11} & p f''_{12} \\ p f''_{21} & p f''_{22} \end{bmatrix} \begin{bmatrix} dx_1/dw_1 \\ dx_2/dw_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{dx_1}{dw_1} = \frac{\begin{vmatrix} 1 & p f''_{12} \\ 0 & p f''_{22} \end{vmatrix}}{\begin{vmatrix} p f''_{11} & p f''_{12} \\ p f''_{21} & p f''_{22} \end{vmatrix}} = \frac{p f''_{22}}{\Delta} \quad (\text{see definition of } \Delta \text{ above})$$

$$\frac{dx_2}{dw_1} = \frac{\begin{vmatrix} Pf_{11}'' & 1 \\ Pf_{21}'' & 0 \end{vmatrix}}{\begin{vmatrix} Pf_{11}'' & Pf_{12}'' \\ Pf_{21}'' & Pf_{22}'' \end{vmatrix}} = \frac{-Pf_{21}''}{\Delta}$$

Finally, $y = f(x_1, x_2)$. The Chain Rule yields

$$\frac{dy}{dw_1} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dw_1} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dw_1}$$

$$= f'_1 \frac{Pf_{22}''}{\Delta} + f'_2 \frac{(-P)f_{21}''}{\Delta}$$

$$= \underbrace{\frac{P}{\Delta} (f'_1 f_{22}'' - f'_2 f_{21}'')}_{\substack{\text{positive} \\ \text{from S.O.C.}}}$$

\uparrow
positive
from S.O.C.

\rightarrow if this is positive, when $w_1 \uparrow$, y will \uparrow ;
so if this is negative, when $w_1 \uparrow$, y will \downarrow .

By symmetry with f_{11}'' , f_{22}'' should be negative
(it's arbitrary which input is called '1) and
which is called '2'). Presumably $f'_1 > 0$
and $f'_2 > 0$. The sign of f_{21}'' is unknown.

A sufficient (but not necessary) condition for $dy/dw_1 < 0$ is that $f_{21}'' > 0$.

Final Exam

1999

Answer 3 cont...

Fall 2005
Final

3. [17 points] Suppose a cost-minimizing firm uses two inputs to produce output (using a production function with smooth isoquants). If a tax is imposed on one of the inputs, does the firm's use of the other input go up or down? By how much?

(3)

$$\min \underbrace{p_1 x_1 + p_2 x_2}_{\text{Inputs}} \quad \text{prices (could also have used } w_1, w_2)$$

s.t. $f(x_1, x_2) = y$ fixed

↑
production
function

↑
output

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda [f(x_1, x_2) - y].$$

Model this as an ↑ in p_1 and determine the effect that has on x_2 .

F.O.C. :

$$0 = \partial \mathcal{L} / \partial x_1 = p_1 + \lambda f'_1$$

$$0 = \partial \mathcal{L} / \partial x_2 = p_2 + \lambda f'_2$$

$$0 = \partial \mathcal{L} / \partial \lambda = f(x_1, x_2) - y.$$

Find the differential of each :

Alternatively, with tax "t" this could be modeled as changing p_1 into $(1+t)p_1$ or into $p_1 + t$. The first is an ad valorem tax and the second is a specific tax. Then you'd find $\partial x_2 / \partial t$.

$$\begin{array}{cccccc}
 x_1 & x_2 & \lambda & p_1 & \text{setting } dp_2 = 0 \text{ and} \\
 0 = \lambda f_{11}'' dx_1 + \lambda f_{12}'' dx_2 + f'_1 d\lambda + 1 dp_1 & & & & dy = 0 \\
 0 = \lambda f_{21}'' dx_1 + \lambda f_{22}'' dx_2 + f'_2 d\lambda \\
 0 = f'_1 dx_1 + f'_2 dx_2
 \end{array}$$

or

$$\begin{bmatrix}
 \lambda f_{11}'' & \lambda f_{12}'' & f'_1 \\
 \lambda f_{21}'' & \lambda f_{22}'' & f'_2 \\
 f'_1 & f'_2 & 0
 \end{bmatrix}
 \begin{bmatrix}
 dx_1 \\
 dx_2 \\
 d\lambda
 \end{bmatrix}
 =
 \begin{bmatrix}
 -1 \\
 0 \\
 0
 \end{bmatrix} dp_1. \text{ By Cramer's Rule:}$$

$$\frac{\partial x_2}{\partial p_1} = \frac{
 \begin{vmatrix}
 \lambda f_{11}'' & -1 & f'_1 \\
 \lambda f_{21}'' & 0 & f'_2 \\
 f'_1 & 0 & 0
 \end{vmatrix}
 }{
 \begin{vmatrix}
 \lambda f_{11}'' & \lambda f_{12}'' & f'_1 \\
 \lambda f_{21}'' & \lambda f_{22}'' & f'_2 \\
 f'_1 & f'_2 & 0
 \end{vmatrix}
 }$$

Numerator: cofactor expansion along second column yields
 $(-1)^{1+2} (-1) (0 - f'_1 f'_2) = -f'_1 f'_2 < 0$

since f'_1 and f'_2 are usually assumed to be positive

denominator: This would have been in standard form for S.O.C. if we'd made $\partial \lambda / \partial \lambda$ the first FOC instead of the third one. So

That's what we should've done. As it is, if you exchange the first and third columns of the determinant we have, then the first and third rows, then the second and third columns, then the second and third rows, these 4 exchanges result in

$$\begin{vmatrix} \lambda f_{11}'' & \lambda f_{12}'' & f'_1 \\ \lambda f_{21}'' & \lambda f_{22}'' & f'_2 \\ f'_1 & f'_2 & 0 \end{vmatrix} = (-1)^4 \underbrace{\begin{vmatrix} 0 & f'_1 & f'_2 \\ f'_1 & \lambda f_{11}'' & \lambda f_{12}'' \\ f'_2 & \lambda f_{21}'' & \lambda f_{22}'' \end{vmatrix}}_{z+1}$$

The SOC for a minimization with
 $m=1$ constraint say this should have
 have the same sign as $(-1)^m = (-1)^1 < 0$.

Hence

$$\frac{\partial x_2}{\partial p_1} = \frac{-f'_1 f'_2}{-f'_1 (f'_1 \lambda f_{22}'' - f'_2 \lambda f_{21}'') + f'_2 (f'_1 \lambda f_{12}'' - f'_2 \lambda f_{11}'')}$$

should be $\frac{\Theta}{\Theta} > 0$.

This is rather obvious : an \uparrow in p_1 will $\downarrow x_1$, because input demand curves are downward sloping. With output fixed (as in all cost-minimization problems), if x_1 falls, x_2 has to rise to keep output fixed (assuming positive marginal products).

Fall 2004
Ex. 1

2. [11 points] A competitive profit-maximizing firm uses two inputs, x_1 and x_2 , to produce output according to the production function $f(x_1, x_2)$, where $f'_1 > 0$ and $f'_2 > 0$.

Find an expression for the effect of an increase in the price of the second input on the firm's output. Determine the sign of this expression if it is possible to do so.

$$2) \pi = p f(x_1, x_2) - w_1 x_1 - w_2 x_2$$

$$\text{F.O.C.'s: } 0 = \pi'_1 = p f'_1 - w_1$$

Full 2004
Ex. 1

$$0 = \pi'_2 = p f'_2 - w_2$$

Take differentials: $x_1 \quad x_2 \quad w_2 \quad dw_1 = 0$

$$0 = p f''_{11} dx_1 + p f''_{12} dx_2 + 0 dw_2$$

$$0 = p f''_{21} dx_1 + p f''_{22} dx_2 - dw_2$$



$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{dw_2} = \begin{bmatrix} p f''_{11} & p f''_{12} \\ p f''_{21} & p f''_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

$$\begin{bmatrix} p f''_{11} & p f''_{12} \\ p f''_{21} & p f''_{22} \end{bmatrix} \begin{bmatrix} dx_1/dw_2 \\ dx_2/dw_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad . \text{Cramer's Rule} \Rightarrow$$

$$\frac{dx_1}{dw_2} = \frac{\begin{vmatrix} 0 & p f''_{12} \\ 1 & p f''_{22} \end{vmatrix}}{\begin{vmatrix} p f''_{11} & p f''_{12} \\ p f''_{21} & p f''_{22} \end{vmatrix}} \leftarrow -p f''_{12}$$

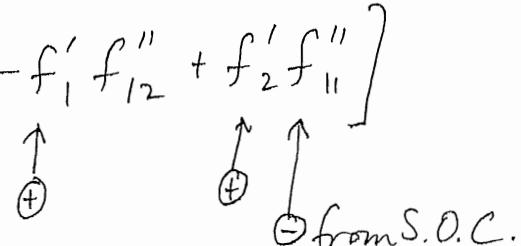
\leftarrow call this "D"

$$\frac{dx_2}{dw_2} = \frac{\begin{vmatrix} p f''_{11} & 0 \\ p f''_{21} & 1 \end{vmatrix}}{D} \leftarrow p f''_{11}$$

$$\frac{\partial f}{\partial w_2} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dw_2} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dw_2} \quad \text{chain rule}$$

$$= f'_1 \frac{-p f''_{12}}{D} + f'_2 \frac{p f''_{11}}{D} = f'_1 \left[-f'_1 f''_{12} + f'_2 f''_{11} \right]$$

The S.O.C. for a max $\Rightarrow D \oplus$



So as long as f''_{12} is \oplus or not too \ominus , $\partial f / \partial w_2 < 0$ as expected for a max.

2005
Qualifier
Sec. 2

3. Suppose a price-taking profit-maximizing firm uses two inputs, x_1 and x_2 , to produce output according to the production function $f(x_1, x_2)$.
 - (a) What are the second-order conditions for a profit maximum?
 - (b) How will the firm's demand for x_1 change when all the exogenous variables experience small changes simultaneously?
 - (c) Attempt to sign the components of your answer to (b), assuming that the conditions you found in (a) hold. If a sign is determinate, explain the intuition behind the sign, otherwise state the underlying conditions which would lead to an indeterminate sign being positive or negative and explain why those conditions are not surprising.

$$(3) \text{ a) } \max_p f(x_1, x_2) - w_1 x_1 - w_2 x_2 = \max \pi$$

↑ output price ↑
 input prices

$$\text{F.O.C. : } 0 = p f'_1 - w_1$$

$$0 = p f'_2 - w_2$$

$$\text{S.O.C. : } \nabla^2 \pi = \begin{bmatrix} \pi_{11}'' & \pi_{12}'' \\ \pi_{21}'' & \pi_{22}'' \end{bmatrix} = \begin{bmatrix} p f_{11}'' & p f_{12}'' \\ p f_{21}'' & p f_{22}'' \end{bmatrix}$$

$m=0$
 $D_{2m+1} = D_1$
 $(-1)^{m+1} = \Theta$

The conditions are

$$0 > \pi_{11}'' = p f_{11}''$$

$$0 < |\nabla^2 \pi| = p^2 f_{11}'' f_{22}'' - p^2 (f_{12}'')^2 = p^2 [f_{11}'' f_{22}'' - (f_{12}'')^2] \Leftrightarrow$$

$$0 < f_{11}'' f_{22}'' - (f_{12}'')^2$$

b) Find the differential of the FOC's :

$$x_1 \quad x_2 \quad w_1 \quad w_2 \quad p$$

$$\partial = pf''_{11} dx_1 + pf''_{12} dx_2 - dw_1 - 0 dw_2 + f'_1 dp$$

$$\partial = pf''_{21} dx_1 + pf''_{22} dx_2 - 0 dw_1 - dw_2 + f'_2 dp$$

$$\begin{bmatrix} pf''_{11} & pf''_{12} \\ pf''_{21} & pf''_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} dw_1 - f'_1 dp \\ dw_2 - f'_2 dp \end{bmatrix} . \text{ By Cramer's Rule :}$$

$$dx_1 = \frac{\begin{vmatrix} dw_1 - f'_1 dp & pf''_{12} \\ dw_2 - f'_2 dp & pf''_{22} \end{vmatrix}}{\begin{vmatrix} pf''_{11} & pf''_{12} \\ pf''_{21} & pf''_{22} \end{vmatrix}} = \frac{pf''_{22} (dw_1 - f'_1 dp) - pf''_{12} (dw_2 - f'_2 dp)}{p^2 f''_{11} f''_{22} - p^2 (f''_{12})^2}$$

$$= \underbrace{\frac{1}{p^2 f''_{11} f''_{22} - p^2 (f''_{12})^2}}_{\oplus \text{ from part (a)}} \left[\underbrace{pf''_{22} dw_1 - pf''_{12} dw_2}_{\text{open, but typically } f''_{22} < 0, \text{ making }} + \underbrace{(pf''_{22} f'_1 + pf''_{12} f'_2) dp}_{\text{ambiguous because } f''_{12} \text{ has an unknown sign.}} \right].$$

c)

\oplus from part (a)

open, but typically
 $f''_{22} < 0$, making

$\partial x_1 / \partial w_1 < 0$, downward
 sloping input demand
 curves, the usual result.
 (This also follows from: the

profit function is convex \Rightarrow its Hessian is positive semi-definite \Rightarrow
 the diagonal terms of the Hessian are positive.)

\uparrow in w_2 might \downarrow or \uparrow
 demand for x_1 , depending
 on whether x_1 and x_2
 are substitutes or complements
 in production. If $f''_{12} > 0$,

as $w_2 \uparrow$, we expect $x_2 \downarrow$, then $f'_1 \downarrow$, so x_1 's marginal product falls, so the firm's demand for x_1 would fall : $\partial x_1 / \partial w_2 < 0$. Sure enough, if $f''_{12} > 0$, the math above gives $\partial x_1 / \partial w_2 < 0$.

concerning $\frac{\partial x_1}{\partial p_1} = \left(-pf_{22}''f'_1 + pf_{12}''f'_2 \right) / (\text{positive denominator}),$

Henderson and Quandt (1980 p. 81) write in their textbook that "Normally an increase in output price will cause an increase in input demand, and this derivative is positive. For it to be negative it is necessary that $f_{12}'' < 0$

[which they earlier say is unlikely], and that $f_{12}''f'_2$ be greater in absolute value than $f_{22}''f'_1$."

4. [17 points] Suppose a competitive, profit-maximizing firm uses one input x to produce one output according to a production function $f(x)$.
- What is the first-order condition for optimality?
 - Using the first-order condition, verify that the input demand has the correct level of homogeneity.
 - Find an equation describing how the endogenous variable changes when all of the exogenous variables change simultaneously.
 - Prove that the homogeneity found in (b) actually is evident in (c).

Fall 2011, Final Exam Qu. 4
For answer see Fall 2005 Exam 1 Qu. 1

Answers to 7005 Midterm, Fall 2005

①

a) $\pi = p f(x) - w x$

$$0 = \frac{d\pi}{dx} = p f'(x^*) - w \quad (1)$$

b) If $p \rightarrow tp$ and $w \rightarrow tw$ then

$$0 = (tp) f'(x^*) - (tw) : \text{dividing by } t,$$

$$0 = p f'(x^*) - w. \quad (2)$$

So the same x which satisfies (1) also satisfies (2) : input demand
 x is homogeneous of degree zero in (p, w) .
 \uparrow
 x^* from (1) is equal to
 \hat{x} from (2)

c) From (1) :

$$0 = p f''(x) dx + f'(x) dp - dw$$

$$\Rightarrow p f''(x) dx = -f'(x) dp + dw \quad (3)$$

$$dx = \frac{-f'(x)}{p f''(x)} dp + \frac{1}{p f''(x)} dw. \quad (4)$$

d) From (3), setting $dp = dw = t$, as you might think would describe changing p to tp and w to tw , does not lead to the expected result of $dx = 0$.

Correct results are obtained by observing that when p changes to tp , the change in p is $tp - p = (t-1)p$; when w changes to tw , the change in w is $tw - w = (t-1)w$. Therefore substitute $dp = (t-1)p$ and $dw = (t-1)w$ into the RHS of (3) :

$$\begin{aligned} p f''(x) dx &= -f'(x) (t-1)p + (t-1)w \\ &= (t-1) \left[-f'(x)p + w \right] \\ &= (t-1) [\quad \circ \quad] \quad \text{from (2)} \\ &= 0 \end{aligned}$$

as desired.

Final Exam
1995
Question 5

(2)

5. Suppose a firm's profit depends on its choice of inputs x_1 and x_2 and also on some exogenous parameter "a." Find dx_2/da and tell me everything you know about its sign. (Hint: think about second-order conditions.)

Answer 5

(5) Let subscripts denote partial differentiation; for example, let π_1 mean $\frac{\partial \pi}{\partial x_1}$, and let π_{2a} mean $\frac{\partial^2 \pi}{\partial x_2 \partial a}$.

The firm wishes to $\max_{x_1, x_2} \pi(x_1, x_2; a)$. The first order conditions are

$O = \pi_1$, and $O = \pi_2$. Take the total derivative of each first order condition:

$$\begin{aligned} O = \pi_1 &\Rightarrow O = \pi_{11} dx_1 + \pi_{12} dx_2 + \pi_{1a} da \\ O = \pi_2 &\Rightarrow O = \pi_{21} dx_1 + \pi_{22} dx_2 + \pi_{2a} da \end{aligned} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} + \begin{bmatrix} \pi_{1a} \\ \pi_{2a} \end{bmatrix} da.$$

So $\begin{bmatrix} -\pi_{1a} \\ -\pi_{2a} \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \begin{bmatrix} dx_1/da \\ dx_2/da \end{bmatrix}$ after dividing by da.

Using Cramer's Rule,

$$\frac{dx_2}{da} = \frac{\begin{vmatrix} \pi_{11} & -\pi_{1a} \\ \pi_{21} & -\pi_{2a} \end{vmatrix}}{\begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix}} = \frac{-\pi_{11}\pi_{2a} + \pi_{21}\pi_{1a}}{\pi_{11}\pi_{22} - \pi_{12}^2} \quad \text{undetermined sign}$$

$\curvearrowleft \quad \curvearrowright$

$D^2 L = D^2 \pi \text{ since } m=0$

Second-order conditions for a maximum

are: D_1 has the same sign as $(-1)^1 \times O$, then $D_2 > 0$.

Since $\begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = D_2$, this determinant should be positive

if the second-order conditions hold.

- | |
|-----------------------------|
| pts |
| 2 F.O.C. |
| 3 total differential |
| 2 Cramer's Rule |
| 3 S.O.C. \Rightarrow sign |

2015 Final Exam Qu. 3

3. [18 points] Suppose a competitive profit-maximizing firm uses gasoline and engines to produce output “y.” Suppose a new type of engine is developed which can produce the same output using less gasoline. The best way to model this “technological progress” would be to assume that the production function changed from

$$y = \hat{f}(\text{gasoline, old engines})$$

to

$$y = \hat{g}(\text{gasoline, old engines, new engines})$$

for old and new production functions \hat{f} and \hat{g} . However, an easier way to model this technological change would be to use one production function “ f ,”

$$y = f(\alpha \cdot \text{gasoline, engines}),$$

where α is a parameter indicating the state of technology, and where the distinction between old and new engines is ignored (they are both “engines”). Use this production function “ f ” when answering this question.

- (a) Would technological progress be indicated by α rising or falling? Why?
- (b) Assuming all prices remain unchanged, will technological progress (as measured by α) cause the demand for gasoline to rise or to fall?

