## Section 3:

## Consumer Theory

Fall 2021 Final Exam Question 2
Consumer Theory

## 2. [16 points]

Suppose a consumer consumes only two goods, $x$ and $y$. In this case, if the consumer has lexicographic preferences, then the consumer will choose between bundles $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ in one of the following two ways. The first is that if $x_{1} \neq x_{2}$, the consumer prefers the bundle whose $x$ is larger; but if $x_{1}=x_{2}$, the consumer prefers the bundle whose $y$ is larger. The second is that if $y_{1} \neq y_{2}$, the consumer prefers the bundle whose $y$ is larger; but if $y_{1}=y_{2}$, the consumer prefers the bundle whose $x$ is larger.
If a consumer has lexicographic preferences, there is no utility function which represents his or her preferences.
Figure 1 is one author's attempt to describe a particular situation involving lexicographic preferences, with $Y$ (measured on the vertical axis) representing the consumer's consumption of food and $E$ (measured on the horizontal axis) representing the state of the natural environment. (I know that calling food " $Y$ " is unusual, but the graph is not mine so I did not choose the notation.)
(a) Why can the solid lines in Figure 1 not be indifference curves?
(b) The author calls the solid lines in Figure 1 "quasi-indifference curves." Describe in words the preferences of the consumer depicted in Figure 1, making a plausible conjecture about the meaning of the quasi-indifference curves. (If your conjecture turns out to be wrong, you nevertheless may earn many points if your conjecture is reasonable and your interpretation of the graph is consistent with your conjecture.)
(c) Sketch the income expansion path in Figure 1.
(d) Show that there exists a (different) consumer whose preferences can be represented by a utility function who has the same income expansion path as the path you drew in part (c). Do this by sketching this consumer's indifference curves.


Figure 1.

Answer to Fall 2021 Econ. 7005 Final Exam, Question 2
a) Indifference curves are contour lines of the utility function. But there is no utility function which represents lexicog graphic preferences. (We showed in class that lexirofraphic preferences violate continuity of preferences.)

So with lexico graphic preferences, there are no indifference curves.
b) For $Y<y^{*}$ : prefer the bundle whose $Y$ is larger. If they hare the same $Y$, prefer the bundle whose $E$ is larger.

For $y>y^{*}$ : prefer the bundle whose $E$ is larger. If they hare the same E, prater the bundle whose y is larger.

If you have inadequate food, you only care about food, not enssonmental protection. But once you have adequate food, you only care about入 unless there is a tie in food environmental protection, not food (unless there is a tie in food).
c)

income go to buying more food.

With bexicofraphic preferences, all in come increases go to buying the good the consumer cares most about.
d)


$$
U_{1}<U_{2}<U_{3}<U_{4}<U_{5}<U_{6}<U_{7}
$$

For $y<y^{*}$, the consume only cores about $y, n_{0} t E$. For $y>y *$, the consume only cares about $E$, not $Y$.
(For $y<y^{*}$, if two bundles have the same $y$, the consumer is indifferent be tween there, even it one of them has more $E$ than the other. This is unlike in parts (a)- (c). For $y>y^{*}$, analogous commats can be made.)

Optional: This suggests that lexicographic preferences ane obscrva tonally equivalent to certain non-lexirographic preferences in a market environment.
Note: Figure from "Post Keynesian consumer choice theory and ecolopical economics," Marc Lavoie. In "Post Keynesian and Ecological Economics: Con fronting Environmental lsves," edited by Richard P.F. Holt, Steven Pressman, \& cire L. Spash. Edward Elgar, 2009.

## 2018 Exam 1 Qu. 3

## 3. [11 points]

Suppose a price-taking utility-maximizing consumer has a utility function of $u(\mathbf{x})=\ln x_{1}+\ln x_{2}$ and faces prices $p_{1}$ and $p_{2}$ for commodities $x_{1}$ and $x_{2}$, respectively; and has income $m$.
(a) Find this consumer's optimal consumption for $x_{1}$ and $x_{2}$ (called $x_{1}^{*}$ and $x_{2}^{*}$, respectively).
(b) Find $\partial x_{1}^{*} / \partial p_{1}$ and $\partial x_{2}^{*} / \partial p_{1}$.
(c) Find this consumer's income expansion path as a function of $x_{1}^{*}$. (Imagine a graph where $x_{1}^{*}$ is on the horizontal axis). Hint: one way to proceed is to consider $x_{2}^{*} / x_{1}^{*}$.
(d) How will the income expansion path shift when $p_{1}$ changes?

Answer to Weestion 3, Exam1, Fall 2018, Econ. 7005

$$
u(x)=\ln x_{1}+\ln x_{2}
$$

a) $\max _{x} u(\underset{\sim}{x})$ s.t. $p, x_{1}+P_{2} X_{2}=m$. The Lagrangian:

$$
\mathscr{L}=\ln x_{1}+\ln x_{2}+\lambda\left(m-p_{1} x_{1}-p_{2} x_{2}\right) \text {. First-vrder unditions: }
$$

(1) $0=\frac{\partial \mathscr{L}}{\partial \lambda}=m-p_{1} x_{1}-p_{2} x_{2}$ (or justify this as a repetition of the constrant wather then as $0=\partial \mathscr{Z} / \partial \lambda$ )
(2) $0=\frac{\partial \mathscr{L}}{\partial x_{1}}=\frac{1}{x_{1}}-\lambda p_{1}$
(3) $\left.0=\frac{\partial \mathscr{L}}{\partial x_{2}}=\frac{1}{x_{2}}-\lambda p_{2}\right\} \begin{aligned} & \Rightarrow \lambda=\frac{1}{p_{1} x_{1}}=\frac{1}{p_{2} x_{2}} \\ & \Rightarrow p_{1} x_{1}=p_{2} x_{2}\end{aligned}$

$$
\Rightarrow p_{1} x_{1}=p_{2} x_{2} \Rightarrow x_{1}=\frac{p_{2} x_{2}}{p_{1}} .
$$

Substitue this into (1):

$$
\begin{aligned}
0=m-p_{2} x_{2}-p_{2} x_{2}=m-2 p_{2} x_{2} \Rightarrow & 2 p_{2} x_{2}=m \\
& x_{2}=\frac{m}{2 p_{2}}
\end{aligned}
$$

and $\left[x_{1}=\frac{p_{2}}{p_{1}} \frac{m}{2 p_{2}}=\frac{m}{2 p_{1}}\right]$.

Optional: $\lambda=\frac{1}{p_{1} x_{1}}=\frac{1}{p_{1}} \frac{2 p_{1}}{m}=\frac{2}{m}$.
b) The easy way (since one is able, in part ia), to solve explicitly for

$$
\begin{aligned}
\left.x_{1}^{*} \text { and } x_{2}^{*}\right) & : \\
\frac{\partial x_{1}^{*}}{\partial p_{1}} & =\frac{\partial}{\partial p_{1}} \frac{m}{2 p_{1}} \text { from part (a) } \\
& =\frac{-m}{2 p_{1}^{2}} \\
\frac{\partial x_{2}^{*}}{\partial p_{1}} & =\frac{\partial}{\partial p_{1}} \frac{m}{2 p_{2}} \text { from part }(a) \\
& =0 .
\end{aligned}
$$

The hardway (whit hworks regardless of whether explicit solutions for $x_{1}^{*}$ and $x_{2}{ }^{*}$ were obtainable): This is a comparative statics problem with endogenous variables $\lambda, x_{1}$, and $x_{2}$, and exogenous variables $m, p_{1}$, and $p_{2}$; but here the only exogenous vanizble that changes is $p_{1}$, so $d m=0$ and $d p_{2}=0$.
The differentials of (1), (2), and (3) are, if $d m=0$ and $d p_{2}=0$ :

$$
\begin{array}{lll}
0= & d \lambda \quad d x_{1} & d x_{2} \\
0 & -p_{1} d x_{1} & -p_{2} d x_{2}
\end{array}-x_{1} d p_{1}
$$

$\underset{\sim}{O}=\left[\begin{array}{ccc}0 & -p_{1} & -p_{2} \\ -p_{1} & -1 / x_{1}^{2} & 0 \\ -p_{2} & 0 & -1 / x_{2}^{2}\end{array}\right]\left[\begin{array}{l}d \lambda \\ d x_{1} \\ d x_{2}\end{array}\right]+\left[\begin{array}{c}-x_{1} \\ -\lambda \\ 0\end{array}\right] d p_{1}$. Using Cranmer's Rule after
rewriting as $\left[\begin{array}{l}x_{1} \\ \lambda \\ 0\end{array}\right]=\left[\begin{array}{ccc}0 & -p_{1} & -p_{2} \\ -p_{1} & -1 / x_{1}^{2} & 0 \\ -p_{2} & 0 & -1 / x_{2}^{2}\end{array}\right]\left[\begin{array}{l}d \lambda / d p_{1} \\ d x_{1} / d p_{1} \\ d x_{2} / d p_{1}\end{array}\right]$ yields

$$
\begin{aligned}
\left.\frac{d x_{1}}{d p_{1}}=\frac{\left|\begin{array}{ccc}
0 & x_{1} & -p_{2} \\
-p_{1} & \lambda & 0 \\
-p_{2} & 0 & -1 / x_{2}^{2}
\end{array}\right|}{\left|\begin{array}{ccc}
0 & -p_{1} & -p_{2} \\
-p_{1} & -1 / x_{1}^{2} & 0 \\
-p_{2} & 0 & -1 / x_{2}^{2}
\end{array}\right| \quad \begin{array}{c}
\text { Numerator: solve by cofactor expansion, here } \\
\text { along the first row : }
\end{array}} \begin{array}{l}
\left.x_{1}(-1)^{1+2}\left|\begin{array}{cc}
-p_{1} & 0 \\
-p_{2} & -1 / x_{2}^{2}
\end{array}\right|-p_{2}(-1)^{1+3} \right\rvert\, \begin{array}{l}
-p_{1} \\
-p_{2}
\end{array} \\
\\
=-x_{1}\left[\frac{p_{1}}{x_{2}^{2}}\right]-p_{2}\left[+p_{2} \lambda\right] \\
\\
\end{array}\right]-\frac{x_{1} p_{1}}{x_{2}^{2}}-\lambda p_{2}^{2}
\end{aligned}
$$

Denominator: also expand along the first row:

$$
\begin{aligned}
& -p_{1}(-1)^{1+2}\left|\begin{array}{cc}
-p_{1} & 0 \\
-p_{2} & -1 / x_{2}^{2}
\end{array}\right|-p_{2}(-1)^{1+3}\left|\begin{array}{cc}
-p_{1} & -1 / x_{1}^{2} \\
-p_{2} & 0
\end{array}\right| \\
& =p_{1}\left(\frac{p_{1}}{x_{2}^{2}}\right)-p_{2}\left(-\frac{p_{2}}{x_{1}^{2}}\right)=\frac{p_{1}^{2}}{x_{2}^{2}}+\frac{p_{2}^{2}}{x_{1}^{2}} .
\end{aligned}
$$

so $\frac{\partial x_{1}}{\partial p_{1}}=\frac{\frac{-p_{1} x_{1}}{x_{2}^{2}}-\lambda p_{2}^{2}}{\frac{p_{1}^{2}}{x_{2}^{2}}+\frac{p_{2}^{2}}{x_{1}^{2}}}$.

Also,
$\frac{\partial x_{2}}{\partial p_{1}}=\frac{\left|\begin{array}{ccc}0 & -p_{1} & x_{1} \\ -p_{1} & -1 / x_{1}^{2} & \lambda \\ -p_{2} & 0 & 0\end{array}\right|}{\text { same denominator as } \partial x_{1} \partial p_{1}}$. The numerator is. expanding along its
frat row, $-p_{1}(-1)^{1+2}\left|\begin{array}{ll}-p_{1} & \lambda \\ -p_{2} & 0\end{array}\right|+x_{1}(-1)^{1+3}\left|\begin{array}{cc}-p_{1} & -1 / x_{1}^{2} \\ -p_{2} & 0\end{array}\right|$

$$
\begin{aligned}
& =p_{1}\left(\lambda p_{2}\right)+x_{1}\left(\frac{-p_{2}}{x_{1}^{2}}\right)=\lambda p_{1} p_{2}-p_{2} / x_{1} \text {. Thur } \\
\frac{\partial x_{2}}{\partial p_{1}} & =\frac{\lambda p_{1} p_{2}-p_{2} / x_{1}}{\frac{p_{1}^{2}}{x_{2}^{2}}+\frac{p_{2}^{2}}{x_{1}^{2}}} .
\end{aligned}
$$

Optional:
As in "the easy way," we could at this point avail ourselves of the tact that we were able to solve for $x_{1}^{*}, x_{2}^{*}$, and $\lambda^{*}$ in part (a) by substiating $\chi_{1}^{*}=m /\left(2 p_{1}\right)$, $x_{2}^{*}=m /\left(2 p_{2}\right)$, and $\lambda^{*}=2 / m$ in to the expressions we derived for $\partial x_{1} / \partial p_{1}$ and $\partial x_{2} / \partial p_{1}$ :

$$
\begin{aligned}
& \frac{\partial x_{1}}{\partial p_{1}}=\frac{\frac{-p_{1} x_{1}}{x_{2}^{2}}-\lambda p_{2}^{2}}{\frac{p_{1}^{2}}{x_{2}^{2}}+\frac{p_{2}^{2}}{x_{1}^{2}}}=\frac{\frac{\left.-p_{1}\right) m}{2\left(p_{1}\right)} \frac{4 p_{2}^{2}}{m^{2}}-\frac{2}{m} \frac{p_{2}^{2}}{1}}{\frac{p_{1}^{2} 4 p_{2}^{2}}{m^{2}}+\frac{p_{2}^{2} 4 p_{1}^{2}}{m^{2}}}=\frac{\frac{-2 p_{2}^{2}}{m}-\frac{2 p_{2}^{2}}{m}}{\frac{8 p_{1}^{2} p_{2}^{2}}{m^{2}}} \\
&=\frac{-4\left(p_{2}^{2}\right)}{(\sqrt[m]{2}} \cdot \frac{m p^{(2)}}{8 p_{1}^{2}\left(p_{2}^{2}\right)}=\frac{-m}{2 p_{1}^{2}} \quad \text { (whichmatches the answer obtained } \\
& \text { in "the easy way") }
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial x_{2}}{\partial p_{1}}=\frac{\lambda p_{1} p_{2}-p_{2} / x_{1}}{\frac{p_{1}^{2}}{x_{2}^{2}}+\frac{p_{2}^{2}}{x_{1}^{2}}} \text { has a numerator of } & \frac{2}{m} p_{1} p_{2}-p_{2} \frac{2 p_{1}}{m} \\
& =\frac{2 p_{1} p_{2}}{m}-\frac{2 p_{1} p_{2}}{m}=0
\end{aligned}
$$

so $\partial x_{2} / \partial p_{1}=0$ (which matches the answer obtained in "the easy way").
c) The income expansion path shows how the optimal consumption bundle in the $x_{2}$ plane changes as income changes. From part (a),

$$
\frac{x_{2}^{*}}{x_{1}^{*}}=\frac{\frac{m}{2 p_{2}}}{\frac{m}{2 p_{1}}}=\frac{m}{2 p_{2}} \cdot \frac{2 p_{1}}{m}=\frac{p_{1}}{p_{2}} \text { so } x_{2}^{*}=\frac{p_{1}}{p_{2}} x_{1}^{*} \text { is }
$$

the in come expansion path. It happens to be a straight line through the origin with slope $p_{1} / p_{2}$.
d) The easy way:

Nothing in part (c)'s calculations depends on a partiwlar valve of $p_{1}$, so the income expansion path is always described by $x_{2}^{*}=\frac{p_{1}}{p_{2}} x_{1}^{*}$, and a nee in $p_{1}$ with (since $\frac{x_{1}^{*}}{p_{2}} \geqslant 0$ ) raise $x_{2}^{*}$ for any given $x_{1}^{*}>0$ (although for $x_{1}^{*}=0, x_{2}^{*}$ will remain at zero).


Income Expansion Path

The hard way:

$$
\frac{\partial x_{1}^{*}}{\partial p_{1}}=\left\{\begin{array}{l}
\cdot \frac{\partial}{\partial p_{1}} \frac{m}{2 p_{1}}=\frac{-m}{2 p_{1}^{2}} \text { from part (a) } \\
\cdot \frac{\frac{-p_{1} x_{1}}{x_{2}^{2}}-\lambda p_{2}^{2}}{\frac{p_{1}^{2}}{x_{2}^{2}}+\frac{p_{2}^{2}}{x_{1}^{2}}} \text { from part (b) }
\end{array}\right.
$$



As $p_{1}$ increases, $x_{2}^{*}$ shifts by $\frac{\partial x_{2}^{*}}{\partial p_{1}} d p_{1}$ (length " $t$ ") and $x_{1}^{*}$ shifts by $\frac{\partial x_{1}^{*}}{\partial p_{1}} d p_{1}$ (length "w").
$x_{1}^{*} \quad$ In our specificiase we know that $\frac{\partial x_{2}^{*}}{\partial p_{1}}=0$
so length " $t$ "is zero, and $\frac{\partial x_{1}^{*}}{\partial p_{1}}=\frac{-m}{2 p_{1}^{2}}<0$ so $x_{1}^{*}$ falls and the graph's
" $w$ " does go to the left as illustrated (al though if $m=0, x_{1}^{*}=0$ and $\partial x_{1}^{*} \mid \partial p_{1}=0$, so at $x_{1}^{*}=0$, (length " $w$ " is zero). It follows that the
income expansion path pirots as shown under "the easy way."
Remark. The first billet point for $\partial x_{2}^{*} / \partial p_{1}$ says it equals zero; the third says it equals $\frac{x_{1}^{*}}{p_{2}}+\frac{p_{1}}{p_{2}} \frac{\partial x_{1}^{*}}{\partial p_{1}}$. Are these compatible? Answer:

$$
\begin{aligned}
\frac{x_{1}^{*}}{p_{2}}+\frac{p_{1}}{p_{2}} \frac{\partial x_{1}^{*}}{\partial p_{1}} & =\frac{\left(m / 2 p_{1}\right)}{p_{2}}+\frac{p_{1}}{p_{2}} \frac{\partial}{\partial p_{1}}\left(\frac{m}{2 p_{1}}\right)=\frac{m}{2 p_{1} p_{2}}+\frac{p_{1}}{p_{2}} \frac{-m}{2 p_{1}^{2}} \\
& =\frac{m}{2 p_{1} p_{2}}-\frac{m}{2 p_{1} \cdot p_{2}}=0, \text { yes. }
\end{aligned}
$$

2. [14 points] Suppose a price-taking consumer has utility function

$$
u(\mathbf{x})=x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{n}^{\alpha_{n}}
$$

for commodities $x_{1}, x_{2}, \ldots, x_{n}$, where $n$ is a positive integer greater than two. Find this consumer's demand for each of the commodities. (You need not check the second-order conditions.)
Final Exam, Fall 2013
$\max \underbrace{x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{n}^{\alpha_{n}}}_{" u "}$ s.t. $p_{1} x_{1}+p_{2} x_{2}+\cdots+p_{n} x_{n}=m_{\nearrow}$

$$
\begin{aligned}
& \text { (Formally: max } \left.\prod_{i=1}^{n} x_{i}^{\alpha_{i}} \text { s.t. } \sum_{i=1}^{n} p_{i} x_{i}=m .\right) \\
& \mathscr{L}=x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{n}^{\alpha_{n}}+\lambda\left[m-p_{1} x_{1}-p_{2} x_{2}-\cdots-p_{n} x_{n}\right] . \\
& 0=\partial \mathscr{L} / \partial \lambda=m-p_{1} x_{1}-p_{2} x_{2}-\cdots-p_{n} x_{n} \\
& 0=\partial \mathscr{1}
\end{aligned}
$$

$0=\partial \not / \partial x_{i}=\alpha_{i} \frac{u}{x_{i}}-\lambda p_{i} \quad$ or see alternative solution on next page $\rightarrow$
This can also bewnitten $\alpha_{i} x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{i}^{\alpha_{i}-1} \cdots x_{n}^{\alpha_{n}}$

$$
\begin{aligned}
& =\alpha_{i} x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{i}^{\alpha_{i}} x_{i}^{-1} \cdots x_{n}^{\alpha_{n}} \\
& =\alpha_{i} u x_{i}^{-1} .
\end{aligned}
$$

So

$$
\begin{aligned}
& \lambda p_{i}=\frac{\alpha_{i}}{x_{i}} u \\
& \Rightarrow x_{i}=\frac{u}{\lambda} \frac{\alpha_{i}}{p_{i}} \text { and } \\
& m=\sum_{i} p_{r} x_{i}=\sum_{i} p_{i} \cdot \frac{u}{\lambda} \frac{\alpha_{i r}}{p_{i}}=\sum_{i} \frac{u}{\lambda} \alpha_{i}=\frac{u}{\lambda} \sum_{i} \alpha_{i}
\end{aligned}
$$

$$
\Rightarrow \frac{u}{\lambda}=\frac{m}{\sum_{i} \alpha_{i}} . \quad \text { (Note that often, } \sum_{i} \alpha_{i}=1 \text {, in which }
$$

$$
\text { Hence } X_{i}^{*}=\frac{u}{\lambda} \frac{\alpha_{1}}{P_{i}}=\frac{m}{\sum_{i} \alpha_{i}} \frac{\alpha_{L^{\prime}}}{P_{i}}
$$ case $X_{i}{ }^{*}=m \alpha_{i} / p_{i}$.)

Alternative solution: F.O.C.'s ave

$$
\begin{aligned}
0 & =\frac{\partial \psi}{\partial \lambda}=m-p_{1} x_{1}-p_{2} x_{2}-\cdots-p_{n} x_{n} \\
0 & =\frac{\partial \psi}{\partial x_{i}}=\alpha_{i} x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{i}^{\alpha_{i}-1} \cdots x_{n}^{\alpha_{n}}-\lambda_{i} a_{n d} \text { similarly } \\
0 & =\alpha_{j} x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{j}^{\alpha_{j}-1} \cdots x_{n}^{\alpha_{n}}-\lambda p_{j} \\
\lambda & =\frac{\alpha_{i}}{p_{i}} x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{i}^{\alpha_{i}-1} \cdots x_{n}^{\alpha_{n}} \\
\lambda & =\frac{\alpha_{j}}{p_{j}} x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{j}^{\alpha_{j}-1} \cdots x_{n}^{\alpha_{n}} \Rightarrow \\
\frac{\alpha_{i}}{p_{i}} x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{i}^{\alpha_{i}-1} \cdots x_{n}^{\alpha_{n}} & =\frac{\alpha_{j}}{p_{j}} x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{j}^{\alpha_{j}-1} \cdots x_{n}^{\alpha_{n}} \\
\frac{\alpha_{i}}{p_{i}} x_{i}^{\alpha_{i}-1} x_{j}^{\alpha_{j}} & =\frac{\alpha_{j}}{p_{j}} x_{i}^{\alpha_{i}} x_{j}^{\alpha_{j}-1} \\
\frac{\alpha_{i}}{p_{i}} x_{i}^{-1} & =\frac{\alpha_{j}}{p_{j}} x_{j}^{-1} \\
\frac{\alpha_{i}}{p_{i} x_{i}} & =\frac{\alpha_{j}}{p_{j} x_{j}} \text { or } \frac{p_{i} x_{i}}{\alpha_{i}}=\text { a constant } \forall i ;
\end{aligned}
$$

Then $m=\sum_{i} p_{i} x_{i}=\sum_{i} \alpha_{i} \cdot L=L \sum_{i} \alpha_{i}$ and $L=m / \sum_{i} \alpha_{i}$.
From the definition of $L$,

$$
x_{i}{ }^{*}=\frac{\alpha_{i}}{p_{i}} L=\frac{\alpha_{i}}{p_{i}} \frac{m}{\sum_{i} \alpha_{i}}
$$

as before.

## 2. [12 points]

Suppose the consumers in this problem are competitive. This is true for both parts (a) and (b) of this question! For simplicity, assume there is only one consumer (this just saves on notation). This consumer only consumes two goods, $X$ and $Y$. Suppose this consumer's income is $\$ 10$, the price of good $X$ is $\$ 2 /$ unit, and the price of good $Y$ is $\$ 1 /$ unit. Suppose the consumer's utility function is $X^{1 / 2} Y^{1 / 2}$.
(a) How much $X$ and $Y$ will this consumer buy? Be sure to verify the second-order conditions.
(b) Suppose when this consumer goes to the store to buy $X$ and $Y$, he can only find 4 units of $Y$ in the store. (This should be less than the amount you calculated that he desired to buy in part (a).) What do you think will happen? Bidding the price of good $Y$ up? In the end, how much $X$ and $Y$ will he end up with?

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Section 2 Questorin 2.
(a) max $x^{1 / 2} y^{1 / 2}$ s.t. $2 x+1 y=10$

$$
\begin{aligned}
& \mathcal{L}=x^{1 / 2} y^{1 / 2}+\lambda(10-2 x-y) \\
& 0=\frac{\partial \mathcal{L}}{\partial \lambda}=10-2 x-y \\
& \left.\left.0=\frac{\partial \mathcal{L}}{\partial x}=\frac{1}{2} \frac{x^{1 / 2} y^{1 / 2}}{x}-2 \lambda\right] \begin{array}{rl}
0=\frac{\partial \mathcal{L}}{\partial y}=\frac{1}{2} \frac{x^{1 / 2 y^{1 / 2}}}{y}-\lambda
\end{array}\right\} \lambda=\frac{1}{4} \frac{x^{1 / 2} y^{1 / 2}}{x}=\frac{1}{2} \frac{x^{1 / 2} y^{1 / 2}}{y}
\end{aligned} \quad \begin{aligned}
& \Rightarrow \frac{1}{4 x}=\frac{1}{2 y} \Rightarrow y=2 x \text { and } 10=2 x+y \\
&=2 x+2 x=4 x \Rightarrow x^{x}=\frac{10}{4}=\frac{5}{2} \\
& \\
& y^{*}=2 x=5 .
\end{aligned}
$$

S.O.C. number of constrants $m=1$
number of varables $n=2$

$$
\begin{aligned}
& 2 m+1=3 \\
& m+n=3
\end{aligned}
$$

So we need $D_{3}$ of $\nabla^{2} \mathscr{L}$ to harc the sipn of $(-1)^{m+1}=1>0$.

$$
\nabla^{2} \mathcal{L}=\left[\begin{array}{lll}
\mathscr{L}_{\lambda \lambda}^{\prime \prime} & \mathscr{L}_{\lambda x}^{\prime \prime} & \mathscr{L}_{\lambda y}^{\prime \prime} \\
\mathscr{L}_{x \lambda}^{\prime \prime} & \mathscr{L}_{x}^{\prime \prime} & \mathscr{L}_{x y}^{\prime \prime} \\
\mathscr{L}_{y: \lambda}^{\prime \prime} & \mathcal{L}_{y, x}^{\prime \prime} & \mathscr{L}_{y}^{\prime \prime} y_{x}
\end{array}\right]=\left[\begin{array}{ccc}
0 & -2 & -1 \\
-2 & -\frac{1}{4} x^{-3 / 2} y^{1 / 2} & \frac{1}{4} x^{-1 / 2} y^{-1 / 2} \\
-1 & \frac{1}{4} x^{-1 / 2} y^{-1 / 2} & \frac{-1}{4} x^{1 / 2} y^{-3 / 2}
\end{array}\right]
$$

$D_{3}$ of $D^{2} \mathscr{A}$ is it determinant. Expanding ahoy the frost mow,

$$
\begin{aligned}
\left|\nabla^{2} y\right|= & (-1)^{1+2}(-2)\left[-2 \cdot \frac{-1}{4} x^{1 / 2} y^{-3 / 2}-(-1) \frac{1}{4} x^{-1 / 2} y^{-1 / 2}\right] \\
& +(-1)^{1+3}(-1)\left[-2 \cdot \frac{1}{4} x^{-1 / 2} y^{-1 / 2}-(-1) \frac{-1}{4} x^{-3 / 2} y^{1 / 2}\right] \\
= & 2\left[\frac{1}{2} x^{1 / 2} y^{-3 / 2}+\frac{1}{4} x^{-1 / 2} y^{-1 / 2}\right] \\
& -\left[-\frac{1}{2} x^{-1 / 2} y^{-1 / 2}-\frac{1}{4} x^{-3 / 2} y^{1 / 2}\right] \\
= & x^{1 / 2} y^{-3 / 2}+\frac{1}{2} x^{-1 / 2} y^{-1 / 2}+\frac{1}{2} x^{-1 / 2} y^{-1 / 2}+\frac{1}{4} x^{-3 / 2} y^{1 / 2} \\
= & x^{1 / 2} y^{-3 / 2}+x^{-1 / 2} y^{-1 / 2}+\frac{1}{4} x^{-3 / 2} y^{1 / 2}
\end{aligned}
$$

$>0$. (Note that it's not even ne cessary to substitute in $x=\frac{5}{2}$
and $y^{\prime \prime}=5$ lace.) (One could also instead prove that this utility function is prasiconcave, then appeal to the result that the S. O. C.foramber are satisfied if the objective function is zuasiconcare and thee is exactly one, linear constrinat.)
(b)

the relevant part of the budget constraint is $y \leqslant 4$

The consumer is competitive and so cant bid the prize up.
So he cont get to the optracl point, $\left(\frac{5}{2}, 5\right)$, and indifference carve $U_{2}$. He'd hove to settle for the point $(3,4)$, with maifference are $U_{1}$, which maximizes his utility given the additional $y \leq 4$ constraint.

Optional:
The idea lore is that in a situation of access demand, a competitive consumer cant bid the price up, so


Le wore just take what he con get and spend more money than le'd like on the other good.

## 2019 Qualifying Exam Sec. 3 Qu. 2

## 2. [16 points]

[Completely optional introduction: This will show that it is incorrect to use the change in "consumer surplus" as a measure of welfare change in the general case of goods having arbitrary income effects.]
(a) Suppose a consumer has a utility function $u=x_{1}^{1 / 2} x_{2}^{1 / 2}$ and income $m=2$ and takes the prices $p_{1}$ and $p_{2}$ as given. If $x_{1}$ is "cheese," find the consumer's (Marshallian) demand curve for cheese.
(b) Make a rough, somewhat large sketch of this consumer's demand curve for cheese for $0<x_{1}=1$ and identify the quantity demanded of cheese for prices $p_{1}$ of $1,2,3$, and 4 dollars per pound (" $\$ / l \mathrm{lb} "$ ) of cheese.
(c) Consider the following explanation of consumer surplus, which resembles what one might find in an undergraduate microeconomics textbook.

Consumer surplus, which is the area under the demand curve, measures how much a consumer would be willing and able to spend to buy cheese. To illustrate this, consider how much money the consumer whose demand curve you drew in part (b) would be willing and able to spend to buy a certain total amount of cheese. If the price of cheese were $\$ 4 / \mathrm{lb}$, he would be willing to buy [fill in this blank, which is part (i) of this sub-part] pounds of cheese, and so would spend the amount of money shown by area [fill in this blank, which is part (ii) of this sub-part] in the diagram. [Designate areas in your graph by giving labels such as $A, B, C$, etc. to the vertices of those geometric areas, rather than say by shading the areas, because shading may make part (d) harder to superimpose onto this graph.]

If after making this transaction the price of cheese were to fall to $\$ 3 / \mathrm{lb}$, he would be willing to buy more cheese, raising his total cheese purchases to [fill in this blank, which is part (iii) of this sub-part] pounds of cheese, and so would in total spend the amount of money shown by area [fill in this blank, which is part (iv) of this sub-part] in the diagram.

If after making this transaction the price of cheese were to fall further, to $\$ 2 / \mathrm{lb}$, he would be willing to buy more
cheese, raising his total cheese purchases to [fill in this blank, which is part (v) of this sub-part] pounds of cheese, and so would in total spend the amount of money shown by area [fill in this blank, which is part (vi) of this sub-part] in the diagram.

If, finally, after making this transaction the price of cheese were to fall even further, to $\$ 1 / l \mathrm{~b}$, he would be willing to buy more cheese, raising his total cheese purchases to [fill in this blank, which is part (vii) of this subpart] pounds of cheese, and so would in total spend the amount of money shown by area [fill in this blank, which is part (viii) of this sub-part] in the diagram. This amount of money is approximately equal to consumer surplus and thus shows that [fill in this blank with the conclusion of this argument, which is part (ix) of this sub-part].
(d) In this part you have to show that the explanation in part (c) is wrong. To do this, suppose the consumer has already spent the money to purchase, at a price of $\$ 4 / \mathrm{lb}$, the amount of cheese you answered in sub-part (i) of part (c). Suppose the consumer has taken ownership of this amount of cheese but has not eaten it yet. Before eating this cheese and before buying any $x_{2}$, the consumer gets the opportunity to buy more cheese at a price of $\$ 3 / \mathrm{lb}$.
i. Show that he will not buy the total amount of cheese given in sub-part (iii) of part (c) by showing that the total amount of cheese he will actually buy is $7 / 24 \approx 0.29$ (where " $\approx$ " means "is approximately equal to"). Hint: first calculate how much extra cheese he will buy.
ii. Superimpose onto your prior graph this consumer's new demand curve for cheese for prices of 3,2 , and 1 dollars per pound, giving a numerical value for the amount of cheese demanded at each of these prices.
iii. Construct an argument that the consumer surplus described in part (c) is not actually "how much a consumer would be willing and able to spend to buy cheese." Include a conceptual explanation of why the the demand curve you derived in part (a) generated a misleading answer to part (c).

Answer to 2019 Miro Qualifying Exam, Section 3 Question 2
a)

$$
\begin{aligned}
& u=x_{1}^{1 / 2} x_{2}^{1 / 2} \quad \text { Buffet constraint } m=p_{1} x_{1}+p_{2} x_{2} \\
& \mathscr{L}=x_{1}^{1 / 2} x_{2}^{1 / 2}+\lambda\left(m-p_{1} x_{1}-p_{2} x_{2}\right)
\end{aligned}
$$

First-order conditions:

$$
\left.\begin{array}{c}
0=\mathscr{L}_{1}^{\prime}=\frac{1}{2} \frac{x_{1}^{1 / 2} x_{2}^{1 / 2}}{x_{1}}-\lambda p_{1} \Rightarrow \lambda=\frac{1}{2 p_{1}} \frac{x_{1}^{1 / 2} x_{2}^{1 / 2}}{x_{1}} \\
0=\mathscr{L}_{2}^{\prime}=\frac{1}{2} \frac{x_{1}^{\prime \prime 2} x_{2}^{1 / 2}}{x_{2}}-\lambda p_{2} \Rightarrow \lambda=\frac{1}{2 p_{2}} \frac{x_{1}^{1 / 2} x_{2}^{1 / 2}}{x_{2}}
\end{array}\right\} \begin{aligned}
& \lambda=\frac{1}{2 p_{1}} \frac{x_{1}^{1 / 2} x_{2}^{1 / 2}}{x_{1}}=\frac{1}{2 p_{2}} \frac{x_{1}^{1 / 2} x_{2}^{1 / 2}}{x_{2}} \\
& \frac{1}{p_{1} x_{1}}=\frac{1}{p_{2} x_{2}} \\
& x_{2}=\frac{x_{1} p_{2} x_{2}}{p_{2}} \Rightarrow \\
& m=p_{1} x_{1}+p_{2} x_{2}=p_{1} x_{1}+p_{2} \frac{p_{1} x_{1}}{p_{2}}=2 p_{1} x_{1} \\
& \Rightarrow x_{1}=\frac{m}{2 p_{1}}=\frac{2}{2 p_{1}}=\frac{1}{p_{1}} \text { and } \\
& x_{2}=\frac{p_{1} x_{1}}{p_{2}}=\frac{p_{1}}{p_{2}} \frac{1}{p_{1}}=\frac{1}{p_{2}} .
\end{aligned}
$$

b) over $\rightarrow$

c) (i) $1 / 4 \mathrm{lb}$.
(ii) $O A B C$
(iii) $1 / 3 / b$.
(iv) $O A B C+C E F G=O A B E F G$
(v) $1 / 2 \mathrm{~B}$
(vi) answer to (iv) plus GHIJ $=$ OABEFHIJ
(vii) 1 lb
(viii) answer to (vi) plus JKLM $=$ OABEFHIKLM
(ix) consumer surplus (the area under the demand curve) is approximately equal to the amount of money this consumer would be willing and
able to spend to buy 1 16. of cheese.
d) (i) He has already bought 1/4 1 b. of cheese. Let $t$ be the amount of extra cheese he buys at $\$ 3 / \mathrm{lb}$. Then hisutility is

$$
u=\left(\frac{1}{4}+t\right)^{1 / 2} x_{2}^{1 / 2}
$$

He has already spent ${ }^{1} 1=$ area $O A B C$ in the diagram. From part $(a)$, he started with $m=\$ 2$. So he has $\$ 1$ left now.
thus his problem is to maximize $\left(\frac{1}{4}+t\right)^{1 / 2} x_{2}^{1 / 2}$ s.t. $1=3 t+p_{2} x_{2}$.
 extracheese extracheese

$$
\left.\begin{array}{c}
\mathscr{L}=\left(\frac{1}{4}+t\right)^{1 / 2} x_{2}^{1 / 2}+\lambda\left[1-3 t-p_{2} x_{2}\right] \\
\left.0=\mathscr{L}_{t}^{\prime}=\frac{1}{2}\left(\frac{1}{4}+t\right)^{-1 / 2} x_{2}^{1 / 2}-3 \lambda \Rightarrow \lambda=\frac{1}{6}\left(\frac{1}{4}+t\right)^{-1 / 2} x_{2}^{1 / 2}\right] \\
0=\mathcal{L}_{2}^{\prime}=\frac{1}{2}\left(\frac{1}{4}+t\right)^{1 / 2} x_{2}^{-1 / 2}-\lambda p_{2} \Rightarrow \lambda=\frac{1}{2 p_{2}}\left(\frac{1}{4}+t\right)^{1 / 2} x_{2}^{-1 / 2}
\end{array}\right\}-.
$$

Substituting into the budget constraint,

$$
\begin{aligned}
& 1=3 t+p_{2} \cdot \frac{3}{p_{2}}\left(\frac{1}{4}+t\right)=3 t+\frac{3}{4}+3 t=6 t+\frac{3}{4} \\
& \frac{1}{4}=6 t \Rightarrow t=\frac{1}{24} .
\end{aligned}
$$

This is the amount of extracheese he will buy. The amount of total cheese he
will buy is thus $\frac{1}{4}+\frac{1}{24}=\frac{6+1}{24}=\frac{7}{24} \approx 0,29$.
(ii) Re-working the optimization problem in part (i) with "p, "replacing "3," the first-vrder conditions would be come

$$
\left.\left.\begin{array}{c}
0=\mathscr{L}_{t}^{\prime}=\frac{1}{2}\left(\frac{1}{4}+t\right)^{-1 / 2} x_{2}^{1 / 2}-p_{1} \lambda \Rightarrow \lambda=\frac{1}{2 p_{1}}\left(\frac{1}{4}+t\right)^{-1 / 2} x_{2}^{1 / 2} \\
0=\mathscr{L}_{2}^{\prime}=\frac{1}{2}\left(\frac{1}{4}+t\right)^{1 / 2} x_{2}^{-1 / 2}-p_{2} \lambda \Rightarrow \lambda=\frac{1}{2 p_{2}}\left(\frac{1}{4}+t\right)^{1 / 2} x_{2}^{-1 / 2}
\end{array}\right\}\right)
$$

Substituting into the budget constraint,

$$
\begin{aligned}
1 & =p_{1} t+p_{2} \cdot \frac{p_{1}}{p_{2}}\left(\frac{1}{4}+t\right)=p_{1} t+p_{1}\left(\frac{1}{4}+t\right)=p_{1} t+\frac{1}{4} p_{1}+p_{1} t \\
& =2 p_{1} t+\frac{1}{4} p_{1} \\
1-\frac{1}{4} p_{1} & =2 p_{1} t \Rightarrow t=\frac{1-\frac{1}{4} p_{1}}{2 p_{1}}=\frac{1}{2 p_{1}}-\frac{1}{8}
\end{aligned}
$$

For contrimation: $p_{1}=3 \Rightarrow t=\frac{1}{2 \cdot 3}-\frac{1}{8}=\frac{1}{6}-\frac{1}{8}=\frac{4-3}{24}=\frac{1}{24}$ as before.

$$
p_{1}=2 \Rightarrow t=\frac{1}{2 \cdot 2}-\frac{1}{8}=\frac{1}{4}-\frac{1}{8}=\frac{2-1}{8}=\frac{1}{8}
$$

Total cheese is $\frac{1}{8}+\frac{1}{4}=\frac{1}{8}+\frac{2}{8}=\frac{3}{8}=0.375$

$$
P_{1}=1 \Rightarrow t=\frac{1}{2.1}-\frac{1}{8}=\frac{1}{2}-\frac{1}{8}=\frac{4-1}{8}=\frac{3}{8}
$$

Total cheese is $\frac{3}{8}+\frac{1}{4}=\frac{3+2}{8}=\frac{5}{8}=0.625$

These points generate the dotted line and the small arcles in the graph drawn a few pages ago.
(iii) It is true that at a price of $\$ 4 / \mathrm{lb}$. The consumer would be willing and able to spend the area under $A B$ to buy the $1 / 4 \mathrm{lb}$. of cheese.

However if after doing that the price tell to $\$ 3 / 1 b$., the demand arne will shift down to RW, and the extra amount of money spat on cheese $\operatorname{un}^{-}$ll be the area under $E R$, not $E F$.
the demand wore derived in port (a) assumed the consumer paid ore price for all units of cheese. If that assumption is violated then that demand curve is invalid.

The new dur and curve is lows than the old one be cause the consumer had to spend, in order to buy the first 1/4 lb . of cheese, more than $\$ 3 / \mathrm{lb} \cdot \frac{1}{4} \mathrm{lb}=\$ 3 / 4$, meaning his remaining income after buying the first $1 / 4 \mathrm{lb}$. of cheese is lower than it would otter wise have been, and this lows remaining income has an income effect which reduces the demand for cheese, meaning that cheese is a normal good for this consumer, and that the demand curve shifts down. It would shift clown again if after buying ER the price tell fur the, to $\$ 2 \mathrm{lb}$; then shift down yet again if the price fell to $\$ 1 / \mathrm{lb}$.

We are of course used to demand curves not shifting when prices change, but that assumes a uniform price for all units of cheese and a demand curve obtained by assuming a uniform price for all units of cheese. If there is a non-unitorm prize for all units of cheese, a demand curve drawn assuming a uniform price for all units of cheese will not behave as expected.

Exam 1
1994

## Answer all of the following five questions.

1. Suppose a consumer's utility function $u$ is given by $u(x)=x_{1} x_{2}^{2}$ where $x_{1}$ and $x_{2}$ are amounts of two commodities consumed.
(a) Is this utility function concave?
(b) Find the utility-maximizing demand for $x_{1}$ and for $x_{2}$.
(c) Are the second-order sufficient conditions for a maximum satisfied in part (b)?
2. a. $u(\underline{x})=x_{1} x_{2}^{2}$.

$$
\left.\begin{array}{ll}
\frac{\partial u}{\partial x_{1}}=x_{2}^{2} & \frac{\partial u}{\partial x_{2}}=2 x_{1} x_{2} \\
\frac{\partial^{2} u}{\partial x_{1}^{2}}=0 & \frac{\partial^{2} u}{\partial x_{2}^{2}}=2 x_{1}
\end{array}\right\} S_{0} \text { the Hessian } \nabla^{2} u(x)=\left[\begin{array}{cc}
0 & 2 x_{2} \\
2 x_{2} & 2 x_{1}
\end{array}\right]
$$

$$
\frac{\partial^{2} u}{\partial x_{1} \partial x_{2}}=2 x_{2}
$$

Newer notation:
$\tilde{D}_{1}=0$ and $2 x,>0$ so not concave $=\Delta_{1}$

$$
\tilde{D}_{2}=-4 x_{2}^{2}<0 \text { (also not convex) }=\Delta_{2}
$$

(The fact that $\tilde{D}_{2}=0$ is enough ito conclude that $u$ is not concave.)
b. $\max x_{1} x_{2}^{2}$ st. $p_{1} x_{1}+p_{2} x_{2}=m$

Remarks. Demand is unchanged by positive nonotour transformations of the utility function. That is why these demand curves ore the some as those from the staubiard Cobb-Dorglas example $\hat{u}=x_{1}^{1 / 3} x_{2}^{2 / 3}$ (since $u$ here is simply $(\hat{u})^{3}$ ).

$$
\begin{aligned}
& \mathscr{L}=x_{1} x_{2}^{2}+\lambda\left(m-p_{1} x_{1}-p_{2} x_{2}\right) \\
& \left.\begin{array}{l}
\frac{\partial y}{\partial x_{1}}=0=x_{2}^{2}-\lambda p_{1} \\
\frac{\partial x}{\partial x_{2}}=0=2 x_{1} x_{2}-\lambda p_{2}
\end{array}\right\} \Rightarrow \frac{x_{2}^{2}}{2 x_{1} x_{2}}=\frac{p_{1}}{p_{2}} \text { so } \frac{x_{2}}{2 x_{1}}=\frac{p_{1}}{p_{2}} \Rightarrow x_{2}=\frac{p_{1}}{p_{2}} 2 x_{1} \\
& \frac{\partial \mathscr{L}}{\partial \lambda}=0=m-p_{1} x_{1}-p_{2} x_{2} \text { and theretive } m=p_{1} x_{1}+p_{2}\left(\frac{p_{1}}{p_{2}} 2 x_{1}\right) \\
& =p_{1} x_{1}+2 p_{1} x_{1}=3 p_{1} x_{1} \\
& \frac{m}{3 p_{1}}=x_{1} \text { and } x_{2}=\frac{p_{1}}{p_{2}} 2\left[\frac{m}{3 p_{1}}\right] \\
& \Rightarrow x_{2}=\frac{2 m}{3 p_{2}} \text {. }
\end{aligned}
$$

$$
\begin{array}{r}
\mathscr{L}_{\lambda \lambda}=0 \quad \mathscr{L}_{\lambda 1}=-p_{1} \mathscr{L}_{\lambda 2}=-p_{2} \\
\mathscr{L}_{11}=0 \quad \mathscr{L}_{12}=2 x_{2} \\
\mathscr{L}_{22}=2 x_{1}
\end{array}
$$

Exam l 1994 Answer 1 cont.

$$
\nabla^{2} \mathscr{L}=\left[\begin{array}{ccc}
0 & -p_{1} & -p_{2} \\
-p_{1} & 0 & 2 x_{2} \\
-p_{2} & 2 x_{2} & 2 x_{1}
\end{array}\right]
$$

$$
m=1 \text { (\# of constraints) }
$$

$$
2 m+1=3 \text {. So check }\left|\nabla^{2} \mathscr{L}\right| \text {. }
$$

$$
\text { For a maximum, this should have signs }(-1)^{m+1}=(-1)^{2}>0 \text {. }
$$

$$
\begin{aligned}
\left|\nabla^{2} \mathscr{L}\right| & =+p_{1}\left|\begin{array}{cc}
-p_{1} & 2 x_{2} \\
-p_{2} & 2 x_{3}
\end{array}\right|-p_{2}\left|\begin{array}{cc}
-p_{1} & 0 \\
-p_{2} & 2 x_{2}
\end{array}\right| \\
& =p_{1}\left[-2 p_{1} x_{1}+2 p_{2} x_{2}\right]-p_{2}\left(-2 p_{1} x_{2}\right)
\end{aligned}
$$

$=-2 p_{1}^{2} x_{1}+2 p_{1} p_{2} x_{2}+2 p_{1} p_{2} x_{2}$ and substituting the optimal values,
(It would have been better to combine the last two terms into $4 p_{1} p_{2} x_{2}$.)

$$
\begin{aligned}
& =-2 p_{1}^{2}\left(\frac{m}{3 p_{1}}\right)+2 p_{1} p_{2}\left(\frac{2 m}{3 p_{2}}\right)+2 p_{1} p_{2}\left(\frac{2 m}{3 p_{2}}\right) \\
& =\frac{-2}{3} p_{1} m+\frac{4}{3} p_{1} m+\frac{4}{3} p_{1} m
\end{aligned}
$$

$=2 p, m>0$ so the $2^{N D}$ order condition for a maxinumis fulfilled.

Final Exam
2000
(1)

Question 5
5. If a consumer has a standard budget constraint and a utility function $u(\mathrm{x})_{1}=x_{1}^{\alpha}+x_{2}^{\alpha}$, what conditions on $\alpha$ have to be satisfied if the consumer is to be able to maximize utility at an interior point? (Hint: second-order conditions.)
5)

$$
\begin{aligned}
& \max u(x)=x_{1}^{\alpha}+x_{2}^{\alpha} \text { s.t. } p_{1} x_{1}+p_{2} x_{2}=m \\
& \mathcal{L}=x_{1}^{\alpha}+x_{2}^{\alpha}+\lambda\left(m-p_{1} x_{1}-p_{2} x_{2}\right) \\
& \text { F.O.C. }\left\{\begin{array}{l}
0=\partial \alpha / \partial \lambda=m-p_{1} x_{1}-p_{2} x_{2} \\
0=\frac{\partial \mathcal{L}}{\partial x_{1}}=\alpha x_{1}^{\alpha-1}-p_{1} \lambda \\
0=\frac{\partial \mathscr{L}}{\partial x_{2}}=\alpha x_{2}^{\alpha-1}-p_{2} \lambda
\end{array}\right.
\end{aligned}
$$



$$
(-1)^{1+1}=(-1)^{2}=+1
$$

So we want $\left|\nabla^{2} y\right|>0$.

$$
\begin{aligned}
\left|\nabla^{2} \mathcal{D}\right|= & -p_{1}^{2} \alpha(\alpha-1) x_{2}^{\alpha-2}-p_{2}^{2} \alpha(\alpha-1) x_{1}^{\alpha-2} \text { expanding abory the first column } \\
= & { }^{\alpha}(1-\alpha) \underbrace{\left[p_{1}^{2} x_{2}^{\alpha-2}+p_{2}^{2} x_{1}^{\alpha-2}\right]}_{+}
\end{aligned}
$$

Therefore, $\left|\nabla^{2} \mathcal{L}\right|$ will satisfy the S.O.C. when $\alpha \in(0,1)$ because then $\alpha(1-\alpha)$ will be positive.
2. I suggest you read both parts of this question before you begin to work on the first part.
(a) Suppose a consumer has a quasiconcave utility function $u(\mathbf{x})$ where $\mathbf{x} \in \mathbf{R}^{n}$. Prove that if
$\mathbf{x}^{*}$ satisfies the first-order conditions for the problem

$$
\begin{equation*}
\max _{\mathbf{x}} u(\mathbf{x}) \quad \text { s.t. } \mathbf{p} \cdot \mathbf{x}=m \tag{1}
\end{equation*}
$$

where $\mathbf{p}$ are the prices (which the consumer takes as given) and $m$ is income,
then
$\mathrm{x}^{*}$ must actually solve (1).
(b) Show that if the consumer does not take $\mathbf{p}$ as given, but rather has some influence over it-say he receives lower prices for a particular commodity if he buys a great deal of it-then it is no longer necessarily true that "if $\mathbf{x}^{*}$ satisfies the first-order conditions for (1) then $\mathbf{x}^{*}$ must actually solve (1)."

[^0](2) a) The Laprougian is $\mathcal{L}=u(\underline{x})+\lambda(m-\underset{\sim}{p} \cdot \underset{\sim}{x})$. At though the quation only
\[

F.O.C.: $$
\begin{aligned}
& O=\partial z / \partial \lambda=m-p \cdot x \\
& 0=\partial L / \partial x_{1}=u_{1}^{\prime}-\lambda p_{1} \\
& \vdots \\
& O=\partial z / \partial x_{n}=u_{n}^{\prime}-\lambda p_{n}
\end{aligned}
$$
\]

An $\underset{\sim}{x}$ satisfying the F.O.C. will actually so he the optimization problem. if it satisfies the second-order suttivient conditions for on optimum:

$$
\nabla^{2} \mathscr{L}^{\prime}=\left[\begin{array}{cccc}
\mathscr{L}_{x \lambda}^{\prime \prime} & \mathscr{L}_{\lambda x_{1}}^{\prime \prime} & \cdots & \mathscr{L}_{\lambda x_{n}}^{\prime \prime} \\
\mathscr{L}_{x_{1}^{\prime \prime}}^{\prime} & \mathscr{L}_{x_{1} x_{1}}^{\prime} & \cdots & \mathscr{L}_{x_{1} x_{n}}^{\prime \prime} \\
\vdots & & & \mathscr{L}^{\prime \prime} \\
\mathscr{L}_{x_{n} \lambda}^{\prime \prime} & \mathscr{L}_{x_{n} x_{1}}^{\prime \prime} & \cdots & \mathcal{L}_{x_{n} x_{n}}
\end{array}\right]=\left[\begin{array}{cccc}
0 & -p_{1} & \cdots & -p_{n} \\
-p_{1} & u_{1 \prime}^{\prime \prime} & \cdots & u_{1 n}^{\prime \prime} \\
\vdots & \vdots & & \vdots \\
-p_{n} & u_{n 1}^{\prime \prime} & \cdots & u_{n n}^{\prime \prime}
\end{array}\right]
$$

but since from the F.O.C., $p_{i}=u_{i}^{\prime} / \lambda$, we have

$$
\nabla^{2} \mathscr{L}=\left[\begin{array}{cccc}
0 & -u_{1}^{\prime} / \lambda & \cdots & u_{n}^{\prime} / \lambda \\
-u_{1}^{\prime} / \lambda & u_{11}^{\prime \prime} & \cdots & u_{i n}^{\prime \prime} \\
\vdots & \vdots & & \vdots \\
-u_{n}^{\prime} / \lambda & u_{n 1}^{\prime \prime} & & u_{n n}^{\prime \prime}
\end{array}\right]
$$

The S.O.C. are (with $m=1$ ): $D_{2 m+1}$ of $\nabla^{2} \mathscr{L}$ has the sion of $(-1)^{m+1} \Leftrightarrow$

$$
D_{3} \text { of } \nabla^{2} \mathcal{L}
$$

then $D_{4}<0, D_{5}>0$, etc.:

$$
O<D_{3} \text { of } \nabla^{2} \alpha=\left|\begin{array}{ccc}
0 & -u_{1}^{\prime} / \lambda & -u_{2}^{\prime} / \lambda \\
-u_{1}^{\prime} / \lambda & u_{11}^{\prime \prime} & u_{17}^{\prime \prime} \\
-u_{2}^{\prime} / \lambda & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}
\end{array}\right|=\underbrace{\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)}_{\uparrow}\left|\begin{array}{lll}
0 & u_{1}^{\prime} & u_{2}^{\prime} \\
u_{1}^{\prime} & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
u_{2}^{\prime} & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}
\end{array}\right|
$$

positive

$$
\begin{aligned}
0>D_{4} \text { of } \nabla^{2} \mathcal{L} & =\left|\begin{array}{cccc}
0 & -u_{1}^{\prime} / \lambda & -u_{2}^{\prime} / \lambda & -u_{3}^{\prime} / \lambda \\
-u_{1}^{\prime} / \lambda & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} & u_{13}^{\prime \prime} \\
-u_{2}^{\prime} / \lambda & u_{21}^{\prime \prime} & u_{22}^{\prime \prime} & u_{23}^{\prime \prime} \\
-u_{3}^{\prime} / \lambda & u_{31}^{\prime \prime} & u_{32}^{\prime \prime} & u_{33}^{\prime \prime}
\end{array}\right| \\
& =\underbrace{\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)}_{\text {positive }}\left|\begin{array}{llll}
0 & u_{1}^{\prime} & u_{2}^{\prime} & u_{3}^{\prime} \\
u_{1}^{\prime} & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} & u_{13}^{\prime \prime} \\
u_{2}^{\prime} & u_{21}^{\prime \prime} & u_{22}^{\prime \prime} & u_{23}^{\prime \prime} \\
u_{3}^{\prime} & u_{31}^{\prime \prime} & u_{32}^{\prime \prime} & u_{33}^{\prime \prime}
\end{array}\right|
\end{aligned}
$$

etc.
But since $\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)>0$, these are just the conditions for $u$ to be guasitioncone, which was already assumed in the question.
b) The F.O.C. are

$$
\begin{aligned}
0 & =\partial \mathscr{L} / \partial \lambda \\
0 & =m-p \cdot x \text { asbefre, but } \\
0 & =\partial \mathscr{L} / \partial x_{i}
\end{aligned}=u_{i}^{\prime}-\lambda p_{i}-\lambda \frac{\partial p_{i}}{\partial x_{i}} x_{i} .
$$

$$
=\left[\begin{array}{ccc}
0 & -u_{1}^{\prime} / \lambda & \cdots \\
-u_{1}^{\prime} / \lambda & u_{1}^{\prime \prime}-2 \lambda p_{1}^{\prime}-\lambda p_{1}^{\prime \prime} x_{1} & \cdots \\
\vdots & \vdots &
\end{array}\right]
$$

Unlike before, this has no simple relation to the conditions for a to be quasizoncare, so assuming $u$ to be quasizoncare does mot assume that $x^{x}$ satisfies the 5.O.C.
2. [14 points] Suppose a person consumes two goods, $x$ and $y$. The price of $x$ is $\$ 0.50$ per unit (that is, $\$ 1 / 2$ per unit). The price of $y$ reflects a volume discount, and is

$$
1-0.12 y
$$

as long as that is positive. The consumer's income is $\$ 2$. This consumer's affordable set is the shaded area in the graph below.

Suppose this consumer's utility function is

$$
x+y+\frac{1}{10} \ln x+\frac{1}{10} \ln y .
$$

Some of this consumer's indifference curves are shown in the graph below.

(a) Show that the utility function is strictly concave.
(b) Show that any $\left(x^{*}, y^{*}, \lambda^{*}\right)$ satisfying

$$
\begin{aligned}
& 2=-\frac{1}{10} \cdot \frac{2.4 y^{2}-10 y}{4.8 y^{2}-10 y+1}+y-0.12 y^{2} \\
& x=-\frac{1}{5} \cdot \frac{2.4 y^{2}-10 y}{4.8 y^{2}-10 y+1} \\
& \lambda=2+\frac{1}{5 x}
\end{aligned}
$$

(or an equivalent set of equations) satisfies the first-order conditions for utility maximization. (These equations should have been written as functions of $x^{*}, y^{*}$, and $\lambda^{*}$, but I omitted the asterisks for enhanced legibility.) Do not try to solve the system for $y^{*}, x^{*}$, or $\lambda^{*}$.
(c) What sufficient condition would ensure that a vector $\left(x^{*}, y^{*}, \lambda^{*}\right)$ satisfying the conditions of part (b) actually is a maximum? Your answer should be a function of $x^{*}, y^{*}$, and $\lambda^{*}$, but you can omit the asterisks for enhanced legibility.
(d) It can be shown that $\left(x^{*}, y^{*}, \lambda^{*}\right)=(0.704259,2.26171,2.28399)$ satisfies the conditions of part (b). This point is marked as a dot on the graph. However, it violates the condition of part (c) (do not prove this; take my word for it). What is the implication of this violation? Could you have predicted this violation?
(e) What do you guess the consumer's utility-maximizing bundle is? Why? (I am asking for a guess here, not a mathematical investigation.)

Summer 2012, Qualifying Exam, Section 1 Qu. 2
(2)

$$
\begin{aligned}
& P_{x}=\frac{1}{2} \\
& P_{y}=1-0.12 y \text { as long as this is positive } \\
& u(x, y)=x+y+\frac{1}{10} \ln x+\frac{1}{10} \ln y
\end{aligned}
$$

a)

$$
\begin{array}{cc}
u_{x}^{\prime}=1+\frac{1}{10 x} & u_{y}^{\prime}=1+\frac{1}{10 y} \\
u_{x_{x}}^{\prime \prime}=\frac{-1}{10 x^{2}} & u_{y y}^{\prime \prime}=\frac{-1}{10 y^{2}} \\
u_{x y}^{\prime \prime}=0 & 0 \\
\nabla^{2} u(x, y)=\left[\begin{array}{cc}
-1 /\left(10 x^{2}\right) & -1 /\left(10 y^{2}\right)
\end{array}\right]
\end{array}
$$

A sufficient condition for $u$ to be stinity concave is that

$$
\begin{aligned}
& D_{1} \text { of } \nabla^{2} u \text { be }<0 \text { and } \\
& D_{2} \text { of } \nabla^{2} u \text { be }>0 .
\end{aligned}
$$

Here $D$, of $\nabla^{2} u$ is $\frac{-1}{10 x^{2}}<0$ and

$$
D_{2} \text { of } D_{u}^{2} \text { is } \frac{-1}{10 x^{2}} \cdot \frac{-1}{10 y^{2}}-0=\frac{1}{100 x^{2} y^{2}}>0
$$

So 4 is strictly concave. Optional: so it is also concave, and quasicuncave.
b) The consume's publem is to

$$
\begin{aligned}
\max u(x, y) \text { s.t. } \begin{aligned}
& 2=p_{x} x+p_{y} y \\
&=\frac{1}{2} x+(1-0.12 y) y \\
&=\frac{1}{2} x+y-0.12 y^{2} . \\
& \mathscr{L}=x+y+\frac{1}{10} \ln x+\frac{1}{10} \ln y+\lambda\left[2-\frac{1}{2} x-y+0.12 y^{2}\right]
\end{aligned} \\
\text { O.C. }
\end{aligned}
$$

F.O.C.

$$
\begin{align*}
& 0=\partial \mathscr{L} / \partial \lambda=2-\frac{1}{2} x-y+0.12 y^{2}  \tag{1}\\
& 0=\partial z / \partial x=1+\frac{1}{10 x}+\lambda\left[-\frac{1}{2}\right]  \tag{2}\\
& 0=\partial z / \partial y=1+\frac{1}{10 y}+\lambda[-1+0.24 y] \tag{3}
\end{align*}
$$

(2) only involves $\lambda$ and $x$, like the third equation given in the question, so Let's sa if we can obtain it:

$$
(2) \Rightarrow \quad \frac{\lambda}{2}=1+\frac{1}{10 x}
$$

(4) $\lambda=2+\frac{1}{5 x}$. Yes, this confirms the third equation.