(3)

a) Holding gasoline "g" and engines "e" constant,

$$\frac{\partial y}{\partial \alpha} = \frac{\partial y}{\partial (\alpha \cdot \text{gasoline})} \cdot \frac{\partial (\alpha \cdot \text{gasoline})}{\partial \alpha}$$

$$= f'_1 \cdot \underbrace{\text{gasoline}}_{\substack{\uparrow \\ \text{y}}} + \underbrace{\text{ }}_{+} \quad > 0. \text{ So } \underline{\text{rising } \alpha} \text{ gives more } y \text{ with constant } g \text{ and } e.$$

Since it is

the marginal product of the first input

Optional : Example : $y = \alpha g e$. As $\alpha \uparrow$ with g and e fixed, $y \uparrow$.

$$\text{b) } \pi = p_y f(\alpha g, e) - p_g g - p_e e$$

↑ ↑ ↓ price of g ↑ price of e
 profit price of y

Maximizing π w.r.t. g and e gives first-order conditions of :

$$0 = \frac{\partial \pi}{\partial g} = p_y \frac{\partial f}{\partial (\alpha g)} \cdot \frac{\partial (\alpha g)}{\partial g} - p_g = p_y f'_1 \alpha - p_g$$

$$0 = \frac{\partial \pi}{\partial e} = p_y \frac{\partial f}{\partial e} - p_e = p_y f'_2 - p_e$$

Take differentials, with differentials of exogenous variables being $dp_g = 0$, $dp_e = 0$, and $d\alpha$:

$$0 = p_y \frac{\partial f'_1}{\partial g} \alpha dg + p_y \frac{\partial f'_1}{\partial e} \alpha de + p_y \left[\frac{\partial f'_1}{\partial \alpha} \alpha + f'_1 \underbrace{\frac{\partial \alpha}{\partial \alpha}}_1 \right] d\alpha$$

$$0 = p_y \frac{\partial f'_2}{\partial g} dg + p_y \frac{\partial f'_2}{\partial e} de + p_y \frac{\partial f'_2}{\partial \alpha} d\alpha.$$

But $\frac{\partial f'_1}{\partial g} = \frac{\partial f'_1}{\partial(\alpha g)}$ $\frac{\partial(\alpha g)}{\partial g} = \underbrace{\frac{\partial}{\partial(\alpha g)} \frac{\partial f}{\partial(\alpha g)}}_{f''_{11}} \cdot \underbrace{\frac{\partial(\alpha g)}{\partial g}}_{\alpha}$

and $\frac{\partial f'_1}{\partial \alpha} = \frac{\partial f'_1}{\partial(\alpha g)} \quad \frac{\partial(\alpha g)}{\partial \alpha} = \underbrace{\frac{\partial}{\partial(\alpha g)} \frac{\partial f}{\partial(\alpha g)}}_{f''_{12}} \cdot \underbrace{\frac{\partial(\alpha g)}{\partial \alpha}}_g$

so the first equation is

$$0 = p_y \alpha^2 f''_{11} dg + p_y \alpha f''_{12} de + p_y [\alpha g f''_{11} + f'_1] d\alpha.$$

Also

$$\frac{\partial f'_2}{\partial g} = \frac{\partial}{\partial g} \frac{\partial f}{\partial e} = \underbrace{\frac{\partial}{\partial(\alpha g)} \frac{\partial f}{\partial e}}_{f''_{12}} \cdot \underbrace{\frac{\partial(\alpha g)}{\partial g}}_{\alpha} = \underbrace{\frac{\partial}{\partial(\alpha g)} \frac{\partial f(\alpha g, e)}{\partial e}}_{f''_{22}} \cdot \underbrace{\frac{\partial(\alpha g)}{\partial g}}_{\alpha}$$

and

$$\frac{\partial f'_2}{\partial \alpha} = \frac{\partial}{\partial \alpha} \frac{\partial f}{\partial e} = \underbrace{\frac{\partial}{\partial(\alpha g)} \frac{\partial f}{\partial e}}_{f''_{12}} \cdot \underbrace{\frac{\partial(\alpha g)}{\partial \alpha}}_g = \underbrace{\frac{\partial}{\partial(\alpha g)} \frac{\partial f(\alpha g, e)}{\partial e}}_{f''_{22}} \cdot \underbrace{\frac{\partial(\alpha g)}{\partial \alpha}}_g$$

so the second equation is

$$0 = p_y \alpha f''_{12} dg + p_y f''_{22} de + p_y f''_{12} g d\alpha.$$

Hence

$$\tilde{O} = \begin{bmatrix} P_y \alpha^2 f_{11}'' & P_y \alpha f_{12}'' \\ P_y \alpha f_{12}'' & P_y f_{22}'' \end{bmatrix} \begin{bmatrix} dg \\ de \end{bmatrix} + \begin{bmatrix} P_y (\alpha g f_{11}'' + f_1') \\ P_y f_{12}'' g \end{bmatrix} d\alpha.$$

$$\begin{bmatrix} P_y \alpha^2 f_{11}'' & P_y \alpha f_{12}'' \\ P_y \alpha f_{12}'' & P_y f_{22}'' \end{bmatrix} \begin{bmatrix} dg/d\alpha \\ de/d\alpha \end{bmatrix} = \begin{bmatrix} -P_y (\alpha g f_{11}'' + f_1') \\ -P_y f_{12}'' g \end{bmatrix}.$$

Since P_y is common to both the left and right of both equations, it can be canceled out :

$$\begin{bmatrix} \alpha^2 f_{11}'' & \alpha f_{12}'' \\ \alpha f_{12}'' & f_{22}'' \end{bmatrix} \begin{bmatrix} dg/d\alpha \\ de/d\alpha \end{bmatrix} = \begin{bmatrix} -(\alpha g f_{11}'' + f_1') \\ -f_{12}'' g \end{bmatrix}. \text{ Using Cramer's Rule,}$$

$$\frac{dg}{d\alpha} = \frac{\begin{vmatrix} -\alpha g f_{11}'' - f_1' & \alpha f_{12}'' \\ -f_{12}'' g & f_{22}'' \end{vmatrix}}{\begin{vmatrix} \alpha^2 f_{11}'' & \alpha f_{12}'' \\ \alpha f_{12}'' & f_{22}'' \end{vmatrix}}$$

} This is $|\nabla^2 \pi| / P_y^2$; with the number of constraints $m=0$, the S.O.C. for maximizing

it are that this is positive and that

$$\alpha^2 f_{11}'' < 0 \Leftrightarrow f_{11}'' < 0, \text{ a typical assumption.}$$

Overall, $\frac{dg}{d\alpha} = \frac{\text{ambiguous}}{\oplus}$.

Exam 1

1995

Question 3

(2)

3. Suppose a profit-maximizing firm uses inputs x_1 and x_2 to make output according to the production function $y = x_1^{1/3} x_2^{1/3}$ where y is output. Let p be the price of output.
- What are the firm's first-order conditions?
 - By differentiating the firm's first-order conditions (*not* by solving the first-order conditions), determine whether the firm's purchases of x_1 increase or decrease if output price falls (say, as a result of a tax).

Exam 1
1995

Answer 3

$$\textcircled{3} \quad \pi = p f(\underline{x}) - \underline{\omega} \cdot \underline{x} = p x_1^{1/3} x_2^{1/3} - \omega_1 x_1 - \omega_2 x_2$$

$$a) \quad O = \frac{\partial \pi}{\partial x_1} = \frac{1}{3} p x_1^{-2/3} x_2^{1/3} - \omega_1$$

$$O = \frac{\partial \pi}{\partial x_2} = \frac{1}{3} p x_1^{1/3} x_2^{-2/3} - \omega_2$$

$$b) d(O) = d\left(\frac{\partial \pi}{\partial x_1}\right)$$

$$\begin{aligned} O &= \frac{\partial}{\partial x_1} \frac{\partial \pi}{\partial x_1} dx_1 + \frac{\partial}{\partial x_2} \frac{\partial \pi}{\partial x_1} dx_2 + \frac{\partial}{\partial w_1} \frac{\partial \pi}{\partial x_1} dw_1 + \frac{\partial}{\partial w_2} \frac{\partial \pi}{\partial x_1} dw_2 \\ &= -\frac{2}{9} p x_1^{-5/3} x_2^{1/3} dx_1 + \frac{1}{9} p x_1^{1/3} x_2^{-2/3} dx_2 - dw_1 + dw_2 + \underbrace{\frac{1}{3} x_1^{-2/3} x_2^{1/3} dp}_{\frac{1}{3} x_1^{-2/3} x_2^{1/3} dp} \end{aligned}$$

$$\text{Similarly, } d(O) = d\left(\frac{\partial \pi}{\partial x_2}\right) \Rightarrow$$

$$O = \frac{1}{9} p x_1^{-2/3} x_2^{-1/3} dx_1 - \frac{2}{9} p x_1^{1/3} x_2^{-5/3} dx_2 + dw_1 - dw_2 + \frac{1}{3} x_1^{1/3} x_2^{-2/3} dp.$$

Set $dw_1 = dw_2 = 0$:

$$\begin{bmatrix} -\frac{1}{3} x_1^{-2/3} x_2^{1/3} \\ -\frac{1}{3} x_1^{1/3} x_2^{-2/3} \end{bmatrix} dp = \begin{bmatrix} -\frac{2}{9} p x_1^{-5/3} x_2^{1/3} & \frac{1}{9} p x_1^{-2/3} x_2^{-2/3} \\ \frac{1}{9} p x_1^{-2/3} x_2^{-1/3} & -\frac{2}{9} p x_1^{1/3} x_2^{-5/3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \text{ and}$$

$$\frac{dx_1}{dp} = \begin{vmatrix} -\frac{1}{3} x_1^{-2/3} x_2^{1/3} & \frac{1}{9} p x_1^{-2/3} x_2^{-2/3} \\ -\frac{1}{3} x_1^{1/3} x_2^{-2/3} & -\frac{2}{9} p x_1^{1/3} x_2^{-5/3} \end{vmatrix} \div \begin{vmatrix} -\frac{2}{9} p x_1^{-5/3} x_2^{1/3} & \frac{1}{9} p x_1^{-2/3} x_2^{-2/3} \\ \frac{1}{9} p x_1^{-2/3} x_2^{-1/3} & -\frac{2}{9} p x_1^{1/3} x_2^{-5/3} \end{vmatrix}$$

↑
⊕
↑
 $\left(\frac{4}{81} - \frac{1}{81}\right) p^2 x_1^{-4/3} x_2^{-4/3} > 0$

$$\therefore dx_1/dp > 0.$$

pts
 part(a) 5
 differentiation 10
 dx_1/dp 10
 sign 8

Exam 1
 1995
 Answer 3 cont...

3. [11 points] A competitive firm buys inputs $x_1 > 0$ and $x_2 > 0$ at exogenously given (that is, "fixed") prices w_1 and w_2 , and combines them in order to produce good y according to the production function

$$y = x_1^\alpha x_2^\beta$$

where $\alpha > 0$ and $\beta > 0$. It then sells y at a fixed price p .

Fall
2006
Ex. 1

- (a) What is the effect of a change in w_1 on y ? (Hint: You may want to find out what happens to x_1 and to x_2 but you do not have to.) Do not expand any determinants for this part of the question; you may leave the determinants unevaluated.
- (b) For what values of α and β is your answer to part (a) valid? You should not leave determinants unevaluated in this part of the question.

(3)

$$a) \pi = p x_1^\alpha x_2^\beta - w_1 x_1 - w_2 x_2$$

Fall 2006
Ex. 1

$$F.O.C. \quad 0 = \frac{\partial \pi}{\partial x_1} = \alpha p x_1^{\alpha-1} x_2^\beta - w_1$$

$$0 = \frac{\partial \pi}{\partial x_2} = \beta p x_1^\alpha x_2^{\beta-1} - w_2$$

Find the differential of the F.O.C.'s, but set $dp=0$ and $d w_2=0$ because they don't change:

$$0 = \begin{bmatrix} \alpha(\alpha-1) p x_1^{\alpha-2} x_2^\beta & \alpha \beta p x_1^{\alpha-1} x_2^{\beta-1} \\ \beta \alpha p x_1^{\alpha-1} x_2^{\beta-1} & \beta(\beta-1) p x_1^\alpha x_2^{\beta-2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} - \begin{bmatrix} dw_1 \\ 0 \end{bmatrix}$$

Call this matrix "A."

Using Cramer's Rule,

$$\frac{dx_1}{dw_1} = \begin{vmatrix} 1 & \alpha \beta p x_1^{\alpha-1} x_2^{\beta-1} \\ 0 & \beta(\beta-1) p x_1^\alpha x_2^{\beta-2} \end{vmatrix} \div |A| \quad (1)$$

$$\frac{dx_2}{dw_1} = \begin{vmatrix} \alpha(\alpha-1) p x_1^{\alpha-2} x_2^\beta & 1 \\ \beta \alpha p x_1^{\alpha-1} x_2^{\beta-1} & 0 \end{vmatrix} \div |A| \quad (2)$$

Since by the Chain Rule

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2,$$

$$\frac{\partial y}{\partial w_1} = \underbrace{\frac{\partial y}{\partial x_1}}_{\alpha x_1^{\alpha-1} x_2^\beta} \underbrace{\frac{\partial x_1}{\partial w_1}}_{(1)} + \underbrace{\frac{\partial y}{\partial x_2}}_{\beta x_1^\alpha x_2^{\beta-1}} \underbrace{\frac{\partial x_2}{\partial w_1}}_{(2)}$$

$y = x_1^\alpha x_2^\beta$

b) # constraints $m = 0$

variables $n = 2$

$$2m+1 = 1$$

$$m+n = 2$$

$D^2 \mathcal{L} = D^2 \pi = \text{matrix "A" from part (a).}$

S.O.C. for \max : D_{2m+1}, \dots, D_m of $D^2 \mathcal{L}$ should alternate in sign beginning with \odot . So $D_1 < 0, D_2 > 0$ is what we hope for.

$D_1 = \alpha(\alpha-1) p x_1^{\alpha-2} x_2^\beta$, which will be < 0 on $(x_1, x_2) \in \overbrace{\mathbb{R}_+^2}$
if $\alpha < 1$.

This means $x_1 \geq 0, x_2 \geq 0$

$$\begin{aligned} D_2 &= [\alpha(\alpha-1)\beta(\beta-1) - \alpha^2\beta^2] p^2 x_1^{2\alpha-2} x_2^{2\beta-2} \\ &= \alpha\beta [(\alpha-1)(\beta-1) - \alpha\beta] p^2 x_1^{2\alpha-2} x_2^{2\beta-2} \\ &= \alpha\beta [\alpha\beta - \alpha - \beta + 1 - \alpha\beta] p^2 x_1^{2\alpha-2} x_2^{2\beta-2} \\ &= \alpha\beta [1 - (\alpha + \beta)] p^2 x_1^{2\alpha-2} x_2^{2\beta-2} \text{ which will be positive as} \end{aligned}$$

long as $1 - (\alpha + \beta) > 0 \iff$

$$\alpha + \beta < 1$$

(note that $\alpha > 0$ and $\beta > 0$ have already been assumed in the problem).

Remark: If $\alpha + \beta > 1$, one has increasing returns to scale, which leads to nonexistence of a profit-maximizing competitive plan.

Qualifying Exam
1995
Question 2

621 Portion of 1995 Qualifying Exam

(2)

2. Suppose a competitive firm uses inputs x_1 and x_2 to produce output according to the production function

$$f(x_1, x_2) = x_1^{5/10} + x_1^{4/10}x_2^{4/10} + x_2^{4/10}$$

- (a) What second-order conditions would have to be satisfied in order for this firm to have a profit-maximizing production plan? Are these conditions satisfied for the above function? (The answer to this requires some messy algebra.)
- (b) Assume that the second order conditions discussed in part (a) are satisfied. Calculate how the firm's demand for second input x_2 changes when the price of the *first* input x_1 changes, and determine the sign of this derivative.

Qualifying Exam
1995

Answer 2

Answers to 621 Portion of the 1995 Micro Qualifying Exam

Long Question (2). $f(x_1, x_2) = x_1^{5/10} + x_1^{4/10} x_2^{4/10} + x_2^{4/10}$

a) profit $\pi(x_1, x_2) = p f(x_1, x_2) - w_1 x_1 - w_2 x_2$ where p is the price of output, w_1 is the price of input x_1 , and w_2 is the price of x_2 .

If ' L ' is the Lagrangian and ' m ' is the number of constraints, then the second-order sufficient conditions for a maximum are that the D 's

(successive principal minors) of $\nabla^2 L$ alternate in sign, starting with

D_{2m+1} , which should have the same sign as $(-1)^{m+1}$. In this problem,

$L = \pi(x_1, x_2)$ and $m = 0$, so we'd need D_1 and D_2 of $\nabla^2 \pi(x_1, x_2)$ to be negative and positive, respectively.

$$\frac{\partial \pi}{\partial x_1} = p \frac{\partial f}{\partial x_1} - w_1 \quad \frac{\partial \pi}{\partial x_2} = p \frac{\partial f}{\partial x_2} - w_2$$

$$\frac{\partial^2 \pi}{\partial x_1^2} = p \frac{\partial^2 f}{\partial x_1^2} \quad \frac{\partial^2 \pi}{\partial x_2^2} = p \frac{\partial^2 f}{\partial x_2^2}$$

$$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = p \frac{\partial^2 f}{\partial x_1 \partial x_2}.$$

Using subscripts on f to denote partial differentiation, this means that

$$\nabla^2 \pi(x_1, x_2) = \begin{bmatrix} p f_{11} & p f_{12} \\ p f_{12} & p f_{22} \end{bmatrix}, \text{ which has a } D_1 \text{ of } p f_{11} \text{ and}$$

a D_2 of $p^2 f_{11} f_{22} - p^2 f_{12}^2$. For the second-order conditions to be

satisfied, we therefore need $f_{11} < 0$ and $f_{11}f_{22} - f_{12}^2 > 0$.

Answer 2 cont...

$$f_1 = \frac{5}{10} x_1^{-5/10} + \frac{4}{10} x_1^{-6/10} x_2^{4/10}$$

$$f_{11} = \frac{-25}{100} x_1^{-15/10} - \frac{24}{100} x_1^{-16/10} x_2^{4/10}, \text{ which is } < 0 \text{ for positive } x_1, x_2.$$

$$f_2 = \frac{4}{10} x_1^{4/10} x_2^{-6/10} + \frac{4}{10} x_2^{-6/10}$$

$$f_{22} = \frac{-24}{100} x_1^{4/10} x_2^{-16/10} - \frac{24}{100} x_2^{-16/10}$$

$$f_{21} = f_{12} = \frac{16}{100} x_1^{-6/10} x_2^{-6/10}$$

$$f_{11} f_{22} = \left[\frac{-25}{100} x_1^{-15/10} - \frac{24}{100} x_1^{-16/10} x_2^{4/10} \right] \left[\frac{-24}{100} x_1^{4/10} x_2^{-16/10} - \frac{24}{100} x_2^{-16/10} \right]$$

$$= \frac{25 \cdot 24}{100^2} x_1^{-11/10} x_2^{-16/10} + \frac{25 \cdot 24}{100^2} x_1^{-15/10} x_2^{-16/10} + \frac{24^2}{100^2} x_1^{-12/10} x_2^{-12/10}$$

$$+ \frac{24^2}{100^2} x_1^{-16/10} x_2^{-12/10}. \text{ Subtracting } f_{12}^2 = \frac{16^2}{100^2} x_1^{-12/10} x_2^{-12/10} \text{ from this}$$

$$\text{yields } f_{11} f_{22} - f_{12}^2 = \frac{25 \cdot 24}{100^2} x_1^{-11/10} x_2^{-16/10} + \frac{25 \cdot 24}{100^2} x_1^{-15/10} x_2^{-16/10} +$$

$$\frac{24^2 - 16^2}{100^2} x_1^{-12/10} x_2^{-12/10} + \frac{24^2}{100^2} x_1^{-16/10} x_2^{-12/10}, \text{ which is } > 0 \text{ for}$$

positive x_1, x_2 because each of its four terms are positive for

positive x_1, x_2 . Hence the second-order conditions for a maximum hold.

Answer 2 cont..

- b) $\pi(x_1, x_2) = p f(x_1, x_2) - w_1 x_1 - w_2 x_2$. Maximize this with respect to x_1, x_2 .

First order conditions :

$$0 = \frac{\partial \pi}{\partial x_1} = p \frac{\partial f}{\partial x_1} - w_1$$

$$0 = \frac{\partial \pi}{\partial x_2} = p \frac{\partial f}{\partial x_2} - w_2$$

Find the total differential of each first-order condition :

$$0 = p \frac{\partial^2 f}{\partial x_1^2} dx_1 + p \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_2 + \frac{\partial f}{\partial x_1} dp - 1 dw_1 + 0 dw_2$$

$$0 = p \frac{\partial^2 f}{\partial x_2 \partial x_1} dx_1 + p \frac{\partial^2 f}{\partial x_2^2} dx_2 + \frac{\partial f}{\partial x_2} dp + 0 dw_1 - 1 dw_2$$

or

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p f_{11} & p f_{12} \\ p f_{12} & p f_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} dp - \begin{bmatrix} 1 \\ 0 \end{bmatrix} dw_1 - \begin{bmatrix} 0 \\ 1 \end{bmatrix} dw_2.$$

The question asks for dx_2/dw_1 , holding $dp = dw_2 = 0$:

$$\begin{bmatrix} dw_1 \\ 0 \end{bmatrix} = \begin{bmatrix} p f_{11} & p f_{12} \\ p f_{12} & p f_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p f_{11} & p f_{12} \\ p f_{12} & p f_{22} \end{bmatrix} \begin{bmatrix} dx_1/dw_1 \\ dx_2/dw_1 \end{bmatrix}$$

by dividing both equations by dw_1 . Cramer's Rule gives

$$\frac{dx_2}{dw_1} = \frac{\begin{vmatrix} p f_{11} & 1 \\ p f_{12} & 0 \end{vmatrix}}{\begin{vmatrix} p f_{11} & p f_{12} \\ p f_{12} & p f_{22} \end{vmatrix}} = \frac{-p f_{12}}{p^2(f_{11}f_{22} - f_{12}^2)} =$$

$\frac{-f_{12}}{p(f_{11}f_{22} - f_{12}^2)}$ Part (a) showed that $f_{11}f_{22} - f_{12}^2 > 0$, and

$$f_{12} = \frac{16}{100} x_1^{-6/10} x_2^{-6/10} > 0 \text{ for positive } x\text{'s. So } \frac{\partial x_2}{\partial w_1} < 0.$$

The expressions two pages ago can be substituted into $\frac{\partial x_2}{\partial w_1} = \frac{-f_{12}}{p(f_{11}f_{22} - f_{12}^2)}$

to give a more explicit form of this comparative-statics derivative.

Summer 2007 Qualifying Exam Section 2 Qu. 1

1. [12 points]

Suppose a profit-maximizing firm produces apples from two inputs, x_1 and x_2 , according to the production function $f(x_1, x_2)$. Suppose the government imposes a specific tax of t dollars per pound on apples. Suppose the demand curve for apples is horizontal.

- (a) Derive an expression showing how the firm's purchases of x_1 change when the tax goes up infinitesimally.
- (b) Find the sign of the expression you derived in part (a). Hint: It may be easiest to do this in an indirect way, arguing the standard Hotelling's Lemma with the standard profit function (whose concavity or convexity you may merely assert; you do not have to prove it).
- (c) Derive an expression showing how the firm's profit changes when the tax goes up infinitesimally. Find the sign of this expression.