The next -simplest equation is (3), be carse unlike (1), (3) does not contain any squared terms. So weill work on it next, substituting in $\lambda$ from (4):

$$
0=1+\frac{1}{10 y}+\left(2+\frac{1}{5 x}\right)(-1+0.24 y)
$$

It should be simple to solve this for $x$ in terms of $y$-obtaining the

Second equation of the answer:

$$
\begin{align*}
& 0=\frac{10 y}{10 y}+\frac{1}{10 y}+\left(2+\frac{1}{5 x}\right)(0.24 y-1) \\
&-\frac{1+10 y}{10 y}=\left(2+\frac{1}{5 x}\right)(0.24 y-1) \\
& \frac{-1}{10 y} \frac{1+10 y}{0.24 y-1}=2+\frac{1}{5 x} \\
& \frac{-1}{5 x}=2+\frac{1}{10 y} \frac{1+10 y}{0.24 y-1}=2+\frac{1+10 y}{2,4 y^{2}-10 y} \\
&=\frac{4.8 y^{2}-20 y}{2.4 y^{2}-10 y}+\frac{1+10 y}{2,4 y^{2}-10 y}=\frac{4.8 y^{2}-10 y+1}{2.4 y^{2}-10 y} \\
& \Rightarrow x=\frac{-1}{5} \underbrace{2.4 y^{2}-10 y}_{\text {this is } 212}  \tag{5}\\
& 2.8 y^{2}-10 y+1
\end{align*},
$$

Confirming the second equation.
The remaining equation must come from (1) since we have not used it yet.
(1) implies

$$
\begin{aligned}
& 2=\frac{1}{2} x+y-0.12 y^{2} ; \text { from (5), this is } \\
& 2=\frac{-1}{10} \frac{2.4 y^{2}-10 y}{4.8 y^{2}-10 y+1}+y-0.12 y^{2}
\end{aligned}
$$

confirming the question's remaining equation.
c) The sinus of $D_{2 m+1}$ of $\nabla^{2} \mathcal{Z}, \ldots, D_{m+n}$ of $\nabla^{2} \mathcal{L}$ should alternate, station with the sign of $(-1)^{m+1}$. Here $n=2 a_{n} m=1$, so the sign of $D_{3}$ of $\nabla^{2}$ should be $(-1)^{1+1}>0$.
From (1), (2), and (3):

$$
\left.\begin{array}{l}
\mathscr{L}_{\lambda \lambda}^{\prime \prime}=0 \\
\mathscr{L}_{x \lambda}^{\prime \prime}=\frac{-1}{2} \\
\mathscr{L}_{y \lambda}^{\prime \prime}=-1+0.24 y \\
\mathscr{L}_{x x}^{\prime \prime}=\frac{-1}{10 x^{2}} \\
\Rightarrow \mathcal{L}_{y x}^{\prime \prime}=0
\end{array} \mathscr{L}_{y y}^{\prime \prime}=\frac{-1}{10 y^{2}}+0.24 \lambda\right]\left[\begin{array}{ccc}
0 & -1 / 2 & 0.24 y-1 \\
-\frac{1}{2} & \frac{-1}{10 x^{2}} & 0 \\
0.24 y-1 & 0 & \frac{-1}{10 y^{2}}+0.24 \lambda
\end{array}\right] .
$$

$$
D_{3} \text { of } \nabla^{2} \mathscr{L}=\left|\nabla^{2} \mathcal{L}\right|=\text { (expanding by the first row) }
$$

$$
(-1)^{1+2}\left(\frac{-1}{2}\right)\left[\frac{-1}{2}\left(\frac{-1}{10 y^{2}}+0.24 \lambda\right)-0\right]
$$

$$
+(-1)^{1+3}(0.24 y-1)\left[0-\frac{-1}{10 x^{2}}(0.24 y-1)\right]
$$

$$
=\frac{1}{2} \cdot \frac{-1}{2}\left(0.24 \lambda-\frac{1}{10 y^{2}}\right)+(0.24 y-1) \frac{1}{10 x^{2}}(0.24 y-1)
$$

$$
=\frac{1}{4}\left(\frac{1}{10 y^{2}}-0.24 \lambda\right)+\frac{1}{10 x^{2}}(0.24 y-1)^{2}
$$

This should be positive for the candidate $(x, y, \lambda)$ to be a maximum.
d) Optional: confirmation that this point satisfies the Frost Order Condifions (part b) is given in Out [6], Out [7], and out [8] of the accompanying hethenatica program. Confirmation that if violates the Second Order Conditions (part c) is given in Out [10] and Out [11] of that program.

Not optional: The implication is that this point is a local utility minimum. This cold have been predicted just by looking at the graph, where motion along the budget constraint in any direction away from the black circle leads to a higher indifference curve.
e) From the graph, it looks like it is a corner solution at the lover ight-hand corner. ("Out [37]" of the Nathenstica program shows this point is $(4,0)$.)

Optional: The thathenatica program shows that appearances are somewhat deceiving. $u(4,0)=-\infty($ Out $[38]$ "). The graph on p. 6 of the Mathematica printout shows three ${ }_{\lambda}^{\text {relevant }}$ points satistying the FOC's (the fourth point has $x<0$ ); I labeled them $A, B$, and $C$. A and $C$ satisfy the S.O.C. (Out [ap] and Out [31]), so are local maxima. Point $C$ is a global max be cause it has the highest utility level (Out [33] exceeds out $[35]$ ). It is at $(3.80,0.099651)$, so close to (4.0) but not at $(4,0)$.


740,
ng
$=:[I] u_{I}$ [1]:=


$=:[z] u_{I}$

Show[BudgetConstraintGraph, IndifferenceCurveGraph];




でした9ぁ・0 Out $[30]=$
-0.0900087
Out $[31]=$ 2.40096
Out $[30]=$ Out［29］$=$

$$
\begin{array}{r}
\text { Out[28]= } \\
\{\{3 . \\
\{- \\
\text { In[29]: }=
\end{array}
$$


$\square \quad \frac{3}{i}$

ssible spelling error: new symbol name "tcpartial"
is similar to existing symbol. "BCpartial". Show [BCpartial, ICpartial,DisplayFunction->\$DisplayFunction];
General::spelli: Contours->\{3.80571\}, ContourShading->False,PlotPoints->15];
Contourplot $[\log [x] / 10+\log [y] / 10+x+y,\{x, 3.6,4\},\{y, 0.001,0.2$ ICpartial=
ColorFunction-> (GrayLevel[(\#, 6) A(0.2)/1.1]\&)
BCpartial=
ContourPlot
$=:\lceil 6 \varepsilon\rceil u_{I}$
2. [11 points] Suppose a utility-maximizing consumer does not take prices as given. You should represent the prices of the two commodities $x_{1}$ and $x_{2}$ he may consume by $p_{1}\left(x_{1}\right)$ and $p_{2}\left(x_{2}\right)$, respectively.
(a) What are the sufficient conditions for a utility maximum? Your answer may include $\lambda^{*}, x_{1}^{*}$, and $x_{2}^{*}$.
(b) Presuming the conditions in (a) are satisfied, what else needs to be true in order for $x_{1}$ to be a normal good?

Fall 2010 Ex. 1 Qu. 2
(2)

The budgot constraint be comes $\begin{gathered}m \\ p\end{gathered}=p_{1}\left(x_{1}\right) x_{1}+p_{2}\left(x_{2}\right) x_{2}$, $\stackrel{\uparrow}{\text { ncome }}$
and the Lagrangion is

$$
\mathcal{L}=u\left(x_{1}, x_{2}\right)+\lambda\left[m-p_{1}\left(x_{1}\right) x_{1}-p_{2}\left(x_{2}\right) x_{2}\right] .
$$

F.O.C.

$$
\begin{aligned}
& O=\frac{\partial z}{\partial \lambda}=m-p_{1}\left(x_{1}\right) x_{1}-p_{2}\left(x_{2}\right) x_{2} \\
& 0=\frac{\partial u}{\partial x_{1}}=u_{1}^{\prime}-\lambda \frac{d p_{1}}{d x_{1}} x_{1}-\lambda p_{1}\left(x_{1}\right) \text { where } u_{1}^{\prime}=\frac{\partial u}{\partial x_{1}}
\end{aligned}
$$

and abbrevation forther,

$$
\begin{aligned}
& =u_{1}^{\prime}-\lambda p_{1}^{\prime} x_{1}-\lambda p_{1}=u_{1}^{\prime}-\lambda\left(p_{1}^{\prime} x_{1}+p_{1}\right) \\
0=\frac{\partial y}{\partial x_{2}} & =u_{2}^{\prime}-\lambda p_{2}^{\prime} x_{2}-\lambda p_{2}=u_{2}^{\prime}-\lambda\left(p_{2}^{\prime} x_{2}+p_{2}\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
\nabla^{2} \mathscr{L} & =\left[\begin{array}{lll}
\mathscr{L}_{\lambda \lambda}^{\prime \prime} & \mathscr{L}_{\lambda x_{1}}^{\prime \prime} & \mathcal{L}_{\lambda x_{2}}^{\prime \prime} \\
\mathscr{L}_{x_{1} \lambda}^{\prime \prime} & \mathcal{L}_{x_{1} x_{1}}^{\prime \prime} & \mathcal{L}_{x_{1} x_{2}}^{\prime \prime} \\
\mathscr{L}_{x_{2} \lambda}^{\prime \prime} & \mathscr{L}_{x_{2} x_{1}}^{\prime \prime} & \mathcal{L}_{x_{2} x_{2}}^{\prime \prime}
\end{array}\right] \\
& =\left[\begin{array}{ccc}
0 & -\left(p_{1}^{\prime} x_{1}+p_{1}\right) & -\left(p_{2}^{\prime} x_{2}+p_{2}\right) \\
-\left(p_{1}^{\prime} x_{1}+p_{1}\right) & u_{11}^{\prime \prime}-\lambda\left(p_{1}^{\prime \prime} x_{1}+p_{1}^{\prime}+p_{1}^{\prime}\right) & u_{12}^{\prime \prime} \\
-\left(p_{2}^{\prime} x_{2}+p_{2}\right) & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}-\lambda\left(p_{2}^{\prime \prime} x_{2}+\right. \\
\left.p_{2}^{\prime}+p_{2}^{\prime}\right)
\end{array}\right]
\end{aligned}
$$

a)

$$
\begin{array}{ll}
n=2 & \text { \& variables } \\
m=1 & \text { \# constraints }
\end{array}
$$

$D_{2 m+1} \ldots . . D_{m+n}$ of $\nabla^{2} \mathcal{L}$ need to alternate in sign beriiming with $(-1)^{m+1}$ to satisfy the sulfiricat condition for amoximum
So we need $D_{3} \quad\left(=D_{2 m+1}=D_{m+n}\right)$ to hare the sign of

$$
(-1)^{1+1}>0 .
$$

For example, expanding cloy the frit row, we want

$$
\begin{aligned}
O< & (-1)^{1+2}(-1)\left(p_{1}^{\prime} x_{1}+p_{1}\right)\left[-\left(p_{1}^{\prime} x_{1}+p_{1}\right)\left[u_{22}^{\prime \prime}-\lambda\left(p_{2}^{\prime \prime} x_{2}+2 p_{2}^{\prime}\right)\right]+\right. \\
& +(-1)^{1+3}(-1)\left(p_{2}^{\prime} x_{2}+p_{2}\right)\left[-\left(p_{1}^{\prime} x_{1}+p_{1}\right) u_{21}^{\prime \prime}+\right. \\
& \left(u_{12}^{\prime}\left(p_{2}^{\prime} x_{2}+p_{2}\right)\left[p_{2}^{\prime \prime}\right)\right] \\
= & \left.\left(p_{1}^{\prime} x_{1}+p_{1}\right)\left\{-\left(p_{1}^{\prime \prime} x_{1}^{\prime}+2 p_{1}^{\prime}\right)\right]\right] \\
& -\left(p_{2}^{\prime} x_{2}+p_{1}\right)\left[p_{22}^{\prime \prime}\right)\left\{-\lambda\left(p_{2}^{\prime \prime} x_{2}+2 p_{2}^{\prime}\right)\right]+u_{12}^{\prime \prime}\left(p_{2}^{\prime} x_{2}+p_{1}\right) u_{21}^{\prime \prime}+\left(p_{2}^{\prime} x_{2}+p_{2}\right)\left\{u_{11}^{\prime \prime}-\lambda\left(p_{1}^{\prime \prime} x_{1}+2 p_{1}^{\prime}\right)\right\} .
\end{aligned}
$$

b) Take the differential of the F.O.C.' ( $m$ is the only exogenous variable and is the variable of interest):

$$
\begin{aligned}
& 0=\mathcal{L}_{\lambda \lambda}^{\prime \prime} d \lambda+\mathcal{L}_{\lambda x_{1}}^{\prime \prime} d x_{1}+\mathcal{L}_{\lambda x_{2}}^{\prime \prime} d x_{2}+1 d m \\
& 0=\mathscr{L}_{x_{1} \lambda}^{\prime \prime} d \lambda+\mathscr{L}_{x_{1} x_{1}}^{\prime \prime} d x_{1}+\mathscr{L}_{x_{1} x_{2}}^{\prime \prime} d x_{2}+0 d m \\
& 0=\mathscr{L}_{x_{2} \lambda}^{\prime \prime} d \lambda+\mathscr{L}_{x_{2} x_{1}}^{\prime \prime} d x_{1}+\mathscr{L}_{x_{2} x_{2}}^{\prime \prime} d x_{2}+0 d m \\
& \Rightarrow \nabla^{2} \mathcal{L}\left[\begin{array}{l}
d \lambda \\
d x_{1} \\
d x_{2}
\end{array}\right]=\left[\begin{array}{c}
-d m \\
0 \\
0
\end{array}\right] \text { or } \\
& \nabla^{2} \mathcal{L}\left[\begin{array}{l}
d \lambda / d m \\
d x_{1} / d m \\
d x_{2} / d m
\end{array}\right]=\left[\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right] \text {. Solve for } \partial x_{1} / \partial m(b y \text { Crime's Rule } \\
& \frac{\partial x_{1}}{\partial m}=\frac{\left|\begin{array}{ccc}
0 & -1 & -\left(p_{2}^{\prime} x_{2}+p_{2}\right) \\
-\left(p_{1}^{\prime} x_{1}+p_{1}\right) & 0 & u_{12}^{\prime \prime} \\
-\left(p_{2}^{\prime} x_{2}+p_{2}\right) & 0 & u_{22}^{\prime \prime}-\lambda\left(p_{2}^{\prime \prime} x_{2}+2 p_{2}^{\prime}\right)
\end{array}\right|}{\left|\nabla^{2} \mathcal{L}\right|} \\
& =\frac{(-1)^{1+2}(-1)\left[-\left(p_{1}^{\prime} x_{1}+p_{1}\right)\left[u_{22}^{\prime \prime}-\lambda\left(p_{2}^{\prime \prime} x_{2}+2 p_{2}^{\prime}\right)\right]+\left(p_{2}^{\prime} x_{2}+p_{2}\right) u_{12}^{\prime \prime}\right]}{\left|\nabla^{2} \mathcal{L}\right|} \\
& =\frac{-\left(p_{1}^{\prime} x_{1}+p_{1}\right)\left[u_{22}^{\prime \prime}-\lambda\left(p_{2}^{\prime \prime} x_{2}+2 p_{2}^{\prime}\right)\right]-\left(p_{2}^{\prime} x_{2}+p_{2}\right) u_{12}^{\prime \prime}}{\left|\nabla^{2} \mathcal{L}\right|} .
\end{aligned}
$$

The denominator is positive from part $(\underset{a}{ })$; if the numerator is positive also, then $\partial x_{1}, 2 m>0$ and the good is normal.
3. Suppose a consumer buys only two goods, $x_{1}$ and $x_{2}$, from his fixed income $m$. The prices for the two goods are $p_{1}$ and $p_{2}$, respectively. Under what conditions will a rise in $p_{2}$ cause the consumer's purchases of $x_{1}$ to fall? (Your answer may contain $\lambda^{*}, x_{1}^{*}$, and $x_{2}^{*}$.).

1999, Exam1, Qu. 3
(3)

$$
\begin{align*}
& \max _{x_{1}, x_{2}} u\left(x_{1}, x_{2}\right) \text { s.t } p_{1} x_{1}+p_{2} x_{2}=m \\
& \mathscr{L}=u\left(x_{1}, x_{2}\right)+\lambda_{1}\left(m-p_{1} x_{1}-p_{2} x_{2}\right) \\
& 0=\mathscr{L}_{\lambda}^{\prime}=m-p_{1} x_{1}-p_{2} x_{2}  \tag{1}\\
& 0=\mathscr{L}_{x_{1}}^{\prime}=u_{1}^{\prime}\left(x_{1}, x_{2}\right)-\lambda p_{1} \\
& 0=\mathscr{L}_{x_{2}}^{\prime}=u_{2}^{\prime}\left(x_{1}, x_{2}\right)-\lambda p_{2}
\end{align*}
$$

endogenous: $\lambda, x_{1}, x_{2}$
exoginous: $p_{1}, p_{2}, m$

1 unchanging, so set $d p_{1}=0$ and $d_{m}=0$

$$
\begin{aligned}
& (1) \Rightarrow \quad 0 d \lambda-p_{1} d x_{1}-p_{2} d x_{2}-x_{2} d p_{2}=0 \\
& (2) \Rightarrow \quad-p_{1} d \lambda+u_{11}^{\prime \prime} d x_{1}+u_{12}^{\prime \prime} d x_{2}+0 d p_{2}=0 \\
& (3) \Rightarrow \quad-p_{2} d \lambda+u_{21}^{\prime \prime} d x_{1}+u_{22}^{\prime \prime} d x_{2}-\lambda d p_{2}=0
\end{aligned}
$$

$$
\left[\begin{array}{ccc}
0 & -p_{1} & -p_{2} \\
-p_{1} & u_{11}^{\prime \prime} & \prime \prime \prime \prime u_{12}^{\prime \prime} \\
-p_{2} & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda \\
d x_{1} \\
d x_{2}
\end{array}\right]=\left[\begin{array}{c}
x_{2} \\
0 \\
\lambda
\end{array}\right] d p_{2}
$$

Exam 1 1999
Answer 3 cont...

$$
\begin{aligned}
& {\left[\begin{array}{ccc}
0 & -p_{1} & -p_{2} \\
-p_{1} & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
-p_{2} & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda / d p_{2} \\
d x_{1} / d p_{2} \\
d x_{2} / d p_{2}
\end{array}\right]=\left[\begin{array}{c}
x_{2} \\
0 \\
\lambda
\end{array}\right] \text {. By Cranes's Rule, }} \\
& \frac{d x_{1}}{d p_{2}}=\frac{\left|\begin{array}{ccc}
0 & x_{2} & -p_{2} \\
-p_{1} & 0 & u_{12}^{\prime \prime} \\
-p_{2} & \lambda & u_{22}^{\prime \prime}
\end{array}\right| \leftarrow+\left(\left.\begin{array}{ccc}
(-1) \\
-p_{1} & -p_{2} \\
-p_{1} & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
-p_{2} & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}
\end{array} \right\rvert\, \longleftrightarrow\right.}{\text { Note that the S.O }} \\
& +(-1)^{1+3}\left(-p_{2}\right)\left(-p_{1} \lambda-0\right) \\
& =-x_{2}\left(p_{2} u_{12}^{\prime \prime}-p_{1} u_{22}^{\prime \prime}\right)+p_{1} p_{2} \lambda \\
& =\underbrace{-x_{2} p_{2} u_{12}^{\prime \prime}}_{(?}+\underbrace{x_{2} p_{1} u_{22}^{\prime \prime}}_{\Theta}+\underbrace{+p_{1} p_{2} \lambda}_{\oplus} \\
& \begin{array}{l}
\text { Inf } u \text { is strictly } \\
\text { concave, whine }
\end{array} \\
& \begin{array}{l}
\text { concave, which } \\
\text { it might not be }
\end{array}
\end{aligned}
$$

Note that the S.O.C. for a maximum, since $2 m+1=3$, are; $D_{3}$ of $\nabla^{2} \mathscr{L}$ has the sign of $(-1)^{m+1}=\oplus \quad$ (sufficient condition); $\hat{\Delta}_{3}$ of $\nabla^{2} \mathscr{L}$ $\qquad$
$\qquad$ (necessary condition).
Since $D_{3}$ of $\nabla^{2} \mathscr{L}=\hat{\Delta}_{3}$ of $\nabla^{2} \mathscr{L}=\left|\nabla^{2} \mathscr{L}\right|=$ the denominator of $d x_{1} / d p_{2}$, this denominator is $\oplus$. (Only when $\nabla^{2} L$ is a $3 \times 3$ matrix is $D_{3}$ of $\nabla^{2} \mathscr{L}=\hat{\Delta}_{3}$ of $\nabla^{2} \mathscr{L}=\left|\nabla^{2} \alpha\right|$.) The exam asks when $d x_{1} / d p_{2}$ will be negative. Given what we now know about
the numerator and denominator of $d x_{1} / d p_{2}$, this will be five when

$$
-x_{2} p_{2} u_{12}^{\prime \prime}+x_{2} p_{1} u_{22}^{\prime \prime}+p_{1} p_{2} \lambda<0 .
$$

Exam 1

$$
1999
$$

Answer 3 cont...

1. [11 points] Suppose a price-taking consumer receives utility from two goods, $x_{1}$ and $x_{2}$. How does the consumer's demand for good 1 change when the consumer's income changes infinitesimally?

Answers to Econ. 7005 Rival Exam, Fall 2006
(1)

$$
\begin{aligned}
& \max u\left(x_{1}, x_{2}\right) \text { set. } p_{1} x_{1}+p_{2} x_{2}=m \\
& \mathscr{\alpha}=u\left(x_{1}, x_{2}\right)+\lambda\left(m-p_{1} x_{1}-p_{2} x_{2}\right) \\
& 0=\frac{\partial \mathscr{L}}{\partial \lambda}=m-p_{1} x_{1}-p_{2} x_{2} \\
& \left.\left.\left.0=\frac{\partial x}{\partial x_{1}}=u_{1}^{\prime}-\lambda p_{1}\right\} \begin{array}{l}
\text { take } \\
0=\frac{\partial \mathscr{L}}{\partial x_{2}}=u_{2}^{\prime}-\lambda p_{2}
\end{array}\right\} \begin{array}{l}
0=-p_{1} d x_{1}-p_{2} d x_{2}+d m \leftarrow \text { if } d p_{1}=d p_{2}=0 \\
0=-p_{1} d \lambda+u_{11}^{\prime \prime} d x_{1}+u_{12}^{\prime \prime} d x_{2}+0 d m
\end{array}\right] \\
& \underset{\sim}{O}=\left[\begin{array}{ccc}
0 & -p_{1} & -p_{2} \\
-p_{1} & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
-p_{2} & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda \\
d x_{1} \\
d x_{2}
\end{array}\right]+\left[\begin{array}{l}
d m \\
0 \\
0
\end{array}\right] \\
& {\left[\begin{array}{ccc}
0 & -p_{1} & -p_{2} \\
-p_{1} & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
-p_{2} & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda / d m \\
d x_{1} / d m \\
d x_{2} / d m
\end{array}\right]=\left[\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right] . \text { UseCrames's Rule: }} \\
& \left|\begin{array}{ccc}
0 & -1 & -p_{2} \\
-p_{1} & 0 & u_{12}^{\prime \prime}
\end{array}\right| \leftarrow(-1)^{1+2}(-1)\left(-p_{1} u_{22}^{\prime \prime}+p_{2} u_{12}^{\prime \prime}\right) \\
& =-p_{1} u_{22}^{\prime \prime}+p_{2} u_{12}^{\prime \prime}
\end{aligned}
$$

$\longleftarrow$ from $2 N P$ OrderCondífions, \#variables $=2$, \# constraints $=1, D_{3}$ of $\nabla^{2} \alpha$ should, for a maximum, be $(-1)^{1+2}=(-1)^{2}=+1$, positive.

$$
\left(D_{2 m+1}=D_{3} \text { and } D_{m+n}=D_{3}\right)
$$

Optional: From the F.O.C., $p_{1}=u_{1}^{\prime} / \lambda$ and $p_{2}=u_{2}^{\prime} / \lambda$, so the numerator of the answer equals $\left(-u_{1}^{\prime} u_{22}^{\prime \prime}+u_{2}^{\prime} u_{12}^{\prime \prime}\right) / \lambda$. Presumably $u_{1}^{\prime}>0$ and $u_{2}^{\prime}>0$. Also, from the Envelope Theorem, $\frac{\partial v}{\partial m}=\frac{\partial L^{*}}{\partial m}=\lambda^{*}$, and since as $m \uparrow$, indirect utility vi, we infer that $\lambda^{*}>0$. However, $u_{22}^{\prime \prime}$ has an unknown sign (it'd be negative if $u$ were concave but we usually only assume quasiconcarity), and $u_{12}^{\prime \prime}$ is certainly of unknown sign. $\left(\frac{\partial x_{1}}{\partial m} \geq 0\right.$ as $x_{1}$ is $\left\{\begin{array}{l}\text { normal } \\ \text { inferior }\end{array}\right)$

Also from the F.O.C., the denominator of $d x_{1} / d m$ is

$$
\left|\begin{array}{ccc}
0 & -u_{1}^{\prime} / \lambda & -u_{2}^{\prime} / \lambda \\
-u_{1}^{\prime} / \lambda & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
-u_{2}^{\prime} / \lambda & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}
\end{array}\right|=\underbrace{\binom{-1}{\lambda}\left(\frac{-1}{\lambda}\right)}_{\oplus}\left|\begin{array}{ccc}
0 & u_{1}^{\prime} & u_{2}^{\prime} \\
u_{1}^{\prime} & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
u_{2}^{\prime} & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}
\end{array}\right|>0 \text { if } u \text { is }
$$

( $($ ) if $u$ is quasiconcave
Denominator. Calculation the determinant:

$$
\begin{aligned}
& (-1)^{2+1}\left(-p_{1}\right)\left(-p_{1} u_{22}^{\prime \prime}+p_{2} u_{21}^{\prime \prime}\right)+(-1)^{3+1}\left(-p_{2}\right)\left(-p_{1} u_{12}^{\prime \prime}+p_{2} u_{11}^{\prime \prime}\right) \\
& =p_{1}\left(-p_{1} u_{22}^{\prime \prime}+p_{2} u_{21}^{\prime \prime}\right)-p_{2}\left(-p_{1} u_{12}^{\prime \prime}+p_{2} u_{11}^{\prime \prime}\right) \\
& =-p_{1}^{2} u_{22}^{\prime \prime}+p_{1} p_{2} u_{21}^{\prime \prime}+p_{1} p_{2} u_{12}^{\prime \prime}-p_{2}^{2} u_{11}^{\prime \prime} \\
& =-p_{1}^{2} u_{22}^{\prime \prime}+2 p_{1} p_{2} u_{12}^{\prime \prime}-p_{2}^{2} u_{11}^{\prime \prime} .
\end{aligned}
$$

Winter 1994 Exam 1 Qu. 5
5. Suppose a price-taking utility-maximizing consumer receives utility from two goods $x_{1}$ and $x_{2}$.
(a) How does this consumer's demand for good 1 change when $p_{1}$ changes infinitesimally?
(b) How does this consumer's demand for good 1 change when $p_{2}$ changes infinitesimally?
(c) Express, as a function of the change in $p_{1}$, how this consumer's demand for good 1 changes when: " $p_{1}$ and $p_{2}$ change simultaneously in such a way that $p_{1}+p_{2}$ is unchanged."

Exam 1
1994
Answer 5
5.

$$
\begin{aligned}
& \mathscr{L}=u\left(x_{1}, x_{2}\right)+\lambda\left(m-p_{1} x_{1}-p_{2} x_{2}\right) \\
& \frac{\partial \mathscr{L}}{\partial \lambda}=0=m-p_{1} x_{1}-p_{2} x_{2} \quad\left[\begin{array}{lll}
\mathscr{L}_{\lambda \lambda} & \mathcal{L}_{\lambda x_{1}} & \mathcal{L}_{\lambda x_{2}} \\
\mathscr{L}_{x_{1} \lambda} & \mathscr{L}_{x_{1} x_{1}} & \mathcal{L}_{x_{1} x_{2}} \\
\mathscr{L}_{x_{2} \lambda} & \mathcal{L}_{x_{2} x_{1}} & \mathscr{L}_{x_{2} x_{2}}
\end{array}\right]\left[\begin{array}{l}
d \lambda \\
d x_{1} \\
d x_{2}
\end{array}\right]=-\left[\begin{array}{c}
-x_{1} \\
-\lambda \\
0
\end{array}\right] d p_{1}-\left[\begin{array}{c}
-x_{2} \\
0 \\
-\lambda
\end{array}\right] d p_{2} \\
& {\left[\begin{array}{ccc}
0 & -p_{1} & -p_{2} \\
-p_{1} & u_{11} & u_{12} \\
-p_{2} & u_{21} & u_{22}
\end{array}\right]\left[\begin{array}{l}
d \lambda \\
d x_{1} \\
d x_{2}
\end{array}\right]=\left[\begin{array}{l}
x_{1} \\
\lambda \\
0
\end{array}\right] d p_{1}+\left[\begin{array}{c}
x_{2} \\
0 \\
\lambda
\end{array}\right] d p_{2}+\left[\begin{array}{cc}
1 & 0 \\
0
\end{array}\right]^{\top} d m \Rightarrow}
\end{aligned}
$$

Long way: Take the total differential of each F. O.C.

$$
\begin{aligned}
& 0=m-p_{1} x_{1}-p_{2} x_{2} \Rightarrow 0=-p_{1} d x_{1}-p_{2} d x_{2}+d m-x_{1} d p_{1}-x_{2} d p_{2} \\
& 0=u_{1}-\lambda p_{1} \Rightarrow 0=-p_{1} d \lambda+u_{11} d x_{1}+u_{12} d x_{2}-\lambda d p^{2} \\
& 0=u_{2}-\lambda p_{2} \Rightarrow 0=-p_{2} d \lambda+u_{12} d x_{1}+u_{22} d x_{2}-\lambda d p_{2}
\end{aligned}
$$

Rearranging jives (1).

$$
\text { a. }\left[\begin{array}{ccc}
0 & -p_{1} & -p_{2} \\
-p_{1} & u_{11} & u_{12} \\
-1 p_{2} & u_{21} & u_{22}
\end{array}\right]\left[\begin{array}{l}
d \lambda \\
d x_{1} \\
d x_{2}
\end{array}\right]=\left[\begin{array}{c}
x_{1} \\
\lambda \\
0
\end{array}\right] d p_{1}
$$

$$
\frac{d x_{1}}{d p_{1}}=\frac{\left|\begin{array}{ccc}
0 & x_{1} & -p_{2} \\
-p_{1} & \lambda & u_{12} \\
-p_{2} & 0 & u_{22}
\end{array}\right|}{\left|\begin{array}{ccc}
0 & -p_{1} & -p_{2} \\
-p_{1} & u_{11} & u_{12} \\
-p_{2} & u_{21} & u_{22}
\end{array}\right|}=
$$

Note: Second Order conditions for a
maximum are: $m=1$, prim. minor of $D^{2} \mathscr{L}$ of $\operatorname{order} 2 m+1=3$ has sigh $(-1)^{m+1}>0$. S. the determinant 3 to.

$$
\text { b. } \begin{aligned}
& {\left[\begin{array}{ccc}
0 & -p_{1} & -p_{2} \\
-p_{1} & u_{11} & u_{12} \\
-p_{2} & u_{21} & u_{22}
\end{array}\right]\left[\begin{array}{l}
d \lambda / d p_{2} \\
d x_{1} / d p_{2} \\
d x_{2} / d p_{2}
\end{array}\right]=\left[\begin{array}{l}
x_{2} \\
0 \\
\lambda
\end{array}\right] } \\
& \frac{d x_{1}}{d p_{2}}=\left|\begin{array}{ccc}
0 & x_{2} & -p_{2} \\
-p_{1} & 0 & u_{12} \\
-p_{2} & \lambda & u_{22}
\end{array}\right|=\frac{-x_{2}\left(-p_{1} u_{22}+p_{2} u_{12}\right)+p_{2} p_{1} \lambda}{\operatorname{sancosin}(a)}
\end{aligned}
$$

$$
=\frac{p_{1} x_{2} u_{22}-p_{2} x_{2} u_{12}-p_{1} p_{2} \lambda}{-p_{1}^{2} u_{22}+2 p_{1} p_{2} u_{12}-p_{2}^{2} u_{11}}
$$

$A$ is the answer to port (a), namely $d x_{1} / d p_{1}$ when $d p_{2}=d m=0$. $B$ is the answer to port ( $b$ ), namely $d x_{1} / d p_{2}$ when $d p_{1}=d m=0$.
c. Let $\frac{d x_{1}}{d p_{1}}=A$ and $\frac{d x_{1}}{d p_{2}}=B$. Here, $p_{1}+p_{2}=$ const. so $d p_{1}+d p_{2}=0$ and $d p_{2}=-d p_{1}$. Hence $d x_{1}=A d p_{1}+B d p_{2}$

$$
\begin{aligned}
& =A d p_{1}+B\left(-d p_{1}\right) \\
& =(A-B) d p_{1} .
\end{aligned}
$$

So $\frac{d x_{1}}{d p_{1}}=A-B \cdot$ when $d p_{2}=-d p_{1}$.
2. [10 points] Suppose a consumer has a standard budget constraint and has a utility function of the form

$$
u\left(x_{1}, x_{2}\right)=x_{1}^{\beta \alpha} x_{2}^{\beta(1-\alpha)}
$$

where $\alpha \in(0,1)$ and $\beta>0$.
(a) Find the Marshallian demand for $x_{1}$.
(b) How does the Marshallian demand for $x_{1}$ change as $\beta$ changes? Why?
(c) Find the Hicksian demand for $x_{1}$.
(d) How does the Hicksian demand for $x_{1}$ change as $\beta$ changes? Why?
(2)
a) $\mathcal{L}=x_{1}^{\beta \alpha} x_{2}^{\beta(1-\alpha)}+\lambda\left(m-p, x_{1}-p_{2} x_{2}\right)$ for $\max _{x_{x}} u(\underline{x})$ s.t. $p \cdot x=m$. Answer 2

$$
\text { F.O.C.: } \begin{align*}
0 & =\frac{\partial \mathscr{Z}}{\partial \lambda}=m-p_{1} x_{1}-p_{2} x_{2}  \tag{1}\\
0 & =\frac{\partial \mathscr{L}}{\partial x_{1}}=\beta \alpha \frac{x_{1}^{\beta_{\alpha}} x_{2}^{\beta(1-\alpha)}}{x_{1}}-\lambda p_{1}  \tag{2}\\
0 & =\frac{\partial \mathscr{Z}}{\partial x_{2}}=\beta(1-\alpha) \frac{x_{1}^{\beta \alpha} x_{2}^{\beta(1-\alpha)}}{x_{2}}-\lambda p_{2} \tag{3}
\end{align*}
$$

$$
\text { Divide (2)\& (3) } \Rightarrow \frac{\beta \alpha}{\beta(1-\alpha)} \frac{x_{1}^{\beta \alpha} x_{2}^{\beta(1-\alpha)}}{x_{1}} \frac{x_{2}}{x_{1}^{\beta_{\alpha}} x_{2}^{\beta(1-\alpha)}}=\frac{\lambda p_{1}}{\lambda p_{2}}
$$

$$
\frac{\alpha}{1-\alpha} \quad \frac{x_{2}}{x_{1}}=\frac{p_{1}}{p_{2}}
$$

$$
X_{2}^{*}=\frac{p_{1}}{p_{2}} \frac{1-\alpha}{\alpha} x_{1}^{*} \text {. Subshtute into budfet constrant: }
$$

$$
\begin{aligned}
& m=p_{1} x_{1}^{*}+p_{2}\left(\frac{p_{1}}{p_{2}} \frac{1-\alpha}{\alpha} x_{1}^{*}\right) \\
&=p_{1} x_{1}^{*}+p_{1} \frac{1-\alpha}{\alpha} x_{1}^{*}=p_{1}\left(1+\frac{1-\alpha}{\alpha}\right) x_{1}^{*}=p_{1}\left(\frac{\alpha}{\alpha}+\frac{1-\alpha}{\alpha}\right) x_{1}^{*}=\frac{p_{1}}{\alpha} x_{1}^{*} \\
& \Rightarrow x_{1}^{*}=\frac{\alpha m_{1}}{p_{1}}
\end{aligned}
$$

b) $x_{1}^{*}$ is unaffected by $\beta$. The vitity function is $\left[x_{1}^{\alpha} x_{2}^{1-\alpha}\right]^{\beta}$, whith Ba movistome transtormation of $x_{1}^{\alpha} x_{2}^{1-\alpha}$, so they have the same demands.
c) $\min _{x} p \cdot x$ st. $u(x)=u_{0} \Rightarrow$

$$
\left.\begin{array}{l}
\min _{x_{1}, x_{2}} p_{1} x_{1}+p_{2} x_{2} \text { st. } x_{1}^{\beta_{\alpha}} x_{2}^{\beta(1-\alpha)}=u_{0} \\
\mathscr{L}=p_{1} x_{1}+p_{2} x_{2}+\lambda\left(x_{1}^{\beta_{\alpha}} x_{2}^{\beta(1-\alpha)}-u_{0}\right) \\
\text { F.O.C. } \quad 0=\frac{\partial \mathscr{L}}{\partial \lambda}=x_{1}^{\beta_{\alpha}} x_{2}^{\beta(1-\alpha)-u_{0}} \\
0=\frac{\partial \mathscr{L}}{\partial x_{1}}=p_{1}+\lambda \beta \alpha \frac{x_{1}^{\beta_{\alpha}} \chi_{2}^{\beta(1-\alpha)}}{x_{1}}=p_{1}+\lambda \beta_{\alpha} \frac{u}{x_{1}} \\
0=\frac{\partial \mathscr{L}}{\partial x_{2}}=p_{2}+\lambda \beta(1-\alpha) \frac{x_{1}^{\beta \alpha} \chi_{2}^{\beta(1-\alpha)}}{x_{2}}=p_{2}+\lambda \beta(1-\alpha) \frac{p_{1}}{x_{2}}
\end{array}\right\}
$$

d) B only affects the cardinal magnitude of the utility target laurel.