Secton 2 Question 1

Summer 200X
Qual.

p6

$$(a) \pi = p f(x_1, x_2) - \omega_1 x_1 - \omega_2 x_2 - t f(x_1, x_2)$$

$$= (p-t) f(x_1, x_2) - \omega_1 x_1 - \omega_2 x_2$$

F.O.C.

$$0 = \frac{\partial \pi}{\partial x_1} = (p-t) f'_1 - \omega_1$$

$$0 = \frac{\partial \pi}{\partial x_2} = (p-t) f'_2 - \omega_2$$

Differentials of both sides, with $dp = d\omega_1 = d\omega_2 = 0$:

$$0 = (p-t) f''_{11} dx_1 + (p-t) f''_{12} dx_2 - f'_1 dt$$

$$0 = (p-t) f''_{21} dx_1 + (p-t) f''_{22} dx_2 - f'_2 dt$$

$$\Rightarrow \begin{bmatrix} f'_1 \\ f'_2 \end{bmatrix} = \begin{bmatrix} (p-t) f''_{11} & (p-t) f''_{12} \\ (p-t) f''_{21} & (p-t) f''_{22} \end{bmatrix} \begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix}$$

Use Cramer's Rule:

$$\frac{dx_1}{dt} = \frac{\begin{vmatrix} f'_1 & (p-t) f''_{12} \\ f'_2 & (p-t) f''_{22} \end{vmatrix}}{\begin{vmatrix} (p-t) f''_{11} & (p-t) f''_{12} \\ (p-t) f''_{21} & (p-t) f''_{22} \end{vmatrix}} = \frac{\begin{vmatrix} f'_1 & f''_{12} \\ f'_2 & f''_{22} \end{vmatrix}}{\begin{vmatrix} f''_{11} & (p-t) f''_{12} \\ f''_{21} & (p-t) f''_{22} \end{vmatrix}}$$

(I took $(p-t)$ out of one column
in both numerator and denominator,
then canceled them)

$$\frac{f'_1 f''_{22} - f'_2 f''_{12}}{(p-t) f''_{11} f''_{22} - (p-t) f''_{21} f''_{12}} = \frac{1}{p-t} \frac{f'_1 f''_{22} - f'_2 f''_{12}}{f''_{11} f''_{22} - f''_{21} f''_{12}}$$

(b) As seen in part (a), a specific tax works just like a decrease in price. Using standard arguments, the profit function $\pi(p, w_1, w_2)$ (where I omit t because it works just like p except in the opposite direction) is convex in $p = (p, w_1, w_2)$. By the Envelope Theorem,

$$\underline{\nabla}_p \pi = \underline{\nabla}_p \mathcal{L}^* = \underline{\nabla}_p p \cdot \underline{\gamma} = \underline{\gamma} \text{ - the input-output vector}$$

(so $\underline{\gamma}$ is $(\text{apples}, x_1, x_2)$). Then

$$\underline{\nabla}_p^2 \pi = \underline{\nabla}_p \underline{\gamma}$$

But the left-hand side is positive semi-definite by the convexity of π .

So $\underline{\nabla}_p \underline{\gamma}$ must be positive semi-definite too. Hence the main diagonal of $\underline{\nabla}_p \underline{\gamma}$ has only non-negative entries. Hence

$$\frac{\partial \text{apples}}{\partial p} > 0, \text{ so as } t \uparrow, \text{ apples} \downarrow: \frac{dx_1}{dt} \text{ must be negative.}$$

(c) By the Envelope Theorem,

$$\frac{\partial \pi^*}{\partial t} = \frac{\partial \mathcal{L}^*}{\partial t} \Rightarrow \frac{\partial}{\partial t} ((p-t)f(x_1^*, x_2^*) - w_1 x_1^* - w_2 x_2^*)$$

$= -f^* < 0$ since f^* is the optimal amount of apples.

3. [17 points] Suppose a competitive firm transforms a single input (z) into two outputs (q_1 and q_2) according to a well-behaved, fully differentiable inverse production function.

Further suppose that the government introduces an *ad valorem* tax (t) on each unit of q_1 sold; the definition of an *ad valorem* tax is that the firm has to pay tp_1q_1 in taxes, where p_1 is the price of good 1. (In other words, the tax is *not* a “specific tax,” which would have a tax payment of tq_1 .)

Derive an expression for how this tax will change each of the following, and sign the expression if possible.

- (a) the firm’s demand for the input z ;
- (b) the supply of the taxed commodity q_1 ; and
- (c) the supply of the untaxed commodity q_2 .

Fall 2012 Final Exam

Answers to Econ. 7005 Final, Fall 2012

(3)

$Z = f(q_1, q_2)$, an example of joint production (like wool and mutton). Let w be the price of Z ,

let p_1 be the price of q_1 , and let p_2 be the price of q_2 .

$$\text{Profit } \pi = p_1 q_1 + p_2 q_2 - wZ - t p_1 q_1$$

an ad valorem tax, as opposed to the similar previous question, which was a specific tax " $t q_1$ "

$$= (1-t) p_1 q_1 + p_2 q_2 - wZ.$$

Endogenous : q_1, q_2

Exogenous : p_1, t, p_2, w

$$\text{F.O.C. : } 0 = \frac{\partial \pi}{\partial q_1} = (1-t)p_1 - w \frac{\partial f}{\partial q_1}$$

$$0 = \frac{\partial \pi}{\partial q_2} = p_2 - w \frac{\partial f}{\partial q_2}.$$

For comparative statics, find the differential :

$$0 = (1-t) dp_1 - p_1 dt + 0 dp_2 - \frac{\partial f}{\partial q_1} dw - w f''_{11} dq_1 - w f''_{12} dq_2$$

$$\Rightarrow 0 = 0 dp_1 + 0 dt + (1) dp_2 - \frac{\partial f}{\partial q_2} dw - w f''_{21} dq_2 - w f''_{22} dq_2$$

$$\begin{bmatrix} w f''_{11} & w f''_{12} \\ w f''_{21} & w f''_{22} \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} = \begin{bmatrix} 1-t \\ 0 \end{bmatrix} dp_1 - \begin{bmatrix} p_1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dp_2 - \begin{bmatrix} f'_1 \\ f'_2 \end{bmatrix} dw$$

\uparrow endogenous \nwarrow exogenous

$$a) \frac{dz}{dt} = \frac{df(g_1, g_2)}{dt} = f'_1 \frac{dg_1}{dt} + f'_2 \frac{dg_2}{dt}.$$

With $dp_1 = dp_2 = dw = 0$, the last equation on the previous page becomes

$$\begin{bmatrix} wf''_{11} & wf''_{12} \\ wf''_{21} & wf''_{22} \end{bmatrix} \begin{bmatrix} dg_1/dt \\ dg_2/dt \end{bmatrix} = \begin{bmatrix} -p_1 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} dg_1/dt \\ dg_2/dt \end{bmatrix} = \begin{bmatrix} wf''_{11} & wf''_{12} \\ wf''_{21} & wf''_{22} \end{bmatrix}^{-1} \begin{bmatrix} -p_1 \\ 0 \end{bmatrix}$$

and using the formula for the
inverse of a 2×2

$$\text{matrix, } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

$$\begin{bmatrix} dg_1/dt \\ dg_2/dt \end{bmatrix} = \frac{1}{wf''_{11}wf''_{22} - wf''_{12}wf''_{21}} \begin{bmatrix} wf''_{22} & -wf''_{12} \\ -wf''_{21} & wf''_{11} \end{bmatrix} \begin{bmatrix} -p_1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{w^2 f''_{11} f''_{22} - w^2 (f''_{12})^2} \begin{bmatrix} -p_1 w f''_{22} \\ p_1 w f''_{21} \end{bmatrix}$$

this is $\begin{bmatrix} wf''_{11} & wf''_{12} \\ wf''_{21} & wf''_{22} \end{bmatrix}$, while the Hessian of the objective function is

$$\nabla^2 \pi = \begin{bmatrix} -wf''_{11} & -wf''_{12} \\ -wf''_{21} & -wf''_{22} \end{bmatrix}. \quad \text{The S.O.C. for a maximum of } \pi \text{ are}$$

$$D_1 \text{ of } \nabla^2 \pi < 0$$

$$D_2 \text{ of } \nabla^2 \pi > 0 \text{ therefore}$$

$$-wf''_{11} < 0$$

(1)

$$w^2 f''_{11} f''_{22} - w^2 f''_{12} f''_{21} > 0 \text{ therefore} \quad (2)$$

$$\begin{bmatrix} dg_1/dt \\ dg_2/dt \end{bmatrix} = \frac{1}{\oplus} \begin{bmatrix} -p_1 w f''_{22} \\ p_1 w f''_{21} \end{bmatrix} \text{ and}$$

$$\frac{dz}{dt} = f'_1 \perp_{\oplus} (-p_1 w f''_{22}) + f'_2 \oplus (p_1 w f''_{21})$$

$$= -p_1 w \perp_{\oplus} f'_1 f''_{22} + p_1 w \oplus f'_2 f''_{21}.$$

\oplus since more output requires more input

$(1) \Rightarrow f''_{11} > 0$, and since it's immaterial which output is called "1" and which is called "2", $f''_{22} > 0$.

ambiguous

$$= -\oplus \perp_{\oplus} \oplus + \oplus \oplus \perp \oplus ?$$

$$= \oplus + ? . \text{ If } f''_{21} \leq 0, \text{ or } f''_{21} > 0 \text{ but small in absolute value,}$$

$$\text{then } dz/dt < 0.$$

b) Let Δ be defined to be $\begin{vmatrix} wf''_{11} & wf''_{12} \\ wf''_{21} & wf''_{22} \end{vmatrix}$. From (2), $\Delta > 0$. From (a),

$$\frac{dg_1}{dt} = \frac{-p_1 w f''_{22}}{\Delta} = \frac{-\oplus \oplus \oplus}{\oplus} < 0 : \text{the supply of the taxed commodity falls.}$$

c)

$$\frac{dg_2}{dt} = \frac{p_1 w f_{21}''}{\Delta} = \frac{\oplus \oplus \textcircled{?}}{\oplus}$$

so the sign of dg_2/dt is
the sign of f_{21}'' , which is ambiguous.

Answer all of the following three questions.

1. [11 points] Suppose a competitive firm produces output q from inputs x_1 and x_2 according to the production function

$$q = x_1^\alpha x_2^\beta$$

where $\alpha > 0$ and $\beta > 0$.

- What level of output will the firm produce? You do *not* need to simplify your answer! Leave your answer as unsimplified as you'd like.
- When are the second-order sufficient conditions satisfied?
- When are the second-order necessary conditions satisfied?
- If the second-order conditions are violated either always or sometimes, what do you think the firm's level of output will be in those cases?

2007
midterm

This is related to
the next question in
this packet.

Answers to Econ. 7005 Midterm, Fall 2007

① a) $\pi = pq - w_1 x_1 - w_2 x_2 \quad \text{s.t. } q = x_1^\alpha x_2^\beta$

$$\pi = p x_1^\alpha x_2^\beta - w_1 x_1 - w_2 x_2 \quad . \quad (\text{Must do profit maximization, not cost minimization, since output } q \text{ is not fixed.})$$

F.O.C.

$$0 = \pi'_1 = \frac{\alpha p x_1^\alpha x_2^\beta}{x_1} - w_1$$

$$0 = \pi'_2 = \frac{\beta p x_1^\alpha x_2^\beta}{x_2} - w_2$$

$$\left\{ \begin{array}{l} \frac{w_1}{w_2} = \frac{\alpha p x_1^\alpha x_2^\beta}{x_1} \cdot \frac{x_2}{\beta p x_1^\alpha x_2^\beta} = \frac{\alpha x_2}{\beta x_1} \\ \end{array} \right. \quad \text{Here there's no choice.}$$

So $x_2^* = \frac{w_1}{\alpha} x_1^*$. Substitute this into the first F.O.C., dropping the * for convenience:

$$0 = \alpha p x_1^{\alpha-1} \left[\frac{\beta}{\alpha} \frac{w_1}{w_2} x_1 \right]^\beta - w_1$$

$$\alpha p \left[\frac{\beta}{\alpha} \frac{w_1}{w_2} \right]^{\beta-1} = x_1^{\alpha+\beta-1}$$

$$\left[w_1^{1-\beta} w_2^\beta \alpha^{\beta-1} \beta^{-\beta-1} p \right]^{\frac{1}{\alpha+\beta-1}} = x_1.$$

Then $x_2 = \frac{\beta}{\alpha} \frac{w_1}{w_2} x_1 = \alpha^{\beta-1} w_1 w_2^{-1} \left[w_1^{1-\beta} w_2^\beta \alpha^{\beta-1} \beta^{-\beta-1} p^{-1} \right]^{\frac{1}{\alpha+\beta-1}}$

$$= \left[\begin{matrix} 1-\alpha-\beta & \alpha+\beta-1 & \alpha+\beta-1 & 1-\alpha-\beta \\ \alpha & \beta & w_1 & w_2 \\ & & w_1 & w_2 \\ \alpha^{\beta-1} & \beta^{-\beta} & w_1^{1-\beta} & w_2^\beta \\ & & w_1 & p^{-1} \end{matrix} \right]^{\frac{1}{\alpha+\beta-1}}$$

$$= \left[\begin{matrix} -\alpha & \alpha-1 & \alpha & 1-\alpha & -1 \\ \alpha & \beta & w_1 & w_2 & p \\ & & w_1 & w_2 & p^{-1} \end{matrix} \right]^{\frac{1}{\alpha+\beta-1}}$$

$$x_1 = \left[\begin{matrix} \alpha^{\beta-1} & \beta^{-\beta} & w_1^{1-\beta} & w_2^\beta & p^{-1} \\ \alpha & \beta & w_1 & w_2 & p^{-1} \end{matrix} \right]^{\frac{1}{\alpha+\beta-1}}$$

and rewriting x_1 ,

Then $g = x_1^\alpha x_2^\beta$; optionally,

$$\begin{aligned} g &= \left[\begin{matrix} \alpha(\beta-1) & \alpha\beta & \alpha(1-\beta) & \alpha\beta & -\alpha \\ \alpha & \beta & w_1 & w_2 & p \end{matrix} \right] \frac{1}{\alpha+\beta-1} \\ &\ast \left[\begin{matrix} -\alpha\beta & \beta(\alpha-1) & \alpha\beta & \beta(1-\alpha) & -\beta \\ \alpha & \beta & w_1 & w_2 & p \end{matrix} \right] \frac{1}{\alpha+\beta-1} \\ &= \left[\begin{matrix} -\alpha & \beta^{-\beta} & w_1^\alpha & w_2^\beta & p^{-\alpha-\beta} \\ \alpha & \beta^{-\beta} & w_1 & w_2 & p \end{matrix} \right] \frac{1}{\alpha+\beta-1} \\ &= \left[\begin{matrix} \left(\frac{w_1}{\alpha}\right)^\alpha & \left(\frac{w_2}{\beta}\right)^\beta & p^{-\alpha-\beta} \end{matrix} \right] \frac{1}{\alpha+\beta-1}. \end{aligned}$$

b) $\nabla^2 \mathcal{L} = \nabla^2 \pi = \begin{bmatrix} \alpha(\alpha-1)p x_1^{\alpha-2} x_2^\beta & \alpha\beta p x_1^{\alpha-1} x_2^{\beta-1} \\ \alpha\beta p x_1^{\alpha-1} x_2^{\beta-1} & \beta(\beta-1)p x_1^\alpha x_2^{\beta-2} \end{bmatrix}$

S.O. sufficient conditions for a max:

$D_{2m+1}, D_{2m+2}, \dots, D_{m+n}$ of $\nabla^2 \mathcal{L}$ should alternate in sign starting with $(-1)^{m+1}$.

Here $m=0$ and $n=2$: D_1 same sign as $(-1)^{0+1} = -1 \Rightarrow D_1 < 0$

and $D_2 > 0$.

$$0 > D_1 = \alpha(\alpha-1)p x_1^{\alpha-2} x_2^\beta \Leftrightarrow 0 > \alpha-1 \Leftrightarrow \boxed{\alpha < 1}$$

$$0 < D_2 = \alpha\beta(\alpha-1)(\beta-1)p^2 x_1^{2\alpha-2} x_2^{2\beta-2}$$

$$-\alpha\beta \quad \alpha \quad \beta \quad p^2 x_1^{2\alpha-2} x_2^{2\beta-2}$$

$$= [(\alpha-1)(\beta-1) - \alpha\beta] \alpha\beta p^2 x_1^{2\alpha-2} x_2^{2\beta-2}$$

$$= [-\alpha - \beta + 1] \alpha\beta p^2 x_1^{2\alpha-2} x_2^{2\beta-2}$$

$$\Leftrightarrow 0 < 1 - \alpha - \beta$$

$\boxed{\alpha + \beta < 1}$. (Optional: since $\alpha > 0$ and $\beta > 0$ by assumption, both α and β have to be < 1 .)

c) S.O. necessary conditions for a max:

$\hat{\Delta}_{2m+1}, \hat{\Delta}_{2m+2}, \dots, \hat{\Delta}_{m+n}$ of $\nabla^2 L$ should alternate in sign starting with $(-1)^{m+1}$ or be zero.

Here: $m=0$, $\hat{\Delta}$'s are the same as Δ 's (principal minors);

$$\Delta_1 \leq 0, \Delta_2 \geq 0.$$

$$\Delta_1 = \left\{ \alpha(\alpha-1) p x_1^{\alpha-2} x_2^\beta, \beta(\beta-1) p x_1^\alpha x_2^{\beta-2} \right\}$$

$$\Rightarrow \alpha-1 \leq 0 \text{ and } \beta-1 \leq 0$$

$$\Leftrightarrow \boxed{\alpha \leq 1} \text{ and } \boxed{\beta \leq 1}.$$

$$\Delta_2 = D_2 = [-\alpha-\beta+1] \alpha \beta p^2 x_1^{2\alpha-2} x_2^{2\beta-2} \geq 0. \Leftrightarrow$$

from p. 2

$$-\alpha - \beta \geq -1$$

$$\Leftrightarrow \boxed{\alpha + \beta \leq 1}.$$

d) If $\alpha + \beta > 1$ (which happens if, but not only if, $\alpha > 1$ or $\beta > 1$),

the firm has increasing returns to scale. Proof: $f(\lambda \tilde{x}) = (\lambda x_1)^\alpha (\lambda x_2)^\beta =$
production function

$$\lambda^{\alpha+\beta} x_1^\alpha x_2^\beta = \lambda^{\alpha+\beta} f(x)$$

so the production function is homogeneous of degree $\alpha + \beta$.

In such a circumstance, this competitive firm's profit-maximizing level of output is infinity. Clearly there is no competitive equilibrium.

2007
Midterm

This is related to
the previous question
in this packet.

2. [11 points] Suppose a competitive firm produces output q from inputs x_1 and x_2 according to the production function

$$q = x_1^\alpha x_2^\beta$$

where $\alpha > 0$ and $\beta > 0$. Assume the second-order sufficient conditions for the firm's problem are satisfied.

- By how much would the firm's choice of q change if α rose slightly?
- Do you think the firm's choice of q would rise or fall if α rose slightly?

Hint 1: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Hint 2:

$$\frac{d}{dx} a^x = a^x \ln a, \text{ not } x a^{x-1}.$$

Hint 3: It might be easiest to use Cramer's Rule at some point in answering this question.

(2) a)

From (b) above, the differentials of the two FOC's are :

$$\left[D^2 \pi \right] \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} + \begin{bmatrix} px_1^{\alpha-1} x_2^\beta + \alpha p x_1^{\alpha-1} x_2^\beta \ln x_1 \\ \beta p x_1^\alpha x_2^{\beta-1} \ln x_1 \end{bmatrix} d\alpha = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha(\alpha-1)p x_1^{\alpha-2} x_2^\beta & \alpha\beta p x_1^{\alpha-1} x_2^{\beta-1} \\ \alpha\beta p x_1^{\alpha-1} x_2^{\beta-1} & \beta(\beta-1)p x_1^\alpha x_2^{\beta-2} \end{bmatrix} \begin{bmatrix} dx_1/d\alpha \\ dx_2/d\alpha \end{bmatrix} = \begin{bmatrix} -px_1^{\alpha-1} x_2^\beta - \alpha p x_1^{\alpha-1} x_2^\beta \ln x_1 \\ -\beta p x_1^\alpha x_2^{\beta-1} \ln x_1 \end{bmatrix}$$

$$\frac{\partial x_1}{\partial \alpha} = \frac{\begin{vmatrix} -px_1^{\alpha-1} x_2^\beta - \alpha p x_1^{\alpha-1} x_2^\beta \ln x_1 & \alpha\beta p x_1^{\alpha-1} x_2^{\beta-1} \\ -\beta p x_1^\alpha x_2^{\beta-1} \ln x_1 & \beta(\beta-1)p x_1^\alpha x_2^{\beta-2} \end{vmatrix}}{|D^2 \pi|}$$

$$\frac{\partial x_2}{\partial \alpha} = \frac{\begin{vmatrix} \alpha(\alpha-1)p x_1^{\alpha-2} x_2^\beta & -px_1^{\alpha-1} x_2^\beta - \alpha p x_1^{\alpha-1} x_2^\beta \ln x_1 \\ \alpha\beta p x_1^{\alpha-1} x_2^{\beta-1} & -\beta p x_1^\alpha x_2^{\beta-1} \ln x_1 \end{vmatrix}}{|D^2 \pi|}$$

where $|D^2 \pi| = D_2$ given in part (b), and the numerators would be calculated

using $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC$. Part (b) has $D_2 > 0$, but these numerators are not easy to sign.

Then if "f" is the production function $x_1^\alpha x_2^\beta$,

$$\frac{dq}{d\alpha} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial \alpha} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial \alpha}$$

\uparrow \uparrow \uparrow \uparrow
 $\alpha x_1^{\alpha-1} x_2^\beta$ above $\beta x_1^\alpha x_2^{\beta-1}$ above

Optional :

The numerator of $\frac{\partial x_1}{\partial \alpha}$ is

$$\begin{vmatrix} -p x_1^{\alpha-1} x_2^\beta (1 + \alpha \ln x_1) & 2\beta p x_1^{\alpha-1} x_2^{\beta-1} \\ -\beta p x_1^\alpha x_2^{\beta-1} \ln x_1 & \beta(\beta-1) p x_1^\alpha x_2^{\beta-2} \end{vmatrix}$$

$$= -p x_1^{\alpha-1} x_2^\beta (1 + \alpha \ln x_1) *$$

$$+ p x_1^\alpha x_2^{\beta-2} \beta(\beta-1) + \beta p x_1^\alpha x_2^{\beta-1} \ln x_1 *$$

$$\alpha \beta p x_1^{\alpha-1} x_2^{\beta-1}$$

$$= -p^2 x_1^{2\alpha-1} x_2^{2\beta-2} \beta(\beta-1) (1 + \alpha \ln x_1) + \alpha \beta p^2 x_1^{2\alpha-1} x_2^{2\beta-2} \ln x_1,$$

$$= p^2 x_1^{2\alpha-1} x_2^{2\beta-2} \beta(1-\beta) (1 + \alpha \ln x_1) + \alpha \beta p^2 x_1^{2\alpha-1} x_2^{2\beta-2} \ln x_1,$$

> 0 since $\beta < 1$ is needed for the S.O.C. (assuming $x_1 > 1$ so $\ln x_1 > 0$).