## Final Exam 2004 Question 2

$\sqrt{\text { 2. [16 points] Suppose a consumer's utility function takes the form } u(x, y)}$ and is quasiconcave. Suppose the consumer's income is fixed at $m$. Under what conditions on $u$ will $x$ be an inferior good?
To receive full credit, your answer should involve only $u$ or its derivalives.
(You lose five points if you have to ask me to give you the definition of an "inferior good.")
(2)

$$
\begin{aligned}
& \max u(x, y) \text { s.t. } p_{x} x+p_{y} y=m \\
& \mathscr{L}=u(x, y)+\lambda\left(m-p_{x} x-p_{y} y\right) \\
& 0=\partial \mathcal{L} / \partial \lambda=m-p_{x} x-p_{y} y \\
& 0=\partial \mathcal{L} / \partial x=u_{x}^{\prime}-\lambda p_{x} \\
& 0=\partial \mathscr{L} / \partial y=u_{y}^{\prime}-\lambda p_{y}
\end{aligned}
$$

$x$ an inferior food $\Leftrightarrow \partial x / 2 m<0$
so this is a comparative statics problem

Take differentials of the FOC's: $\quad d p_{x}=d p_{y}=0$
$d \lambda d x d y d m$
$0 d \lambda-p_{x} d^{\prime} x-p_{y} d y+d m=0$
$-p_{x} d \lambda+u_{x x}^{\prime \prime} d x+u_{x y}^{\prime \prime} d y+0 e_{m}=0$
$-p_{y} d \lambda+u_{y x}^{\prime \prime} d x+u_{y y}^{\prime \prime} d y+0 d m=0$

Final Exam
2004
Answer 2

$$
\begin{aligned}
& {\left[\begin{array}{ccc}
0 & -p_{x} & -p_{y} \\
-p_{x} & u_{x x}^{\prime \prime} & u_{x y}^{\prime \prime} \\
-p_{y} & u_{y x}^{\prime \prime} & u_{y y}^{\prime \prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda \\
d x \\
d y
\end{array}\right]+\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right] d m=0 . } \\
\Rightarrow & \text { either }\left[\begin{array}{ccc}
0 & -p_{x} & -p_{y} \\
-p_{x} & u_{x x}^{\prime \prime} & u_{x y}^{\prime \prime} \\
-p_{y} & u_{y x}^{\prime \prime} & u_{y y}^{\prime \prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda \\
d x \\
d y
\end{array}\right]=\left[\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right] d m \text { or }\left[\begin{array}{ccc}
0 & p_{x} & p_{y} \\
p_{x} & u_{x x}^{\prime \prime} & u_{x y}^{\prime \prime} \\
p_{y} & u_{y x}^{\prime \prime} & u_{y y}^{\prime \prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda \\
d x \\
d y
\end{array}\right]=\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]
\end{aligned}
$$

$$
\left[\begin{array}{ccc}
0 & -p_{x} & -p_{y} \\
-p_{x} & u_{x x}^{\prime \prime} & u_{x y}^{\prime} \\
-p_{y} & u_{y x}^{\prime \prime} & u_{y y}^{\prime \prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda / d m \\
d x / d m \\
d y / d m
\end{array}\right]=\left[\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right]
$$

Cranmer's Rule:
this case is easier ; Ill omit its continuation, since it's just live the other case wi Thous all the negative signs to worry about.

Final Exam
2004
Answer 2 cont...
From FOC: $-p_{x}=-u_{x}^{\prime} / \lambda$

$$
-p_{y}=-u_{y}^{\prime} / \lambda
$$

So the numerator of $\partial x / \partial \mathrm{m}$ is

$$
\begin{aligned}
\left|\begin{array}{ccc}
0 & -1 & -u_{y}^{\prime} / \lambda \\
-u_{x}^{\prime} / \lambda & 0 & u_{x y}^{\prime \prime} \\
-u_{y}^{\prime} \lambda & 0 & u_{y y}^{\prime \prime}
\end{array}\right| & =(-1)^{1+2}(-1)\left[-\frac{u_{x}^{\prime}}{\lambda} u_{y y}^{\prime \prime}+\frac{u_{y}^{\prime}}{\lambda} u_{x y}^{\prime \prime}\right] \begin{array}{l}
\text { by cotuctor } \\
\text { expannsom aby } \\
\text { second column }
\end{array} \\
& =\frac{1}{\lambda}\left[u_{y}^{\prime} u_{x y}^{\prime \prime}-u_{x}^{\prime} u_{y y}^{\prime \prime}\right] .
\end{aligned}
$$

The demominator of $\partial x / \partial m$ is $\left|\begin{array}{ccc}0 & -u_{x}^{\prime} / \lambda & -u_{y}^{\prime} / \lambda \\ -u_{x}^{\prime} / \lambda & u_{x x}^{\prime \prime} & u_{x y}^{\prime \prime} \\ -u_{y}^{\prime} / \lambda & u_{y x}^{\prime \prime} & u_{y y}^{\prime \prime}\end{array}\right|=$

$$
\underbrace{\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)}_{\substack{(+) \\
\text { sinceths } \\
\text { is }(-1 / \lambda)^{2}}} \underbrace{\left|\begin{array}{ccc}
0 & u_{x}^{\prime} & u_{y}^{\prime} \\
u_{x}^{\prime} & u_{x x}^{\prime \prime} & u_{x y}^{\prime \prime} \\
u_{y}^{\prime} & u_{y x}^{\prime \prime} & u_{y y}^{\prime \prime}
\end{array}\right|}_{\text {abordered Hessima det }}
$$

Final Exam
2004
Answer 2 cont...
a bordered Hessima detiminant, called $\delta_{2}$ in my handort; $u$ quasiconcure $\Rightarrow \delta_{2}>0$.

So the denominuter of $\partial x / \partial m$ is partive.
Furthermore; $\lambda$ is positive since it's $\partial u^{*} / \partial m$ (more monay " $m$ " $\Rightarrow$ hifter $u^{*}$ ).
So for $\partial x / \partial m<0$ we'd need

$$
u_{y}^{\prime} u_{x y}^{\prime \prime}-u_{x}^{\prime} u_{y y}^{\prime \prime}<0
$$

2016 Exam 1 Qu. 3
3. [11 points] Suppose a consumer's utility function takes the form $u(x, y)$ and is quasiconcave. Suppose the consumer's income is fixed at $m$. Under what conditions on $u$ will $x$ be a Giffen good?
To receive full credit, your answer should involve only $u$ or its derivatives.
(You lose three points if you have to ask me to give you the definition of a "Giffen good.")
(3)

$$
\begin{aligned}
& \max u(x, y) \text { s.t. } p_{x} x+p_{y} y=m \\
& \mathscr{L}=u(x, y)+\lambda\left(m-p_{x} x-p_{y} y\right) \\
& 0=\frac{\partial \mathscr{L}}{\partial \lambda}=m-p_{x} x-p_{y} y \\
& 0=\frac{\partial \mathscr{L}}{\partial x}=u_{x}^{\prime}-\lambda p_{x} \\
& 0=\frac{\partial \mathscr{L}}{\partial y}=u_{y}^{\prime}-\lambda p_{y}
\end{aligned}
$$

Take differentials of the F.O.C.'s with $d m=0$ and $d p_{y}=0$ :

$$
\begin{aligned}
& d \lambda \quad d x \quad d y \quad d \rho x \\
& 0 d \lambda-p_{x} d x-p_{y} d y-x d p_{x}=0 \\
& -p_{x} d \lambda+u_{x_{x}}^{\prime \prime} d x+u_{x y}^{\prime \prime} d y-\lambda d p_{x}=0 \\
& -p_{y} d \lambda+u_{y x}^{\prime \prime} d x+u_{y y}^{\prime \prime} d y+0 d p_{x}=0 \\
& {\left[\begin{array}{ccc}
0 & -p_{x} & -p_{y} \\
-p_{x} & u_{x x}^{\prime \prime} & u_{x y}^{\prime \prime} \\
-p_{y} & u_{y x}^{\prime \prime} & u_{y y}^{\prime \prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda \\
d x \\
d y
\end{array}\right]+\left[\begin{array}{c}
-x \\
-\lambda \\
0
\end{array}\right] d p_{x}=0} \\
& {\left[\begin{array}{ccc}
0 & -p_{x} & -p_{y} \\
-p_{x} & u_{x x}^{\prime \prime} & u_{x y}^{\prime \prime} \\
-p_{y} & u_{y x}^{\prime \prime} & u_{y y}^{\prime \prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda \\
d x \\
d y
\end{array}\right]=\left[\begin{array}{c}
x \\
\lambda \\
0
\end{array}\right] d p_{x}} \\
& {\left[\begin{array}{ccc}
0 & -p_{x} & -p_{y} \\
-p_{x} & u_{x x}^{\prime \prime} & u_{x y}^{\prime \prime} \\
-p_{y} & u_{y x}^{\prime \prime} & u^{\prime \prime} y y
\end{array}\right]\left[\begin{array}{l}
d \lambda / d p_{x} \\
d x / d p_{x} \\
d y / d p_{x}
\end{array}\right]=\left[\begin{array}{c}
x \\
\lambda \\
0
\end{array}\right] \text {. Use Crater's Rule to }}
\end{aligned}
$$

Solve for $d x / d p_{x}$ :

$$
\frac{d x}{d p_{x}}=\frac{\left|\begin{array}{ccc}
0 & x & -p_{y} \\
-p_{x} & \lambda & u_{x y}^{\prime \prime} \\
-p_{y} & 0 & u_{y y}^{\prime \prime}
\end{array}\right|}{\left|\begin{array}{ccc}
0 & -p_{x} & -p_{y} \\
-p_{x} & u_{x x}^{\prime \prime} & u_{x y}^{\prime \prime} \\
-p_{y} & u_{y x}^{\prime \prime} & u_{y y}^{\prime \prime}
\end{array}\right|} .
$$

Using the F.O.C.'s, the denominator is $\left|\begin{array}{ccc}0 & -u_{x}^{\prime} / \lambda & -u_{y}^{\prime} / \lambda \\ -u_{x}^{\prime} / \lambda & u_{x x}^{\prime \prime} & u_{x y}^{\prime \prime} \\ -u_{y}^{\prime} / \lambda & u_{y x}^{\prime \prime} & u_{y y}^{\prime \prime}\end{array}\right|$

$$
=\underbrace{\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)}\left|\begin{array}{lll}
0 & u_{x}^{\prime} & u_{y}^{\prime} \\
u_{x}^{\prime} & u_{x x}^{\prime \prime} & u_{x y}^{\prime \prime} \\
u_{y}^{\prime} & u_{y x}^{\prime \prime} & u_{y y}^{\prime \prime}
\end{array}\right|
$$

a bordered Hessian determinant, called $\delta_{2}$ in my handout;

$$
\frac{1}{\lambda^{2}}>0 \quad \text { u quariconcare } \Rightarrow \delta_{2}>0 \text {. }
$$

So the denominator of $d x / d p_{x}$ is positive. Hs numerator is

$$
\begin{aligned}
& \left\lvert\, \begin{array}{ccc|cc}
0 & x & -p_{y} & 0 & x \\
-p_{x} & \lambda & u_{x y}^{\prime \prime} & -p_{x} & \lambda \\
-p_{y} & 0 & u_{y y}^{\prime \prime} & -p_{y} & 0
\end{array}\right. \\
& \begin{aligned}
\text { determinant } & =x u_{x y}^{\prime \prime}\left(-p_{y}\right)-x\left(-p_{x}\right) u_{y y}^{\prime \prime}-\left(-p_{y}\right) \lambda\left(-p_{y}\right) \\
& =-x u_{x y}^{\prime \prime} p_{y}+x p_{x} u_{y y}^{\prime \prime}-p_{y}^{2} \lambda .
\end{aligned}
\end{aligned}
$$

From the F.O.C.'s, we cold use either $\lambda=\frac{u_{x}^{\prime} x}{p_{x}}$ or $\lambda=\frac{u_{y}^{\prime}}{p_{y}}$ :

$$
\text { Numerator of } \begin{aligned}
d x 1 d p_{x} & =-x u_{x y}^{\prime \prime} p_{y}+x p_{x} u_{y y}^{\prime \prime}-p_{y}^{2}\left(\frac{u_{y}^{\prime}}{p_{y}}\right) \\
& =x\left(p_{x} u_{y y}^{\prime \prime}-p_{y} u_{x y}^{\prime \prime}\right)-p_{y} u_{y}^{\prime} .
\end{aligned}
$$

In order for $x$ to be bitten, tho has to be positive:

$$
\begin{aligned}
& x\left(p_{x} u_{y y}^{\prime \prime}-p_{y} u_{x y}^{\prime \prime}\right)-p_{y} u_{y}^{\prime}>0 \\
& x p_{x} u_{y y}^{\prime \prime}-p_{y} u_{y}^{\prime}>p_{y} u_{x y}^{\prime \prime} \\
& \oplus \oplus
\end{aligned}
$$

$$
\begin{array}{lll}
\oplus \oplus \begin{array}{lll}
\oplus & \oplus & \\
\\
& \oplus+\tan \Theta & \oplus \\
\hline
\end{array} \\
\hline
\end{array}
$$

$\underbrace{\operatorname{stth} \theta}_{\text {often } \Theta} \oplus$ so $u_{x y}^{\prime \prime}$ would have to be very negative.
Alternatively: for $x$ to be bitten,

$$
x \frac{u_{x}^{\prime}}{\lambda} u_{y y}^{\prime \prime}-\frac{u_{y}^{\prime}}{\lambda} u_{y}^{\prime}>\frac{u_{y}^{\prime}}{\lambda} u_{x y}^{\prime \prime} .
$$

$\lambda>0$ because using the envelope theorem, $\frac{\partial v}{\partial m}=\lambda$ and $\frac{\partial v}{\partial m}>0$ (move in come $\Rightarrow$ greater utiticf).
So the condition becomes

$$
x^{u_{x}^{\prime}} u_{y y}^{\prime \prime}-\left(u_{y}^{\prime}\right)^{2}>u_{y}^{\prime} u_{x y}^{\prime \prime}
$$

 often as before.
5. [14 points] Consider a consumer who has a quasi-concave utility function defined over two goods. Determine, if possible, the sign of the
Fall2004 slope of this consumer's Hicksian demand curve for good 1 if the conFinal sumer does not take the price of good 2 as given, but rather considers the price of good 2 to be a function of how much of good 2 he consumes.

The expenditure-minimization problem is to
Fall 2004 Trinal
$\min p_{1} x_{1}+p_{2}\left(x_{2}\right) x_{2}$ s.t. $u\left(x_{1}, x_{2}\right)=\bar{u}$

$$
\mathscr{L}=p_{1} x_{1}+p_{2}\left(x_{2}\right) x_{2}+\lambda\left(\bar{u}-u\left(x_{1}, x_{2}\right)\right)
$$

F.O.C.'s: $0=\partial \mathscr{L} / \partial \lambda=\bar{u}-u\left(x_{1}, x_{2}\right)$

$$
0=\partial \mathscr{L} / \partial x_{1}=p_{1}-\lambda u_{1}^{\prime}
$$

$$
0=\partial \mathscr{L} / \partial x_{2}=p_{2}^{\prime} x_{2}+p_{2}-\lambda u_{2}^{\prime} \text { where } p_{2}^{\prime} \text { is } d p_{2} / d x_{2} \text {. }
$$

Endogenous variables: $\lambda_{1} x_{1}, x_{2}$
Exogenous variables: $p_{1}$ is the only one which changes (the others are $p_{2}$ and $\bar{u}$ ).
Find the differential of each F.O.C.:

$$
\begin{aligned}
& d \lambda \\
& 0= \\
& 0
\end{aligned} \begin{array}{ccc}
d x_{1} & d x_{2} & d p_{1} \quad d p_{2}=d u=0 \\
0=-u_{1}^{\prime} d x_{1} & -u_{2}^{\prime} d x_{2} & -\lambda u_{11}^{\prime \prime} d x_{1}-\lambda u_{12}^{\prime \prime} d x_{2}+d p_{1} \\
0=-u_{2}^{\prime} d \lambda-\lambda u_{21}^{\prime \prime} d x_{1}+\left(p_{2}^{\prime \prime} x_{2}+p_{2}^{\prime}+p_{2}^{\prime}-\lambda u_{22}^{\prime \prime}\right) d x_{2}+0 d p_{1} \\
0 & =\left[\begin{array}{ccc}
0 & -u_{1}^{\prime} & -u_{2}^{\prime} \\
-u_{1}^{\prime} & -\lambda u_{11}^{\prime \prime} & -\lambda u_{12}^{\prime \prime} \\
-u_{2}^{\prime} & -\lambda u_{21}^{\prime \prime} & p_{2}^{\prime \prime} x_{2}+2 p_{2}^{\prime}-\lambda u_{22}^{\prime \prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda \\
d x_{1} \\
d x_{2}
\end{array}\right]+\left[\begin{array}{c}
0 \\
d p_{1} \\
0
\end{array}\right]
\end{array}
$$

$$
\left[\begin{array}{lll}
0 & -u_{1}^{\prime} & -u_{2}^{\prime} \\
-u_{1}^{\prime} & -\lambda u_{11}^{\prime \prime} & -\lambda u_{12}^{\prime \prime} \\
-u_{2}^{\prime} & -\lambda u_{21}^{\prime \prime} & -\lambda u_{22}^{\prime \prime}+p_{2}^{\prime \prime} x_{2}+2 p_{2}^{\prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda \\
d x_{1} \\
d x_{2}
\end{array}\right]=\left[\begin{array}{c}
0 \\
-d p_{1} \\
0
\end{array}\right]
$$

or

$$
\left[\begin{array}{lll}
0 & -u_{1}^{\prime} & -u_{2}^{\prime} \\
-u_{1}^{\prime} & -\lambda u_{11}^{\prime \prime} & -\lambda u_{12}^{\prime \prime} \\
-u_{2}^{\prime} & -\lambda u_{21}^{\prime \prime} & -\lambda u_{22}^{\prime \prime}+p_{2}^{\prime \prime} x_{2}+2 p_{2}^{\prime}
\end{array}\right]\left[\begin{array}{l}
d \lambda / d p_{1} \\
d x_{1} / d p_{1} \\
d x_{2} / d p_{1}
\end{array}\right]=\left[\begin{array}{c}
0 \\
-1 \\
0
\end{array}\right]
$$

Creamer's Rule $\Rightarrow$

$$
\frac{d x_{1}}{d p_{1}}=\frac{\left|\begin{array}{ccl}
0 & 0 & -u_{2}^{\prime} \\
-u_{1}^{\prime} & -1 & -\lambda u_{12}^{\prime \prime} \\
-u_{2}^{\prime} & 0 & -\lambda u_{22}^{\prime \prime}+p_{2}^{\prime \prime} x_{2}+2 p_{2}^{\prime}
\end{array}\right|}{\left|\begin{array}{ccl}
0 & -u_{1}^{\prime} & -u_{2}^{\prime} \\
-u_{1}^{\prime} & -\lambda u_{11}^{\prime \prime} & -\lambda u_{12}^{\prime \prime} \\
-u_{2}^{\prime} & -\lambda u_{21}^{\prime \prime} & -\lambda u_{22}^{\prime \prime}+p_{2}^{\prime} x_{2}+2 p_{2}^{\prime}
\end{array}\right|}
$$

Numerator: expand along second column to get

$$
\begin{gathered}
(-1)^{2+2}(-1)\left(0-\left(-u_{2}^{\prime}\right)\left(-u_{2}^{\prime}\right)\right)=-1\left[-u_{2}^{\prime} u_{2}^{\prime}\right] \\
=\left(u_{2}^{\prime}\right)^{2}>0
\end{gathered}
$$

Denominator: This is $\left|\nabla^{2} \mathcal{L}\right|$, whet h for a minimum should hare the sign of $(-1)^{m}=-1^{\prime}=-1$, negative. assume it is a minimum So $d x_{1} / d p_{1}<0$, the usual down-ward slopiry Hizksian demand curve.

$$
\begin{aligned}
& e\left(\lambda p_{1 a}+(1-\lambda) p_{1 b}, \bar{u}\right)=\min _{x_{1}, x_{2}}\left(\lambda p_{1 a}+(1-\lambda) p_{1 b}\right) x_{1}+p_{2}\left(x_{2}\right) x_{2} \text { s.t. } u(\underset{\sim}{x})=\bar{u} \\
& =\min _{x_{1}, x_{2}} \lambda p_{1 a} x_{1}+(1-\lambda) p_{1 b} x_{1}+\underbrace{[\lambda+(1-\lambda)}_{=1}] p_{2}\left(x_{2}\right) x_{2} \text { s.t. } u(x)=\bar{u} \\
& =\min _{x_{1}, x_{2}} \lambda p_{1 a} x_{1}+\lambda p_{2}\left(x_{2}\right) x_{2}+(1-\lambda) p_{1 b} x_{1}+(1-\lambda) p_{2}\left(x_{2}\right) x_{2} \text { s.t. } u(\underset{\sim}{x})=\bar{u} \\
& =\min _{x_{1}, x_{2}} \lambda\left(p_{1 a} x_{1}+p_{2}\left(x_{2}\right) x_{2}\right)+(1-\lambda)\left(p_{1 b} x_{1}+p_{2}\left(x_{2}\right) x_{2}\right) \text { s.t. } u(x)=\bar{u} \\
& \geqslant \min _{x_{1}, x_{2}} \lambda\left(p_{1 a} x_{1}+p_{2}\left(x_{2}\right) x_{2}\right) \text { s.t. } u(x)=\bar{u} \\
& +\min _{x_{1}, x_{2}}(1-\lambda)\left(p_{1 b} x_{1}+p_{2}\left(x_{2}\right) x_{2}\right) \text { s.t. } u\left({\underset{\sim}{x}}_{x}\right)=\bar{u} \\
& =\lambda \min _{x_{1}, x_{2}} p_{1 a} x_{1}+p_{2}\left(x_{2}\right) x_{2} \text { s.t. } u(x)=\bar{u} \\
& +(1-\lambda) \min _{x_{1}, x_{2}} p_{1 b}+p_{2}\left(x_{2}\right) x_{2} \text { s,t. } u(\underset{\sim}{x})=\bar{u} \\
& =\lambda e\left(p_{1 a}, \bar{u}\right)+(1-\lambda) e\left(p_{1 b}, \bar{u}\right) \text {. Then } 0 \geqslant \frac{\partial^{2} e}{\partial p_{1}^{2}}=\frac{\partial}{\partial p_{1}} \frac{\partial e}{\partial p_{1}} \\
& =\frac{\partial}{\partial p_{1}} h_{1} \text { so }{ }^{\text {sownward }} \text { is } \\
& \text { downward- } \\
& \text { sloping. }
\end{aligned}
$$

## Answer all of the following three questions.

1. [11 points] Suppose a consumer's utility function is given by a quasiconcave function $u\left(x_{1}, x_{2}\right)$.
(a) Suppose the consumer takes the price of $x_{1}$ as given and the price of $x_{2}$ as given. Call these prices $p_{1}$ and $p_{2}$. Let income be $m$. Implicitly find the consumer's demand for $x_{1}$ and $x_{2}$ and verify that these demands actually do maximize the consumer's utility.
(b) Suppose the consumer takes the price of $x_{1}$ as given, but the consumer faces a price of $x_{2}$ which declines the more of $x_{2}$ the consumer buys. Implicitly find the consumer's demand for $x_{1}$ and $x_{2}$ and try to verify that these demands actually do maximize the consumer's utility.

Proposition 2. [Test of Pseudoconvexity.] Let $f$ be a $C^{2}$ function defined in an open, convex set $S$ in $R^{n}$. Define the "bordered Hessian" determinants $\delta_{r}(\mathrm{x}), r=1, \ldots, n$ by

$$
\delta_{r}(\mathrm{x})=\left|\begin{array}{cccc}
0 & f_{1}^{\prime}(\mathrm{x}) & \cdots & f_{r}^{\prime}(\mathrm{x}) \\
f_{1}^{\prime}(\mathrm{x}) & f_{11}^{\prime \prime}(\mathrm{x}) & \cdots & f_{1 r}^{\prime \prime}(\mathrm{x}) \\
\vdots & \vdots & \ddots & \vdots \\
f_{r}^{\prime}(\mathrm{x}) & f_{r 1}^{\prime \prime}(\mathrm{x}) & \cdots & f_{r r}^{\prime \prime}(\mathrm{x})
\end{array}\right|
$$



A sufficient condition for $f$ to be pseudoconvex is that $\delta_{r}(\mathrm{x})<0$ for $r=2$, $\ldots, n$, and all $\mathrm{x} \in S$.
[Proposition 2': Similarly, a sufficient condition for $f$ to be pseudoconcave is that $\delta_{r}(\mathrm{x})$ alternate in sign beginning with $>0$ for $r=2, \ldots, n$, and all $\mathrm{X} \in S$.]

Answers to Econ 7005 Midterm, Fall 2006
(1) a) max $u\left(x_{1}, x_{2}\right)$ s.t. $p_{1} x_{1}+p_{2} x_{2}=m$

$$
\begin{align*}
& \mathscr{L}=u\left(x_{1}, x_{2}\right)+\lambda\left(m-p_{1} x_{1}-p_{2} x_{2}\right) \\
& \text { F.O.C. } \quad 0=\partial \mathscr{L} / \partial \lambda=m-p_{1} x_{1}-p_{2} x_{2}  \tag{1}\\
& 0=\partial \mathscr{L} / \partial x_{1}=u_{1}^{\prime}-\lambda p_{1}  \tag{2}\\
& 0=\partial \mathscr{L} / \partial x_{2}=u_{2}^{\prime}-\lambda p_{2} \tag{3}
\end{align*}
$$

S.O.C.

$$
\nabla^{2} \mathscr{L}=\left[\begin{array}{lll}
\mathscr{L}_{\lambda \lambda}^{\prime \prime} & \mathcal{L}_{\lambda x_{1}}^{\prime \prime} & \mathscr{L}_{\lambda x_{2}}^{\prime \prime} \\
\mathscr{L}_{x_{1} \lambda}^{\prime \prime} & \mathscr{L}_{x_{1} x_{1}}^{\prime \prime} & \mathscr{L}_{x_{1} x_{2}}^{\prime \prime} \\
\mathscr{L}_{x_{2, \lambda} \prime \prime}^{\prime \prime} & \mathcal{L}_{x_{2} x_{1}}^{\prime \prime} & \mathcal{L}_{x_{2} x_{2}}^{\prime \prime}
\end{array}\right]=\left[\begin{array}{ccc}
0 & -p_{1} & -p_{2} \\
-p_{1} & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
-p_{2} & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}
\end{array}\right]
$$

but from (d), $-p_{1}=-u_{1}^{\prime} / \lambda ;$ from (3), $-p_{2}=-u_{2}^{\prime} / \lambda$.
Substituting,

$$
\nabla^{2} \mathscr{L}=\left[\begin{array}{ccc}
0 & -u_{1}^{\prime} / \lambda & -u_{2}^{\prime} / \lambda \\
-u_{1}^{\prime} / \lambda & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
-u_{2}^{\prime} / \lambda & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}
\end{array}\right]
$$

The S.O.C. for a maximum are that $D_{2 m+1} \cdots D_{m+n}$ of $\nabla^{2} \mathcal{L}$ alternate in sign starting with $(-1)^{m+1}$. Here $m=1$ (constraint)) and $n=2$ (variables), so $2 m+1=3$ and, $m+n=3$ and $m+1=2$, So we need $D_{3}$ of $V^{2} \mathcal{L}$ to be the same sign as $(-1)^{2}>0$.

$$
D_{3} \text { of } \nabla^{2} \mathscr{L}=\left|\nabla^{2} \alpha\right|=\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)\left|\begin{array}{ccc}
0 & u_{1}^{\prime} & u_{2}^{\prime} \\
u_{1}^{\prime} & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
u_{2}^{\prime} & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}
\end{array}\right|
$$

Obviously $\left(\frac{-1}{\lambda}\right)\left(-\frac{1}{\lambda}\right)=\left(\frac{-1}{\lambda}\right)^{2}>0$. In adaction, $\left|\begin{array}{ccc}0 & u_{1}^{\prime} & u_{2}^{\prime} \\ u_{1}^{\prime} & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\ u_{2}^{\prime} & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}\end{array}\right|=\delta_{2}$, the "bordered Hessian" of $u$. Since a is quasiconcave, we know that $\delta_{2}>0$. Thus $D_{3}$ of $D^{2} \mathcal{L}$ is $>0$, fulfilling the S.O.C.
b) $\mathcal{L}$ is the same as in (a), but now $p_{2}$ i a function of $x_{2}$ :
F.O.C.

$$
\begin{align*}
& 0=\partial \mathscr{L} / \partial \lambda=m-p_{1} x_{1}-p_{2} x_{2}  \tag{1}\\
& 0=\partial \mathscr{L} / \partial x_{1}=u_{1}^{\prime}-\lambda p_{1} \\
& 0=\partial \mathscr{L} / \partial x_{2}=u_{2}^{\prime}-\lambda p_{2}-\lambda p_{2}^{\prime} x_{2} \text { with } p_{2}^{\prime}=\frac{d p_{2}}{d x_{2}} .
\end{align*}
$$

$$
\begin{aligned}
\nabla^{2} \alpha & =\left[\begin{array}{lll}
0 & -p_{1} & -p_{2}-p_{2}^{\prime} x_{2} \\
-p_{1} & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
-p_{2}-p_{2}^{\prime} x_{2} & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}-\lambda p_{2}^{\prime}-\lambda p_{2}^{\prime \prime} x_{2}-\lambda p_{2}^{\prime}
\end{array}\right] ;(2) \\
& =\left[\begin{array}{ccc}
0 & -u_{1}^{\prime} / \lambda & -u_{2}^{\prime} / \lambda \\
\text { and } \\
\left(3^{\prime}\right) \\
-u_{1}^{\prime} / \lambda & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
-u_{2}^{\prime} / \lambda & u_{21}^{\prime \prime} & u_{22}^{\prime \prime}-2 \lambda p_{2}^{\prime}-\lambda p_{2}^{\prime \prime} x_{2}
\end{array}\right]
\end{aligned}
$$

and as before,

$$
\left|\nabla^{2} \alpha\right|=\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)\left|\begin{array}{lll}
0 & u_{1}^{\prime} & u_{2}^{\prime} \\
u_{1}^{\prime} & u_{11}^{\prime \prime} & u_{12}^{\prime \prime} \\
u_{2}^{\prime} & u_{22}^{\prime \prime} & u_{22}^{\prime \prime}-2 \lambda p_{2}^{\prime}-\lambda p_{2}^{\prime \prime} x_{2}
\end{array}\right| .
$$

This is not a bordered Hessian, and while $p_{2}^{\prime}<0$ and $\lambda$ is (typically) positive, not knowing $P_{2}^{\prime \prime}$ makes it clear we will be unable to sign $\left|D^{2} \not \partial\right|$. So we don't know if the $x_{1}{ }^{*}$ and $x_{2}{ }^{*}$ imploitly defined in (1')-(3') satisfy the S.O.C.