Similarly, the numerator of $\frac{\partial x_2}{\partial \alpha}$ is

$$\begin{aligned} & \alpha(\alpha-1) p x_1^{\alpha-2} x_2^\beta * \\ & (-\beta) p x_1^\alpha x_2^{\beta-1} \ln x_1 + \alpha \beta p x_1^{\alpha-1} x_2^{\beta-1} * p x_1^{\alpha-1} x_2^\beta (1 + \alpha \ln x_1) \\ & = \alpha \beta (1-\alpha) p^2 x_1^{2\alpha-2} x_2^{2\beta-1} \ln x_1 + \alpha \beta p^2 x_1^{2\alpha-2} x_2^{2\beta-1} (1 + \alpha \ln x_1) \\ & > 0 \text{ since } \alpha < 1 \text{ is needed for the S.O.C. (assuming } x_1 > 1 \text{).} \end{aligned}$$

Since $|\nabla^2 \pi| = D_2 > 0$, all four parts of the expression for $\frac{\partial f}{\partial \alpha}$ are positive, so so is $\frac{\partial g}{\partial \alpha}$.

Intuition: As $\alpha \uparrow$, production becomes easier, so the firm produces more.

↑
Answer to last part

Alternative Approach : from part (a),

$$g = \left[\left(\frac{w_1}{\alpha} \right)^\alpha \left(\frac{w_2}{\beta} \right)^\beta p^{-\alpha-\beta} \right] \frac{1}{\alpha+\beta-1}$$

Abbreviate this by $g = h(\alpha)^{\frac{j}{\alpha}}$. Then

$$\frac{dg}{d\alpha} = j h^{j-1} \frac{\partial h}{\partial \alpha} + h^j (\ln h) \frac{\partial j}{\partial \alpha}.$$

Then

$$\frac{\partial j}{\partial \alpha} = \frac{\partial}{\partial \alpha} (\alpha+\beta-1)^{-1} = -(\alpha+\beta-1)^{-2} = \frac{-1}{(\alpha+\beta-1)^2} < 0.$$

Also,

$$\begin{aligned} \frac{\partial h}{\partial \alpha} &= \frac{\partial}{\partial \alpha} \left[w_1^\alpha \alpha^{-\alpha} \left(\frac{w_2}{\beta} \right)^\beta p^{-\alpha-\beta} \right] \\ &= \left(\frac{\partial}{\partial \alpha} w_1^\alpha \right) \alpha^{-\alpha} \left(\frac{w_2}{\beta} \right)^\beta p^{-\alpha-\beta} + w_1^\alpha \left(\frac{\partial}{\partial \alpha} \alpha^{-\alpha} \right) \left(\frac{w_2}{\beta} \right)^\beta p^{-\alpha-\beta} \\ &\quad + w_1^\alpha \alpha^{-\alpha} \left(\frac{w_2}{\beta} \right)^\beta \left(\frac{\partial}{\partial \alpha} p^{-\alpha-\beta} \right) \\ &= w_1^\alpha (\ln w_1) \alpha^{-\alpha} \left(\frac{w_2}{\beta} \right)^\beta p^{-\alpha-\beta} \\ &\quad + w_1^\alpha \left[-\alpha \cdot \alpha^{-\alpha-1} - \alpha^{-\alpha} \ln \alpha \right] \left(\frac{w_2}{\beta} \right)^\beta p^{-\alpha-\beta} \\ &\quad + w_1^\alpha \alpha^{-\alpha} \left(\frac{w_2}{\beta} \right)^\beta p^{-\alpha-\beta} (\ln p) \frac{\partial (-\alpha-\beta)}{\partial \alpha} \\ &= w_1^\alpha \alpha^{-\alpha} \left(\frac{w_2}{\beta} \right)^\beta p^{-\alpha-\beta} \left[\ln w_1 - (1+\ln \alpha) - \ln p \right] \\ &= w_1^\alpha \alpha^{-\alpha} \left(\frac{w_2}{\beta} \right)^\beta p^{-\alpha-\beta} \left[1 + \ln \frac{\alpha(p)}{w_1} \right] (-1) \end{aligned}$$

$$\begin{aligned}
 \frac{dq}{d\alpha} &= h^{j-1} \left[j \frac{\partial h}{\partial \alpha} + h \frac{\partial j}{\partial \alpha} \right] \\
 &= h^{j-1} \left[\frac{1}{\alpha+\beta-1} \left(\frac{w_1}{\alpha} \right)^\alpha \left(\frac{w_2}{\beta} \right)^\beta p^{-\alpha-\beta} \left(1 + \ln \frac{\alpha P}{w_1} \right) (-1) + \right. \\
 &\quad \left. \left(\frac{w_1}{\alpha} \right)^\alpha \left(\frac{w_2}{\beta} \right)^\beta p^{-\alpha-\beta} \frac{-1}{(\alpha+\beta-1)^2} \right] \\
 &= h^{j-1} \frac{-1}{(\alpha+\beta-1)^2} \left(\frac{w_1}{\alpha} \right)^\alpha \left(\frac{w_2}{\beta} \right)^\beta p^{-\alpha-\beta} \left[(\alpha+\beta-1) \left(1 + \ln \frac{\alpha P}{w_1} \right) + 1 \right] \text{ (*)} \\
 &= " \left[(\alpha+\beta) + (\alpha+\beta) \ln \frac{\alpha P}{w_1} - 1 - \ln \frac{\alpha P}{w_1} + 1 \right] \\
 &= " \left[(\alpha+\beta) + (\alpha+\beta) \ln \frac{\alpha P}{w_1} - \ln \frac{\alpha P}{w_1} \right] \\
 &= " \left[(\alpha+\beta) + (\alpha+\beta-1) \ln \frac{\alpha P}{w_1} \right] \text{ (**)}
 \end{aligned}$$

Despite knowing that $\alpha+\beta-1 < 0$ from the second-order conditions, it does not seem possible to sign either (*) or (**). For example, the sign of $\ln \frac{\alpha P}{w_1}$ is not known. So this approach does not seem to work.

Exam 1
1998
Question 3

(7)

3. Suppose a firm uses inputs x_1 and x_2 to produce output y according to the production function $y = g(x_1) + h(x_2)$. Let C denote the firm's cost function.

Show that the firm's marginal cost curve, dC/dy , is given by

$$\frac{w_1g'(x_1)h''(x_2) + w_2g''(x_1)h'(x_2)}{[g'(x_1)]^2h''(x_2) + g''(x_1)[h'(x_2)]^2}.$$

Also, speculate about the sign of this expression.

(Ideally, this would be a function of w and y instead of a function of w and x_1 and x_2 . However, that would be more work, so I do not want you to try to figure that out.)

Hint: Find dx_1/dy and dx_2/dy , then think about how $C = w_1x_1 + w_2x_2$ changes with y .

③ The firm's problem is to

$$\min_{\mathbf{x}} w_1 x_1 + w_2 x_2 \text{ s.t. } y = g(x_1) + h(x_2)$$

so

$$\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda [y - g(x_1) - h(x_2)].$$

The first-order conditions are

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = y - g(x_1) - h(x_2)$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \lambda g'(x_1)$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_2} = w_2 - \lambda h'(x_2).$$

Exam 1

1998

Answer 3

Taking the total differential of each of these, but setting $dw_1 = 0$ and $dw_2 = 0$ since these do not change in this problem, we have

$$0 = \begin{bmatrix} d\lambda & dx_1 & dx_2 \\ 0 & -g'(x_1) & -h'(x_2) \\ -g'(x_1) & -\lambda g''(x_1) & 0 \\ -h'(x_2) & 0 & -\lambda h''(x_2) \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} dy \Rightarrow$$

$$\begin{bmatrix} 0 & g'(x_1) & h'(x_2) \\ g'(x_1) & \lambda g''(x_1) & 0 \\ h'(x_2) & 0 & \lambda h''(x_2) \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} dy \Rightarrow (\text{dividing each equation by } dy)$$

$$\begin{bmatrix} 0 & g'(x_1) & h'(x_2) \\ g'(x_1) & \lambda g''(x_1) & 0 \\ h'(x_2) & 0 & \lambda h''(x_2) \end{bmatrix} \begin{bmatrix} d\lambda/dy \\ dx_1/dy \\ dx_2/dy \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Use Cramer's Rule:

$$\frac{dx_1}{dy} = \frac{\begin{vmatrix} 0 & 1 & h' \\ g' & 0 & 0 \\ h' & 0 & \lambda h'' \end{vmatrix}}{\begin{vmatrix} 0 & g' & h' \\ g' & \lambda g'' & 0 \\ h' & 0 & \lambda h'' \end{vmatrix}}, \quad \frac{dx_2}{dy} = \frac{\begin{vmatrix} 0 & g' & 1 \\ g' & \lambda g'' & 0 \\ h' & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & g' & h' \\ g' & \lambda g'' & 0 \\ h' & 0 & \lambda h'' \end{vmatrix}}.$$

Then since $C = w_1 x_1 + w_2 x_2$, with w_1 and w_2 taken as given,

$$\frac{dC}{dy} = w_1 \frac{dx_1}{dy} + w_2 \frac{dx_2}{dy}. \text{ Substituting for } \frac{dx_1}{dy} \text{ and } \frac{dx_2}{dy},$$

$$\frac{dC}{dy} = \left\{ w_1 \begin{vmatrix} 0 & 1 & h' \\ g' & 0 & 0 \\ h' & 0 & \lambda h'' \end{vmatrix} + w_2 \begin{vmatrix} 0 & g' & 1 \\ g' & \lambda g'' & 0 \\ h' & 0 & 0 \end{vmatrix} \right\} \Big/ \begin{vmatrix} 0 & g' & h' \\ g' & \lambda g'' & 0 \\ h' & 0 & \lambda h'' \end{vmatrix}.$$

$$\text{The numerator is } w_1 (-g' \lambda h'') + w_2 (-\lambda h' g'') = -\lambda (w_1 g' h'' + w_2 h' g'').$$

The denominator is related to the determinant of the Hessian of the Lagrangian,

$$|\nabla^2 \mathcal{L}| = \begin{vmatrix} 0 & -g' & -h' \\ -g' & -\lambda g'' & 0 \\ -h' & 0 & -\lambda h'' \end{vmatrix} = (-1)^3 \begin{vmatrix} 0 & g' & h' \\ g' & \lambda g'' & 0 \\ h' & 0 & \lambda h'' \end{vmatrix} =$$

$$(-1)^3 [-g' (\lambda g' h'') + h' (-\lambda h' g'')] = \underbrace{(-1)^3 (-\lambda)}_{=\lambda} [g' g' h'' + h' h' g''].$$

The second-order sufficient conditions for a minimum for the problem, which

has $m=1$ constraint and $n=2$ unknowns, is D_{2m+1} of $\nabla^2 \mathcal{L} = D_3$ of $\nabla^2 \mathcal{L}$

to D_{m+n} of $\nabla^2 \mathcal{L} = D_3$ of $\nabla^2 \mathcal{L}$ has the same sign as $(-1)^m = -1$. Therefore

$$|\nabla^2 \mathcal{L}| = D_3 \text{ of } \nabla^2 \mathcal{L} < 0 \Leftrightarrow \lambda (g' g' h'' + h' h' g'') < 0.$$

Combining the numerator and denominator,

$$\frac{dC}{dy} = \frac{-\lambda (w_1 g' h'' + w_2 h' g'')}{-\lambda (g' g' h'' + h' h' g'')} = \frac{w_1 g' h'' + w_2 h' g''}{(g')^2 h'' + (h')^2 g''}.$$

↑
this minus sign comes from
the $(-1)^3$ term on the last
page; "the denominator" is
 $|D^2L| / (-1)^3$

We know that

$\lambda(g' g' h'' + h' h' g'') < 0$,
but that does not help much
in signing dC/dy , since we do
not know the sign of λ . (In

fact, the Envelope Theorem can be used to prove that $dC/dy = \lambda$, so the sign of λ is the unknown.) However, it seems reasonable to speculate that both h' and g' are positive (positive marginal products) and that both h'' and g'' are negative (diminishing returns). In that case,

$$\frac{dC}{dy} = \frac{\overset{+}{w}_1 \overset{+}{g}' \overset{-}{h}'' + \overset{+}{w}_2 \overset{+}{h}' \overset{-}{g}''}{(\overset{+}{g}')^2 \overset{-}{h}'' + (\overset{+}{h}')^2 \overset{-}{g}''} > 0.$$

This is consistent with the second-order condition $0 > \lambda(g' g' h'' + h' h' g'') = (\frac{dC}{dy})(\underbrace{g' g' h''}_{\overset{+}{+} \overset{-}{-}} + \underbrace{h' h' g''}_{\overset{+}{+} \overset{-}{-}}) = \overset{-}{-}$.

In actuality, dC/dy cannot be negative (assuming free disposal), because $dC/dy < 0$ would imply that as $y \uparrow$, $C \downarrow$:

y	C
y_1	C_1
y_2 with $y_2 > y_1$	$C_2 < C_1$

. The firm

is being irrational because when it wanted to produce y_1 , instead of incurring costs C_1 , it could have produced y_2 , thrown $y_2 - y_1$ away, and incurred costs of $C_2 < C_1$, increasing its profit.

Fall 2019 Final Exam Qu 3:

old: 1998 Exam 1 Qu 3, but with an added part (b).

3. [17 points]

Suppose a firm uses inputs x_1 and x_2 to produce output y according to the production function $y = g(x_1) + h(x_2)$. Let C denote the firm's cost function.

(a) Show that the firm's marginal cost curve is given by

$$\frac{dC}{dy} = \frac{w_1 g'(x_1) h''(x_2) + w_2 g''(x_1) h'(x_2)}{[g'(x_1)]^2 h''(x_2) + g''(x_1) [h'(x_2)]^2}$$

where $\mathbf{w} = (w_1, w_2)$ is the vector of input prices. Also, speculate about the sign of this expression.

Hint: Find dx_1/dy and dx_2/dy , then think about how $C = w_1 x_1 + w_2 x_2$ changes with y .

(b) If $g(x_1) = 2\sqrt{x_1}$ and $h(x_2) = 2\sqrt{x_2}$, find dC/dy as a function of \mathbf{w} and y instead of a function of \mathbf{w} , x_1 , and x_2 .

Answer to Econ.7005 Fall 2019 Final Exam, Qu. 3

a) See the answer to the previous problem.

b) From part (a), the first-order conditions are

$$0 = y - g(x_1) - \lambda h(x_2)$$

$$0 = w_1 - \lambda g'(x_1)$$

$$0 = w_2 - \lambda h'(x_2).$$

These cannot be solved explicitly for x_1 and x_2 unless one uses functions like "the inverse function of $g'(x)$ ", which are complicated. However, in this part of the problem, we are given ^{simple} explicit forms for g and h , so solving the first-order conditions explicitly for x_1 and x_2 becomes possible:

$$\begin{aligned} 0 &= y - 2\sqrt{x_1} - 2\sqrt{x_2} \\ 0 &= w_1 - \lambda x_1^{-1/2} \\ 0 &= w_2 - \lambda x_2^{-1/2} \end{aligned} \quad \left\{ \begin{array}{l} g(x_1) = 2\sqrt{x_1} = x_1^{1/2} \\ g'(x_1) = x_1^{-1/2} \\ h(x_2) = 2\sqrt{x_2} \\ h'(x_2) = x_2^{-1/2} \end{array} \right.$$

$$\text{so } \lambda = w_1 x_1^{1/2} = w_2 x_2^{1/2} \Rightarrow x_2^{1/2} = \frac{w_1}{w_2} x_1^{1/2} \text{ and } x_2 = \left(\frac{w_1}{w_2}\right)^2 x_1. \text{ Then substitute into the first F.O.C.: } y = 2\sqrt{x_1} + 2\sqrt{\left(\frac{w_1}{w_2}\right)^2 x_1} = 2\sqrt{x_1} + \frac{w_1}{w_2} 2\sqrt{x_1} = \left(1 + \frac{w_1}{w_2}\right) 2\sqrt{x_1}$$

$$= \frac{w_1 + w_2}{w_2} 2\sqrt{x_1} \Rightarrow \sqrt{x_1} = \frac{w_2}{w_1 + w_2} \frac{y}{2} \text{ and } x_1 = \left(\frac{w_2}{w_1 + w_2} \frac{y}{2}\right)^2.$$

$$\text{Then since } x_2 = \left(\frac{w_1}{w_2}\right)^2 x_1, \text{ we have } x_2 = \frac{w_1^2}{w_2^2} \frac{w_2^2}{(w_1 + w_2)^2} \left(\frac{y}{2}\right)^2 = \left(\frac{w_1}{w_1 + w_2} \frac{y}{2}\right)^2.$$

There are now two alternative ways to proceed. Method 1 makes use of the result of Part (a):

Method 1

$$g'(x_1) = x_1^{-\frac{1}{2}} = \left(\frac{w_2}{w_1+w_2} - \frac{y}{2} \right)^{-1} = \frac{w_1+w_2}{w_2} \frac{2}{y}$$

$$g''(x_1) = -\frac{1}{2} x_1^{-\frac{3}{2}} = -\frac{1}{2} \left(\frac{w_2}{w_1+w_2} - \frac{y}{2} \right)^{-3} = -\frac{1}{2} \left(\frac{w_1+w_2}{w_2} - \frac{2}{y} \right)^3$$

$$h'(x_2) = x_2^{-\frac{1}{2}} = \left(\frac{w_1}{w_1+w_2} - \frac{y}{2} \right)^{-1} = \frac{w_1+w_2}{w_1} \frac{2}{y}$$

$$h''(x_2) = -\frac{1}{2} x_2^{-\frac{3}{2}} = -\frac{1}{2} \left(\frac{w_1}{w_1+w_2} - \frac{y}{2} \right)^{-3} = -\frac{1}{2} \left(\frac{w_1+w_2}{w_1} - \frac{2}{y} \right)^3$$

and substituting into the answer to part (a),

$$\frac{dc}{dy} = \frac{\frac{w_1}{w_2} \frac{2}{y} - \frac{1}{2} \left(\frac{w_1+w_2}{w_1} - \frac{2}{y} \right)^3 + w_2 \frac{2}{y} - \frac{1}{2} \left(\frac{w_1+w_2}{w_2} - \frac{2}{y} \right)^3 \frac{w_1+w_2}{w_1} \frac{2}{y}}{\left(\frac{w_1+w_2}{w_2} - \frac{2}{y} \right)^2 - \frac{1}{2} \left(\frac{w_1+w_2}{w_1} - \frac{2}{y} \right)^3 + \frac{1}{2} \left(\frac{w_1+w_2}{w_2} - \frac{2}{y} \right)^3 \left(\frac{w_1+w_2}{w_1} - \frac{2}{y} \right)^2}$$

$$= \frac{-w_1^{1-3} (w_1+w_2)^{1+3} w_2^{-1} 2^{-2} y^{-1-1+3} - w_2^{-1-3} (w_1+w_2)^{1-3} 2^{3+1} y^{-1+3+1} w_1^{-1}}{-(w_1+w_2)^{2+3} w_2^{-2} 2^{2-1+3} y^{-2-3} w_1^{-2} - w_2^{-1+3+2} (w_1+w_2)^{3+2} w_2^{-3} y^{-3-2} w_1^{-2}}$$

$$= \frac{+w_1^{-2} (w_1+w_2)^4 w_2^{-1} 2^3 y^{-4} + w_2^{-2} (w_1+w_2)^4 2^3 y^{-4} w_1^{-1}}{+(w_1+w_2)^5 w_2^{-2} 2^4 y^{-5} w_1^{-3} + 2^4 (w_1+w_2)^5 w_2^{-3} y^{-5} w_1^{-2}}$$

$$= \frac{w_1^{-1} (w_1+w_2)^4 w_2^{-1} 2^3 y^{-4} [w_1^{-1} + w_2^{-1}]}{(w_1+w_2)^5 w_2^{-2} 2^4 y^{-5} w_1^{-2} [w_1^{-1} + w_2^{-1}]} = \frac{(w_1+w_2)^4 2^3 w_2^2 y^5 w_1^2}{w_1 w_2 y^4 (w_1+w_2)^5 2^4}$$

↑ I eliminated all negative exponents

$$= \boxed{\frac{w_1 w_2 y}{(w_1+w_2) 2}}.$$

Method 2

$$\begin{aligned} \text{Total cost is } w_1 x_1^* + w_2 x_2^* &= w_1 \left(\frac{w_2}{w_1+w_2} \frac{y}{2} \right)^2 + w_2 \left(\frac{w_1}{w_1+w_2} \frac{y}{2} \right)^2 \\ &= \frac{w_1 w_2^2 y^2}{2^2 (w_1+w_2)^2} + \frac{w_1^2 w_2 y^2}{2^2 (w_1+w_2)^2} = \frac{w_1 w_2 y^2 (w_1+w_2)}{2^2 (w_1+w_2)^2} = \frac{w_1 w_2 y^2}{4(w_1+w_2)} \end{aligned}$$

and hence marginal cost is

$$\frac{\partial C}{\partial y} = \frac{\partial}{\partial y} \frac{w_1 w_2 y^2}{4(w_1+w_2)} = \boxed{\frac{w_1 w_2 y}{2(w_1+w_2)}}, \text{ agreeing with the answer via Method 1.}$$

So in part (b), total cost is quadratic in y and marginal cost is linear in y .

1. [11 points] The most commonly used approximation to an arbitrary function $f(x)$ about the point $x = a$ is given by

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n,$$

which is called the “ n th-order Taylor polynomial for f about $x = a$ ” (where the ! operator denotes the factorial).

- (a) Suppose a firm has a cost function $C(y)$ where y is output. Prove that the second-order Taylor polynomial for C around $y = 0$ is

Fall 2008

Ex. 1

$$C(y) \approx C(0) + C'(0)y + \frac{C''(0)}{2}y^2.$$

- (b) What are the second-order conditions for profit maximization for a competitive firm if you replace its true cost function $C(y)$ with the second-order Taylor polynomial for C around $y = 0$? If you wish, you can make the notation easier by using α as an abbreviation for $C'(0)$ and use β as an abbreviation for $C''(0)/2$.
- (c) What are the second-order conditions for profit maximization for a competitive firm if you replace its true cost function $C(y)$ with the *first*-order Taylor polynomial for C around $y = 0$? If you wish, you can make the notation easier by using the same abbreviations suggested in part (b). Please explain your answer.
- (d) What are the second-order conditions for profit maximization for a competitive firm if you replace its true cost function $C(y)$ with the *third*-order Taylor polynomial for C around $y = 0$? If you wish, you can make the notation easier by using the same abbreviations suggested in part (b), and by using a new abbreviation for γ as is most convenient for you. Your answer should be a condition on the parameters of the cost function, so it should not involve y or the optimal level of y , denoted y^* .

You may need to use the fact that the roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Answers to Econ 7005 Midterm Exam, Fall 2008

$$\textcircled{1} \quad f(x) \approx f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n$$

a) The second-order Taylor polynomial is

$$f(x) \approx f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2$$

since $n! = n(n-1)(n-2)\dots(2)(1)$ if n is an integer (in other words,
 $n! = \prod_{i=0}^{n-1} (n-i)$). Set $f = C$, $x = y$ (output), $a = 0$:

$$C(y) \approx C(0) + C'(0)y + \frac{C''(0)}{2} y^2.$$

b) $\pi = py - C(y)$ where y is output (profit = total revenue - total cost)
 ↑ ↑
 profit price, fixed because the firm is competitive

F.O.C. $0 = \frac{d\pi}{dy} = p - C'(y)$ would be the usual approach, but here

we are asked to use, not $C(y)$, but the second-order Taylor
 polynomial approximation to it. So

$$\pi = py - [C(0) + C'(0)y + \frac{C''(0)}{2} y^2]$$

and the first-order condition is

$$0 = \frac{d\pi}{dy} = p - [C'(0) + C''(0)y].$$

[Optimal: Solving for y yields

$$C''(0) \cdot y = p - C'(0) \Rightarrow$$

$$y = [p - C'(0)] / C''(0).$$

The second-order condition for a maximum is

$$0 > \frac{d^2\pi}{dy^2} = -C''(0) \Leftrightarrow \underline{C''(0) > 0}.$$

c) Using the first-order Taylor approximation gives

$$\begin{aligned}\pi &= py - [c(0) + c'(0)y] \\ &= py - c(0) - c'(0)y \\ &= [p - c'(0)]y - c(0).\end{aligned}$$

The F.O.C. for an interior solution is

$$0 = \frac{d\pi}{dy} = p - c'(0)$$

and the S.O.C. for a maximum is

$$0 > \frac{d^2\pi}{dy^2} = 0.$$

The S.O.C. is not satisfied because π is linear in y . If the F.O.C. were satisfied, then $p = c'(0)$ and $\pi = -c(0) \forall y$.

If $p > c'(0)$ then the optimal y is ∞ .

If $p < c'(0)$ then the optimal y is 0.

This is just like constant returns to scale behavior, and is probably a poor approximation whenever the total cost function is not linear.

d)

$$\pi = py - [c(0) + c'(0)y + \frac{c''(0)}{2}y^2 + \underbrace{\frac{c'''(0)}{6}y^3}]$$

$6=3!$

F.O.C.

$$0 = \frac{d\pi}{dy} = p - [c'(0) + c''(0)y + \frac{1}{2}c'''(0)y^2]$$

S.O.C.

$$0 > \frac{d^2\pi}{dy^2} = -c''(0) - \frac{1}{2}c'''(0)y$$

$$0 < c''(0) + c'''(0)y$$

↑ actually, y^*

To eliminate y^* , use the F.O.C.:

$$\begin{aligned}
 0 &= -p + [c'(0) + c''(0)y + \frac{1}{2}c'''(0)y^2] \quad \leftarrow "y \text{ here is actually } y^*\right. \\
 &= \frac{1}{2}c'''(0)y^2 + c''(0)y + (c'(0) - p) \leftarrow \\
 \Rightarrow y^* &= \frac{-c''(0) \pm \sqrt{[c''(0)]^2 - 4 \cdot \frac{1}{2}c'''(0)(c'(0) - p)}}{c'''(0)} \\
 &= \frac{-c''(0) \pm \sqrt{[c''(0)]^2 - 2c'''(0)(c'(0) - p)}}{c'''(0)}
 \end{aligned}$$

Substitute this into the S.O.C. $0 < c''(0) + c'''(0)y^*$.

Section 1.
Answer all of the following three questions.

1. [14 points] Suppose a competitive firm produces output q and incurs total costs given by the following total cost function:

$$c(q) = \frac{1}{3}q^3 - q^2 + 2q + 1.$$

- (a) On one graph, sketch this firm's Marginal Cost curve, its Average Cost curve, and its Average Variable Cost curve.

Hint: "Average Variable Cost" means the average of: "total cost minus fixed cost," where "Fixed Cost" means $c(0)$.

Give numerical coordinates for:

- i. the minimum of the Marginal Cost Curve;
- ii. the minimum of the Average Variable Cost Curve;
- iii. the vertical-axis intercept of the Marginal Cost Curve, if it has such an intercept;
- iv. the vertical-axis intercept of the Average Cost Curve, if it has such an intercept;
- v. the vertical-axis intercept of the Average Variable Cost Curve, if it has such an intercept.

- (b) Explicitly find the supply curve of this firm for *all non-negative* values of the price of q , which is denoted by " p ." (Note that I am not asking you to find the *inverse* supply curve.) Be sure to check second-order conditions.

Hint: if $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Summer 2013 Qualifying Exam Section 1 Question 1

Section 1 Question 1.

$$c(q) = \frac{1}{3}q^3 - q^2 + 2q + 1$$

a) Marginal Cost $c'(q) = q^2 - 2q + 2 = "MC(q)"$

$$\text{Average Cost } "AC(q)" = \frac{c(q)}{q} = \frac{1}{3}q^2 - q + 2 + \frac{1}{q}$$

$$\text{Fixed Cost } c(0) = 1 = "FC"$$

$$\begin{aligned}\text{Variable Cost} &= \text{Total Cost} - \text{Fixed Cost} = \frac{1}{3}q^3 - q^2 + 2q + 1 - 1 \\ &= \frac{1}{3}q^3 - q^2 + 2q = "VC(q)"\end{aligned}$$

$$\text{Average Variable Cost } "AVC(q)" = \frac{VC(q)}{q} = \frac{1}{3}q^2 - q + 2$$

(i) minimum of MC : where is the derivative of MC equal to zero?

$$0 = \frac{d}{dq} MC(q) = \frac{d}{dq} c'(q) = c''(q) = 2q - 2 \Rightarrow$$

$$2 = 2q \Rightarrow q = 1. \text{ Here, } MC(1) = 1^2 - 2(1) + 2 = 1 - 2 + 2 = 1.$$

(ii) minimum of AVC : where is the derivative of AVC equal to zero?

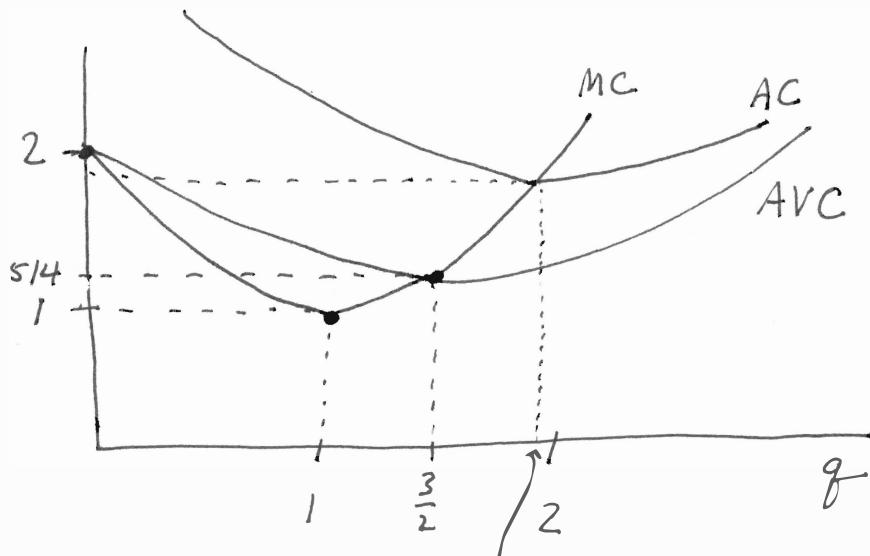
$$0 = \frac{d}{dq} AVC(q) = \frac{2}{3}q - 1 \Rightarrow 1 = \frac{2}{3}q \Rightarrow q = \frac{3}{2}. \text{ Here, }$$

$$AVC\left(\frac{3}{2}\right) = \frac{1}{3}\left(\frac{3}{2}\right)^2 - \frac{3}{2} + 2 = \frac{1}{3} \cdot \frac{9}{4} - \frac{6}{4} + \frac{8}{4} = \frac{3-6+8}{4} = \frac{5}{4}.$$

$$(iii) MC(0) = 0^2 - 2 \cdot 0 + 2 = 2$$

$$(iv) AC(0) = \frac{1}{3} \cdot 0^2 - 0 + 2 + \frac{1}{0} = +\infty$$

$$(v) AVC(0) = \frac{1}{3} \cdot 0^2 - 0 + 2 = 2$$



Optional : If AVC 's minimum is interior, then

at that point, $AVC = MC$. Proof : At AVC 's

minimum, assuming it is interior : $0 =$

$$\frac{d}{dq} AVC = \frac{d}{dq} \frac{c(q) - FC}{q} = \frac{c'(q)}{q} - \frac{c(q) - FC}{q^2}$$

$$\Rightarrow 0 = q c'(q) - c(q) + FC$$

$$\Rightarrow \frac{c(q) - FC}{q} = c'(q)$$

$$\Rightarrow AVC = MC.$$

optional : min of AC occurs at

$$0 = \frac{d}{dq} AC = \frac{2}{3}q - 1 - \frac{1}{q^2} \Leftrightarrow$$

$$0 = \frac{2}{3}q^3 - q^2 - 1 \Rightarrow$$

$q \approx 1.91$ according to Mathematica ;

$AC(q=1.91) \approx 1.83$ according to
Mathematica

$$b) \pi = pq - c(q) ; \max \pi \Rightarrow \boxed{0 = \pi'_q = p - c'(q)} \Rightarrow$$

↑
fixed output

price

$$\boxed{p = c'(q)} \Rightarrow$$

$$p = q^2 - 2q + 2 \Rightarrow$$

$$0 = q^2 - 2q + (2-p) \Rightarrow$$

$$q = \frac{2 \pm \sqrt{4 - 4(1)(2-p)}}{2}$$

$$= 1 \pm \frac{1}{2} \sqrt{4 - 4(2-p)}$$

$$= 1 \pm \sqrt{1 - 1(2-p)}$$

$$= 1 \pm \sqrt{1 - 2 + p}$$

$$= 1 \pm \sqrt{p-1}. \quad (1)$$

To investigate second-order conditions, calculate

$$\pi''_{qq} = -c''(q) = -(2q-2) \text{ from part (a)(i)}$$

$$= 2 - 2q.$$



The S.O.C. for a maximum is $\pi_{qq}'' < 0$, so

$$2 - 2q < 0$$

$$2 < 2q$$

$$1 < q.$$

Therefore in (1) above, $q = 1 \pm \sqrt{p-1}$, choosing the negative sign would violate the S.O.C. (it would cause q to be less than one). We conclude that the supply curve is

$$q = 1 + \sqrt{p-1} \quad (2)$$

where this is a real number, that is, for $p \geq 1$. [From (2), note that the inverse supply curve is

$$q - 1 = \sqrt{p-1}$$

$$(q-1)^2 = p-1$$

$$q^2 - 2q + 1 = p - 1$$

$$q^2 - 2q + 2 = p$$

$$MC(q) = p \text{ from part (a)}, \quad (3)$$

i.e., price equals marginal cost, as in the first line of part (b)'s answer.]^{end of optional part}

This conclusion is tentative, though, because we have ignored the $q \geq 0$ constraint. The corner solution $q = 0$ will be chosen instead of (2) if

it brings more profit than (2). In general, the $q = 0$ solution brings profit

$$\text{of } \pi(q) \Big|_{q=0} = p_f - VC(q) - FC \Big|_{q=0} = -FC, \text{ while the interior solution's}$$

profit is $\pi_i = \pi(q_i) = pq_i - VC(q_i) - FC$ using " q_i " to denote

the interior solution. The interior solution is strictly better if and only if

$$\pi_i > \pi(0)$$

$$pq_i - VC(q_i) - FC > -FC$$

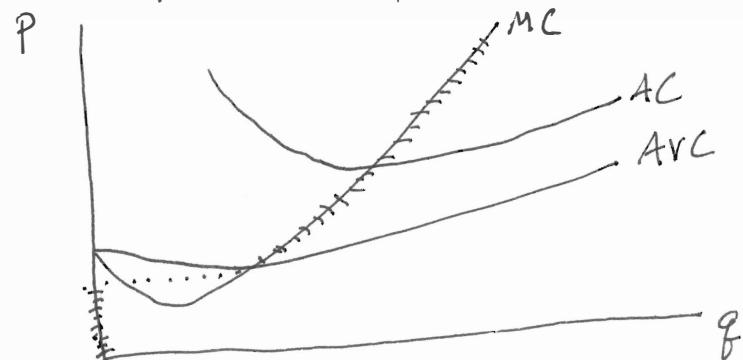
$$pq_i > VC(q_i)$$

$$p > AVC(q_i). \quad (4)$$

Hence the solution is given by q_i satisfying (2) (equivalently, (3)) if

$p \geq AVC(q_i)$, else, $q^* = 0$. (If $p = AVC(q_i)$, then both q_i and 0 maximize profit.)

Optional: the graph of the supply curve is



1. [12 points]

Suppose a competitive firm earns profit π by producing output Q using a cost function $C(Q)$ with $Q \geq 0$.

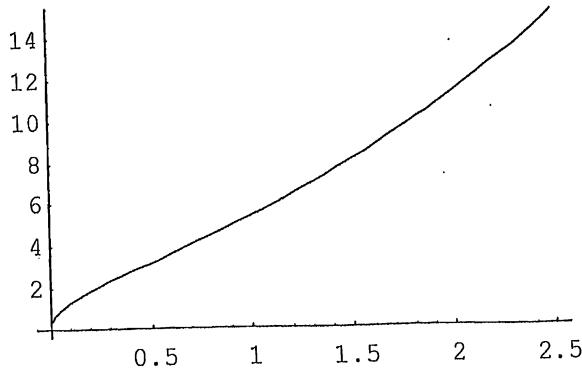
- (a) Under what conditions is this firm's supply curve upward-sloping?
(b) Suppose the cost function has the form

Summer
2007
Qual.

$$C(Q) = 4\sqrt{Q} + \frac{\sqrt{8}}{2} Q^2$$

(a graph of this function is attached to this exam). Find the firm's supply curve. You do not have to find it explicitly; implicitly is enough. Be sure you work through the following steps to help ensure your answer is right.

- i. For what values of Q are the appropriate second-order conditions satisfied?
- ii. For the values of Q you found in (i), what are the corresponding values of p ?
- iii. For other values of p , what is the optimal quantity supplied?
(Hint: at other values of p , does the Q which satisfies the first-order condition yield a maximum or a minimum?)



This is a graph of the cost function of Part 1's Question 1.

Answers to 7005 portion of the 2007
Microeconomics Qualifying Exam

Section 4 Question 1:

$$\pi = pQ - C(Q) \text{ with "p" for price}$$

a) F.O.C. $O = \frac{d\pi}{dQ} = p - C'(Q) \Rightarrow$ take differential :

$$O = dp - C''(Q) dQ$$

$$C'' dQ = dp$$

$$\frac{dQ}{dp} = \frac{1}{C''}$$

which is positive as long as $C'' > 0$ (as long as $C(Q)$ is convex).

b) From part (a), the F.O.C. is

$$O = p - C'(Q)$$

$$= p - [2Q^{-1/2} + \sqrt{8} Q] \Leftrightarrow p = 2Q^{-1/2} + \sqrt{8} Q \text{ implicitly}$$

defines the supply curve $Q(p)$.

S.O.C. : $\frac{d^2}{dQ^2} [p - C(Q)] < 0$ for a maximum

$$\Leftrightarrow \frac{d}{dQ} [p - C'(Q)] < 0$$

$$\Leftrightarrow -C''(Q) < 0 \Leftrightarrow C''(Q) > 0.$$

Here, $C''(Q) = \frac{d}{dQ} [2Q^{-1/2} + \sqrt{8} Q]$

$$= -Q^{-3/2} + \sqrt{8}, \text{ which is positive if}$$

$$-Q^{-\frac{3}{2}} + \sqrt{8} > 0$$

$$\sqrt{8} > Q^{-\frac{3}{2}}$$

$$8 > Q^{-3} = \frac{1}{Q^3}$$

$$\sqrt[3]{8} > \frac{1}{Q}$$

$$2 > \frac{1}{Q} \Rightarrow Q > \frac{1}{2}. \text{ This answers (i).}$$

(ii) The corresponding value of price is

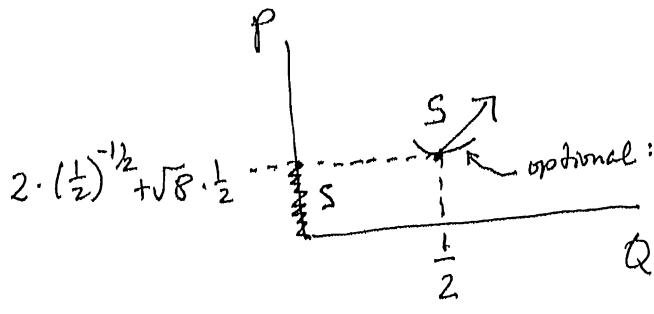
$$p = 2 \cdot \left(\frac{1}{2}\right)^{-\frac{1}{2}} + \sqrt{8} \cdot \frac{1}{2}$$

optional: this is $2\sqrt{2} + \sqrt{4 \cdot 2} \cdot \frac{1}{2}$
 $= 2\sqrt{2} + 2\sqrt{2} \cdot \frac{1}{2} = \sqrt{2}(2+1)$
 $= 3\sqrt{2}.$

(for $Q=\frac{1}{2}$); when $Q > \frac{1}{2}$, p is greater than this. (The supply curve is upward-sloping when $Q > \frac{1}{2}$ from part (a) and the fact that $Q > \frac{1}{2} \Rightarrow C'' > 0.$)

(iii) When $p < 2 \cdot \left(\frac{1}{2}\right)^{-\frac{1}{2}} + \sqrt{8} \cdot \frac{1}{2}$, C'' is negative, the S.O.C. for a (local)

minimum one satisfied, which is not what we'd want. The alternative is to produce $Q^* = 0$. This could be shown formally by using Kuhn-Tucker conditions. The supply curve looks like



optional: this is average cost, "AC"; the supply curve is the marginal cost curve above the bottom of average variable cost, which is AC here, so the AC curve must look something like this

Fall 2022 Final Exam Qu. 2 is new

2. [20 points] Although as discussed in class the notion of using “capital” K as an input for a production function is problematic (because it invariably entails aggregation by value), here we ignore this difficulty and assume output is given by a production function f of labor “ L ” and “capital” K , so output is $f(L, K)$. Let the price of output be p , the price of labor be w , and the price of capital be r . Assume $L \geq 0$ and $K \geq 0$.

- (a) Suppose the objective of the firm is to maximize profit π .
- i. State the first-order conditions for a solution to the firm’s problem. State these in terms of L , K , and f and its derivatives, not solely in terms of the derivatives of π such as π'_K . Assume an interior solution.
 - ii. State the second-order sufficient conditions for a point satisfying those first-order conditions to be a solution to the firm’s problem. State these in terms of L , K , and f and its derivatives, not solely in terms of the derivatives of π such as π''_{KL} .
 - iii. Show that if the sufficient conditions for production function f to be strictly concave are satisfied, then the second-order conditions you derived in part (ii) are satisfied.
 - iv. If f has the form $L^\alpha K^\beta$ with $\alpha > 0$ and $\beta > 0$, what conditions on α and on β would ensure satisfaction of the sufficient conditions for f to be strictly concave?
 - v. What type of returns to scale does $f = L^\alpha K^\beta$ have if α and β satisfy the conditions which you derived in part (iv)?
- (b) Now suppose that the objective of the firm is not to maximize profit, but is instead to maximize something called “return to capital,” which is π/K . In the rest of this problem, assume $p = w = r = 1$ for simplicity. Write the production function as $f(L, K)$ as before, but do *not* assume that $f = L^\alpha K^\beta$.
- i. Show that the first-order conditions for a solution to the firm’s problem, assuming an interior solution, are

$$0 = \frac{f'_L - 1}{K} \quad \text{and}$$

$$0 = \frac{f'_K}{K} - \frac{f - L}{K^2}.$$

- ii. State the second-order sufficient conditions for a point satisfying those first-order conditions to be a solution to the firm's problem. State these in terms of L , K , and f and its derivatives, not solely in terms of the derivatives of π such as π''_{KL} . You do not have to otherwise simplify the expression you get.
- iii. Now *do* assume that the production function takes the Cobb-Douglas form, $f = L^\alpha K^\beta$ with $\alpha > 0$ and $\beta > 0$.