Fall 2006
Final
4. [11 points] Compose a problem of economic importance which involves quasiconcavity. Then work the problem you composed and demonstrate the importance of quasiconcavity in your mathematical working-out of that problem.

Answers will vary.
2. [11 points] Suppose a price-taking consumer has income $m$ and utility function $u=\ln y+\ln z$ where $y$ and $z$ are the two goods which the consumer buys. For this consumer, calculate separately the left:hand side and right-hand side of the Slutsky Equation

$$
\frac{\partial x_{y}(\mathrm{p}, m)}{\partial p_{z}}=\frac{\partial h_{y}(\mathrm{p}, v(\mathbf{p}, m))}{\partial p_{z}}-\frac{\partial x_{y}(\mathrm{p}, m)}{\partial m} x_{z}(\mathrm{p}, m) \quad \text { Fall } 2006
$$

(where $x$ denotes the Marshallian demand curve and $h$ the Hicksian demand curve), and show that the left-hand side is equal to the righthand side. (In the process, it helps to calculate the consumer's indirect utility function.)
(2)

Faw ${ }^{2006} \operatorname{cind}^{2 l}$
Or covld stat by taking a mionotionc transtormation of the vility fonction;

$$
v=\ln y^{*}+\ln z^{*}=\ln \frac{m}{2 p y}+\ln \frac{m}{2 p_{z}}
$$

$$
=\ln \frac{m^{2}}{4 p_{y} p_{z}}=v\left(p_{1} m\right) .
$$

$v(p, e(p, u)) \equiv u$ so from above,

$$
\left.\begin{array}{l}
\ln \frac{e\left(p_{p} u\right)^{2}}{4 p_{y} p_{z}}=u x \\
\frac{e\left(p_{1} u\right)^{2}}{4 p_{y} p_{z}}=e^{u} \\
e\left(p_{1} u\right)^{2}=4 p_{y} p_{z} e^{u} \\
e\left(p_{1} u\right)=2 \sqrt{p_{y} p_{z}} e^{u / 2} \\
\sim \\
h_{y}=\frac{\partial e}{\partial p_{y}}=\sqrt{\frac{p_{z}}{p_{y}}} e^{u / 2}
\end{array}\right]
$$

Marshallion Demand Curres

$$
\begin{aligned}
& \text { one such traw tormation is } \\
& \mathscr{L}=\ln y+\ln z+\lambda\left(m-p_{y} y-p_{z} z\right) \\
& 0=\frac{\partial \mathscr{L}}{\partial \lambda}=m-p_{z} z-p_{y} y \\
& 0=\frac{\partial z}{\partial y}=\frac{1}{y}-\lambda p_{y} \Rightarrow \lambda=\frac{1}{y p_{y}} \\
& \left.0=\frac{\partial x}{\partial z}=\frac{1}{z}-\lambda p_{z} \Rightarrow \lambda=\frac{1}{z p_{z}}\right\} \frac{1}{y p_{y}}=\frac{1}{z p_{z}} \\
& z=\frac{y p_{y}}{p_{z}} \text { and } \\
& 0=m-p_{z}\left(\frac{y p_{y}}{p_{z}}\right)-p_{y} y \\
& =m-y p_{y}-y p_{y}=m-2 y p_{y} \\
& 2 y p_{y}=m \\
& y=\frac{m}{2 p_{y}} \text { and } z=\frac{p_{y}}{p_{z}} \cdot \frac{m}{2 p_{y}}=\frac{m}{2 p_{z}} .
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial h_{y}}{\partial p_{z}}=\frac{1}{2} \sqrt{\frac{1}{p_{y} p_{z}}} e^{u / 2} \\
& \begin{aligned}
& \frac{\partial h_{y}\left(p_{1}, v(p, m)\right)}{\partial p_{z}}=\frac{1}{2} \sqrt{\frac{1}{p_{y} p_{z}}} e^{v\left(p_{1}, m\right) / 2} \\
&=\frac{1}{2} \sqrt{\frac{1}{p_{y} p_{z}}} e^{\frac{1}{2} \ln \frac{m^{2}}{4 p_{y} p_{z}}} \\
&=\frac{1}{2} \sqrt{\frac{1}{p_{y} p_{z}}} e^{\ln \left(\frac{m^{2}}{4 p_{y} p_{z}}\right)^{1 / 2}}=\frac{1}{2} \sqrt{\frac{1}{p_{y} p_{z}}} e^{\ln \frac{m}{2 \sqrt{p_{y} p_{z}}}} \\
&=\frac{1}{2} \sqrt{\frac{1}{p_{y} p_{z}}} \frac{m}{2 \sqrt{p_{y} p_{z}}}=\frac{m}{4 p_{y} p_{z}} \\
& \begin{aligned}
\frac{\partial x_{y}}{\partial m} & =\frac{\partial}{\partial m} \frac{m}{2 p_{y}}
\end{aligned}=\frac{1}{2 p_{y}} \\
& \frac{\partial x_{y}}{\partial m} x_{z}=\frac{1}{2 p_{y}} \frac{m}{2 p_{z}}=\frac{m}{4 p_{y} p_{z}}
\end{aligned}
\end{aligned}
$$

RHS of Slutsky Equation: $\frac{\partial h_{y}\left(p_{1} v\right)}{\partial p_{z}}-\frac{\partial x_{y}}{\partial m} x_{z}=\frac{m}{4 p_{y} p_{z}}-\frac{m}{4 p_{y} p_{z}}=0$ LHS of slutiky Equation: $\frac{\partial x_{y}}{\partial p_{z}}=\frac{\partial}{\partial p_{z}} \quad \frac{m}{2 p_{y}}=0$.

Since $O=0$, verification is complete.
3. [11 points] Suppose a consumer consumes two goods, $x_{1}$ and $x_{2}$, and has a utility function of

$$
u(\mathbf{x})=x_{1}^{1 / 2} x_{2}^{1 / 2}
$$

This consumer takes the prices of the goods $p_{1}$ and $p_{2}$ as given, and has a fixed income $m$.
(a) Find the consumer's Hicksian demand curve for good $1, h_{1}(\mathbf{p}, \hat{u})$, without explicitly solving a utility-maximization problem.
(b) Find this consumer's expenditure function. Hint: one way of writing this consumer's expenditure function is

$$
\left(\frac{p_{2}}{p_{1}}\right)^{1 / 2} p_{1} \hat{u}+\left(\frac{p_{1}}{p_{2}}\right)^{1 / 2} p_{2} \hat{u}
$$

(c) Find this consumer's indirect utility function from this consumer's expenditure function. Hint: one way of writing this consumer's indirect utility function is

$$
m p_{1}^{-1 / 2} p_{2}^{-1 / 2}
$$

(d) Derive this consumer's Marshallian demand function for good 1 from the consumer's indirect utility function.
(e) Using your answers to parts (a) and (d), verify the Slutsky equation

$$
\frac{\partial x_{1}}{\partial p_{1}}=\frac{\partial h_{1}}{\partial p_{1}}-x_{1} \frac{\partial x_{1}}{\partial m}
$$

for this consumer. (In other words, calculate each side of this equation separately, then show that they are equal to each other.) If you need this consumer's Marshallian demand function for good 2 in order to solve this problem, you may deduce its form by symmetry from the answer to part (d) instead of deriving it.

Fall 2012 Exam 1

Answer to Question 3 of Exam 1, Fall 2012, Econ. 7005
a)

$$
\begin{align*}
& \min _{x} p \cdot x \text { st. } u(x)=\hat{u} \\
& \mathscr{L}=p_{1} x_{1}+p_{2} x_{2}+\lambda\left[\hat{u}-x_{1}^{1 / 2} x_{2}^{1 / 2}\right] \\
& 0=\partial \mathscr{L} / \partial \lambda=\hat{u}-x_{1}^{1 / 2} x_{2}^{1 / 2}  \tag{1}\\
& 0=\partial \mathscr{L} / \partial x_{1}=p_{1}-\lambda\left(\frac{1}{2}\right) x_{1}^{-1 / 2} x_{2}^{1 / 2}  \tag{2}\\
& 0=\partial \mathscr{L} / \partial x_{2}=p_{2}-\lambda\left(\frac{1}{2}\right) x_{1}^{1 / 2} x_{2}^{-1 / 2} . \tag{3}
\end{align*}
$$

(2) $\Rightarrow \lambda=2 p_{1} x_{1}^{1 / 2} x_{2}^{-1 / 2}$; substitution into (3),

$$
\begin{aligned}
p_{2} & =2 p_{1} x_{1}^{1 / 2} x_{2}^{-1 / 2} \cdot\left(\frac{1}{2}\right) x_{1}^{11} x_{2}^{-1 / 2} \\
& =p_{1} x_{1} x_{2}^{-1} \Rightarrow x_{1}=\left(p_{2} / p_{1}\right) x_{2} \text {. Substititerto (1): } \\
\hat{u} & =\left(\frac{p_{2}}{p_{1}}\right)^{1 / 2} x_{2}^{1 / 2} \cdot x_{2}^{1 / 2}=x_{2} \sqrt{\frac{p_{2}}{p_{1}}} \Rightarrow x_{2}=\hat{u} \sqrt{\frac{p_{1}}{p_{2}}} \text {. Then from }
\end{aligned}
$$

the sentence before last, $x_{1}=\left(\frac{p_{2}}{p_{1}}\right) x_{2}=\left(\frac{p_{2}}{p_{1}}\right) \hat{u} \sqrt{\frac{p_{1}}{p_{2}}}=\hat{u} \sqrt{\frac{p_{2}}{p_{1}}}$.
Hence

$$
h_{1}(p, \hat{u})=\hat{u} \sqrt{\frac{p_{2}}{p_{1}}} .
$$

b)

$$
\begin{aligned}
& =\hat{u} \sqrt{p_{1} p_{2}}+\hat{u} \sqrt{p_{1} p_{2}}=2 \hat{u} \sqrt{p_{1} p_{2}} .
\end{aligned}
$$

c)

$$
m=e(p, v(p, m))=e(p, v)=2 v \sqrt{p_{1} p_{2}} \text { from }(b) \text {; }
$$

so $v=\frac{m}{2 \sqrt{p_{1} p_{2}}}=\frac{m}{2} p_{1}^{-1 / 2} p_{2}^{-1 / 2}$.
d) $x_{1}=\frac{-\partial v / \partial p_{1}}{\partial v / \partial m}=-\frac{\frac{m}{2} \cdot \frac{-1}{2} p_{1}^{-3 / 2} p_{2}^{-1 / 2}}{\frac{1}{2} p_{1}^{-1 / 2} p_{2}^{-1 / 2}}=\frac{m}{2} p_{1}^{-1}=\frac{m}{2 p_{1}}$.
e) $\frac{\partial x_{1}}{\partial p_{1}}=\frac{\partial}{\partial p_{1}} \frac{m}{2} p_{1}^{-1}=\frac{-m}{2 p_{1}^{2}}$. This is the Left-hand sidle.

$$
\frac{\partial h_{1}}{\partial p_{1}}=\frac{\partial}{\partial p_{1}} \hat{u} \sqrt{\frac{p_{2}}{p_{1}}}=\frac{\partial}{\partial p_{1}} \hat{u} p_{2}^{1 / 2} p_{1}^{-1 / 2}=\frac{-1}{2} \hat{u} p_{2}^{1 / 2} p_{1}^{-3 / 2} .
$$

$\frac{\partial x_{1}}{\partial m}=\frac{\partial}{\partial m} \frac{m}{2 p_{1}}=\frac{1}{2 p_{1}}$. So the right-hand side is

$$
\frac{-1}{2} \hat{u} p_{1}^{-3 / 2} p_{2}^{1 / 2}-\frac{m}{2 p_{1}} \frac{1}{2 p_{1}}=\frac{-1}{2}\left(x_{1}^{\left.1 / 2 x_{2}^{1 / 2}\right)} p_{1}^{-3 / 2} p_{2}^{1 / 2}-\frac{m}{4 p_{1}^{2}}\right.
$$

$$
=\frac{-1}{2}\left[\left(\frac{m}{2 p_{1}}\right)^{1 / 2}\left(\frac{m}{2 p_{2}}\right)^{1 / 2}\right] p_{1}^{-3 / 2} p_{2}^{1 / 2}-\frac{m}{4 p_{1}^{2}}
$$

Or instead from (c) use $\hat{u}=m /\left(2 \sqrt{P_{1} p_{2}}\right)$.
$x_{2}$ by symmetry

$$
=\frac{-1}{2} \frac{m}{2 \sqrt{p_{1} p_{2}}} p_{1}^{-\frac{3}{2}} p_{2}^{1 / 2}-\frac{m}{4 p_{1}^{2}}=\frac{-m}{2} \cdot \frac{1}{2} p_{1}^{-\frac{3}{2}-\frac{1}{2}} p_{2}^{-\frac{1}{2}+\frac{1}{2}}-\frac{m}{4 p_{1}^{2}}
$$

$=\frac{-m}{4 p_{1}^{2}}-\frac{m}{4 p_{1}^{2}}=\frac{-m}{2 p_{1}^{2}}$, which agrees with the
left-hand side.

$$
\begin{gathered}
\text { Exam 1 } \\
1997 \\
\text { Question } 2
\end{gathered}
$$

(1).
' 2. Suppose a consumer's expenditure function $e(\mathbf{p}, u)$ is equal to $p_{1}^{a} p_{2}^{1-a} u$, where the $p$ 's denote prices and $u$ denotes a utility level.
(a) Find the consumer's Hicksian demand curve for $\operatorname{good} 2, h_{2}(\mathbf{p}, u)$.
(b) Find the consumer's Marshallian demand curve for $\operatorname{good} 2, x_{2}(\mathbf{p}, m)$, where $m$ is income.
(c) Verify the following Slutsky equation for this consumer:

$$
\frac{\partial x_{2}(\mathbf{p}, m)}{\partial p_{2}}=\frac{\partial h_{2}(\mathbf{p}, u)}{\partial p_{2}}-x_{2}(\mathbf{p}, m) \frac{\partial x_{2}(\mathbf{p}, m)}{\partial m} .
$$

(2)

$$
e(p, u)=p_{1}^{a} p_{2}^{1-a} u
$$

Exam 1
a) $h_{2}=\frac{\partial e}{\partial p_{2}}=(1-a) p_{1}^{a} p_{2}^{-a} u$

$$
1997
$$

Answer 2
b) Since $e(p, r(p, m))=m$ where $m$ is incorie,

$$
\begin{aligned}
& p_{1}^{a} p_{2}^{l-a} v(p, m)=m \Rightarrow \\
& v(p, m)=m p_{1}^{-a} p_{2}^{a-1} .
\end{aligned}
$$

Then by Roy's Identity,

$$
x_{2}=-\frac{\partial v / \partial p_{2}}{\partial v / \partial m}=-\frac{(a-1) m p_{1}^{-a} p_{2}^{a-2}}{p_{1}^{-a} p_{2}^{a-1}}=(1-a) m p_{2}^{-1} ;
$$

- alternetrely,

$$
x_{2}(p, m)=h_{2}\left(p_{1}, v(p, m)\right)=(1-a) p_{1}^{a} p_{2}^{-a}\left(m p_{1}^{-a} p_{2}^{a-1}\right)=(1-a) m p_{2}^{-1} \text {. }
$$

c) The left-hand side is

$$
\frac{\partial x_{2}}{\partial p_{2}}=\frac{\partial}{\partial p_{2}}(1-a) m p_{2}^{-1}=(a-1) m p_{2}^{-2} .
$$

Exam 1
Answer 2 cont...
The nght-hand side is

$$
\begin{aligned}
& \frac{\partial h_{2}}{\partial p_{2}}-x_{2} \frac{\partial x_{2}}{\partial m}=\frac{\partial\left[(1-a) p_{1}^{a} p_{2}^{-a} u\right]}{\partial p_{2}}-\frac{(1-a) m}{p_{2}} \frac{\partial}{\partial m} \frac{(1-a) m}{p_{2}} \\
& =-a(1-a) p_{1}^{a} p_{2}^{-a-1} u-\frac{(1-a) m}{p_{2}} \frac{(1-a)}{p_{2}} ; \text { since } u=v\left(p_{1}, m\right), \\
& =-a(1-a) p_{1}^{a} p_{2}^{-a-1} m p_{1}^{-a} \underline{p_{2}}-\frac{(1-a)^{2} m}{p_{2}^{2}} \\
& =\frac{-a(1-a) m}{p_{2}^{2}}-\frac{(1-a)^{2} m}{p_{2}^{2}}=\frac{-(1-a) m}{p_{2}^{2}}[a+1-a] \\
& =(a-1) m p_{2}^{-2}, \text { which is the same as the left-hand side. }
\end{aligned}
$$

Finial Exam. 2000 (7) Question 4
4. Suppose a consumer has a standard budget constraint and a utility function $u(\mathrm{x})=x_{1}+\frac{1}{2} x_{2}$. Find this consumer's indirect utility functron.
(4) $u(\underset{\sim}{x})=x_{1}+\frac{1}{2} x_{2}$. The budget constraint is $p_{1} x_{1}+p_{2} x_{2}=m$.

$$
\begin{aligned}
& \mathscr{L}=x_{1}+\frac{1}{2} x_{2}+\lambda\left(m-p_{1} x_{1}-p_{2} x_{2}\right) \\
& 0=\frac{\partial \mathcal{L}}{\partial x_{1}}=1-\lambda p_{1} \Rightarrow \lambda=\frac{1}{p_{1}} \\
& \left.0=\frac{\partial \mathscr{L}}{\partial x_{2}}=\frac{1}{2}-\lambda p_{2} \Rightarrow \lambda=\frac{1}{2 p_{2}}\right\} \frac{1}{p_{11}}=\frac{1}{2 p_{2}} \Rightarrow p_{1}=2 p_{2} \text { for }
\end{aligned}
$$

Suppose $p_{1}=2 p_{2}$. Then let $x_{1}=\alpha$. We would have $x_{2}=\frac{m-p_{1} x_{1}}{p_{2}}=\frac{m-p_{1} \alpha}{p_{2}}$ and vilify would be $\alpha+\frac{1}{2} \frac{m-p_{1} \alpha}{p_{2}}=\alpha+\frac{m}{2 p_{2}}-\frac{p_{1} \alpha}{2 p_{2}}=\frac{m}{2 p_{2}}+\frac{2 p_{2} \alpha-p_{1} \alpha}{2 p_{2}}$

$$
\begin{aligned}
=\frac{m+\left(2 p_{2}-p_{1}\right) \alpha}{2 p_{2}} & =\frac{m}{2 p_{2}} \text { since } 2 p_{2}-p_{1}=0 \\
& =m / p_{1}
\end{aligned}
$$

$$
>\text { or, note that } \frac{p_{1}}{2 p_{2}}=1,50
$$

If $p_{1} \neq 2 p_{2}$, there is a corner solution, not an interior solution. (The indiffence corves in this problem are linear.) If $p_{1}<2 p_{2}$, the consumer spends all 'ci's income on good 1, and utility is $m / p_{1} \quad\left(\right.$ since $\left.x_{1}=m / p_{1}\right)$. If $p_{1}>2 p_{2}$, the consumer spends all his income on' good 2 , and utility is $\frac{1}{2} \frac{m}{p_{2}}\left(\sin c e, x_{2}=\frac{m}{p_{2}}\right)$. So to summarize,

$$
v\left(p_{1}, p_{2}, m\right)= \begin{cases}m / p_{1} & \text { if } p_{1}<2 p_{2} \\ \frac{m}{p_{1}}=\frac{m}{2 p_{2}} & \text { if } p_{1}=2 p_{2} \\ \frac{m}{2 p_{2}} & \text { if } p_{1}>2 p_{2}\end{cases}
$$

This can also be written $v\left(p_{1}, p_{2}, m\right)=\max \left\{\frac{m}{p_{1}}, \frac{m}{2 p_{2}}\right\}$.

Answer all of the following five questions.
Fall 2000 Fuel

1. [8 points] Give a two-dimensional graphical interpretation of the result that the indirect utility function is quasiconvex in prices.

Answers to Final Exam, Econ. 7005. Fall 2005
(1) "quasiconvex" means "convex lower level sets"


In the diagram, the lower level set needs utivify): call this level "vo" to be a convex set sinceris guasiconvex.
So the indicated contour line, if thought of as being a function of $\rho_{1}$, reeds to be convex.

A "convex set." has the property that a straight line drown between any two members of the set stays within the set.

1. [11 points] Suppose a consumer receives utility from consumption of two goods, $x_{1}$ and $x_{2}$, according to the utility function $x_{1}^{a} x_{2}^{(1-a)}$, where $a \in(0,1)$.
(a) Find this consumer's indirect utility function.

Fall 2004 Ex. 1
(b) Verify that this consumer's indirect utility function is quasiconvex in $\mathbf{p}$. Directly check for quasiconvexity; do not use the result that a convex function is quasiconvex, and do not use the result that a strictly convex function is quasiconvex. (If you could not solve part (a), then work part (b) with the indirect utility function $\ln m-a \ln p_{1}-(1-a) \ln p_{2}$, which may or may not be the correct answer to part (a).)

Answers to Exam l, Econ 7005, Fall 2004
(1) See Variamp.II.
a) $\max u \Leftrightarrow \max \ln u$ since $\ln u$ is ax increasing function of $u$.

$$
\therefore \max x_{1}^{a} x_{2}^{(1-a)} \Leftrightarrow \max \ln x_{1}^{a} x_{2}^{1-a}
$$

$\max \left(\ln x_{1}^{a}+\ln x_{2}^{1-a}\right)$
$\max \left[a \ln x_{1}+(1-a) \ln x_{2}\right]$ which is easier to work with.

$$
\left.\begin{array}{l}
\max \left[a \ln x_{1}+(1-a) \ln x_{2}\right] \text { sit. } p_{1} x_{1}+p_{2} x_{2}=m \\
\mathscr{L}=a \ln x_{1}+(1-a) \ln x_{2}+\lambda\left[m-p_{1} x_{1}-p_{2} x_{2}\right] \\
0=\partial \mathcal{L} 1 \partial \lambda=m-p_{1} x_{1}-p_{2} x_{2} \\
0=\partial \mathcal{L} 1 \partial x_{1}=\frac{a}{x_{1}}-\lambda p_{1} \\
0=\partial \mathcal{L} / \partial x_{2}=\frac{1-a}{x_{2}}-\lambda p_{2}
\end{array}\right\} \lambda=\frac{a}{p_{1} x_{1}}=\frac{1-a}{p_{2} x_{2}} . \quad \Rightarrow x_{1}=\frac{a p_{2} x_{2}}{(1-a) p_{1}} . \text { substitute into the }
$$

constraint: $m=p_{1} \frac{a p_{2} x_{2}}{(1-a) p_{1}}+p_{2} x_{2}$

$$
\begin{aligned}
& =\frac{a}{1-a} p_{2} x_{2}+p_{2} x_{2}=p_{2} x_{2}\left[\frac{a}{1-a}+1\right]=p_{2} x_{2} \frac{a+1-a}{1-a} \\
& =\frac{p_{2}}{1-a} x_{2} \Rightarrow x_{2}^{*}=(1-a) \frac{m}{p_{2}} \text { and } \\
x_{1}^{*} & =\frac{a p_{2}}{(1-a) p_{1}} \cdot(1-a) \frac{m}{p_{2}}=a \frac{m}{p_{1}} .
\end{aligned}
$$

So $u^{*}=a \ln \frac{a m}{p_{1}}+(1-a) \ln \frac{(1-a) m}{p_{2}}$

$$
=a \ln a+a \ln \frac{m}{p_{1}}+(1-a) \ln (1-a)+(1-a) \ln \frac{m}{p_{2}}
$$

but since constants are not important, the first and third terms are unimportant, and we can write

$$
\begin{aligned}
v(p, m) & =a \ln \frac{m}{p_{1}}+(1-a) \ln \frac{m}{p_{2}} \\
& =a \ln m-a \ln p_{1}+(1-a) \ln m-(1-a) \ln p_{2} \\
& =\ln m-a \ln p_{1}-(1-a) \ln p_{2}
\end{aligned}
$$

b) the bordered Hessian is

$$
\left[\begin{array}{ccc}
0 & v_{1}^{\prime} & v_{2}^{\prime} \\
v_{1}^{\prime} & v_{11}^{\prime \prime} & v_{12}^{\prime \prime} \\
v_{2}^{\prime} & v_{21}^{\prime \prime} & v_{22}^{\prime \prime}
\end{array}\right]=\left[\begin{array}{ccc}
0 & -a / p_{1} & \frac{a-1}{p_{2}} \\
-a / p_{1} & +a / p_{1}^{2} & 0 \\
\frac{a-1}{p_{2}} & 0 & \frac{1-a}{p_{2}^{2}}
\end{array}\right]
$$

For quasiconverity we want $\delta_{2}<0$. In this case, $\delta_{2}$ is the determinant of the entire bordered Hessian. This is (expanding along the frost column):

$$
\begin{aligned}
& (-1)^{2+1}\left(\frac{-a}{p_{1}}\right)\left(\frac{-a}{p_{1}} \cdot \frac{1-a}{p_{2}^{2}}\right)+(-1)^{3+1} \frac{a-1}{p_{2}}\left(-\frac{a}{p_{1}^{2}} \frac{a-1}{p_{2}}\right) \\
& =\frac{a}{p_{1}} \frac{a}{p_{1}} \frac{a-1}{p_{2}^{2}}+\frac{a-1}{p_{2}} \frac{a}{p_{1}^{2}} \frac{1-a}{p_{2}}=\frac{a}{p_{1}^{2} p_{2}^{2}}\left(a^{2}-a+(a-1)(1-a)\right) \\
& =\frac{a}{p_{1}^{2} p_{2}^{2}}\left[a^{2}-a+a-a^{2}-1+a\right]=\frac{a(a-1)}{p_{1}^{2} p_{2}^{2}}<0 .
\end{aligned}
$$

## 2017 Exam 1 Qu. 1

## 1. [11 points]

(a) Suppose a consumer has income $m$ and a standard budget constraint and a utility function $u(x, y)=\alpha \ln x+\beta \ln y$ with $\alpha>0$ and $\beta>0$. Find this consumer's indirect utility function $v$.
(b) From your answer to part (a), find $\partial v / \partial m$ and $\partial^{2} v / \partial m^{2}$. Interpret these in terms of "the marginal utility of money." What is the sign of $\partial v / \partial m$ and $\partial^{2} v / \partial m^{2}$ ? What do these signs imply for the shape of a graph of $v$ versus $m$ ?
(c) Find $\partial v / \partial m$ by using the Envelope Theorem and verify that you get the same answer you got in part (b). Explain why the Envelope Theorem is relevant to the problem.
(d) Suppose another consumer has income $m$ and a standard budget constraint and a utility function $\hat{u}(x, y)=x^{\alpha} y^{\beta}$ with $\alpha>0$ and $\beta>0$. Without solving an optimization problem, explain why this consumer has the same demand curves for $x$ and $y$ as the consumer in the earlier parts of this question.
(e) Find this second consumer's indirect utility function $\hat{v}$ and find $\partial \hat{v} / \partial m$ and $\partial^{2} \hat{v} / \partial m^{2}$.
(f) Is the sign of $\partial \hat{v} / \partial m$ the same as the sign of $\partial v / \partial m$ ? Why or why not?
(g) Is the sign of $\partial^{2} \hat{v} / \partial m^{2}$ the same as the sign of $\partial^{2} v / \partial m^{2}$ ? Why or why not?

Answer to Que 1, Miadtim Exam, Fall 2017 (Econ. 7005)
a) $\mathscr{L}=\alpha \ln x+\beta \ln y+\lambda\left(m-p_{x} x-p_{y} y\right)$ for max $a(x, y)$ st. $p_{x} x+p_{y} y=m$.
F.O.C.

$$
\left.\begin{array}{rl}
0=\mathscr{L}_{x}^{\prime}=\frac{\alpha}{x}-\lambda p_{x} \Rightarrow \lambda p_{x}=\frac{\alpha}{x} \Rightarrow \lambda=\frac{\alpha}{p_{x} x} \\
0=\mathscr{L}_{y}^{\prime}=\frac{\beta}{y}-\lambda p_{y} \Rightarrow \lambda p_{y}=\frac{\beta}{y} \Rightarrow \lambda=\frac{\beta}{p_{y} y}
\end{array}\right\} \Rightarrow \begin{gathered}
\frac{\alpha}{p_{x} x}=\frac{\beta}{p_{y} y} \\
0=m-p_{x} x-p_{y} y
\end{gathered} \quad \begin{aligned}
x & =\frac{\alpha p_{y} y}{\beta p_{x}} \\
m & =p_{x}\left(\frac{\alpha p_{y} y}{\beta p_{x}}\right)+p_{y} y \\
& =\left(\frac{\alpha}{\beta}+1\right) p_{y} y=\left(\frac{\alpha}{\beta}+\frac{\beta}{\beta}\right) p_{y} y=\frac{\alpha+\beta}{\beta} p_{y} y \\
\Rightarrow y^{*} & =\frac{\beta}{\alpha+\beta} \frac{m}{p_{y}} .
\end{aligned}
$$

Them from above, $x^{*}=\frac{\alpha p_{y} y}{\beta p_{x}}=\frac{\alpha p_{y}}{\beta p_{x}} \frac{\beta}{\alpha+\beta} \frac{m}{p_{y}}=\frac{\alpha}{\alpha+\beta} \frac{m}{p_{x}}$.
Then $v=\alpha \ln x^{*}+\beta \ln y^{*}=\alpha \ln \frac{\alpha m}{(\alpha+\beta) p_{x}}+\beta \ln \frac{\beta m}{(\alpha+\beta) p y}$.

$$
\text { b) } \begin{aligned}
\frac{\partial v}{\partial m} & =\alpha \frac{(\alpha+\beta) p_{x}}{\alpha m} \frac{\alpha}{(\alpha+\beta) p_{x}}+\beta \frac{(\alpha+\beta) p_{y}}{\beta m} \frac{\beta}{(\alpha+\beta) p_{y}} \\
& =\frac{\alpha}{m}+\frac{\beta}{m}=\frac{\alpha+\beta}{m}>0 .
\end{aligned}
$$

$$
\frac{\partial^{2} V}{\partial m^{2}}=-\frac{\alpha+\beta}{m^{2}}<0
$$

$$
\text { Note: } x>0, \beta>0, m>0 \text {. }
$$


$\checkmark$ versus in is increasing and concave. It may be in Quadrant II (v maybe negative). (The sign of $v$ is unimportant.)
av/2m is the "marimuevitioty of money." Here, oviom is decreasing on $m$ since the graph $B$ concave.
c)

$$
v=\max _{x, y} u(x, y) \text { s.t. } p_{x} x+p_{y} y=m \text {. }
$$

By the Envelope Theorem, $\frac{\partial v}{\partial m}=\frac{\partial \mathscr{L}^{*}}{\partial m}$, and using LL from the trot line of

$$
\begin{gathered}
\quad p_{a} t(a), \\
=\lambda^{*} . \\
\text { Frmpart(a), } \lambda^{*}=\frac{\alpha}{p_{x} x^{*}}=\frac{\alpha}{p_{x}} \frac{(\alpha+\beta) p_{x}}{\alpha m}=\frac{\alpha+\beta}{m} \text { or } \\
\lambda^{*}=\frac{\beta}{p_{y} y^{*}}=\frac{\beta}{p_{y}} \frac{(\alpha+\beta) p_{y}}{\beta m}=\frac{\alpha+\beta}{m} . \\
\text { So } \partial v / \partial m=\frac{\alpha \beta \beta}{m}, \text { as in part }(b) .
\end{gathered}
$$

d) Notice that $\ln \hat{u}=\ln \left(x^{\alpha} y^{\beta}\right)=\ln x^{\alpha}+\ln y^{\beta}=\alpha \ln x+\beta \ln y=u(x, y)$.