A. Use one of the first-order conditions to show that

$$L^{\alpha-1}K^\beta = \frac{1}{\alpha}$$

and use the other first-order condition to show that

$$L^{\alpha-1}K^\beta = \frac{1}{1-\beta}.$$

- B. If $\alpha + \beta < 1$, what do you conclude from the previous step about the assumption in part (b)(i) that the solution is interior?
- C. What is the amount of output at non-interior points?

Answer to Question 2, Final Exam, Fall 2022, Econ. 7005

a) i) $\pi = p f(L, K) - wL - rK$

F.O.C. $0 = \pi'_L = p f'_L - w$

$0 = \pi'_K = p f'_K - r$

ii) S.O.C. number of constraints $m=0$; $\nabla^2 \mathcal{L} = \nabla^2 \pi$. $n=2$ (#variables).

D_{2m+1}, \dots, D_{m+n} leading principal minors of $\nabla^2 \mathcal{L}$
 $(-1)^{m+1}$ then alternating signs

$$\pi''_{LL} = p f''_{LL} \quad \pi''_{LK} = p f''_{LK} \quad \pi''_{KK} = p f''_{KK}$$

$$\nabla^2 \mathcal{L} = \nabla^2 \pi = \begin{bmatrix} p f''_{LL} & p f''_{LK} \\ p f''_{LK} & p f''_{KK} \end{bmatrix} . \quad \begin{aligned} D_{2m+1} &= D_1, \text{ which should be} \\ (-1)^{m+1} &= -1, \text{ negative.} \end{aligned}$$

Then D_2 should be positive.

S.O.C.:

$$0 > p f''_{LL}$$

$$0 < p f''_{LL} \cdot p f''_{KK} - (p f''_{LK})^2$$

iii) Sufficient conditions for f to be strictly concave: D_r of $\nabla^2 f$ alternate in sign beginning with Θ .

$$\nabla^2 f = \begin{bmatrix} f''_{LL} & f''_{LK} \\ f''_{LK} & f''_{KK} \end{bmatrix} \quad \text{so we want} \quad 0 > f''_{LL}$$

$$0 < f''_{LL} f''_{KK} - (f''_{LK})^2. \quad \text{Mul-}$$

tiplying these inequalities by p (which is positive) yields the same two inequalities that are at the end of part (ii).

$$iv) f = L^\alpha K^\beta$$

$$f'_L = \alpha L^{\alpha-1} K^\beta$$

$$f'_K = \beta L^\alpha K^{\beta-1}$$

$$f''_{LL} = \alpha(\alpha-1) L^{\alpha-2} K^\beta$$

$$f''_{LK} = \alpha\beta L^{\alpha-1} K^{\beta-1}$$

$$f''_{KK} = \beta(\beta-1) L^\alpha K^{\beta-2}$$

Using (iii), we want

$$0 > f''_{LL} = \underbrace{\alpha(\alpha-1)}_{\oplus} \underbrace{L^{\alpha-2}}_{\ominus} \underbrace{K^\beta}_{\oplus}$$

This should be negative, so

$$\alpha-1 < 0$$

$$\boxed{\alpha < 1.}$$

Also, we want

$$0 < f''_L f''_{KK} - (f''_{LK})^2$$

$$0 < \alpha(\alpha-1) L^{\alpha-2} K^\beta \beta(\beta-1) L^\alpha K^{\beta-2} - \alpha^2 \beta^2 L^{2\alpha-2} K^{2\beta-2}$$

$$0 < (\alpha-1)(\beta-1) \alpha \beta L^{2\alpha-2} K^{2\beta-2} - \alpha^2 \beta^2 L^{2\alpha-2} K^{2\beta-2}.$$

Divide both sides by $L^{2\alpha-2} K^{2\beta-2} > 0$:

$$0 < (\alpha-1)(\beta-1) \alpha \beta - \alpha^2 \beta^2 = (\alpha\beta - \alpha - \beta + 1) \alpha \beta - \alpha^2 \beta^2.$$

Divide both sides by $\alpha \beta > 0$:

$$0 < \alpha\beta - \alpha - \beta + 1 - \alpha\beta = -\alpha - \beta + 1$$

$$\boxed{\alpha + \beta < 1.} \quad \text{So } \boxed{\beta < 1.}$$

$$v) f(\lambda L, \lambda K) = (\lambda L)^\alpha (\lambda K)^\beta = \lambda^{\alpha+\beta} L^\alpha K^\beta = \lambda^{\alpha+\beta} f(L, K)$$

and from part (iv), $\alpha + \beta < 1$, so f has decreasing returns to scale.

$$b) \max_{L,K} \frac{\pi}{K} = \max_{L,K} \frac{f(L,K) - L - K}{K} = \max_{L,K} \left[\underbrace{\frac{f(L,K) - L}{K}}_{\pi/K} - 1 \right]$$

$$i) \text{ F.O.C. } O = \frac{\partial}{\partial L} \frac{\pi}{K} = \frac{f'_L - 1}{K} \quad (1)$$

$$O = \frac{\partial}{\partial K} \frac{\pi}{K} = \frac{f'_{KK}}{K} - \frac{f - L}{K^2} \quad (2).$$

Optional: Note that (1) $\Rightarrow O = f'_L - 1$, which is the same as the $O = \pi'_L$ F.O.C. in (a)(i) when $p=w=1$. However, (2) is quite different from the $O = \pi'_{KK}$ F.O.C. in (a)(i).

$$ii) \text{ Symbolically, } \nabla^2 \mathcal{L} = \nabla^2 \left(\frac{\pi}{K} \right) = \begin{bmatrix} \left(\frac{\pi}{K} \right)''_{LL} & \left(\frac{\pi}{K} \right)''_{LK} \\ \left(\frac{\pi}{K} \right)''_{LK} & \left(\frac{\pi}{K} \right)''_{KK} \end{bmatrix} \text{ and the}$$

S.O.C. would be $O > \left(\frac{\pi}{K} \right)''_{LL}$

$$O < \left(\frac{\pi}{K} \right)''_{LL} \left(\frac{\pi}{K} \right)''_{KK} - \left[\left(\frac{\pi}{K} \right)''_{LK} \right]^2$$

as in part (a)(ii). From (1) and (2) :

$$\frac{\partial^2}{\partial L^2} \frac{\pi}{K} = \frac{f''_{LL}}{K}$$

$$\frac{\partial^2}{\partial K \partial L} \frac{\pi}{K} = \frac{f''_{LK}}{K} - \frac{f'_L - 1}{K^2}$$

$$\frac{\partial^2}{\partial K^2} \frac{\pi}{K} = \frac{f''_{KK}}{K} - \frac{f'_{KK}}{K^2} - \frac{f'_{KK}}{K^2} + 2 \frac{f - L}{K^3} = \frac{f''_{KK}}{K} - \frac{2f'_{KK}}{K^2} + 2 \frac{f - L}{K^3}.$$

So the S.O.C. are

$$O > \frac{f''_{LL}}{K}$$

$$O < \frac{f''_{LL}}{K} \left(\frac{f''_{KK}}{K} - \frac{2f'_K}{K^2} + 2 \frac{f-L}{K^3} \right) - \left[\frac{f''_{LK}}{K} - \frac{f'_{L-1}}{K^2} \right]^2.$$

[Optional: the first S.O.C. is like that of (a)(ii), but the second is quite different.]

$$\text{iii)} \quad f = L^\alpha K^\beta$$

$$f'_L = \alpha L^{\alpha-1} K^\beta$$

$$f'_K = \beta L^\alpha K^{\beta-1}$$

$$[\text{A}] \text{ Part (b)(i) equation (1)} : O = \frac{f'_{L-1}}{K} \Leftrightarrow 1 = f'_L = \alpha L^{\alpha-1} K^\beta$$

$$\frac{1}{\alpha} = L^{\alpha-1} K^\beta.$$

$$\text{Part (b)(i) equation (2)} : O = \frac{f'_K}{K} - \frac{f-L}{K^2}. \text{ Multiply by } K^2:$$

$$O = K f'_K - f + L$$

$$O = K \cdot \beta L^\alpha K^{\beta-1} - L^\alpha K^\beta + L$$

$$= \beta L^\alpha K^\beta - L^\alpha K^\beta + L$$

$$= (\beta-1) L^\alpha K^\beta + L$$

$$(1-\beta) L^\alpha K^\beta = L ; \text{ divide by } L :$$

$$(1-\beta) L^{\alpha-1} K^\beta = 1$$

$$L^{\alpha-1} K^\beta = \frac{1}{1-\beta}.$$

[B] From part (b) (iii) [A],

$$\frac{1}{\alpha} = L^{\alpha-1} K^\beta = \frac{1}{1-\beta}$$

So $\alpha = 1-\beta$ and $\alpha + \beta = 1$. So if $\alpha + \beta < 1$, the F.O.C.'s cannot hold.

So the solution must not be interior.

[C] At non-interior points, either $L=0$ or $K=0$, hence $f = L^\alpha K^\beta = 0$ at non-interior points.

Optimal: At non-interior points, $f=0$ and $\frac{\pi}{K} = \frac{0-L-K}{K} = -\frac{L}{K} - 1$. If $L=0$, this is -1 , and if $K=0$, this is $-\infty$. Using a computer, there are interior points making $\frac{\pi}{K} > 0$, so the non-interior points do not contain an optimum. Hence if $\alpha + \beta < 1$, no maximum of $\frac{\pi}{K}$ exists.

2018 Qualifying Exam Sec. 1 Qu. 1

1. **[20 points]** Consider a perfectly competitive firm which purchases labor l and machines m , and with these inputs produces an output in the amount of $\tilde{\chi} \cdot l \cdot m$, where $\tilde{\chi}$ is a random variable representing uncertain knowledge of the production function. (The notation is inspired by the fact that “ χ ” is pronounced in a similar way to the third and fourth letters in “technology.”) Output sells for a price p . Labor costs w per unit and the price of machines is p_m per unit.

Unlike the firms discussed in the textbook, this firm is constrained in how much it can spend on labor and on machines. It can only spend at most CR dollars on labor and on machines, where CR , which is exogenous to the firm, is the amount of credit extended to the firm by its lenders. Its total net costs are the amount it spends on hiring labor, the amount it spends on buying machines, and $r \cdot CR$, where $r > 0$ is the interest rate and hence $r \cdot CR$ is the amount of interest which the firm has to pay back to its lenders. (The loan principal, CR , is both part of total revenue and part of total cost, so it cancels out of the expression for profit.)

- (a) State mathematically the firm’s optimization problem.
- (b) Suppose the owner of the firm believes $\tilde{\chi}$ will take the value one with probability one. Under this assumption, find the optimal amount of labor l^* and machines m^* which the owner will hire. You need not check second-order conditions. Assume the spending constraint is satisfied with equality.
- (c) Prove that the optimized level of profit, π^* , given the firm owner’s expectation that $\tilde{\chi} \equiv 1$, is

$$\pi^* = \left[\frac{p CR}{4p_m w} - (1 + r) \right] CR .$$

- (d) Suppose the owner’s opportunity cost of running this firm is denoted by “ OOC .” What inequality must be satisfied in order for the owner to decide to operate this firm?
- (e) If $OOC = 0$, what values of CR are necessary to induce the owner to operate this firm?
- (f) Assuming that the owner rationally decides that he should operate the firm (rather than engaging in an alternative activity), prove that π^* is increasing in CR . Explain in non-mathematical

terms what this implies for the demand for credit by the owner of the firm.

- (g) Suppose the supplier of credit to the firm is a bank. The bank creates credit “out of thin air” (out of nothing); it incurs no cost in creating credit. (This is because the loan expands both the asset and the liability side of the bank’s balance sheet, but you do not need to know that.) The bank is not as optimistic as the firm owner about the value of $\tilde{\chi}$. Instead, the bank believes $\tilde{\chi}$ will be zero with probability $pr_b > 0$ (“bad probability”) and will be one with probability $1 - pr_b < 1$. Assume that whether $\tilde{\chi}$ is 0 or 1 will only be known after the bank lends CR to the firm and after the firm has used CR to purchase l and m .

Argue that in “the bad state of the world” (that is, when $\tilde{\chi}$ turns out to be zero), the firm will be bankrupt and the bank will lose CR dollars.

- (h) Suppose that the bank maximizes

$$pr_b \ln[\mathcal{W}_0 - CR] + (1 - pr_b) \ln[\mathcal{W}_0 + r CR]$$

where $\mathcal{W}_0 > 0$ is the bank’s initial amount of wealth. (This comes from assuming the bank is risk-averse and maximizes its expected utility, but you do not need to know that.)

- i. Argue from the bank’s objective function that the bank’s optimal level of CR , denoted by CR^* , satisfies $CR^* < \mathcal{W}_0$.

- ii. Prove that

$$CR^* = \left(1 - pr_b - \frac{pr_b}{r}\right) \mathcal{W}_0.$$

You do not have to check second-order conditions.

- iii. Considering parts (h)(ii) and (f), would the firm owner prefer to borrow from a bank with large \mathcal{W}_0 or from a bank with small \mathcal{W}_0 ?

- iv. Prove that $CR^* > 0$ and $r > 0$ imply

$$pr_b < \frac{r}{1+r} < r.$$

- (i) Show that if the owner rationally decides that he should operate the firm (rather than engaging in an alternative activity), and if $OOC = 0$, then

$$\left(1 - pr_b - \frac{pr_b}{r}\right) \mathcal{W}_0 > (1+r) \frac{4wp_m}{p}.$$

(j) The “expected value” (not “expected utility”) of the bank is

$$EV_{bk} = pr_b (\mathcal{W}_0 - CR^*) + (1 - pr_b)(\mathcal{W}_0 + r CR^*).$$

- i. Make a conjecture about the sign of $\partial EV_{bk} / \partial pr_b$. As always, explain your answer.
 - ii. [If you are running short on time, I recommend skipping this part because it takes a while and it's not worth very much.] Prove that the conjecture you just made is correct. Hint 1: it is easier to work with $\partial(EV_{bk}/\mathcal{W}_0) / \partial pr_b$. Hint 2: use (h)(iv).
- (k) Briefly argue *against* the following opinion: “in this model, it is easy to define ‘the rate of profit,’ and ‘the rate of profit’ is equal to ‘the return on lending money’.” (There is more than one correct way of answering this question.)

Answers to Question 1, Section 1, Summer 2018 Micro Qualifying Exam

a) profit is $p \underbrace{\chi l m - w l}_{\text{total revenue}} - \underbrace{p_m m - r}_{\text{- total cost}} CR$

The spending constraint is $w l + p_m m \leq CR$.

The firm's problem is to maximize the above expression for profit subject to the above spending constraint.

b) With $\tilde{\chi} = 1$,

$$\mathcal{L} = p l m - w l - p_m m - r CR + \lambda (CR - w l - p_m m).$$

First order conditions:

$$0 = CR - w l - p_m m, \text{ the constraint (equals } \partial \mathcal{L} / \partial \lambda \text{)}$$

$$0 = \mathcal{L}'_l = p_m - w - \lambda w \Rightarrow p_m = (1+\lambda)w \Rightarrow m = (1+\lambda) \frac{w}{p}$$

$$0 = \mathcal{L}'_m = p l - p_m - \lambda p_m \Rightarrow p l = (1+\lambda)p_m \Rightarrow l = (1+\lambda) \frac{p_m}{p}$$

$$CR = w l + p_m m$$

$$= w (1+\lambda) \frac{p_m}{p} + p_m (1+\lambda) \frac{w}{p}$$

$$= (1+\lambda) \left[\frac{w p_m}{p} + \frac{p_m w}{p} \right] = 2(1+\lambda) \frac{w p_m}{p}$$

So $\frac{1+\lambda}{P} = \frac{CR}{2w p_m}$, and substituting into the expressions for m and l ,

$$m^* = \frac{CR}{2w p_m} w = \frac{CR}{2P_m}$$

$$l^* = \frac{CR}{2w p_m} p_m = \frac{CR}{2W}$$

$$\begin{aligned}
 c) \pi^* &= p \ell^* m^* - w \ell^* - p_m m^* - r CR \\
 &= p \frac{\frac{CR}{2}}{2P_m} \frac{\frac{CR}{2}}{2W} - w \frac{\frac{CR}{2}}{2W} - P_m \frac{\frac{CR}{2}}{2P_m} - r CR \\
 &= \frac{p CR^2}{4P_m W} - \frac{CR}{2} - \frac{CR}{2} - r CR = \frac{p CR^2}{4P_m W} - (1+r) CR \\
 &= \left[\frac{p CR}{4P_m W} - (1+r) \right] CR.
 \end{aligned}$$

d) $\pi^* \geq 0$ OOC with π^* as m(c).

e) Substituting (c)'s last expression for π^* , into (d),

$$\left[\frac{p CR}{4P_m W} - (1+r) \right] CR \geq 0.$$

Clearly $CR > 0$, else the firm cannot buy any ℓ or m . So

$$\frac{p CR}{4P_m W} - (1+r) \geq 0$$

$$\frac{p CR}{4P_m W} \geq 1+r$$

$$CR \geq (1+r) \frac{4P_m W}{p}$$

Optional interpretation: the higher input prices P_m, W , and (in some sense) r are, and the lower output price p is, the more credit is required.

f)

Using the last expression from (c),

$$\frac{\partial \pi^*}{\partial CR} = \frac{P}{4p_m w} \cdot CR + \left[\frac{P}{4p_m w} - (1+r) \right] \cdot 1$$

$$= \frac{P}{4p_m w} \cdot CR + \frac{\pi^*}{CR} > 0 \text{ since both terms are positive}$$

(given (d) for π^* and given the reasoning in (e) for CR).

The owner of this firm has an infinite demand for credit.

Optional: This is probably because the production function has increasing returns to scale while output price is a constant, so the firm wants to produce an infinite amount.

g) If $X = 0$, output is zero, so total revenue is zero, but total costs are not zero, so $\pi^* < 0$: the firm is bankrupt, there is no money to pay back the bank, so the bank loses CR .

h) i) If CR exceeded W_0 then the first term in the bank's objective function, $pr_b \ln(W_0 - CR)$, would be $pr_b \ln(\text{a negative number})$, which is not defined.

ii) Maximizing the bank's objective function with respect to CR

yields $0 = \frac{-pr_b}{W_0 - CR} + \frac{1 - pr_b}{W_0 + r CR} - r \Rightarrow$

$$O = \frac{-pr_b (W_0 + rCR) + (r - rpr_b)(W_0 - CR)}{(W_0 - CR)(W_0 + rCR)} . \quad \text{The denominator is not zero, so}$$

$$O = -pr_b (W_0 + rCR) + (r - rpr_b)(W_0 - CR)$$

$$= -pr_b W_0 \cancel{- pr_b r CR} + r W_0 - r CR - r pr_b W_0 + \cancel{r pr_b CR}$$

$$rCR = -pr_b W_0 + r W_0 - r pr_b W_0$$

$$CR = \left(-\frac{pr_b}{r} + 1 - pr_b \right) W_0 = \left(1 - pr_b - \frac{pr_b}{r} \right) W_0.$$

(iii) From (f), the firm wants as much CR as it can get. From (h)(ii), the bigger W_0 , the bigger CR^* , the amount of credit extended by the bank, is. So the firm owner would want to borrow from a bank with a large W_0 .

(iv) From (h)(ii), and (h)'s assumption that $W_0 > 0$, $CR^* > 0$ implies

$$\text{that } 1 - pr_b - \frac{pr_b}{r} > 0$$

$$r - rpr_b - pr_b > 0$$

$$r > (1+r) pr_b$$

$$\frac{r}{1+r} > pr_b .$$

$$\text{Whenever } r > 0, \frac{r}{1+r} < \frac{r}{1+0} = r .$$

Optional interpretation: if $pr_b > r$, the bank will not lend to the firm.

i)

From (e), for the owner to want to operate the firm,

$$CR \geq (1+r) \frac{4pmw}{P}.$$

From (h)(ii), $CR^* = (1 - pr_b - \frac{pr_b}{r}) W_0$. Substituting this in for CR proves the inequality.

j)

i) Presumably when the probability of the bad state of the world increases, the expected value of the bank will fall. The bank can only make money by making loans, and at high pr_b , it will not make many loans.

ii) Taking CR^* from (h)(ii),

$$\begin{aligned} EV_{bk} &= pr_b \left[W_0 - (1 - pr_b - \frac{pr_b}{r}) W_0 \right] \\ &\quad + (1 - pr_b) \left[W_0 + r (1 - pr_b - \frac{pr_b}{r}) W_0 \right] \end{aligned}$$

$$\begin{aligned} \frac{EV_{bk}}{W_0} &= pr_b \left[1 - 1 + pr_b + \frac{pr_b}{r} \right] + (1 - pr_b) \left[1 + r - r pr_b - pr_b \right] \\ &= pr_b^2 \left(1 + \frac{1}{r} \right) + 1 + r - r pr_b - pr_b - r pr_b + r pr_b^2 + pr_b^2 \\ &= pr_b^2 \left(1 + \frac{1}{r} + r + 1 \right) + pr_b (-r - 1 - 1 - r) + 1 + r \\ &= pr_b^2 \left(2 + r + \frac{1}{r} \right) - pr_b (2 + 2r) + 1 + r \\ &= (2 + r + \frac{1}{r}) pr_b^2 - 2(1+r) pr_b + 1 + r \end{aligned}$$

$$\frac{\partial (EV_{bk}/W_0)}{\partial pr_b} = 2 \left(2 + r + \frac{1}{r} \right) pr_b - 2(1+r). \text{ We would like to}$$

Verify that this is negative:

$$2(2+r+\frac{1}{r})pr_b - 2(1+r) \stackrel{?}{<} 0$$

$$(2+r+\frac{1}{r})pr_b - (1+r) \stackrel{?}{<} 0$$

$$pr_b \stackrel{?}{<} \frac{1+r}{2+r+\frac{1}{r}} = \frac{r+r^2}{2r+r^2+1} = \frac{r^2+r}{r^2+2r+1}$$

$$= \frac{r(r+1)}{(r+1)^2} = \frac{r}{r+1}$$

which is true from (i)(iv).

k) The "return on lending money" (extending credit) is r .