Also,

is an mereasing function of $\hat{u}(x, y)$, and hence $u$ and $\hat{u}$ represent the same preferences.
e) From (d), we know that (a)'s answers for $x^{*}$ and $y^{*}$ are tire for $\hat{u}$. So

$$
\begin{aligned}
& \hat{v}=\left(x^{*}\right)^{\alpha}\left(y^{*}\right)^{\beta}=\left(\frac{\alpha m}{(\alpha+\beta) p_{x}}\right)^{\alpha}\left(\frac{\beta m}{(\alpha+\beta) p_{y}}\right)^{\beta}=\frac{\alpha^{\alpha} \beta^{\beta} m^{\alpha+\beta}}{(\alpha+\beta)^{\alpha+\beta} p_{x}^{\alpha} p_{y}^{\beta}} . \\
& \frac{\partial \hat{v}}{\partial m}=(\alpha+\beta) \frac{\alpha^{\alpha} \beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta} p_{x}^{\alpha} p_{y}^{\beta}} m^{\alpha+\beta-1}=\frac{\alpha^{\alpha} \beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta-1} p_{x}^{\alpha} p_{y}^{\beta}} m^{\alpha+\beta-1}>0
\end{aligned}
$$

and

$$
\frac{\partial^{2} \hat{v}}{\partial m^{2}}=(\alpha+\beta-1) \frac{\alpha^{\alpha} \beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta-1} p_{x}^{\alpha} p_{y}^{\beta}} m^{\alpha+\beta-2} .
$$

f) Yes, both $\partial v / \partial m$ and $\partial \hat{v} / \partial m$ are positive. (See port (e).)
g) From $(b), \partial^{2} v / \partial m^{2}<0$.

However from (e), $\partial^{2} \hat{v} / \partial m^{2}$ has the same sift as $\alpha+\beta-1$, which is positive if $\alpha+\beta>1$ and negative if $\alpha+\beta<1$ (and zero if $\alpha+\beta=1$ ). For the case of $\alpha+\beta>1$, the graph of $\hat{r}$ versus $m$ would look like


Part (b)'s graph, with a falling marginal utility of money, reflects the unvantional idea that a marginal dollar B worth more to a poor person than to a rich person. The graph of $\hat{v}$ versus m shows the apposite behavior. However, we have seen that $v$ and $\hat{v}$ represent the some underlying preteraces. This mean that the "marsinalutility of money" 13 a cardinal, not ordinal, idea, and hence that the notion of a "cimmishing marginal uniting of money" is meaningless in the context of ordinal utility theory.
2. [11. points] Suppose a price-taking consumer has a utility function

$$
u(\mathbf{x})=2 \ln x_{1}+\ln x_{2}
$$

over two goods $x_{1}$ and $x_{2}$.
(a) Show that this consumer's indirect utility function is

$$
v(\mathbf{p}, m)=2 \ln \frac{2 m}{3 p_{1}}+\ln \frac{m}{3 p_{2}}
$$

where $p_{1}$ is the price of the first good, $p_{2}$ is the price of the second good, and $m$ is income.
(b) If the price of the first good rises, what change in income would leave utility unchanged? [Hint: It is possible to use the indirect utility function from part (a) in answering part (b).]

## Fall 2005 Ex. 1

(2)
a)

$$
\begin{aligned}
& u=2 \ln x_{1}+\ln x_{2} \\
& \mathscr{L}=2 \ln x_{1}+\ln x_{2}+\lambda\left(m-p_{1} x_{1}-p_{2} x_{2}\right) \\
& 0=\frac{\partial z}{\partial x_{1}}=\frac{2}{x_{1}}-\lambda p_{1} \\
& 0=\frac{\partial x}{\partial x_{2}}=\frac{1}{x_{2}}-\lambda p_{2} \\
& 0=\frac{\partial z}{\partial \lambda}=m-p_{1} x_{1}-p_{2} x_{2} \\
& \text { \} } \frac{2}{x_{1}} \div \frac{1}{x_{2}}=\frac{p_{1}}{p_{2}} \\
& \frac{2}{x_{1}} \frac{x_{2}}{1}=\frac{p_{1}}{p_{2}} \\
& x_{2}=\frac{p_{1}}{p_{2}} \frac{x_{1}}{2} ; \text { into budget } \\
& \text { constraint } \Rightarrow \\
& m=p_{1} x_{1}+p_{2}\left(\frac{p_{1}}{p_{2}} \frac{x_{1}}{2}\right) \\
& =p_{1} x_{1}+\frac{1}{2} p_{1} x_{1} \\
& =\frac{3}{2} p_{1} x_{1} \Rightarrow x_{1}^{*}=\frac{2}{3} p_{1} \text { and } \\
& x_{2}^{*}=\frac{p_{1}}{p_{2}} \frac{1}{x}\left(\frac{7}{3} \frac{m}{p_{1}}\right)=\frac{m}{3 p_{2}} .
\end{aligned}
$$

So $v=u^{*}=2 \ln x_{1}^{*}+\ln x_{2}^{*}$

$$
=2 \ln \frac{2 m}{3 p_{1}}+\ln \frac{m}{3 p_{2}} .
$$

b) We want utility to be unchanged, so we want $d v=0$. But

$$
\begin{aligned}
d v & =\frac{\partial v}{\partial p_{1}} d p_{1}+\frac{\partial v}{\partial p_{2}} d p_{2}+\frac{\partial v}{\partial m} d m \cdot w_{1} \text { th } d p_{2}=0 \text {, this is } \\
= & 2\left(\frac{3 p_{1}}{2 m}\right)\left(\frac{-2 m^{\prime}}{3 p_{1}^{2}} d p_{1}+0+\left(2 \frac{3 p_{1}}{2 m} \frac{2}{3 p_{1}}+\frac{3 p_{2}}{m} \frac{1}{3 p_{2}}\right) d m\right. \\
= & 2 \frac{-1}{p_{1}} d p_{1}+\left(\frac{2}{m}+\frac{1}{m}\right) d m
\end{aligned}
$$

$$
=\frac{-2}{p_{1}} d p_{1}+\frac{3}{m} d m .
$$

In order for do to be zero, one must have

$$
\begin{aligned}
0 & =\frac{-2}{p_{1}} d p_{1}+\frac{3}{m} d m \\
\frac{2}{p_{1}} d p_{1} & =\frac{3}{m} d m \\
d p_{1} & =\frac{3 p_{1}}{2 m} d m \Rightarrow \frac{d m}{d p_{1}}=\frac{2 m}{3 p_{1}} .
\end{aligned}
$$

Optional: the elasticity required is $\frac{d m / m}{d p_{1} / p_{1}}=\frac{2}{3}$.
For every $1 \%$ in $p_{1}$, you'd need a $\frac{2}{3} \% \uparrow$ in income. A different approach: $v=\ln \left[\left(\frac{2 m}{3 p_{1}}\right)^{2} \frac{m}{3 p_{2}}\right] \cdot v=$ constant $\Rightarrow$

$$
\begin{align*}
\text { Constant } & =\left(\frac{2 m}{3 p_{1}}\right)^{2} \frac{m}{3 p_{2}}=\frac{4 m^{3}}{27 p_{1}^{2} p_{2}}  \tag{*}\\
\Rightarrow m & =\left[\frac{27}{4} \text { constr } \cdot p_{1}^{2} p_{2}\right]^{1 / 3} \\
\frac{d m}{d p_{1}} & =\frac{1}{3}\left[\frac{27}{4} \text { coast. } p_{1}^{2} p_{2}\right]^{-2 / 3} \cdot \frac{27}{4} \text { constr } \cdot\left(2 p_{1}\right) p_{2} \\
& =\frac{1}{3} \quad \frac{1}{m^{2}} \\
& =\frac{27}{6} \quad \text { cost. } \frac{p_{1} p_{2}}{m^{2}}=\frac{27}{6} \frac{4 m^{3}}{6} \frac{4 p_{1}}{27 p_{1}^{2} p_{2}} \frac{p_{1} p_{2}}{m^{2}} \\
& =\frac{2}{3} \frac{m}{p_{1}} \text { as before. }
\end{align*}
$$

A third approach: from (*) , if $m$ joesto $\hat{m}$ and $p_{1}$ to $\hat{p}_{1}$ we need

$$
\begin{aligned}
& \frac{4 m^{3}}{27 \hat{p}_{1}^{2} p_{2}}=\frac{4 \hat{m}^{3}}{27 \hat{p}_{1}^{2} p_{2}} \Rightarrow \hat{m}^{3}=\left(\frac{\hat{p}_{1}}{p_{1}}\right)^{2} m^{3} \text { and } \\
& \hat{m}=\left(\hat{p}_{1} / p_{1}\right)^{2 / 3} m
\end{aligned}
$$

A fourth approach:

$$
v(p, e(\underset{\sim}{p}, u))=u
$$

From earlier, $v=2 \ln \frac{2 m}{3 p_{1}}+\ln \frac{m}{3 p_{2}}=\ln \left(\frac{2 m}{3 p_{1}}\right)^{2}+\ln \frac{m}{3 p_{2}}$

$$
=\ln \frac{4 m^{2}}{9 p_{1}^{2}}+\ln \frac{m}{3 p_{2}}=\ln \frac{4 m^{3}}{27 p_{1}^{2} p_{2}} .
$$

$S_{0} v(p, e)=u=\ln \frac{4 e^{3}}{27 p_{1}^{2} p_{2}} \Rightarrow \quad$ Like in the Second Approach

$$
\exp (u)=\frac{4 e^{3}}{27 p_{1}^{2} p_{2}} \quad \text { " } e \text { "here is } e(p, u) \text {, not the }
$$

- Notation change -

$$
\begin{aligned}
e^{u} & =\frac{4 e(p, u)^{3}}{27 p_{1}^{2} p_{2}} \\
\frac{27 p_{1}^{2} p_{2}}{4} e^{u} & =e\left(p_{1} u\right)^{3} \Rightarrow \\
e\left(p_{1} u\right) & =3\left(p_{1}^{2} p_{2} / 4\right)^{1 / 3} e^{u / 3} \text { and } \\
\left.\frac{\partial e\left(p_{1} u\right)}{\partial p_{1}}\right|_{u} & =\left[\frac{\partial}{\partial p_{1}} p_{1}^{2 / 3}\right] \cdot 3\left(p_{2} / 4\right)^{1 / 3} e^{u / 3} \\
& =\frac{2}{3} p_{1}^{-1 / 3} \cdot 3\left(p_{2} / 4\right)^{1 / 3} e^{u / 3}=\frac{2}{3} p_{1}^{12 / 3} p_{1}^{-1} \cdot 3\left(p_{2} / 4\right)^{1 / 3} e^{u / 3} \\
& =\frac{2}{3 p_{1}} \cdot 3\left(p_{1}^{2} p_{2} / 4\right)^{1 / 3} e^{u / 3}=\frac{2}{3 p_{1}} e\left(p_{1} u\right)=\frac{2 m}{3 p_{1}}
\end{aligned}
$$

A fifthapproach:
Compensation Variation $C V=e(\hat{p}, v(\hat{p}, \hat{m}))-e(\underset{\sim}{p}, v(p, m))$ $=\hat{m}-e(\hat{p}, v(p, m))$. Using $e(p, u)$ from the Fourth Approach,

$$
=\hat{m}-3\left(\hat{p}_{1}^{2} \hat{p}_{2} / 4\right)^{1 / 3} e^{v\left(p_{1} m\right) / 3} ; \text { using } v\left(p_{1} m\right) \text { from either the }
$$

Second Approach or the Fourth Approach,

$$
\begin{aligned}
& =\hat{m}-3\left(\hat{p}_{1}^{2} \hat{p}_{2} / 4\right)^{1 / 3} e^{\left(\ln \frac{4 m^{3}}{27 p_{1}^{2} p_{2}}\right) / 3} \\
& =\hat{m}-3\left(\hat{p}_{1}^{2} \hat{p}_{2} / 4\right)^{1 / 3} e^{\frac{1}{3} \ln \frac{4 m^{3}}{27 p_{1}^{2} p_{2}}} \\
& =\hat{m}-3\left(\hat{p}_{1}^{2} \hat{p}_{2} / 4\right)^{1 / 3} e^{\ln \left[\frac{4 m^{3}}{27 p_{1}^{2} p_{2}}\right]^{1 / 3}} \\
& =\hat{m}-3\left(\frac{\hat{p}_{1}^{2} \hat{p}_{2}}{4}\right)^{1 / 3}\left(\frac{4 m^{3}}{27 p_{1}^{2} p_{2}}\right)^{1 / 3} \\
& =\hat{m}-3\left[\frac{\hat{p}_{1}^{2} \hat{p}_{2}}{p_{1}^{2} p_{2}}\right]^{1 / 3} \frac{m}{3}=\hat{m}-\left[\frac{\hat{p}_{1}^{2} \hat{p}_{2}}{p_{1}^{2} p_{2}}\right]^{1 / 3} m
\end{aligned}
$$

We know that $\hat{p}_{2}=p_{2}$. Find the $\hat{m}$ which makes $C V=0$ : *

$$
0=\hat{m}-\left(\frac{\hat{p}_{1}^{2}}{p_{1}^{2}}\right)^{1 / 3} m \Rightarrow \hat{m}=\left(\hat{p}_{1} / p_{1}\right)^{2 / 3} m
$$

This is the same conclusion as the Third Approach.

$$
\text { *Or calculate }-C V=\left(\hat{p}_{1} / p_{1}\right)^{2 / 3} m-m \text {. }
$$

## 2015 Qualifying Exam Sec. 2 Qu. 2

## 2. [8 points]

(a) If the price vector faced by a price-taking consumer changes from $\mathbf{p}$ to $\gamma \mathbf{p}$, what change in income would leave utility unchanged? Why?
(b) Suppose a price-taking consumer has a utility function

$$
u(\mathbf{x})=2 \ln x_{1}+\ln x_{2}
$$

over two goods $x_{1}$ and $x_{2}$. If the price of only the first good rises, what change in income would leave utility unchanged?

Section 2 Question 2
a) Before the change in prices, $v(\underset{\sim}{p}, m)=\max _{\sim}^{x} u(\underset{\sim}{x})$ s.t. $P \cdot \underset{\sim}{x}=m$.

After the change in prices, suppose income changes from $m$ to $\alpha m$. Then

$$
v\left(\gamma_{\sim}^{p}, a m\right)=\max _{\sim}^{x} u(\underset{\sim}{x}) \text { st. } \gamma_{\sim}^{p} \cdot x=\alpha m .
$$

If $\alpha=\gamma$ then this constriant is $\gamma \underset{\sim}{\gamma} \cdot{ }_{\sim}^{x}=\gamma \mathrm{m}$

$$
\Rightarrow \underset{\sim}{p} \cdot \underset{\sim}{x}=m
$$

which is the same constraint as be fore. The objective function is also the same as before, so $u$ with be the same as before. So the answer is that $m$ also has to change by a factor of $\gamma$.
b) This combines (a) and (b) of the previous problem.
3. Suppose a consumer's utility function $u$ is given by $u(x)=x_{1}^{1 / 2}+2 x_{2}$ where $x_{1}$ and $x_{2}$ are amounts of two commodities consumed. Find this consumer's indirect utility function and expenditure function.
3.

$$
\begin{align*}
& u(\underset{\sim}{x})=x_{1}^{1 / 2}+2 x_{2} \\
& \mathcal{L}=x_{1}^{1 / 2}+2 x_{2}+\lambda\left(m-p_{1} x_{1}-p_{2} x_{2}\right) \\
& \frac{\partial z}{\partial x_{1}}=0=\frac{1}{2} x_{1}^{-1 / 2}-\lambda p_{1} \Rightarrow \frac{2}{p_{2}} p_{1}=\frac{1}{2} x_{1}^{-1 / 2} \Rightarrow x_{1}=\left(\frac{1}{4} \frac{p_{2}}{p_{1}}\right)^{2}  \tag{1}\\
& \frac{\partial L_{1}}{\partial x_{2}}=0=2-\lambda p_{2} \Rightarrow 2=\lambda p_{2} \Rightarrow \lambda=2 / p_{2} \\
& \frac{\partial \mathscr{L}}{\partial \lambda}=0=m-p_{1} x_{1}-p_{2} x_{2} \Rightarrow m=p_{1} x_{1}+p_{2} x_{2} \\
& =p_{1}\left(\frac{1}{4} \frac{p_{2}}{p_{1}}\right)^{2}+p_{2} x_{2} \\
& =\frac{1}{16} \frac{p_{2}^{2}}{p_{1}}+p_{2} x_{2} \\
& \frac{m}{p_{2}}-\frac{1}{16} \frac{p_{2}}{p_{1}}=x_{2} \text {. } \tag{2}
\end{align*}
$$

Substitite (1) and (2) back into $u(x)$ to obtain

$$
\begin{align*}
v\left(p_{2}, m\right) & =\frac{1}{4} \frac{p_{2}}{p_{1}}+2\left[\frac{m}{p_{2}}-\frac{1}{16} \frac{p_{2}}{p_{1}}\right] \\
& =\frac{1}{4} \frac{p_{2}}{p_{1}}+\frac{2 m}{p_{2}}-\frac{1}{8} \frac{p_{2}}{p_{1}} \\
& =\frac{2 m}{p_{2}}+\frac{p_{2}}{8 p_{1}} . \tag{3}
\end{align*}
$$

Finally, $v(\underset{\sim}{p}, e(p, u)) \equiv u$ so from (3)

$$
\begin{align*}
& \frac{2 e(p, u)}{p_{2}}+\frac{p_{2}}{8 p_{1}}=u \\
& 2 e(p, u)=u p_{2}-\frac{p_{2}^{2}}{8 p_{1}} \\
& e(p, u)=\frac{1}{2} u p_{2}-\frac{1}{16} p_{2}^{2} / p_{1} . \tag{4}
\end{align*}
$$

Optional Remark. Imposing $x_{2} \geqslant 0$ in (2) implies that.

$$
\begin{array}{r}
\frac{m}{p_{2}}-\frac{1}{16} \frac{p_{2}}{p_{1}} \geqslant 0 \Leftrightarrow \\
m \geqslant \frac{1}{16} \frac{p_{2}^{2}}{p_{1}} . \tag{5}
\end{array}
$$

If. $(5)$ is violated them $x_{2}^{*}=0$ and all the in come jos to $x_{1}$, la ding. to $x_{i}^{*}=\frac{m}{p_{1}}$. In this case $r(p, m)=\sqrt{m / p_{1}}$ and $\sqrt{e\left(p_{1}, u\right) / p_{1}} \equiv u$

$$
\begin{equation*}
e(p, u)=p, u^{2} . \tag{i}
\end{equation*}
$$

(7)

To summarize, $v(p, m)$ is queen by (6) if $m<\frac{1}{16} \frac{p_{2}^{2}}{p_{1}}$ and by (3) otherwise. This ought to be contrivious at $m=\frac{1}{16} \frac{p_{2}^{2}}{p_{1}}$ :

$$
\begin{align*}
& \text { (3) } \Rightarrow v=\frac{2}{p_{2}}\left[\frac{1}{16} \cdot \frac{p_{2}^{2}}{p_{1}}\right]+\frac{p_{2}}{8 p_{1}}=\frac{p_{2}}{8 p_{1}}+\frac{p_{2}}{8 p_{1}}=\frac{p_{2}}{4 p_{1}} \\
& \text { (6) } \Rightarrow v=\sqrt{\frac{1}{16} \frac{p_{2}^{2}}{p_{1}} / p_{1}}=\frac{1}{4} \frac{p_{2}}{p_{1}} \longleftarrow \text { coandinuty ok! }
\end{align*}
$$

This shows that the utility level at the boundary between the two nimes is $\frac{p_{2}}{4 p_{1}}$ For $u<\frac{p_{2}}{4 p_{1}}, e(\underset{\sim}{p}, u)$ is given by $(7)$ : otherinice 14 is firn $b_{y}(4)$.'To check continuity of $e(p, 4)$ at the point $u=p_{2} /\left(4 p_{1}\right)$ :

$$
(4) \Rightarrow e=\frac{1}{2}\left[\frac{p_{2}}{4 p_{1}}\right] p_{2}-\frac{p_{2}^{2}}{16 p_{1}}=\frac{p_{2}^{2}}{8 p_{1}}-\frac{p_{2}^{2}}{16 p_{1}}=\frac{p_{2}^{2}}{16 p_{1}} .
$$

$$
\begin{equation*}
(7) \Rightarrow e=p_{1}\left[\frac{p_{2}}{4 p_{1}}\right]^{2}=\frac{p_{2}^{2}}{16 p_{1}} \tag{contarityok!}
\end{equation*}
$$

Contivisity of the demand for $x_{2}$ is obvious by construction $($ see $(5))$. To check continue of $x_{1}$, note that at $m=\frac{1}{16} \frac{p_{2}^{2}}{p_{1}}, x_{1}^{*}=\frac{m}{p_{1}}=\frac{1}{16} \frac{p_{2}^{2}}{p_{1}^{2}}$, which is the same as given by (1).

$$
\begin{gathered}
\text { Exam 1 } \\
1996 \\
\text { Question } 2
\end{gathered}
$$

2. Suppose a consumer obtains utility from consumption of two goods, $x_{1}$ and $x_{2}$, according to the utility function $u(\mathbf{x})=x_{1} x_{2}$. This consumer's income is denoted by $m$. The consumer takes the price of the second good as a constant, $p_{2}$. However, the consumer's actions influence $p_{1}$, the price of the first good; $p_{1}$ increases with the consumer's purchases of $x_{1}$ according to the relationship $p_{1}=x_{1} / 3$.
(a) Find this consumer's indirect utility function.
(b) Find this consumer's expenditure function.
(2)

$$
\begin{align*}
& \max u(x) \text { s.t. } m=p_{1} x_{1}+p_{2} x_{2} \\
&=p_{1}\left(x_{1}\right) x_{1}+p_{2} x_{2}=\frac{x_{1}}{3} x_{1}+p_{2} x_{2}=\frac{1}{3} x_{1}^{2}+p_{2} x_{2} . \\
& \mathscr{L}=x_{1} x_{2}+\lambda\left[m-\frac{1}{3} x_{1}^{2}-p_{2} x_{2}\right] . \quad \text { (the Lagrangian) } \\
& \text { F.O.C. : } \begin{aligned}
0 & =\frac{\partial \mathscr{L}}{\partial \lambda}
\end{aligned}=m-\frac{1}{3} x_{1}^{2}-p_{2} x_{2}  \tag{a}\\
& 0=\frac{\partial \mathscr{L}}{\partial x_{1}}=x_{2}-\frac{2}{3} \lambda x_{1}  \tag{b}\\
& 0=\frac{\partial \mathscr{L}}{\partial x_{2}}=x_{1}-\lambda p_{2} \tag{c}
\end{align*}
$$

$$
\left.\begin{array}{l}
(b) \Rightarrow x_{2}^{\prime}=\frac{2}{3} \lambda x_{1} \\
(c) \Rightarrow x_{1}=\lambda p_{2}
\end{array}\right\} \text { divide } \Rightarrow \frac{x_{2}}{x_{1}}=\frac{2}{3} \frac{x_{1}}{p_{2}} \Rightarrow p_{2} x_{2}=\frac{2}{3} x_{1}^{2} \cdot \text { Sub }
$$

stifute this into (a) to obtain

$$
\begin{aligned}
m & =\frac{1}{3} x_{1}^{2}+p_{2} x_{2} \\
& =\frac{1}{3} x_{1}^{2}+\frac{2}{3} x_{1}^{2}=x_{1}^{2} \Rightarrow x_{1}^{*}=\sqrt{m} .
\end{aligned}
$$

Then $x_{2}^{*}=\frac{1}{p_{2}} \cdot \frac{2}{3} x_{1}^{2}=\frac{1}{p_{2}} \cdot \frac{2}{3} \cdot m=\frac{2 m}{3 p_{2}}$.
a)

$$
\begin{aligned}
v(p, m) & =u\left({\underset{\sim}{1}}_{*}^{*}(\underset{\sim}{p}, m),{\underset{\sim}{x}}_{2}^{*}(p, m)\right)=x_{1}^{*} \cdot x_{2}^{*}=\sqrt{m} \cdot \frac{2 m}{3 p_{2}} \\
& =\frac{2 m^{3 / 2}}{3 p_{2}}
\end{aligned}
$$

b) $v(\underset{\sim}{p}, e(\underset{\sim}{p}, u))=u$ is abasic identity. Using part a), this implies

$$
\begin{aligned}
& \frac{2[e(p, u)]^{3 / 2}}{3 p_{2}}=u \Rightarrow[e(p, u)]^{3 / 2}=\frac{3 u p_{2}}{2} \Rightarrow \\
& e\left(p_{\sim}, u\right)=\left[\frac{3}{2} u p_{2}\right]^{2 / 3}
\end{aligned}
$$

4. Suppose a consumer's expenditure function is

$$
\left(\frac{p_{1}}{a}\right)^{a}\left(\frac{p_{2}}{1-a}\right)^{1-a} u^{o} .
$$

# Final Exam <br> 1998 <br> Question 4 

Find the consumer's (direct) utility function.

$$
\text { (4) } \begin{aligned}
& e\left(\underset{\sim}{p}, u^{0}\right)=\left(\frac{p_{1}}{a}\right)^{a}\left(\frac{p_{2}}{1-a}\right)^{1-a} u^{0} \\
m= & e(p, v(p, m))=\left(\frac{p_{1}}{a}\right)^{a}\left(\frac{p_{2}}{1-a}\right)^{1-a} v(p, m) \\
& \Rightarrow v(p, m)=\left(\frac{a}{p_{1}}\right)^{a}\left(\frac{1-a}{p_{2}}\right)^{1-a} m .
\end{aligned}
$$

Final Exam 1998
Answer 4

$$
\begin{aligned}
\text { Then } u(\underset{\sim}{x}) & =\min _{\sim}^{p} v(\underset{\sim}{p} 1) \text { s.t. } \\
u(\underset{\sim}{p}) & =\min _{p_{1} p_{2}}\left(\frac{a}{p_{1}}\right)^{a}\left(\frac{1-a}{p_{2}}\right)^{1-a} \text { st. Therefore } p_{1} x_{1}+p_{2} x_{2}=1 . \\
\mathcal{L} & =\left(\frac{a}{p_{1}}\right)^{a}\left(\frac{1-a}{p_{2}}\right)^{1-a}+\lambda\left(p_{1} x_{1}+p_{2} x_{2}-1\right) \\
& =a^{a}(1-a)^{1-a} p_{1}^{-a} p_{2}^{-(1-a)}+\lambda\left(p_{1} x_{1}+p_{2} x_{2}-1\right)
\end{aligned}
$$

$$
\begin{align*}
0=\frac{\partial \mathcal{L}}{\partial p_{1}} & =a^{a}(1-a)^{1-a}(-a) p_{1}^{-a-1} p_{2}^{-(1-a)}+\lambda x_{1}  \tag{1}\\
0=\frac{\partial \mathcal{L}}{\partial p_{2}} & =a^{a}(1-a)^{1-a} p_{1}^{-a}(-(1-a)) p_{2}^{-(1-a)-1} \\
& =a^{a}(1-a)^{1-a} p_{1}^{-a}(a-1) p_{2}^{a-2}+\lambda x_{2}  \tag{2}\\
\text { (1) and (2) } \Rightarrow \quad & \frac{a^{a}(1-a)^{1-a}(-a) p_{1}^{-a-1} p_{2}^{a-1}}{a^{a}(1-a)^{1-a}(a-1) p_{1}^{-a} p_{2}^{a-2}}=\frac{x_{1}}{x_{2}}
\end{align*}
$$

Final Exam

$$
\begin{aligned}
& \frac{a}{(1-a) p_{2}}=\frac{x_{1}}{x_{2}} \\
& \quad p_{2}=\frac{i-a}{a} \frac{x_{1}}{x_{2}} p_{1}
\end{aligned}
$$

$$
1998
$$

Answer 4 cont...

Substitute this into the constraint, $0=\frac{\partial \mathscr{L}}{\partial \lambda}=p_{1} x_{1}+p_{2} x_{2}-1$ :

$$
\begin{aligned}
1 & =p_{1} x_{1}+\frac{1-a}{a} \frac{x_{1}}{x_{2}} p_{1} x_{2} \\
& =\left(x_{1}+\frac{1-a}{a} \frac{x_{1}}{x_{2}} x_{2}\right) p_{1}=\left(x_{1}+\frac{1-a}{a} x_{1}\right) p_{1}=\left(1+\frac{1-a}{a}\right) p_{1} x_{1} \\
& =\left(\frac{a}{a}+\frac{1-a}{a}\right) p_{1} x_{1}=\frac{1}{a} p_{1} x_{1} \Rightarrow p_{1}^{*}=\frac{a}{x_{1}} .
\end{aligned}
$$

$$
\text { Then } p_{2}^{*}=\frac{1-a}{a} \frac{x_{1}}{x_{2}} p_{1}^{*}=\frac{1-a}{a} \frac{x_{1}}{x_{2}} \frac{a}{x_{1}}=\frac{1-a}{x_{2}} \text {. }
$$

Finally, $u(x)=\left(\frac{a}{p_{1}^{*}}\right)^{a}\left(\frac{1-a}{p_{2}^{*}}\right)^{1-a}=\left(\frac{a}{a / x_{1}}\right)^{a}\left(\frac{1-a}{(1-a) / x_{2}}\right)^{1-a}=x_{1}^{a} x_{2}^{1-a}$.
3. [11 points] Suppose a consumer's indirect utility function is given by

$$
v(\mathbf{p}, m)=\ln m-a \ln p_{1}-(1-a) \ln p_{2} .
$$

Find this consumer's:
(a) expenditure function;
(b) (direct) utility function.

$$
\begin{aligned}
& 2007 \\
& \text { Midterm }
\end{aligned}
$$

(3)

$$
v=\ln m-a \ln p_{1}-(1-a) \ln p_{2}
$$

a)

$$
\begin{aligned}
& v\left(p_{1}, e(p, u)\right)=u \text { so } \\
& \begin{aligned}
& \ln e(p, u)-a \ln p_{1}-(1-a) \ln p_{2}=u ; \text { solve for } e(p, u) . \\
& \ln e\left(p_{1} u\right)=a \ln p_{1}+(1-a) \ln p_{2}+u \\
&=\ln p_{1}^{a}+\ln p_{2}^{1-a}+u \\
&=\ln p_{1}^{a}+\ln p_{2}^{1-a}+\underbrace{\ln e^{u}}
\end{aligned} .
\end{aligned}
$$

a trick; " $h_{n}$ "and" $e$ " are inverse functions, so $\ln e^{u}=u$

$$
\begin{aligned}
& =\ln p_{1}^{a} p_{2}^{1-a} e^{u} \text { so } \\
e(p, u) & =p_{1}^{a} p_{2}^{1-a} e^{u}
\end{aligned}
$$

Note: Don't confuse this " $e$ "), which is
the expenditure function. with this " $e$," which is the base of the natural logarithms, the number $2.71828 \ldots$.
b)

$$
\begin{aligned}
& u(\underline{x})=\min _{p} v(p, 1) \text { set. } p \cdot x=1 \\
& \mathcal{L}=\underset{0}{\left.\ln -1-a \ln p_{1}-(1-a) \ln p_{2}+\lambda\left(p_{1} x_{1}+p_{2} x_{2}-1\right)\right)} \\
& \text { F.O.C. } \left.\begin{array}{rl}
0 & =\partial L / \partial p_{1}=\frac{-a}{p_{1}}+\lambda x_{1} \\
0 & =\partial L \partial p_{2}=-\frac{1-a}{p_{2}}+\lambda x_{2}
\end{array}\right\} \lambda=\frac{a}{p_{1} x_{1}}=\frac{1-a}{p_{2} x_{2}} \\
& \Rightarrow p_{2}=\frac{1-a}{a} \frac{p_{1} x_{1}}{x_{2}} \text { and } \\
& 1=p_{1} x_{1}+p_{2} x_{2}=p_{1} x_{1}+\frac{1-a}{a} p_{1} x_{1}=\left(\frac{a}{a}+\frac{1-a}{a}\right) p_{1} x_{1} \\
& =\frac{p_{1} x_{1}}{a} \Rightarrow p_{1}^{*}=\frac{a}{x_{1}} \text { and } \\
& p_{2}^{*}=\frac{1-a}{a} \frac{a}{x_{2}}=\frac{1-a}{x_{2}} \text {. Then }
\end{aligned}
$$

$$
\begin{aligned}
& u=\ln 1-a \ln p_{1}^{*}-(1-a) \ln p_{2}{ }^{*} \\
& =0-a \ln \frac{a}{x_{1}}-(1-a) \ln \frac{1-a}{x_{2}} \\
& =-a\left(\ln a-\ln x_{1}\right)-(1-a)\left(\ln (1-a)-\ln x_{2}\right) \\
& =-a \ln a+a \ln x_{1}-(1-a) \ln (1-a)+(1-a) \ln x_{2} \\
& =[-a \ln a-(1-a) \ln (1-a)]+a \ln x_{1}+(1-a) \ln x_{2} \\
& =\left[\quad{ }^{\prime} \quad\right]+\ln x_{1}{ }^{a}+\ln x_{2}^{1-a}
\end{aligned}
$$

## Final Exam <br> 1994 <br> Question 2

2. Suppose the indirect utility function of a consumer is given by

$$
v(\mathbf{p}, m)=\frac{4 m^{3}}{27 p_{1} p_{2}^{2}}
$$

(a) Find the (direct) utility function.
(b) Find the Hicksian demand curve for good 1.

Final Exam 1994
Answer 2
(2)

$$
v\left(p_{1} m\right)=\frac{4 m^{3}}{27 p_{1} p_{2}^{2}}
$$

a) $u(x)=\min _{\sim} v(p, 1)$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x}=1$ so set $m=1$ and minimize $v$ w.r.t. $p$ :

$$
\left.\begin{array}{l}
\mathscr{L}=\frac{4}{27 p_{1} p_{2}^{2}}+\lambda\left[1-p_{1} x_{1}-p_{2} x_{2}\right] \\
0=\frac{\partial \mathscr{L}}{\partial p_{1}}=\frac{-4}{27 p_{1}^{2} p_{2}^{2}}-\lambda x_{1} \\
0=\frac{\partial \Psi}{\partial p_{2}}=\frac{-8}{27 p_{1} p_{2}^{3}}-\lambda x_{2}
\end{array}\right\} \frac{x_{1}}{x_{2}}=\frac{-4}{27 p_{1}^{2} p_{2}^{2}} \cdot \frac{27 p_{1} p_{2}^{3}}{-8}=\frac{1}{2} \frac{p_{2}}{p_{1}}
$$

$$
\begin{aligned}
& \Rightarrow p_{2}=2 p_{1} \frac{x_{1}}{x_{2}} \\
& \text { and } 1=p_{1} x_{1}+p_{2} x_{2} \\
& =p_{1} x_{1}+\left[2 p_{1} \frac{x_{1}}{x_{2}}\right] x_{2}=p_{1} x_{1}+2 p_{1} x_{1}=3 p_{1} x_{1} \Rightarrow p_{1}=\frac{1}{3 x_{1}}, \\
& \\
& p_{2}=2\left[\frac{1}{3 x_{1}}\right] \frac{x_{1}}{x_{2}}=\frac{2}{3 x_{2}},
\end{aligned}
$$

$$
\text { So } v\left(p_{\sim}^{*}, 1\right)=\frac{4}{27} \frac{3 x_{1}}{1} \frac{9 x_{2}^{2}}{4}=x_{1} x_{2}^{2}=u(\underset{x}{x}
$$

b)

$$
\begin{align*}
v(\underset{\sim}{p}, \dot{\sim}(p, u)) & \equiv u \\
\frac{4 e^{3}}{27 p_{1} p_{2}^{2}}=u \Rightarrow e^{3} & =\frac{27}{4} p_{1} p_{2}^{2} u, \\
e(p, u) & =\frac{3}{\sqrt[3]{4}} p_{1}^{1 / 3} p_{2}^{2 / 3} u^{1 / 3} \underbrace{2 p p t s}_{\sim} \\
h_{1}(\underset{\sim}{p}, u) & =\frac{\partial e}{\partial p_{1}}=\frac{1}{\sqrt[3]{4}} p_{1}^{-2 / 3} p_{2}^{2 / 3} u^{1 / 3}=\left(\frac{u p_{2}^{2}}{4 p_{1}^{2}}\right)^{1 / 3}
\end{align*}
$$

2. [11 points] Suppose a consumer with income $m$ faces given prices $p_{1}$ and $p_{2}$ for commodities $x_{1}$ and $x_{2}$, respectively. Suppose this consumer's indirect utility function is

$$
v(\mathbf{p}, m)=m\left(p_{1}^{r}+p_{2}^{r}\right)^{-1 / r}
$$

for some positive constant $r$.
(a) Find this consumer's Marshallian demand curves for $x_{1}$ and $x_{2}$.
(b) Prove either that this consumer's (direct) utility function $u\left(x_{1}, x_{2}\right)$ can be written as

$$
\left\{\frac{x_{1}^{\frac{r}{r-1}}}{\left[x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}\right]^{r}}+\frac{x_{2}^{\frac{r}{r-1}}}{\left[x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}\right]^{r}}\right\}^{-1 / r}
$$

or as

$$
\left(x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}}
$$

or, with $\rho=r /(1-r)$, as

$$
\left(x_{1}^{\rho}+x_{2}^{\rho}\right)^{1 / \rho}
$$

(which is called the "Constant Elasticity of Substitution" ("CES") utility function). Hint: at some point you may find it helpful to multiply

$$
\frac{1}{x_{1}+x_{1}^{\frac{-1}{r-1}} x_{2}^{\frac{r}{r-1}}}
$$

by one in the form of $x_{1}^{\frac{1}{r-1}} / x_{1}^{\frac{1}{r-1}}$ in order to obtain

$$
\frac{x_{1}^{\frac{1}{r-1}}}{x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}} .
$$

Fall 2013 Exam 1

Ansur to Questiva 2, Exam 1,
Fall 2013
a) $v=m\left(p_{1}^{r}+p_{2}^{r}\right)^{-1 / r}$. Use Roy's /dentity:

$$
\begin{aligned}
x_{1} & =-\frac{\partial v / \partial p_{1}}{\partial v / \partial m}=-\frac{m\left(\frac{-1}{r}\right)\left(p_{1}^{r}+p_{2}^{r}\right)^{-\frac{1}{r}-1} r p_{1}^{r-1}}{\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r}}}=\frac{4 m}{r}\left(p_{1}^{r}+p_{2}^{r}\right)^{-1} r p_{1}^{r-1} \\
& =\frac{p_{1}^{r-1} m}{p_{1}^{r}+p_{2}^{r}} \cdot \text { Analogovsly, } x_{2}=\frac{p_{2}^{r-1} m}{p_{1}^{r}+p_{2}^{r}} .
\end{aligned}
$$

b)

$$
\begin{aligned}
& u(x)=\min _{\sim} w(p) \text { s.t. } \underset{\sim}{p} \cdot \underset{\sim}{x}=1 \quad(\text { so } m=1) \text {. } \\
& \mathscr{L}=v(p)+\lambda\left(p_{1} x_{1}+p_{2} x_{2}-1\right) \\
& \text { F.O.C. } 0=p_{1} x_{1}+p_{2} x_{2}-1 \\
& 0=\partial \mathcal{Z} / \partial p_{1}=\partial v / \partial p_{1}+\lambda x_{1} ; \text { from part (a) } \\
& =\frac{-m}{r}\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r}-1}\left(r p_{1}^{r-1}\right)^{2}+\lambda x_{i} ; \text { using } m=1 \text {, } \\
& =-\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r}-1} p_{1}^{r-1}+\lambda x_{1} \\
& \Rightarrow \lambda=\frac{1}{x_{1}}\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r-1}} p_{1}^{r-1} \text {. } \\
& 0=\frac{\partial \mathscr{L}}{\partial p_{2}}=\frac{\partial v}{\partial p_{2}}+\lambda x_{2}=\frac{-m}{r}\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r}-1}\left(r p_{2}^{r-1}\right)+\lambda x_{2} \Rightarrow \\
& \left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r}-1} p_{2}^{r-1}=\lambda x_{2}=\frac{x_{2}}{x_{1}}\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r}-1} p_{1}^{r-1} \\
& \text { so } p_{2}^{r-1}=\frac{x_{2}}{x_{1}} p_{1}^{r-1} \text {. }
\end{aligned}
$$

Substitute into the constraint: this is $p_{2}$

$$
\begin{aligned}
1 & =p_{1} x_{1}+p_{2} x_{2}=p_{1} x_{1}+\overbrace{\left(\frac{x_{2}}{x_{1}}\right)^{\frac{1}{r-1}} p_{1}}^{r} \cdot x_{2} \\
& =p_{1}\left[x_{1}+x_{1}^{\frac{-1}{r-1}} x_{2}^{\frac{1}{r-1}} x_{2}\right] ; \frac{1}{r-1}+1=\frac{1}{r-1}+\frac{r-1}{r-1}=\frac{r}{r-1} \Rightarrow \\
& =p_{1}\left[x_{1}+x_{1}^{\frac{-1}{r-1}} x_{2}^{\frac{r}{r-1}}\right] \Rightarrow \\
p_{1} & =\frac{1}{x_{1}+x_{1}^{\frac{-1}{r-1}} x_{2}^{\frac{r}{r-1}}} \cdot \frac{x_{1}^{\frac{1}{r-1}}}{x_{1}^{\frac{1}{r-1}}}=\frac{x_{1}^{\frac{1}{r-1}}}{x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}} ;
\end{aligned}
$$

by analogy, $p_{2}=\frac{x_{2}^{\frac{1}{r-1}}}{x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}}$.
So $v_{\text {min }}=1\left(\left(p_{1}^{*}\right)^{r}+\left(p_{2}^{*}\right)^{r}\right)^{-1 / r}$

$$
\begin{aligned}
& =\left[\frac{x_{1}^{\frac{r}{r-1}}}{\left(x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}\right)^{r}}+\frac{x_{2}^{\frac{r}{r-1}}}{\left(x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}\right)^{r}}\right]^{-1 / r} \\
& =\frac{\left(x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}\right)^{-1 / r}}{\left(x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}\right)^{r \cdot \frac{-1}{r}} ; \frac{-1}{r}-\left(r \cdot \frac{-1}{r}\right)=\frac{-1}{r}+1=} \begin{array}{l}
\frac{-1}{r}+\frac{r}{r}=\frac{r-1}{r} \Rightarrow \\
=\left(x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}}
\end{array} .
\end{aligned}
$$

$$
\begin{gathered}
\text { Qualifying Exam } \\
1996 \\
\text { Question 3 }
\end{gathered}
$$

Question 3. If a consumer's expenditure function is $e(\mathbf{p}, u)=\left(p_{1}^{r}+p_{2}^{r}\right)^{1 / r} u$, find the' consumer's:
a) (direct) utility function;
b) Marshallian demand curve for good 2;
c) Hicksian demand curve for good 2 .

If you decide to solve an optimization problem when you answer this question, you do not have to verify that the second-order conditions hold, but you do, have to state the second-order conditions. In stating these second-order conditions, it is acceptable to leave derivatives unevaluated, as long as the only things you have left undone are simple differentiations.

Optional Question \#3. See the last example in Ch. 7 of Varian. Qualifying Exam

$$
\underset{\sim}{e(p, u)}=\left(p_{1}^{r}+p_{2}^{r}\right)^{1 / r} u .
$$ 1996

First I'll answer part (c): Use Shephard's Lemma.

$$
\begin{aligned}
& h_{2}=\frac{\partial e}{\partial p_{2}}=\frac{1}{r} \cdot\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{1}{r-1}}+p_{2}^{r-1} u=\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{1}{r}-1} p_{2}^{r-1} u \\
& \text { Hhcksiaa demand for good } 2
\end{aligned}
$$

is the thicksian demand for good 2 .
Now here's part (b). Obtain the indirect utility function by applying

$$
\begin{aligned}
& e(p, v(p, m))=m \Rightarrow\left(p_{1}^{r}+p_{2}^{r}\right)^{1 / r} v(p, m)=m \Rightarrow \\
& v(p, m)=m\left(p^{r}+p^{r}\right)^{-1 / r}
\end{aligned}
$$

$$
v\left(p_{1} m\right)=m\left(p_{1}^{r}+p_{2}^{r}\right)^{-1 / r} \text {. From here there are two ways to proceed: }
$$

either use $h\left(p_{1}^{\prime}, v(p, m)\right)=x(p, m)$ with $v$ and $h_{2}$ from above to obtain

$$
\begin{aligned}
x_{2}(p, m) & =\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{1}{r}-1} p_{2}^{r-1} \cdot m\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r}} \\
& =m p_{2}^{r-1} /\left(p_{1}^{r}+p_{2}^{r}\right)
\end{aligned}
$$

or use 'Roy's Identity to obtain

$$
\begin{aligned}
x_{2}= & -\frac{\partial v / \partial p_{2}}{\partial v / \partial m}=-\frac{\left(\frac{\downarrow \downarrow}{\frac{1}{r}}\right) m\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r-1}} \downarrow \downarrow p_{2}^{r-1}}{\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r}}} \\
& -m p_{2}^{r-1}
\end{aligned}
$$

$=\frac{m p_{2}^{r-1}}{p_{1}^{r}+p_{2}^{r}}$ as before, for the Mar shallian demand for yod 2. 1996
Finally, part (a):"nn $(\underset{\sim}{x})=\min _{\sim} v(p)$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x}=1$ Answer 3 cont...
Using $v(p, m)$ from part (b) and settiry $m=1$, the Lagrangien of thas minivization problem is

$$
\begin{equation*}
\mathscr{L}=\left(p_{1}^{r}+p_{2}^{r}\right)^{-1 / r}+\lambda\left(p_{1} x_{1}+p_{2} x_{2}-1\right) \tag{1}
\end{equation*}
$$

Finst-order conditions: $0=\frac{\partial \mathcal{Z}}{\partial \lambda}=p_{1} x_{1}+p_{2} x_{2}-1$

$$
\begin{align*}
0 & =\frac{\partial \mathscr{L}}{\partial p_{1}}=\frac{-1}{r}\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r}-1} r p_{1}^{r-1}+\lambda x_{1}  \tag{2}\\
0 & =\frac{\partial \mathscr{L}}{\partial p_{2}}=\frac{-1}{r}\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r}-1} r p_{2}^{r-1}+\lambda x_{2}  \tag{4}\\
(3) \&(4) \Rightarrow & \frac{p_{1}^{r-1}}{p_{2}^{r-1}}=\frac{x_{1}}{x_{2}} \Rightarrow p_{1}^{r-1}=\frac{x_{1}}{x_{2}} p_{2}^{r-1} \Rightarrow p_{1}=\left(\frac{x_{1}}{x_{2}}\right)^{\frac{1}{r-1}} p_{2}
\end{align*}
$$

Substivite (5) anto (1):

$$
\begin{aligned}
1 & =p_{1} x_{1}+p_{2} x_{2}=\left(\frac{x_{1}}{x_{2}}\right)^{\frac{1}{r-1}} p_{2} x_{1}+p_{2} x_{2}=\left[\left(\frac{x_{1}}{x_{2}}\right)^{\frac{1}{r-1}} x_{1}+x_{2}\right] p_{2} \\
& =\left[\frac{x_{1}^{\frac{1}{r-1}} x_{1}}{x_{2}^{\frac{1}{r-1}}}+\frac{x_{2}}{1} \frac{x_{2}^{\frac{1}{r-1}}}{x_{2}^{\frac{1}{r-1}}}\right] p_{2}=\frac{x_{1}^{\frac{1}{r-1}+1}+x_{2}^{\frac{1}{r-1}+1}}{x_{2}^{\frac{1}{r-1}}} p_{2}
\end{aligned}
$$

and since $\frac{1}{r-1}+1=\frac{1}{r-1}+\frac{r-1}{r-1}=\frac{r}{r-1}$, thas equals

$$
\begin{equation*}
\frac{x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}}{x_{2}^{\frac{1}{r-1}}} p_{2} \Rightarrow p_{2}=\frac{x_{2}^{\frac{1}{r-1}}}{x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}} \tag{6}
\end{equation*}
$$

Then from (5), $p_{1}=\frac{x_{1}^{\frac{1}{r-1}}}{x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}}$. (7)
Qualifying Exam 1996

Substituting (6) and (7) into $u(x)=\min _{\sim}^{p}\left(p_{1}^{r}+p_{2}^{r}\right)^{-1 / r}$ s.t. $p \cdot \underset{\sim}{x}=1$ yields

$$
\begin{aligned}
u(x) & =\left[\frac{x_{1}^{\frac{r}{r-1}}}{\left(x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}\right)^{r}}+\frac{x_{2}^{\frac{r}{r-1}}}{\left(x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}\right)^{r}}\right]^{-1 / r} \\
& =\left[\frac{x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}}{\left(x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}\right)^{r}}\right]^{-1 / r}=\left[\left(x_{1}^{\frac{r}{r-1}}+x_{2}^{\frac{r}{r-1}}\right)^{1-r}\right]^{-1 / r} \\
& =\left[( x _ { 1 } ^ { \frac { r } { r - 1 } } + x _ { 2 } ^ { \frac { r } { r - 1 } } ) ^ { \frac { r - 1 } { r } } \cdot \left(\text { If } \rho=\frac{r-1}{r} \text { then } u(\underset{\sim}{x})=\left(x_{1}^{l}+x_{2}^{e}\right)^{1 / e}\right.\right.
\end{aligned}
$$

the Constant Elasticity of Substation utility function.)
The second-order sufficient conditions for a minimum are that the leading principal minors (the "D's") of the Hessian of the Lagrayion ( $\nabla^{2} \mathcal{L}$ ) have
the same sign as $(-1)^{m}$, starting with $D_{2 m+1}$. In. the case, $m=1$, so the condition is that $D_{3}$ (which equals $\left|D^{2} \mathcal{\alpha}\right|$ ) should hare the same sign as $(-1)^{\prime}=-1<0$. Using subscripts to denote differentiation of $v$ by $p$ :

$$
\nabla^{2} \mathscr{L}=\left[\begin{array}{lll}
\mathscr{L}_{\lambda \lambda} & \mathscr{L}_{\lambda 1} & \mathscr{L}_{\lambda 2} \\
\mathscr{L}_{1 \lambda} & \mathscr{L}_{11} & \mathscr{L}_{12} \\
\mathscr{L}_{2 \lambda} & \mathscr{L}_{21} & \mathscr{L}_{22}
\end{array}\right]=\left[\begin{array}{ccc}
0 & x_{1} & x_{2} \\
x_{1} & v_{11} & v_{12} \\
x_{2} & v_{21} & v_{22}
\end{array}\right]
$$

Qualifying Exam 1996
Answer 3 cont...
Expanding along the first column,

$$
\begin{aligned}
\left|\nabla^{2} \dot{\alpha}\right| & =(-1)^{2+1} x_{1}\left(x_{1} v_{22}-x_{2} v_{21}\right)+(-1)^{2+2} x_{2}\left(x_{1} v_{12}-x_{2} v_{11}\right) \\
& =-x_{1}\left(x_{1} v_{22}-x_{2} v_{21}\right)+x_{2}\left(x_{1} v_{12}-x_{2} v_{11}\right) .
\end{aligned}
$$

Thus should be negative for the second-order condition to hold.
new: 2019 Exam 1, Qu. 1.
Resembles 1997 Exam 1 Question 2 and 1996 Qualifying Exam Question 3

## 1. [11 points]

If a consumer's expenditure function is $e(p, \bar{u})=p_{1}^{a} p_{2}^{1-a} \bar{u}$, find the consumer's:
(a) (direct) utility function;
(b) Marshallian demand curve for good 2;
(c) Hicksian demand curve for good 2.

If you decide to solve an optimization problem when you answer this question, you do not have to verify that the second-order conditions hold, but you do have to state the second-order conditions. In stating these second-order conditions, it is acceptable to leave derivatives unevaluated, as long as the only things you have left undone are simple differentiations.

Answer to Exam 1, Fall 20 19, Question 1
a)

$$
e(p, \bar{u})=p_{1}^{a} p_{2}^{1-a} \bar{u}
$$

To use $u=\min _{\sim} v(P)$ sit. $\underset{\sim}{P} \cdot \underset{\sim}{x}=1$, we need to find $v$ :

$$
\begin{aligned}
e\left(p_{1}, v(p, m)\right)=m \Rightarrow p_{1}^{a} p_{2}^{1-a} v(p, m) & =m \\
& \Rightarrow v(p, m)=m p_{1}^{-a} p_{2}^{a-1} .
\end{aligned}
$$

Now we solve $\min _{\sim} \operatorname{q}_{m}(1) p_{1}^{-a} p_{2}^{a-1}$ s.t. $p_{1} x_{1}+p_{2} x_{2}=1$.

$$
\begin{aligned}
& \mathcal{L}=p_{1}^{-a} p_{2}^{a-1}+\lambda\left[p_{1} x_{1}+p_{2} x_{2}-1\right] \\
& 0=\partial \mathscr{L} / \partial p_{1}=-a p_{1}^{-a} p_{1}^{-1} p_{2}^{a-1}+\lambda x_{1} \\
& 0=\partial 义 / \partial p_{2}=(a-1) p_{1}^{-a} p_{2}^{a} p_{2}^{-2}+\lambda x_{2} \\
& \Rightarrow \lambda=a \underbrace{p_{1}}_{\uparrow} \underbrace{-a}_{\uparrow} p_{1}^{-1} x_{1}^{p_{2}^{a-1}} x_{1}^{-1}=(1-a) \underbrace{p_{1}}_{\uparrow} \underbrace{p_{2}^{a} p_{2}^{-2}}_{\hat{\delta}} x_{2}^{-1} \\
& \text { a } p_{1}^{-1} p_{2} x_{1}^{-1}=(1-a) \quad \text {. } x_{2}^{-1} \\
& \frac{1}{a} p_{1} p_{2}^{-1} x_{1}=\frac{1}{1-a} x_{2} \\
& \frac{1-a}{a} p_{1} p_{2}^{-1} x_{1}=x_{2} \Rightarrow p_{2}=\frac{1-a}{a} p_{1} \frac{x_{1}}{x_{2}} \text { (remember it's the }
\end{aligned}
$$ pries which are ends fenous)

$$
\Rightarrow 1=p_{1} x_{1}+p_{2} x_{2}=p_{1} x_{1}+\frac{1-a}{a} p_{1} x_{1}=\left(\frac{a}{a}+\frac{1-a}{a}\right) p_{1} x_{1}=\frac{p_{1} x_{1}}{a} \Rightarrow p_{1}=\frac{a}{x_{1}}
$$

Then $p_{2}=\frac{1-a}{a} \frac{a}{x_{1}} \frac{x_{1}}{x_{2}}=\frac{1-a}{x_{2}}$ and

To check second-order conditions:

$$
\begin{aligned}
& \partial Z / \partial \lambda=p_{1} x_{1}+p_{2} x_{2}-1 \\
& \text { from above }\left\{\begin{array}{l}
\partial Z / \partial p_{1}=-a p_{1}^{-a-1} p_{2}^{a-1}+\lambda x_{1} \\
\partial \mathscr{L} / \partial p_{2}=(a-1) p_{1}^{-a} p_{2}^{a-2}+\lambda x_{2}
\end{array}\right. \\
& \text { \#Variables } n=2 \\
& \text { \# Construents } m=1 \\
& D_{2 m+1} \ldots D_{m+n} \text { is } D_{3} \ldots D_{3} . \\
& \text { For a minimum, we want } D_{3} \text { of } \nabla^{2} \mathscr{L} \\
& \text { to have the since sign as }(-1)^{m}=-1 \text {. } \\
& \text { So } \nabla^{2} \mathscr{L}=\left[\begin{array}{lll}
\partial^{2} \mathscr{L} / \partial \lambda^{2} & \partial^{2} \mathscr{L} / \partial \lambda \partial p_{1} & \partial^{2} \mathscr{L} / \partial \lambda \partial p_{2} \\
\partial^{2} \mathscr{L} / \partial p_{1} \partial \lambda & \partial^{2} \mathcal{L} / \partial p_{1}^{2} & \partial^{2} \mathscr{L} / \partial p_{1} \partial p_{2} \\
\partial^{2} \mathscr{L} / \partial p_{2} \lambda \lambda & \partial^{2} \mathscr{L} / \partial p_{2} \partial p_{1} & \partial^{2} \mathscr{L} / \partial p_{2}^{2}
\end{array}\right] \\
& =\left[\begin{array}{ccc}
0 & x_{1} & x_{2} \\
x_{1} & a(a+1) p_{1}^{-a-2} p_{2}^{a-1} & a(1-a) p_{1}^{-a-1} p_{2}^{a-2} \\
\chi_{2} & a(1-a) p_{1}^{-a-1} p_{2}^{a-2} & (a-1)(a-2) p_{1}^{-a} p_{2}^{a-3}
\end{array}\right] .
\end{aligned}
$$

the second-onder sufficient condition is that the determinant of this matrix be negative.

Optional: expanding alow the first row, that condition is

$$
\begin{aligned}
& 0>(-1)^{1+2} x_{1}\left[x_{1}(a-1)(a-2) p_{1}^{-a} p_{2}^{a-3}-x_{2} a(1-a) p_{1}^{-a-1} p_{2}^{a-2}\right] \\
& +(-1)^{1+3} x_{2}\left[x_{1 a}(1-a) p_{1}^{-a-1} p_{2}^{a-2}-x_{2} a(a+1)_{1}^{-a-2} p_{2}^{a-1}\right]
\end{aligned}
$$

b) Use Roy's Identity

$$
\left.V^{v}\left(p_{1} m\right)=m p_{1}^{-a} p_{2}^{a-1} \text { from part } a\right)
$$

$$
x_{2}=-\frac{\partial v / \partial p_{2}}{\partial v / \partial m}=-\frac{(a-1) m p_{1}^{-a} p_{2}^{a-2}}{p_{1}^{-a} p_{2}^{a-1}}=(1-a) m / p_{2} .
$$

c) Use Shepherd's Lemma

$$
h_{2}=\frac{\partial e}{\partial p_{2}}=\frac{\partial}{\partial p_{2}} p_{1}^{a} p_{2}^{1-a} \bar{u}=(1-a) p_{1}^{a} p_{2}^{-a} \bar{u} .
$$

Alternate answer for (c):

$$
\begin{aligned}
& h_{2}(\underset{\sim}{p}, \bar{u})=x_{2}(\underset{\sim}{p}, e(\underset{\sim}{p}, m)) ; \text { from }(b), \\
&=\frac{1-a}{p_{2}} e(\underset{\sim}{p}, m) ; \text { from problem } \\
& \text { statemat, } \\
&=\frac{1-a}{p_{2}} p_{1}^{a} p_{2}^{1-a} \bar{u} \\
&=(1-a) p_{1}^{a} p_{2}^{-a} \bar{u} .
\end{aligned}
$$

Alternate answer for (b):

$$
\begin{aligned}
x_{2}(p, m) & =h_{2}\left({\underset{\sim}{p}}_{1} v(\underset{\sim}{p}, m)\right) ; \text { from }(c), \\
& =(1-a) p_{1}^{a} p_{2}^{-a} v(p, m) ; \text { from } v: \\
& =(1-a) p_{1}^{a} p_{2}^{-a} n \cdot p_{1}^{-a} p_{2}^{a-1} \\
& =(1-a) p_{2}^{-1} m .
\end{aligned}
$$

1. [11 points] Suppose the expenditure function of a consumer is $e(\mathbf{p}, u)=$ $\left(p_{1}^{r}+p_{2}^{r}\right)^{1 / r} u$ where $p_{1}$ and $p_{2}$ are prices, $\mathbf{p}$ is the vector $\left(p_{1}, p_{2}\right)$, and $u$ is utility. Find this consumer's Marshallian demand curves.

Fall 2011, Exam 1 Qu 1

Answers to Exam 1, Econ. 7005, Full 2011
(1)

$$
\begin{aligned}
& e\left(p_{1} u\right)=\left(p_{1}^{r}+p_{2}^{r}\right)^{1 / r} u \\
& \left.\begin{array}{l}
e\left(p_{1} v(p, m)\right) \equiv m \\
\underset{\sim}{\|} \underset{\sim}{\left(p_{1}^{r}+p_{2}{ }^{r}\right)^{1 / r}} \underset{v(p, m)}{ }
\end{array}\right\} \Rightarrow\left(p_{1}^{r}+p_{2}^{r}\right)^{1 / r} \underset{\sim}{v}(p, m)=m \\
& \text { replacing } u \\
& \text { by } v(p, m) \\
& \Rightarrow v\left(p_{1} m\right)=m\left(p_{1}^{r}+p_{2}^{r}\right)^{-1 / r} . \\
& \text { either }
\end{aligned}
$$

Roy's Identity

$$
\begin{aligned}
x_{1}\left(p_{1} m\right) & =-\frac{\partial v\left(p_{1} m\right) / \partial p_{1}}{\partial v\left(p_{1} m\right) / \partial m}=-\frac{m\left(\frac{-1}{r}\right)\left(p_{1}^{r}+p_{2}^{r}\right)^{-\frac{1}{r}-1} r p_{1}^{r-1}}{\left(p_{1}^{r}+p_{2}^{r}\right)^{-1 / r}} \\
& =\frac{m p_{1}^{r-1}}{p_{1}^{r}+p_{2}^{r}} \\
x_{2}\left(p_{2}, m\right) & =-\frac{\partial v / \partial p_{2}}{\partial v / \partial m}=-\frac{m\left(\frac{-1}{r}\right)\left(p_{1}^{r}+p_{2}^{r}\right)^{-\frac{1}{r}-1} r p_{2}^{r-1}}{\left(p_{1}^{r}+p_{2}^{r}\right)^{-1 / r}} \\
& =\frac{m p_{2}^{r-1}}{p_{1}^{r}+p_{2}^{r}}
\end{aligned}
$$

Alternatively, find $h_{1}$ and ho from $e$, then use

$$
x_{i}(p, m)=h_{i}(p, v(\underset{\sim}{p}, m)):
$$

${ }^{1}$ Hicksiom demand ave

Shepherd's Lemma

$$
\begin{aligned}
& h_{1}=\frac{\partial e}{\partial p}=\frac{1}{r}\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{1}{r}-1} r p_{1}^{r-1} u=\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{1}{r}-1} p_{1}^{r-1} u \\
& h_{2}=\frac{\partial e}{\partial p_{2}}=\frac{1}{r}\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{1}{r}-1} r p_{2}^{r-1} u=\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{1}{r}-1} p_{2}^{r-1} u
\end{aligned}
$$

and

$$
\begin{aligned}
x_{1}\left(p_{\sim}, m\right) & =h_{1}\left(\underset{\sim}{p}, v\left(\sim_{r}, m\right)\right)=\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{1}{r-1}} p_{1}^{r-1} v\left(p_{1} m\right) \\
& =\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{1}{r-1}} p_{1}^{r-1} m\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r}}=\frac{m p_{1}^{r-1}}{p_{1}^{r}+p_{2}^{r}} \\
x_{2}\left(p_{\sim}, m\right) & =h_{2}\left(\underset{\sim}{p}, v\left(p_{\sim}, m\right)\right)=\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{1}{r}-1} p_{2}^{r-1} v\left(p_{1} m\right) \\
& =\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{1}{r-1}} p_{2}^{r-1} m\left(p_{1}^{r}+p_{2}^{r}\right)^{\frac{-1}{r}}=\frac{m p_{2}^{r-1}}{p_{1}^{r}+p_{2}^{r}}
\end{aligned}
$$

as before.