Opinions will vary about how to define "the rate of profit" in this model.

Presumably its numerator is $p_X l_m - w_l - p_m m$ or

$$p_X l_m - w_l - p_m m - r CR;$$

The first is of more interest to the bank, the second to the firm's owner.

The denominator of "the rate of profit" might be $p_m m$, the value of the firm's machines (its "fixed capital"), or it might be CR .

It is not easy to decide what the most useful definition of "the rate of profit" is, and whatever it is, it does not seem to be equal to r .

2013 Qualifying Exam Sec. 3 Qu. 1

1. **[10 points]** Consider a perfectly competitive firm which purchases labor l and machines m , and with these inputs produces an output in the amount of $f(l, m)$. Output sells for a price p . Labor costs w per unit and the price of machines is p_m per unit.

Unlike the firms discussed in the textbook, this firm is constrained in how much it can spend on labor and on machines. It can only spend at most CR dollars on labor and on machines, where CR , which is exogenous to the firm, is the amount of credit extended to the firm by its lenders. Its total net costs are the amount it spends on hiring labor, the amount it spends on buying machines, and $r \cdot CR$, where $r > 0$ is the interest rate and hence $r \cdot CR$ is the amount of interest which the firm has to pay back to its lenders. (The loan principal, CR , is both part of total revenue and part of total cost, so it cancels out of the expression for profit.)

- Show that the Hessian of the Lagrangian for the profit-maximization problem is

$$\nabla^2 \mathcal{L} = \begin{bmatrix} 0 & -pf'_l + \lambda w & -pf'_m + \lambda p_m \\ -pf'_l + \lambda w & pf''_{ll} & pf''_{lm} \\ -pf'_m + \lambda p_m & pf''_{ml} & pf''_{mm} \end{bmatrix}.$$

Hint: Make use of the first-order conditions.

- Show that if the spending constraint is not binding, the second-order sufficient conditions for a maximum profit for the firm can be expressed in terms of a rather simple property of the production function, and state what the “rather simple property” is.

Answer to Question 1, Section 3, Summer 2018 Micro Qualifying Exam

Maximize $\pi = p f(l, m) - w l - p_m m - r CR$ s.t. $w l + p_m m = CR$.

a) $\mathcal{L} = p f(l, m) - w l - p_m m - r CR + \lambda [CR - w l - p_m m]$

$$0 = \partial \mathcal{L} / \partial \lambda = CR - w l - p_m m$$

$$0 = \partial \mathcal{L} / \partial l = p f'_l - w - \lambda w$$

$$0 = \partial \mathcal{L} / \partial m = p f'_m - p_m - \lambda p_m \text{ are the first order conditions.}$$

$$\nabla^2 \mathcal{L} = \begin{bmatrix} \mathcal{L}_{\lambda\lambda}'' & \mathcal{L}_{\lambda e}'' & \mathcal{L}_{\lambda m}'' \\ \mathcal{L}_{e\lambda}'' & \mathcal{L}_{ee}'' & \mathcal{L}_{em}'' \\ \mathcal{L}_{m\lambda}'' & \mathcal{L}_{me}'' & \mathcal{L}_{mm}'' \end{bmatrix} = \begin{bmatrix} 0 & -w & -p_m \\ -w & p f''_{ee} & p f''_{em} \\ -p_m & p f''_{me} & p f''_{mm} \end{bmatrix}.$$

From the second first-order condition,

$$-w = -p f'_e + \lambda w$$

and from the third first-order condition,

$$-p_m = -p f'_m + \lambda p_m.$$

So

$$\nabla^2 \mathcal{L} = \begin{bmatrix} 0 & -p f'_e + \lambda w & -p f'_m + \lambda p_m \\ -p f'_e + \lambda w & p f''_{ee} & p f''_{em} \\ -p f'_m + \lambda p_m & p f''_{me} & p f''_{mm} \end{bmatrix}.$$

b) The second-order sufficient conditions for a maximum is that

D_{2m+1} of $\nabla^2 \mathcal{L}$, which is D_3 of $\nabla^2 \mathcal{L}$, has the same sign as

$$(-1)^{m+1} = (-1)^{1+1} = (-1)^2 > 0.$$

If the spending constraint is not binding then $\lambda = 0$ and

$$\begin{aligned} D_3 \text{ of } \nabla^2 \mathcal{L} &= \begin{vmatrix} 0 & -pf'_e & -pf'_m \\ -pf'_e & pf''_{ee} & pf''_{me} \\ -pf'_m & pf''_{me} & pf''_{mm} \end{vmatrix} \\ &= (-p)(-p) \underbrace{\begin{vmatrix} 0 & f'_e & f'_m \\ f'_e & f''_{ee} & f''_{me} \\ f'_m & f''_{me} & f''_{mm} \end{vmatrix}}_{\oplus}. \end{aligned}$$

For the second-order condition to hold, this determinant should be positive.

This determinant is the "bordered Hessian" of f . If f is quasiconcave

then this determinant is positive (the determinant is denoted δ_2 in the prerequisite notes). So if the spending constraint is not binding, f being quasiconcave is sufficient for this problem's second-order conditions to hold.

optional

If the spending constraint is binding, however, then $\lambda \neq 0$ and there is no simple relationship between the second-order conditions and quasiconcavity of f .

3. Suppose a firm uses inputs x_1 and x_2 —which we shall collectively call \mathbf{x} —to produce output according to a production function $f(\mathbf{x})$. Further suppose:

- the firm takes the price of output as a constant, p ;
- the firm takes the price of x_1 as a constant, w ; and
- the firm takes the price of x_2 , called “ q ,” as a function of the amount of input 2 which it buys: $q(x_2)$. Suppose the more of x_2 the firm wishes to purchase, the higher the price it must pay to buy x_2 .

Finally, suppose the firm's second-order conditions for profit maximization are fulfilled.

Exam 1

1997

Question 3

(2)

- (a) Find the expression for how the firm's purchases of x_1 vary when x_1 's price varies.
- (b) Under what situations is this firm's input demand curve for input 1 downward-sloping?
- (c) How does your answer to part (b) change if the supply curve which the firm faces for input 2, namely $q(x_2)$, is linear?

③ $\pi = p f(x) - w x_1 - q(x_2) x_2$. To maximize profit, the first-order conditions are

$$0 = \frac{\partial \pi}{\partial x_1} = p \frac{\partial f}{\partial x_1} - w$$

Exam 1
1997

$$0 = \frac{\partial \pi}{\partial x_2} = p \frac{\partial f}{\partial x_2} - \frac{dq}{dx_2} x_2 - q$$

Answer 3

Totally differentiating the first-order conditions and setting $d\rho = 0$:

$$0 = \frac{\partial^2 \pi}{\partial x_1^2} dx_1 + \frac{\partial^2 \pi}{\partial x_2 \partial x_1} dx_2 + \frac{\partial^2 \pi}{\partial x_1 \partial w} dw$$

$$0 = \frac{\partial^2 \pi}{\partial x_1 \partial x_2} dx_1 + \frac{\partial^2 \pi}{\partial x_2^2} dx_2 + \frac{\partial^2 \pi}{\partial x_2 \partial w} dw$$

Exam 1
1997

Answer 3 cont...

$$0 = P \frac{\partial^2 f}{\partial x_1^2} dx_1 + P \frac{\partial^2 f}{\partial x_2 \partial x_1} dx_2 - dw$$

$$0 = P \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_1 + \left(P \frac{\partial^2 f}{\partial x_2^2} - \frac{d^2 g}{dx_2^2} x_2 - \frac{dg}{dx_2} - \frac{dg}{dx_2} \right) dx_2 - 0 dw.$$

Using abbreviations to denote differentiation, one has

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} dw = \begin{bmatrix} P f_{11}'' & P f_{12}'' \\ P f_{21}'' & P f_{22}'' - g'' x_2 - 2g' \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

$$a) \frac{dx_1}{dw} = \frac{\begin{vmatrix} 1 & P f_{12}'' \\ 0 & P f_{22}'' - g'' x_2 - 2g' \end{vmatrix}}{\begin{vmatrix} \pi_{11}'' & \pi_{12}'' \\ \pi_{21}'' & \pi_{22}'' \end{vmatrix}} = \frac{P f_{22}'' - x_2 g'' - 2g'}{\pi_{11}'' \pi_{22}'' - (\pi_{12}'')^2}$$

b) The second-order conditions for a maximum in this problem are

$$\pi_{11}'' < 0 \text{ and } \begin{vmatrix} \pi_{11}'' & \pi_{12}'' \\ \pi_{21}'' & \pi_{22}'' \end{vmatrix} > 0. \text{ So the denominator of } \frac{dx_1}{dw} \text{ is}$$

positive. Hence a downward-sloping demand curve for input 1 occurs when

$$pf_{22}'' - x_2 f'' - 2f'_1 < 0.$$

Note that $f_{22}'' < 0$; this is because we require $0 > \pi_{11}'' = pf_{11}''$, which means that $f_{11}'' < 0$, and the numbering of inputs as "1" and "2" is arbitrary.

Also note that $f'_1 > 0$ because the problem says that "the more of x_2 the firm wishes to purchase, the higher the price it must pay to buy x_2 ".

c) If f is linear then $f'' = 0$, and

$$\text{sign}\left(\frac{dx_1}{dw}\right) = \text{sign}(pf_{22}'' - 0 - 2f'_1) < 0,$$

↑ ↑
 ⊖ ⊕

EXAM 1
1997
Answer 3 cont...

meaning that the demand curve for input 1 is downward-sloping for sure.

Optional Note: If the firm is a price-taker, Varian shows that input demand curves are downward-sloping. This problem shows that if the firm is not a price-taker for input 2, then one cannot guarantee that the firm's demand curve for input 1 is downward-sloping, unless the supply curve which the firm faces for input 2 is linear.

2. [8 points]

- (a) A monopolist produces output “ x ,” faces a demand curve $p(x, \alpha)$, and has a cost function $c(x)$. The parameter “ α ” shifts consumers’ demand for his product up, and the monopolist can choose α , but the cost to the monopolist of α is 5α .

Implicitly find the monopolist’s optimal choices.

- (b) If, in part (a),

$$p(x, \alpha) = 10\alpha - 3x \quad \text{and}$$
$$c(x) = x^2,$$

are the second-order conditions satisfied when the first-order conditions are?

Summer 2013 Qualifying Exam Section 2 Question 2

Section 2 Question 2.

a) $\pi = p(x, \alpha)x - c(x) - 5\alpha$ with $\frac{\partial}{\partial \alpha} p(x, \alpha) > 0$, so that π in α cause an \uparrow in p (that is, in demand), all else equal.

$$\left. \begin{aligned} 0 = \pi'_x &= \frac{\partial p}{\partial x} x + p(x, \alpha)(1) - c'(x) \\ 0 = \pi'_\alpha &= \frac{\partial p}{\partial \alpha} x - 5. \end{aligned} \right\} \text{These implicitly define } x^* \text{ and } \alpha^*.$$

For the second-order sufficient conditions:

$$\begin{aligned} \pi''_{xx} &= \frac{\partial^2 p}{\partial x^2} x + \frac{\partial p}{\partial x}(1) + \frac{\partial p}{\partial x} - c''(x) \\ &= p''_{xx} + 2p'_x - c'' \end{aligned}$$

$$\pi''_{x\alpha} = \frac{\partial^2 p}{\partial x \partial \alpha} x + \frac{\partial p}{\partial \alpha} = p''_{x\alpha} x + p'_\alpha$$

$$(\text{Optional } \pi''_{\alpha x} = \frac{\partial^2 p}{\partial \alpha \partial x} x + \frac{\partial p}{\partial x} = p''_{\alpha x} x + p'_\alpha \stackrel{\curvearrowright}{=} \text{as expected.})$$

$$\pi''_{\alpha\alpha} = \frac{\partial^2 p}{\partial \alpha^2} x = p''_{\alpha\alpha} x.$$

$$\nabla^2 \mathcal{L} = \nabla^2 \pi(x, \alpha) = \begin{bmatrix} \pi''_{xx} & \pi''_{x\alpha} \\ \pi''_{x\alpha} & \pi''_{\alpha\alpha} \end{bmatrix}$$

$$\text{S.O.C. for a maximum: } \pi''_{xx} < 0$$

$$\pi''_{xx} \pi''_{\alpha\alpha} - (\pi''_{x\alpha})^2 > 0.$$

b)

$$\begin{array}{ll}
 p(x,\alpha) = 10\alpha - 3x & c(x) = x^2 \\
 p'_x = -3 & c'(x) = 2x \\
 p''_{xx} = 0 & p''_{\alpha\alpha} = 0 & p''_{x\alpha} = 0 & c''(x) = 2
 \end{array}$$

From the last part of (a) (the optional part of (a) — otherwise the calculations need to be done in this part) :

$$\pi''_{xx} = p''_{xx} + 2p'_x - c'' = 0 + 2(-3) - 2 = -8$$

$$\pi''_{x\alpha} = p''_{\alpha x} x + p'_\alpha = 0 + 10 = 10$$

$$\pi''_{\alpha\alpha} = 0.$$

S.O.C. for a maximum :

$$0 > \pi''_{xx} = -8 \quad \text{OK}$$

$$0 < \pi''_{xx} \pi''_{\alpha\alpha} - (\pi''_{x\alpha})^2 = (-8)(0) - (10)^2 = -100, \text{ not OK.}$$

So the S.O.C. for a maximum fail.

Ideally, one would note here that this is a failure of the second-order sufficient conditions for a maximum ; that the second-order necessary conditions for a maximum are $\pi''_{xx} \leq 0$, $\pi''_{\alpha\alpha} \leq 0$, and $\pi''_{xx} \pi''_{\alpha\alpha} - (\pi''_{x\alpha})^2 \geq 0$; hence here the SONC are $-8 \leq 0$, $0 \leq 0$, and $-100 \geq 0$; failure of the last condition implies failure of the SONC, so this is not a maximum.

OPTIONAL. Further analysis of Part (b) progresses as follows. In Part (b),

$$\pi = (10\alpha - 3x)x - x^2 - 5\alpha$$

$$= 10\alpha x - 3x^2 - x^2 - 5\alpha$$

$$= 10\alpha x - 4x^2 - 5\alpha.$$

$$\text{F.O.C. : } \begin{aligned} 0 &= \pi'_x = 10\alpha - 8x \Rightarrow 0 = 10\alpha - 8\left(\frac{1}{2}\right) = 10\alpha - 4 \Rightarrow 4 = 10\alpha \\ &\quad \uparrow \\ 0 &= \pi'_\alpha = 10x - 5 \Rightarrow 5 = 10x \Rightarrow x = \frac{1}{2} \quad \frac{2}{5} = \frac{4}{10} = \alpha. \end{aligned}$$

So $x = \frac{1}{2}, \alpha = \frac{2}{5}$ satisfies the F.O.C. and is the candidate point for a maximum.

$$\begin{aligned} \pi\left(x = \frac{1}{2}, \alpha = \frac{2}{5}\right) &= 10\left(\frac{2}{5}\right)\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right)^2 - 5\left(\frac{2}{5}\right) \\ &= 2 - 4\left(\frac{1}{4}\right) - 2 = -1. \end{aligned}$$

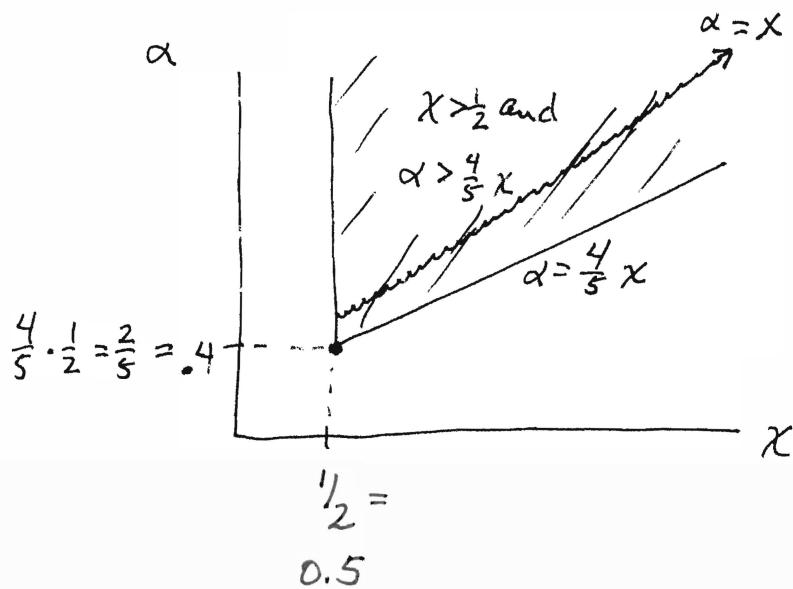
However, this is not a max. min since by inspection π at $x=0, \alpha=0$ is zero, which is bigger than -1.

It turns out that $(0,0)$, which is a corner solution, is not the maximum either. Consider

$$\pi'_x = 10\alpha - 8x$$

$$\pi'_\alpha = 10x - 5.$$

For all $x > \frac{1}{2}$, $\pi'_\alpha > 0$. For any x , we can also make $\pi'_x > 0$ by imposing $0 < 10\alpha - 8x \Leftrightarrow 8x < 10\alpha \Leftrightarrow \frac{8}{10}x < \alpha \Leftrightarrow \alpha > \frac{4}{5}x$. The area of $x > \frac{1}{2}$ and $\alpha > \frac{4}{5}x$, that is, the area of $\pi'_x > 0$ and $\pi'_\alpha > 0$, looks like



In this hatched area, $\pi'_x > 0$ and $\pi'_{\alpha} > 0$, so π_x and π_{α} will $\uparrow \pi$. How high can π get? Can it exceed $\pi(0,0) = 0$?

To investigate, think about the line where $\alpha = x$. Along this line,

$$\begin{aligned}\pi(x, x) &= 10x^2 - 4x^2 - 5x = 6x^2 - 5x \\ &= (6x - 5)x, \text{ and}\end{aligned}$$

$$\lim_{x \rightarrow \infty} \pi(x, x) = \lim_{x \rightarrow \infty} 6x \cdot x = \lim_{x \rightarrow \infty} 6x^2 = \infty.$$

So at least along this line, profit is unbounded. This indicates a flaw in the problem formulation, at least for large α and x .

621 Portion of Firm Qualifying Exam
1996

Req. Question

Required Question:

(2)

Question 3. Suppose a firm uses two inputs x_1 and x_2 to produce output y according to the production function $y = f(x_1, x_2)$.

The firm is a monopsony purchaser of input 1; the supply curve of input 1 is $w_1 = ax_1$ where w_1 is the price of input 1 and a is a constant.

The firm is a competitive purchaser of input 2. The price of input 2 is w_2 .

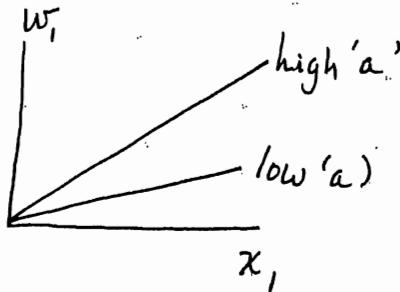
The firm is a monopolist. The demand curve for its output is $p = 10 - y$ where p is the price of its output.

- a) Guess at the sign of $\partial y / \partial a$. Explain the economic interpretation behind your guess.
- b) Calculate $\partial y / \partial a$ and find out as much as you can about its sign. (Hint: If you have calculated $\partial y / \partial a$ but are having trouble determining the sign, try to express the troublesome term as a function of the firm's marginal revenue. You should know the sign of the firm's marginal revenue.)

Answers to 621 Portion of Micro Qualifying Exam, 1996

Required Question

a) Supply Curve of Input 1:



Qualifying Exam
1996
Req. Answer

When $a \uparrow$, $w_i \uparrow$ holding x_i constant. Typically, when an input price like w_i rises, output will fall — or at least that is a good guess. That would mean $\frac{\partial y}{\partial a} < 0$.

b) total revenue = price · output

$$= (10 - y) y = [10 - f(x_1, x_2)] f(x_1, x_2) \text{ or } (10 - f) f \text{ for short.}$$

$$\text{total cost} = w_1 x_1 + w_2 x_2 = (a x_1) x_1 + w_2 x_2 = a x_1^2 + w_2 x_2.$$

$$\text{So profit } \pi(x_1, x_2) = (10 - f) f - a x_1^2 - w_2 x_2.$$

Let $\pi_1 = \frac{\partial \pi}{\partial x_1}$, $\pi_2 = \frac{\partial \pi}{\partial x_2}$, and let other subscripts of π similarly denote partial differentiation. The first-order conditions for profit maximization are

$$0 = \pi_1 \quad (1)$$

$$0 = \pi_2 \quad (2)$$

Taking the total differential of (1) and (2) yields

$$0 = \pi_{11} dx_1 + \pi_{12} dx_2 + \pi_{1a} da \quad (3)$$

$$0 = \pi_{21} dx_1 + \pi_{22} dx_2 + \pi_{2a} da. \quad (4)$$

From the equation on p. 1 for $\pi(x_1, x_2)$, we have

$$\pi_{1a} = \frac{\partial}{\partial x_1} \frac{\partial \pi}{\partial a} = \frac{\partial}{\partial x_1} (-x_1^2) = -2x_1, \quad (5)$$

$$\pi_{2a} = \frac{\partial}{\partial x_2} \frac{\partial \pi}{\partial a} = \frac{\partial}{\partial x_2} (-x_1^2) = 0, \quad (6)$$

so the system (3)-(4) becomes

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} + \begin{bmatrix} -2x_1 \\ 0 \end{bmatrix} da \Rightarrow \quad (7)$$

$$\begin{bmatrix} 2x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \begin{bmatrix} dx_1/da \\ dx_2/da \end{bmatrix}. \text{ Let } A = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}. \quad (8)$$

By Cramer's Rule,

$$\frac{dx_1}{da} = \frac{\begin{vmatrix} 2x_1 & \pi_{12} \\ 0 & \pi_{22} \end{vmatrix}}{|A|} = \frac{2x_1 \pi_{22}}{|A|} \quad \text{and} \quad \frac{dx_2}{da} = \frac{\begin{vmatrix} \pi_{11} & 2x_1 \\ \pi_{21} & 0 \end{vmatrix}}{|A|} = \frac{-2x_1 \pi_{21}}{|A|}.$$

Since $f = f(x_1, x_2) = y$, we have Qualifying Exam
1996

(9)(10)

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 \quad \text{Req. Answer Cont...} \quad (11)$$

$$\frac{\partial y}{\partial a} = \frac{\partial f}{\partial a} = \frac{\partial f}{\partial x_1} \frac{dx_1}{da} + \frac{\partial f}{\partial x_2} \frac{dx_2}{da}$$

\leftarrow (a key step, connecting the change in output with the changes in the two inputs) (12)

$$= f_1 \frac{2x_1 \pi_{22}}{|A|} + f_2 \frac{(-2x_1 \pi_{21})}{|A|} \quad \text{from (9) and (10) and}$$

using subscripts on f to denote its partial derivatives. So

$$\frac{\partial y}{\partial a} = \frac{2x_1}{|A|} \left[f_1 \pi_{22} - f_2 \pi_{21} \right]$$

(13)

|A| is + and π_{22} is - due to A being negative semidefinite for second-order conditions for a max.

$\Rightarrow \frac{\partial y}{\partial a} = + [\Theta - + \pi_{21}]$.

So if $\pi_{21} > 0$, the whole expression would have been negative, as guessed in part (a).

Optional: $\pi_1 = \frac{1}{2} f_1 f + (10-f)f_1 - 2ax_1$

$$= -2f_1 f + 10f_1 - 2ax_1$$

$$\begin{aligned} \pi_{21} &= -2f_{12}f - 2f_1 f_2 + 10f_{12} \\ &= (10-2f)f_{12} - 2f_1 f_2 \end{aligned}$$

$$= MR \cdot f_{12} - 2f_1 f_2 \quad (\text{since total revenue } = (10-f)f = 10f - f^2, \text{ marginal revenue is } 10-2f)$$

Qualifying Exam
1996

Req. Answer Cont...

So a sufficient (but not necessary) condition for $\frac{\partial y}{\partial a} < 0$ is that

$$0 < \pi_{21} = MR \cdot f_{12} - 2f_1 f_2 \Rightarrow \underbrace{2f_1 f_2}_{+} < \underbrace{MR \cdot f_{12}}_{+} \quad ?$$

MR is always positive because if it weren't, reducing output would both increase revenues and decrease total costs — so the position couldn't have maximized profit.

Very Optional: From (9), (10), and the signs in (13), we see that (12) is

$$\frac{\partial y}{\partial a} = \frac{\overset{+}{\partial f}}{\partial x_1} \frac{\overset{\ominus}{dx_1}}{\partial a} + \frac{\overset{+}{\partial f}}{\partial x_2} \frac{\overset{?}{dx_2}}{\partial a}$$

The negative sign of dx_1/da is expected

because 'a' is related to the price of x_1 , so when $a \uparrow$, one guesses that $x_1 \downarrow$.

The only ambiguity is how an \uparrow in 'a' affects x_2 .

Qualifying Exam
1996

Req. Answer Cont...

Two Possibly Correct Alternatives to the Long Problem.

- A worse way to solve the problem is: write production costs as $c(a, w_2, y)$.

The firm wants to maximize $\pi(y) = (10-y)y - c(a, w_2, y)$.

The first-order condition is $0 = \pi_y$ using subscripts to denote differentiation.

Take the total differential: $0 = \pi_{yy} dy + \pi_{ya} da \Rightarrow$

$$\frac{dy}{da} = \frac{-\pi_{ya}}{\pi_{yy}}. \text{ Now } \pi_{yy} < 0 \text{ from second-order conditions, and}$$

$$\pi_{ya} = \frac{\partial}{\partial y} \pi_a = \frac{\partial}{\partial y} \frac{-\partial c}{\partial a} = -\frac{\partial^2 c}{\partial y \partial a}. \text{ Therefore}$$

$$\text{sign}\left(\frac{dy}{da}\right) = \text{sign}\left(-\frac{\partial^2 c}{\partial y \partial a}\right).$$

Qualifying Exam
1996

Reg. Answer Cont...

To minimize costs, the firm must

$$\min_x a x_1^2 + w_2 x_2 \text{ s.t. } f(x_1, x_2) = y.$$

$$\mathcal{L} = a x_1^2 + w_2 x_2 + \lambda [f(x_1, x_2) - y].$$

$$\text{Envelope Theorem} \Rightarrow \frac{dc^*}{da} = \frac{\partial \mathcal{L}}{\partial a} \Big|_* = 2a x_1^* > 0.$$

To find out $\partial^2 c / \partial y \partial a$, differentiate to obtain $\partial^2 c / \partial y \partial a = 2a \frac{\partial x_1^*}{\partial y}$.

So it is necessary to find $\partial x_1^* / \partial y$. This requires taking the total differential of \mathcal{L} with respect to all the endogenous variables (x_1, x_2 , and λ) and with respect to y . One can do this but

it involves a 3×3 matrix and its determinant.

Qualifying Exam
1996

Req. Answer Cont...

- Another bad approach is to

$$\max (10-y) y - \alpha x_1^2 - w_2 x_2 \quad \text{s.t. } f(x_1, x_2) = y \Rightarrow$$

$$\mathcal{L} = (10-y) y - \alpha x_1^2 - w_2 x_2 + \lambda [f(x_1, x_2) - y].$$

Doing comparative statics will require taking the total differential of \mathcal{L} with respect to all the endogenous variables (x_1, x_2, y , and λ) and with respect to ' α '. This will involve the determinant of a 4×4 matrix,

Qualifying Exam
1994
Question 2

621 Portion of 1994 Qualifying Exam

(2)

2. A profit-maximizing firm produces output from inputs x_1 and x_2 according to the following production function:

$$f(x_1, x_2) = \ln(x_1 + 1) + 2x_2^{1/2}.$$

The firm is competitive in its output market; the price of its output is p . The firm is competitive in the market for x_1 ; the price of x_1 is w_1 . The firm is a monopsonist in the market for x_2 ; the supply curve it faces for x_2 is given by $w_2 = 10 + 0.5x_2$ where w_2 is the price of x_2 .

- (a) Show that if $p = 11$ and $w_1 = 1$ then $x_1 = 10$ and $x_2 = 1$.
 (b) Show that for this firm

$$\begin{bmatrix} \frac{-p}{(x_1+1)^2} & 0 \\ 0 & \frac{-p}{2x_2^{3/2}} - 1 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = - \begin{bmatrix} -1 & \frac{1}{x_1+1} \\ 0 & \frac{1}{x_2^{1/2}} \end{bmatrix} \begin{bmatrix} dw_1 \\ dp \end{bmatrix}$$

and therefore

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \left(\frac{p}{(x_1+1)^2} \left(\frac{p}{2x_2^{3/2}} + 1 \right) \right)^{-1} \begin{bmatrix} \frac{-p}{2x_2^{3/2}} - 1 & \frac{1}{x_1+1} \left(\frac{p}{2x_2^{3/2}} + 1 \right) \\ 0 & \frac{p}{(x_1+1)^2} \frac{1}{x_2^{1/2}} \end{bmatrix} \begin{bmatrix} dw_1 \\ dp \end{bmatrix}.$$

- (c) Show that if $p = 11$ and $w_1 = 1$ then

$$\begin{aligned} dx_1 &= -11dw_1 + dp \\ dx_2 &= \frac{2}{13}dp. \end{aligned}$$

- (d) Start from " $p = 11$ and $w_1 = 1$." Then suppose that simultaneously w_1 increases by $\$/\epsilon$ unit and p increases by $\$/\epsilon$ unit where ϵ is a small positive number. Using the result of part (c), determine whether:
 i. the firm's demand for x_1 goes up or down;
 ii. the firm's demand for x_2 goes up or down;
 iii. the firm's output goes up or down.
 (e) Are the firm's second-order conditions for a maximum satisfied when $p = 11$ and $w_1 = 1$?

Answers to 621 Portion of 1994 Qualifying Exam

2. (Long Question)

$$f = \ln(x_1 + 1) + 2x_2^{1/2}$$

$$\pi = p \left[\ln(x_1 + 1) + 2x_2^{1/2} \right] - w_1 x_1 - (10 + \frac{1}{2} x_2) x_2$$

$$\begin{aligned} a) \quad O &= \frac{\partial \pi}{\partial x_1} = \frac{P}{x_1 + 1} - w_1, \quad (1) \\ O &= \frac{\partial \pi}{\partial x_2} = \frac{P}{x_2^{1/2}} - 10 - x_2 \quad (2) \end{aligned} \quad \left. \begin{array}{l} \text{if } p = 11 \\ \text{and } w_1 = 1 \text{ then } O = \frac{11}{x_1 + 1} - 1 \\ O = \frac{11}{x_2^{1/2}} - 10 - x_2; \end{array} \right.$$

To show that $x_1 = 10$ and $x_2 = 1$ are the answers, substitute in:

$$O = \frac{11}{10+1} - 1 \text{ yes}$$

$$O = \frac{11}{1} - 10 - 1 \text{ yes.}$$

b) Totally differentiating (1) and (2),

$$(1) \Rightarrow O = \frac{-P}{(x_1 + 1)^2} dx_1 + 0 dx_2 - 1 dw_1 + \frac{1}{x_1 + 1} dp$$

$$(2) \Rightarrow O = 0 dx_1 + \left[\frac{-P}{2x_2^{3/2}} - 1 \right] dx_2 + 0 dw_1 + \frac{1}{x_2^{1/2}} dp$$

$$\text{or } \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-P}{(x_1 + 1)^2} & 0 \\ 0 & \frac{-P}{2x_2^{3/2}} - 1 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} + \begin{bmatrix} -1 & \frac{1}{x_1 + 1} \\ 0 & \frac{1}{x_2^{1/2}} \end{bmatrix} \begin{bmatrix} dw_1 \\ dp \end{bmatrix}$$

↑ endogenous ↑ exogenous

This gives next the first form of the answer shown on the exam. Then

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} -\frac{p}{(x_1+1)^2} & 0 \\ 0 & \frac{-p}{2x_2^{3/2}} - 1 \end{bmatrix}^{-1} (-1) \begin{bmatrix} -1 & \frac{1}{x_1+1} \\ 0 & \frac{1}{x_2^{1/2}} \end{bmatrix} \begin{bmatrix} dw_1 \\ dp \end{bmatrix} \quad (3)$$

Since $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \div (ad-bc)$, (3) becomes

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \left[\frac{p}{(x_1+1)^2} \left(\frac{p}{2x_2^{3/2}} + 1 \right) \right]^{-1} \begin{bmatrix} \frac{-p}{2x_2^{3/2}} - 1 & 0 \\ 0 & \frac{-p}{(x_1+1)^2} \end{bmatrix} \begin{bmatrix} +1 & \frac{-1}{x_1+1} \\ 0 & \frac{-1}{x_2^{1/2}} \end{bmatrix} \begin{bmatrix} dw_1 \\ dp \end{bmatrix}$$

Multiplying the 2×2 matrices together yields the second form of the answer shown on the exam.

c) From part (a), if $p=11$ and $w_1=1$ then $x_1=10$ and $x_2=1$. Substituting these values into the exam's answer to part (b), one obtains

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \left(\frac{11}{11^2} \left(\frac{11}{2} + 1 \right) \right)^{-1} \begin{bmatrix} \frac{-11}{2} - 1 & \frac{1}{11} \left(\frac{11}{2} + 1 \right) \\ 0 & \frac{-11}{11^2} \cdot \frac{1}{1} \end{bmatrix} \begin{bmatrix} dw_1 \\ dp \end{bmatrix}$$

$$= 11 \left(\frac{11}{2} + \frac{2}{2} \right)^{-1} \begin{bmatrix} -\frac{13}{2} & \frac{1}{11} \cdot \frac{13}{2} \\ 0 & \frac{1}{11} \end{bmatrix} \begin{bmatrix} dw_1 \\ dp \end{bmatrix}$$

Qualifying Exam
1994

Answer 2 cont...

$$= 11 \left(\frac{2}{13} \right) \begin{bmatrix} -\frac{13}{2} & \frac{13}{11 \cdot 2} \\ 0 & \frac{1}{11} \end{bmatrix} \begin{bmatrix} dw_1 \\ dp \end{bmatrix} = \begin{bmatrix} -11 & 1 \\ 0 & \frac{2}{13} \end{bmatrix} \begin{bmatrix} dw_1 \\ dp \end{bmatrix}.$$

$$(d) \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} -11 & 1 \\ 0 & \frac{2}{13} \end{bmatrix} \begin{bmatrix} +\varepsilon \\ +\varepsilon \end{bmatrix} = \begin{bmatrix} -11\varepsilon + \varepsilon \\ \frac{2}{13}\varepsilon \end{bmatrix} = \begin{bmatrix} -10\varepsilon \\ \frac{2}{13}\varepsilon \end{bmatrix}$$

so $x_1 \downarrow$ and $x_2 \uparrow$.

$$\begin{aligned} \text{Output is } f(x_1, x_2), \text{ so } df &= \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 \\ &= \frac{1}{x_1+1} dx_1 + \frac{1}{x_2^{1/2}} dx_2 \\ &= \frac{1}{11} (-10\varepsilon) + \frac{1}{\frac{2}{13}\varepsilon} \left(\frac{2}{13}\varepsilon \right) \\ &= \frac{-10}{11}\varepsilon + \frac{2}{13}\varepsilon = \frac{-130 + 22}{11 \cdot 13}\varepsilon \\ &= \frac{-108}{143}\varepsilon < 0 \text{ and output falls.} \end{aligned}$$

(e) Check if the Hessian of the objective function is negative definite:

$$\begin{bmatrix} \frac{\partial^2 \pi}{\partial x_1^2}, \frac{\partial^2 \pi}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1}, \frac{\partial^2 \pi}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{-p}{(x_1+1)^2} & 0 \\ 0 & \frac{p}{2x_2^{3/2}} - 1 \end{bmatrix} \quad \frac{-p}{(x_1+1)^2} < 0 \text{ OK}$$

$\frac{-p}{(x_1+1)^2} (-1) \left(\frac{p}{2x_2^{3/2}} + 1 \right) > 0 \text{ OK}$

(It's also OK to substitute the values from (a) first, before evaluating.)

so the answer is 'yes.'

Exam 1

1995

Question 1

(2)

Answer all of the following three questions.

- Let y be output, w_1 be the cost of input 1 and x_1 be the amount of input 1, and let w_2 be the cost of input 2 and x_2 be the amount of input 2. Are the following observations consistent with cost-minimizing behavior? Explain thoroughly.

year	y	w_1	w_2	x_1	x_2
1	4	3	1	6	6
2	5	2	2	10	4
3	6	1.5	8	11	2

Exam I

1995

Answers to Exam I, Econ. 621, Winter 1995

Answer I

① Weak Axiom of Cost Minimization: $\underline{w}^t \cdot \underline{x}^t \leq \underline{w}^s \cdot \underline{x}^s$ for all $y^s \geq y^t$.

$t=1: y^s \geq y^t$ for $s = \{2, 3\}$

• check $s=2$: $\underline{w}^1 \cdot \underline{x}^1 \stackrel{?}{\leq} \underline{w}^2 \cdot \underline{x}^2 \Leftrightarrow (3, 1) \cdot (6, 6) \stackrel{?}{\leq} (3, 1) \cdot (10, 4)$

$24 \stackrel{?}{\leq} 34$, yes.

• check $s=3$: $\underline{w}^1 \cdot \underline{x}^1 \stackrel{?}{\leq} \underline{w}^3 \cdot \underline{x}^3 \Leftrightarrow 24 \stackrel{?}{\leq} (3, 1) \cdot (11, 2) = 35$, yes.

$t=2: y^s \geq y^t$ for $s = \{3\}$

• check $s=3$: $\underline{w}^2 \cdot \underline{x}^2 \stackrel{?}{\leq} \underline{w}^3 \cdot \underline{x}^3 \Leftrightarrow (2, 2) \cdot (10, 4) \stackrel{?}{\leq} (2, 2) \cdot (11, 2)$

$28 \stackrel{?}{\leq} 26$, no!

WACM violated here.

$t=3: y^s \geq y^t$ for no other values of s .

Note that the table has been constructed so that $dx_i/dw_i < 0$.

pts.
WACM: 13
violation: 10
other checks: 10

Final Exam
1999
Question 5

②

5. [7 points] Which of the following two environments would a competitive firm prefer to be in?

- (a) The price of input 1 is \$10 half the time and \$20 half the time.
- (b) The price of input 1 is \$15 all the time.

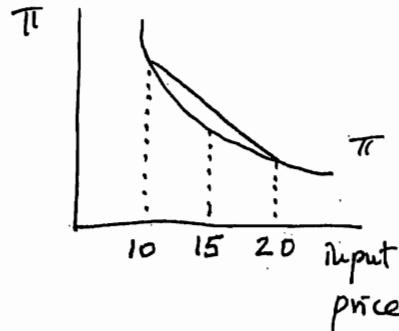
Why?

(5)

Profit at the stable price is $\pi(15)$, where $\pi(p)$ is the profit function (Suppose its relation on other prices, since these do not change)

Expected profit with the fluctuating price is

$$E\pi\left(\frac{1}{2} \circ \$10 \oplus \frac{1}{2} \circ \$20\right) = \frac{1}{2}\pi(10) + \frac{1}{2}\pi(20).$$



π is convex in prices, so $\pi(15) < \frac{1}{2}\pi(10) + \frac{1}{2}\pi(20)$.

Hence the firm would prefer the fluctuating price.

See top of p. 43 of Varian, 3rd edition.

Final Exam

1999

Answer 5

5. Suppose a firm is competitive in its output market. The price of output, p , is 2. This firm produces output using only one input, labor "L." The firm is the single purchaser of labor (it is a "monopsonist"). Therefore, the wage rate " w " (the price of labor) will depend on how much labor the firm buys.
- (a) Suppose the supply curve of labor is $w = L$ where L is the amount of labor the firm hires.
- What is the derivative of the firm's profit function with respect to p ?
 - How does the firm's demand for labor change when p changes?
- (b) Suppose the supply curve of labor is $w = g(L)$ where L is the amount of labor the firm hires.
- What is the derivative of the firm's profit function with respect to p ?
 - How does the firm's demand for labor change when p changes?

Final Exam
1998
Question 5

(5)

Final Exam

1998

⑤ price of output $p=2$ a) supply curve of labor $w=L$

Answer 5

Let production function be $f(L)$.

$$\pi = p f(L) - wL = p f(L) - L \cdot L = p f(L) - L^2.$$

$$\max_L \pi \Rightarrow 0 = \frac{d\pi}{dL} = p f'(L) - 2L \text{ for (ii).}$$

For (i): let the profit function be $\pi^*(p)$. By the Envelope Theorem,

$$\frac{d\pi^*(p)}{dp} = \frac{\partial \pi^*}{\partial p} = \frac{\partial [pf(L) - L^2]}{\partial p}^* = f(L^*).$$

For (ii), take the total differential of the F.O.C.:

$$0 = [p f''(L) - 2] dL + [f'(L)] dp$$

$$\Rightarrow [2 - p f''(L)] dL = f'(L) dp$$

$$\frac{dL}{dp} = \frac{f'(L)}{2 - p f''(L)} = \frac{f'(L)}{2 - 2 f''(L)}.$$

Optional: the S.O.C. is that $0 > \frac{d^2\pi}{dL^2} = p f''(L) - 2 = 2 f''(L) - 2$,so the denominator of dL/dp (which is -1 times $d^2\pi/dL^2$) is positive.Usually $f'(L) > 0$ (else $L^* \equiv 0$), making $dL/dp > 0$: when output price \uparrow , more labor is hired.

$$b) \pi = p f(L) - wL = p f(L) - g(L) L \rightarrow$$

$$(i) \frac{d\pi^*}{dp} = \frac{\partial \varphi^*}{\partial p} = \frac{\partial}{\partial p} [pf(L) - g(L)L]^* \\ = f(L^*).$$

$$(ii) \text{ F.O.C. is } 0 = pf'(L) - g'(L)L - g(L).$$

$$\text{Hence } 0 = [pf'' - g''L - g' - g']dL + [f']dp \\ = (pf'' - g''L - 2g')dL + f'dp$$

$$-f'dp = (pf'' - g''L - 2g')dL$$

$$\Rightarrow \frac{dL}{dp} = \frac{-f'}{pf'' - g''L - 2g'}$$

Final Exam
1998

Answer 5 cont...

Optional: As before, $pf'' - g''L - 2g' < 0$ from the S.O.C., and $f' > 0$,
 $\therefore dL/dp > 0$.