Yet another method starting from $v(p, m)$ would be to obtain $u(x)$ via $\operatorname{mix}_{p} v(p, 1)$, then get $x_{1}$ and $x_{2}$ by $\max _{\sim} u(x)$ sit. $p \cdot x=m$.


2. [11 points] Suppose a consumer has utility function $u\left(x_{1}, x_{2}\right)=\ln x_{1}+$ $\ln x_{2}$ where $x_{1}$ and $x_{2}$ are quantities of two goods. Suppose the consumer's income is $m$ and let the prices of the goods be $p_{1}$ and $p_{2}$ respectively.
(a) Find the consumer's Marshallian demand curves. (Hint: the answers are $x_{1}^{*}=m /\left(2 p_{1}\right)$ and $x_{2}^{*}=m /\left(2 p_{2}\right)$.)
(b) State and verify the second-order conditions.
(c) By using the Slutsky equation, find how the consumer's Hicksian (i.e., "compensated") demand curve for the first commodity varies as $p_{2}$ varies. (I want a formula for the appropriate derivelive.) (Hint: the answer is $m /\left(4 p_{1} p_{2}\right)$.)
(d) Find the indirect utility function. (Hint: one way of writing the answer is $\ln \left[m^{2} /\left(4 p_{1} p_{2}\right)\right]$.)
(e) Find the expenditure function. (Hint: the answer is $2 \sqrt{p_{1} p_{2}} e^{u / 2}$ where " $e$ " is the irrational number 2.718....)
(f) From the expenditure function, derive the consumer's Hicksian demand curve for the first good and find how this demand changes with changes in $p_{2}$. (Hint: the answer is $e^{u / 2} /\left(2 \sqrt{p_{1} p_{2}}\right)$.)
(g) Verify that the answers to parts (c) and (f) are the same. You may want to use part (d) to help.
(h) How does the consumer's Hicksian demand curve for the second commodity vary as $p_{1}$ varies?

Answer Z
（2）a）max $\ln x_{1}+\ln x_{2}$ s．t．$p_{1} x_{1}+p_{2} x_{2}=m$

$$
\left.\begin{array}{rl}
\mathscr{L} & =\ln x_{1}+\ln x_{2}+\lambda\left(m-p_{1} x_{1}-p_{2} x_{2}\right) \\
0 & =\partial \mathscr{L} / \partial \lambda=m-p_{1} x_{1}-p_{2} x_{2} \\
0 & =\frac{1}{x_{1}}-\lambda p_{1} \\
0 & =\frac{1}{x_{2}}-\lambda p_{2}
\end{array}\right\} \lambda=\frac{1}{p_{1} x_{1}}=\frac{1}{p_{2} x_{2}} \Rightarrow x_{2}=\frac{p_{1} x_{1}}{p_{2}}, ~ \Rightarrow m=p_{1} x_{1}+p_{2} \frac{p_{1} x_{1}}{p_{2}} \Rightarrow \text { } \quad \Rightarrow \quad .
$$

$$
m=p_{1} x_{1}+p_{1} x_{1}=2 p_{1} x_{1} \Rightarrow x_{1}=\frac{m}{2 p_{1}} \text { and } x_{2}=\frac{m}{2 p_{2}} .
$$

b) $\nabla^{2} \mathscr{L}=\left[\begin{array}{lll}\mathcal{L}_{\lambda \lambda}^{\prime \prime} & \mathcal{L}_{\lambda x_{1}}^{\prime \prime} & \mathcal{L}_{\lambda x_{2}}^{\prime \prime} \\ \mathcal{L}_{x_{1} \lambda}^{\prime \prime} & \mathcal{L}_{x_{1} x_{1}}^{\prime \prime} & \mathscr{L}_{x_{1} x_{2}}^{\prime \prime} \\ \mathcal{L}_{x_{2} \lambda}^{\prime \prime} & \mathcal{L}_{x_{2} x_{1}}^{\prime \prime} & \mathscr{L}_{x_{2} x_{2}}^{\prime \prime}\end{array}\right]=\left[\begin{array}{ccc}0 & -p_{1} & -p_{2} \\ -p_{1} & \frac{-1}{x_{1}^{2}} & 0 \\ -p_{2} & 0 & \frac{-1}{x_{2}^{2}}\end{array}\right]$

So (for a max: $D_{2 m+1} \cdots D_{n+m}$ alt. indign starting with $(-1)^{m+1} . m=1 n=$ $D_{3}$ has sita of $(-1)^{2}>0$ :

$$
\begin{aligned}
D_{3} \text { of } \nabla^{2} \mathscr{L}=\left|\nabla^{2} \mathscr{L}\right|= & +p_{1}\left(p_{1} \frac{1}{x_{2}^{2}}-0\right)-p_{2}\left(-p_{2} \frac{1}{x_{1}^{2}}\right) \\
& =\frac{p_{1}^{2}}{x_{2}^{2}}+\frac{p_{2}^{2}}{x_{1}^{2}}>0 \text { ok }
\end{aligned}
$$

c)

$$
\begin{aligned}
& \frac{\partial x_{i}}{\partial p_{j}}=\frac{\partial h_{i}}{\partial p_{j}}-\frac{\partial x_{i}}{\partial m} x_{j} \text { Exam 1 } \\
& \Rightarrow \frac{\partial h_{i}}{\partial p_{j}}=\frac{\partial x_{i}}{\partial p_{j}}+\frac{\partial x_{i}}{\partial m} x_{j} \text { Answer } 2 \text { cont... } \\
& \frac{\partial h_{1}}{\partial p_{2}}=\frac{\partial x_{1}}{\partial p_{2}}+\frac{\partial x_{1}}{\partial m} x_{2}=0+\frac{1}{2 p_{1}} \frac{m}{2 p_{2}}=\frac{m}{4 p_{1} p_{2}}
\end{aligned}
$$

d) $v=\ln x_{1}^{*}+\ln x_{2}^{*}=\ln \frac{m}{2 p_{1}}+\ln \frac{m}{2 p_{2}}=\ln \frac{m^{2}}{4 p_{1} p_{2}}$
e)

$$
\begin{aligned}
& v(\underset{\sim}{p, \operatorname{expende}}) \equiv u \\
& \uparrow e(p, u) \\
& \ln \frac{\operatorname{expend}^{2}}{4 p_{1} p_{2}}=u \\
& \frac{\operatorname{expead}^{2}}{4 p_{1} p_{2}}=e^{u} \\
& \text { expend }{ }^{2}=4 p_{1} p_{2} e^{u} \\
& \text { expenditure }=2 \sqrt{p_{1} p_{2}} e^{u / 2}
\end{aligned}
$$

f)

$$
\begin{aligned}
& h_{1}=\frac{\partial e}{\partial p_{1}}=\sqrt{\frac{p_{2}}{p_{1}}} e^{u / 2} \\
& \frac{\partial h_{1}}{\partial p_{2}}=\frac{1}{2} \cdot \frac{1}{\sqrt{p_{1} p_{2}}} e^{u / 2}
\end{aligned}
$$

g)
g)

$$
\begin{aligned}
" & =\frac{1}{2} \frac{1}{\sqrt{p_{1} p_{2}}} e^{\frac{1}{2} \ln \frac{m^{2}}{4 p_{1} p_{2}}}: \\
& =\frac{1}{2} \frac{m}{2 p_{1} p_{2}}=\frac{m}{4 p_{1} p_{2}} O K
\end{aligned}
$$

n)

$$
\frac{\partial h_{2}}{\partial p_{1}}=\frac{\partial h_{1}}{\partial p_{2}} \text { by symmetry }
$$

$$
\hat{\imath}_{\text {give }} \text { in }(f) \text { and }(c)
$$

Exam 1 2004
Answer 2 cent....

1. [20 points] Suppose a price-taking consumer consumes two commodities $x$ and $y$ and has a utility function of the form

$$
u(x, y)=\alpha \ln x+\beta \ln y .
$$

Suppose the price of $x$ is $\bar{p}_{x}$ and the price of $y$ is $\bar{p}_{y}$.
(a) Show that this consumer's indirect utility function can be written as

$$
v\left(\bar{p}_{x}, \bar{p}_{y}, m\right)=\ln \left[\frac{\alpha^{\alpha} \beta^{\beta} m^{\alpha+\beta}}{\bar{p}_{x}^{\alpha} \bar{p}_{y}^{\beta}(\alpha+\beta)^{\alpha+\beta}}\right]
$$

where $m$ is the consumer's income.
(b) Show that this consumer's expenditure function is

$$
e\left(\bar{p}_{x}, \bar{p}_{y}, u\right)=(\alpha+\beta)\left(\frac{\bar{p}_{x}^{\alpha} \bar{p}_{y}^{\beta}}{\alpha^{\alpha} \beta^{\beta}}\right)^{\frac{1}{\alpha+\beta}} \exp \left(\frac{u}{\alpha+\beta}\right)
$$

(c) Form this consumer's money metric indirect utility function,

$$
\mu\left(\hat{p}_{x}, \hat{p}_{y} ; \bar{p}_{x}, \bar{p}_{y}, m\right) \equiv e\left(\hat{p}_{x}, \hat{p}_{y}, v\left(\bar{p}_{x}, \bar{p}_{y}, m\right)\right)
$$

Your final expression should not explicitly involve the indirect utility function $v$.
(d) Briefly explain the economic interpretation of your answer to part (c).

Answers to Econ 7005 Final Exam. Fall 2008
(1)

$$
u(x, y)=\alpha \ln x+\beta \ln y
$$

a) max u s.t. $m=\bar{p}_{x} x+\bar{p}_{y} y$

$$
\left.\begin{array}{l}
\mathscr{L}=\alpha \ln x+\beta \ln y+\lambda\left[m-\bar{p}_{x} x-\bar{p}_{y} y\right] \\
0=\partial \mathscr{L} / \partial \lambda=m-\bar{p}_{x} x-\bar{p}_{y} y \\
0=\partial \mathscr{L} / \partial x=\frac{\alpha}{x}-\lambda \bar{p}_{x} \Rightarrow \lambda=\frac{\alpha}{\bar{p}_{x} x} \\
0=\partial \not \partial / \partial y=\frac{\beta}{y}-\lambda \bar{p}_{y} \Rightarrow \lambda=\frac{\beta}{\bar{p}_{y} y}
\end{array}\right\} \frac{\alpha}{\bar{p}_{x} x}=\frac{\beta}{\bar{p}_{y} y} \Rightarrow \quad \begin{aligned}
& x=\frac{\alpha}{\beta} \frac{\bar{p}_{y}}{\bar{p}_{x}} y ;
\end{aligned}
$$

substizing in to the first F. O.C.:

$$
\begin{aligned}
& m=\bar{p}_{x} \frac{\alpha}{\beta} \frac{\bar{p}_{y}}{\bar{p}_{x}} y+\bar{p}_{y} y=\frac{\alpha}{\beta} \bar{p}_{y} y+\bar{p}_{y} y=\left(\frac{\alpha}{\beta}+1\right) \bar{p}_{y} y=\frac{\alpha+\beta}{\beta} \bar{p}_{y} y \\
& \Rightarrow y^{*}=\frac{\beta}{\alpha+\beta} \frac{m}{\bar{p}_{y}} \text { and } \\
& x^{*}=\frac{\alpha}{\beta} \frac{\bar{p}_{y}}{p_{x}} y=\frac{\alpha}{\beta} \frac{\bar{p}_{y}}{\bar{p}_{x}} \frac{\beta}{\alpha+\beta} \frac{m}{\bar{p}_{y}}=\frac{\alpha}{\alpha+\beta} \frac{m}{\bar{p}_{x}} .
\end{aligned}
$$

Then

$$
\begin{aligned}
v & =u\left(x^{*}, y^{*}\right)=\alpha \ln x^{*}+\beta \ln y^{*} \\
& =\ln \left(x^{*}\right)^{\alpha}+\ln \left(y^{*}\right)^{\beta}=\ln \left(x^{*}\right)^{\alpha}\left(y^{*}\right)^{\beta} \\
& =\ln \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{m}{F_{x}}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\left(\frac{m}{F_{y}}\right)^{\beta} \quad \text { over } \rightarrow
\end{aligned}
$$

$=\ln \frac{\alpha^{\alpha} \beta^{\beta} m^{\alpha+\beta}}{\bar{p}_{x}^{\alpha} \bar{p}_{y}^{\beta}(\alpha+\beta)^{\alpha+\beta}}$ as was to be shown. (This "v "is $v\left(\bar{p}_{x}, \bar{p}_{y}, m\right)$.)
b) Since $u=v(p, e(p, u))$ and we know $v$ from part $(a)$ :

$$
u=v\left(\bar{p}_{x}, \bar{p}_{y}, e\right)=\ln \frac{\alpha^{\alpha} \beta^{\beta} e^{\alpha+\beta}}{\vec{p}_{x}^{\alpha} \bar{p}_{y}^{\beta}(\alpha+\beta)^{\alpha+\beta}} .
$$

This " $e$ " is the expenditure function, not the base of the natural logarithms. To avoid confusion - that is. to avoid having two different " $e$ " symbols in this problem Ill use $\exp (x)$ to denote raising the base of the natural logarithms to the "x" power.
Solve for the expenditure function, $e$ :

$$
\begin{aligned}
& \exp (u)=\frac{\alpha^{\alpha} \beta^{\beta} e^{\alpha+\beta}}{\bar{p}_{x}^{\alpha} \bar{p}_{y}^{\beta}(\alpha+\beta)^{\alpha+\beta}} \\
& \frac{\bar{p}_{x}^{\alpha} \bar{p}_{y}^{\beta}}{\alpha^{\alpha} \beta^{\beta}}(\alpha+\beta)^{\alpha+\beta} \exp (u)=e^{\alpha+\beta} \text { so } \\
& e\left(\bar{p}_{x}, \bar{p}_{y}, u\right)=\left(\frac{\bar{p}_{x}^{\alpha} \bar{p}_{y}^{\beta}}{\alpha^{\alpha} \beta^{\beta}}\right)^{\frac{1}{\alpha+\beta}}(\alpha+\beta) \exp \left(\frac{u}{\alpha+\beta}\right)
\end{aligned}
$$

as was to be shown.
c) Using the answers to ( $a$ ) and (b),

$$
\begin{aligned}
\mu(\hat{p} ; \bar{p}, m) & \equiv e(\underset{\sim}{\hat{p}}, w(\vec{p}, m)) \\
& =\left(\frac{\hat{p}_{x}^{\alpha} \hat{p}_{y}^{\beta}}{\alpha^{\alpha} \beta^{\beta}}\right)^{\frac{1}{\alpha+\beta}}(\alpha+\beta) \exp \left(\frac{v(\bar{p}, m)}{\alpha+\beta}\right) \rightarrow
\end{aligned}
$$

$$
\begin{aligned}
& =\left(\frac{\hat{P}_{x}^{\alpha} \hat{P}_{y}^{\beta}}{\alpha^{\alpha} \beta^{\beta}}\right)^{\frac{1}{\alpha+\beta}}(\alpha+\beta)[\exp v(\bar{p}, m)]^{\frac{1}{\alpha+\beta}} \\
& =(\alpha+\beta)\left(\frac{\hat{P}_{x}^{\alpha} \hat{P}_{y}}{\alpha^{\alpha} \beta^{\beta}}\right)^{\frac{1}{\alpha+\beta}}\left[\frac{\alpha^{\alpha} \beta^{\beta} m^{\alpha+\beta}}{\bar{P}_{x}^{\alpha} \bar{P}_{y}^{\beta}(\alpha+\beta)^{\alpha+\beta}}\right] \frac{1}{\alpha+\beta} \\
& =(\alpha+\beta)\left[\begin{array}{ll}
\hat{P}_{x}^{\alpha} \hat{P}_{y} \\
\frac{P^{\alpha}-\beta}{P_{x}} & \left.\frac{\alpha^{\alpha} \beta^{\beta}}{\alpha}\right]^{\alpha} \beta
\end{array}\right]^{\frac{1}{\alpha+\beta}} \frac{m}{\alpha+\beta}=\left[\frac{\hat{P}_{x}^{\alpha} \hat{P}_{y}^{\beta}}{\vec{P}_{x}^{\alpha} \bar{P}_{y}^{\beta}}\right]^{\frac{1}{\alpha+\beta}} m .
\end{aligned}
$$

Optional: This can easily be checked by noting that if $\hat{p}=\vec{p}$, then $\mu$ should equal $m$, which it does.
d) $\mu(\underset{\sim}{p}: \bar{p}, m) \equiv e(\hat{p}, v(\bar{p}, m))$ is the amount of money which the consumer would require at prices $\hat{p}$ to have the same utility as if the priceswere $\vec{p}$ and his income was $m$.

1. [13 points] Suppose a price-taking consumer consumes two commodities $x$ and $y$ and has an indirect utility function of the form

$$
v\left(\bar{p}_{x}, \bar{p}_{y}, m\right)=\ln \left[\frac{\alpha^{\alpha} \beta^{\beta} m^{\alpha+\beta}}{\bar{p}_{x}^{\alpha} \bar{p}_{y}^{\beta}(\alpha+\beta)^{\alpha+\beta}}\right]
$$

where $m$ is the consumer's income, $\bar{p}_{x}$ is the price of $x$, and $\bar{p}_{y}$ is the price of $y$.
(a) Show that this consumer's expenditure function is

$$
e\left(\bar{p}_{x}, \bar{p}_{y}, u\right)=(\alpha+\beta)\left(\frac{\bar{p}_{x}^{\alpha} \bar{p}_{y}^{\beta}}{\alpha^{\alpha} \beta^{\beta}}\right)^{\frac{1}{\alpha+\beta}} \exp \left(\frac{u}{\alpha+\beta}\right)
$$

where exponentiation is denoted by "exp" to avoid confusion with the notation for the expenditure function $e$.
(b) Form this consumer's money metric indirect utility function,

$$
\mu\left(\hat{p}_{x}, \hat{p}_{y} ; \bar{p}_{x}, \bar{p}_{y}, m\right) \equiv e\left(\hat{p}_{x}, \hat{p}_{y}, v\left(\bar{p}_{x}, \bar{p}_{y}, m\right)\right) .
$$

Your final expression should not explicitly involve the indirect utility function $v$.
(c) Show that this consumer's utility function is either

$$
u(x, y)=\alpha \ln x+\beta \ln y
$$

or a monotonically increasing function of this (such as $x^{\alpha} y^{\beta}$ ).

Summer 2011 qualifying exam, Sec. 1 Qu. 1

Answers to Mire conomics Qualify ny Exam
Questions of Prot.Lozada, Summer 2011

Section 1 Question 1.
[This question is very closely related to Fall 2008 Final Exam Question 1.]
a) We know that $v(p, e(p, u))=u, 0$, abbreviation the expenditure function by "e", $\quad v(\underset{\sim}{p}, e)=u$ so

$$
u=v(p, e)=v\left(\bar{p}_{x}, \bar{p}_{y},{\underset{\uparrow}{\uparrow}}_{\substack{e \\ \text { e in } \\ \text { our case }}} \frac{\alpha^{\alpha} \beta^{\beta} e^{\alpha+\beta}}{\bar{p}_{x}^{\alpha} \bar{p}_{y}^{\beta}(\alpha+\beta)^{\alpha+\beta}} .\right.
$$

We now have to solve for $e$, remembering that $e$ is the expenditure function, not the base of the natural logarithms.

$$
\begin{aligned}
\exp u & =\frac{\alpha^{\alpha} \beta^{\beta} e^{\alpha+\beta}}{\bar{p}_{x}^{\alpha} \bar{p}_{y}^{\beta}(\alpha+\beta)^{\alpha+\beta}} \Rightarrow e^{\alpha+\beta}=(\alpha+\beta)^{\alpha+\beta} \frac{\bar{P}_{x}^{\alpha} \bar{p}_{y}^{-\beta}}{\alpha^{\alpha} \beta^{\beta}}(\exp u) \\
& \Rightarrow e=(\alpha+\beta)\left(\frac{\bar{P}_{x}^{\alpha} \bar{P}_{y}^{\beta}}{\alpha^{\alpha} \beta^{\beta}}\right)^{\frac{1}{\alpha+\beta}} \exp \left(\frac{u}{\alpha+\beta}\right) .
\end{aligned}
$$

$$
\begin{aligned}
& \text { b) Since } e\left(\hat{p}_{x}, \hat{p}_{y}, v\right)=(\alpha+\beta)\left[\frac{\hat{p}_{x}^{\alpha} \hat{p}_{y}^{\beta}}{\alpha^{\alpha} \beta}\right]^{\frac{1}{\alpha+\beta}} \exp \left(\frac{v}{\alpha+\beta}\right), \\
& e\left(\hat{p}_{x}, \hat{p}_{y}, v\left(\bar{p}_{x}, \bar{p}_{y}, m\right)\right)=(\alpha+\beta)\left[\frac{\hat{p}_{x}^{\alpha} \hat{p}_{y}^{\beta}}{\alpha^{\alpha} \beta^{\beta}}\right]^{\frac{1}{\alpha+\beta}} \exp \left[\frac{v\left(\bar{p}_{x}, \bar{p}_{y}, m\right)}{\alpha+\beta}\right] \\
& =(\alpha+\beta)\left[\frac{\hat{p}_{x}^{\alpha} \hat{p}_{y}}{\alpha^{\alpha} \beta^{\beta}}\right]^{\frac{1}{\alpha+\beta}}\left[\exp v\left(\bar{p}_{x}, \bar{p}_{y}, m\right)\right]^{\frac{1}{\alpha+\beta}}=
\end{aligned}
$$

$$
(\alpha+\beta)\left[\frac{\hat{P}_{x}^{\alpha} \hat{P}_{y}^{\beta}}{\alpha^{\alpha} \beta^{\beta}}\right]^{\frac{1}{\alpha+\beta}}\left[\frac{\alpha^{\alpha} \beta^{\beta} m^{\alpha+\beta}}{\vec{P}_{x}^{\alpha} \vec{P}_{y}^{\beta}(\alpha+\beta)^{\alpha+\beta}}\right] \frac{1}{\alpha+\beta}
$$

.This simplefies
c)

$$
\begin{aligned}
u(\underline{x}) & =\min _{p} v(p) \text { s.t. } p \cdot x=1 \\
& \left.=\min \ln \left[\bar{p}_{x}^{-\alpha} \bar{p}_{y}^{-\beta} \frac{\alpha^{\alpha} \beta^{\beta} 1^{\alpha+\beta}}{(\alpha+\beta)^{\alpha+\beta}}\right]_{x}^{\alpha} \hat{p}_{y}^{\beta} / \bar{p}_{x}^{\alpha} \bar{p}_{y}^{\beta}\right]^{\frac{1}{\alpha+\beta}} \\
& =\min -\alpha \ln -1
\end{aligned}
$$

$$
=\min -\alpha \ln \bar{p}_{x}-\beta \ln \vec{p}_{y}+\ln \frac{\alpha^{\alpha} \beta \beta}{(\alpha+\beta)^{\alpha+\beta}}
$$

$$
\begin{aligned}
& \mathscr{L}=-\alpha \ln \bar{p}_{x}-\beta \ln \bar{p}_{y}+\ln \frac{\alpha^{\alpha} \beta \beta}{(\alpha+\beta)^{\alpha+\beta}}+\lambda\left(\bar{p}_{x} x+\bar{p}_{y} y-1\right) \\
& \text { F.O.C.'s }
\end{aligned}
$$

F.O.C.'s

$$
\left.\begin{array}{rl}
0=\frac{\partial x}{\partial \bar{p}_{x}}=\frac{-\alpha}{\bar{p}_{x}}+\lambda x \\
0=\frac{\partial x}{\partial \bar{p}_{y}}=\frac{-\beta}{\bar{p}_{y}}+\lambda y
\end{array}\right\} \Rightarrow \begin{aligned}
& x \bar{p}_{x}=\frac{\beta}{y \bar{p}_{y}} \\
& 0=\frac{\partial \mathcal{L}}{\partial \lambda}=\bar{p}_{x} x+\bar{p}_{y} y-1 \Rightarrow \bar{p}_{x}=\frac{\alpha}{x} \frac{y}{\beta} \bar{p}_{y} \\
&=\left(\frac{\alpha}{x} \frac{y}{\beta} \bar{p}_{y}\right) x+\bar{p}_{y} y \\
&=\frac{\alpha y}{\beta} \bar{p}_{y}+\vec{p}_{y} y=y \bar{p}_{y}\left(\frac{\alpha}{\beta}+1\right) \\
&=y \bar{p}_{y} \frac{\alpha+\beta}{\beta} \Rightarrow \bar{p}_{y}=\frac{\beta}{\alpha+\beta} \frac{1}{y} \text { and }
\end{aligned}
$$

theretire $\bar{P}_{x}=\frac{\alpha}{x} \frac{y}{\beta}\left[\frac{\beta}{\alpha+\beta} \frac{1}{y}\right]=\frac{\alpha}{\alpha+\beta} \frac{1}{x}$. Substiving these minimizing
$\bar{p}_{x}$ and $\bar{p}_{y}$ into $v$ results in

$$
u=\ln \left[\frac{\alpha}{\alpha+\beta} \frac{1}{x}\right]^{-\alpha}\left[\frac{\beta}{\alpha+\beta} \frac{1}{y}\right]^{-\beta} \frac{\alpha^{\alpha} \beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}}=\ln \frac{(\alpha+\beta)^{\alpha}}{\alpha^{\alpha}} x^{\alpha} \frac{(\alpha+\beta)^{\beta}}{\beta^{\beta}} y^{\beta} \frac{\alpha^{\alpha} \beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}}
$$

$$
=\ln x^{\alpha} y^{\beta}=\alpha \ln x+\beta \ln y .
$$

Note that since $\ln x$ is increasing in $x$, ore can, instead of minimizing

$$
\ln \left[\bar{P}_{x}^{-\alpha} \bar{P}_{y}^{-\beta} \frac{\alpha^{\alpha} \beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}}\right] \quad \text { s.t. } p \cdot x=1
$$

minimize instead

$$
\left[\bar{P}_{x}^{-\alpha} \bar{p}_{y}^{-\beta} \frac{\alpha^{\alpha} \beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}}\right] \text { st. } p \cdot x_{x}^{x}=1 \text {. }
$$

If you do this, the result is $u=x^{\alpha} y^{\beta}$, which represent the same preferences as $\ln x^{\alpha} y^{\beta}=\alpha \ln x+\beta \ln y$.

# Final Exam 2004 Question 1 

(1)

## Answer all of the following four questions.

1. [13 points] Let $h_{i}\left(\mathbf{p}, u^{0}\right)$ denote a consumer's Hicksian demand curve for good $i$ when the consumer faces prices $p$ and enjoys utility level $u^{0}$. Prove that

$$
\frac{\partial h_{i}}{\partial p_{j}}=\frac{\partial h_{j}}{\partial p_{i}}
$$

by using the definition of the expenditure function and by using the Envelope Theorem.

Answers to Final Exam, Econ 7005, Spry 2004
(1)

$$
\begin{aligned}
e(p, u)=\min _{\sim} \underset{\sim}{p} \cdot \underset{\sim}{x} \text { st. } u(x)=u(f i x e d) & \text { Final Exam } \\
\mathcal{L}=p \cdot x+\lambda(u-u(\underset{\sim}{x})) & \text { Answer } 1
\end{aligned}
$$

$\frac{\partial e}{\partial p_{i}}=\frac{\partial \mathscr{L}^{*}}{\partial p_{i}}$ from the Envelope Theorem
$=x_{i}$ or, in our usual notation, $h_{i}$. the thicksion demand curve

$$
\frac{\partial e}{\partial p_{i}}{ }^{\prime}=h_{i} \Rightarrow\left\{\begin{array}{l}
\frac{\partial h_{i}}{\partial p_{j}}=\frac{\partial}{\partial p_{i}} \frac{\partial e}{\partial p_{i}}=\frac{\partial^{2} e}{\partial p_{j} \partial p_{i}} \leftrightarrow \rightarrow \text { which ore equal } \\
\frac{\partial h_{j}}{\partial p_{i}}=\frac{\partial}{\partial p_{i}} \frac{\partial e}{\partial p_{j}}=\frac{\partial^{2} e}{\partial p_{i} \partial p_{j}} \leftrightarrow \text { Young's theorem). }
\end{array}\right.
$$



Question 2. Let ' $i$ ' and ' $j$ ' be any two commodities which a consumer buys. If the Hicksian (or "compensated") demand function of a consumer is $\mathbf{h}\left(\mathbf{p}, u_{0}\right)$, prove that $\partial h_{i} / \partial p_{i} \leq 0$ and that $\partial h_{i} / \partial p_{j}=\partial h_{j} / \partial p_{i}$.
You do not have to prove the Envelope Theorem here, but you do have to prove all other results which you use.

Optional Question 2.

Step 1: the analogue of Shepherd's Lemma for consumers, namely $h_{i}=\nabla_{p} e\left(p, u_{0}\right)$. Poof: $e\left(p, u_{0}\right)=\min p \cdot x$ sit. $u(x) \geqslant u_{0}$

The Lagrangian is $\left.\mathscr{L}=p \cdot x+\lambda\left[u(x)-u_{0}\right]\right] \begin{aligned} & \text { Note: Varia has " } u \text { " } \\ & \text { instead of " } u_{0} \text { ", which } \\ & \text { B OK, just a bit } \\ & \text { confusing bit }\end{aligned}$. The Envelope Theorem them implies that

$$
\frac{\partial e\left(p_{1} u_{0}\right)}{\partial p_{i}}=\frac{\partial \mathscr{L}^{*}}{\partial p_{i}}
$$

which is just $x_{i}^{*}$, the demand for commodity $i$. The vector of such demands is $\underset{\sim}{h}$, and the vector form of $\partial e\left(p, u_{0}\right) / \partial p_{i}$ is $\nabla_{p} e\left(p, u_{0}\right)$.

Step 2. Differentiate Step 1 's result with respect to $p$ : Qualifying Exam 1997

$$
\nabla_{p} \underset{\sim}{h}=\nabla_{\sim}^{2} e\left(p ; u_{0}\right)
$$

$$
\text { Answer } 2
$$

The RHS Ba Hessian and so is symmetric; therefore, the LHS is symmetric and $\quad \partial h_{i} / \partial p_{j}=\partial h_{j} / \partial p_{i}$.

Step 3. $e\left(p, u_{0}\right)$ is concave in $p$.
Proof:

$$
e\left(\lambda p_{1}+(1-\lambda) p_{2}, u_{0}\right)=\min _{x}\left(\lambda p_{1}+(1-\lambda) p_{2}\right) \cdot \underset{\sim}{x} \text { st. } u(x) \geqslant u_{0}
$$

$$
\geqslant \min _{x} \lambda_{\sim} p_{1} \cdot \underset{\sim}{x}\left(\text { s.t. } u(x) \geqslant u_{0}\right)+\min _{\sim}(1-\lambda)_{\sim}^{p_{2}} x_{\sim}^{x}\left(\text { st. } u(x) \geqslant u_{0}\right)
$$

because taking two separate miminizations couniot lead to a higher value

$$
\begin{aligned}
= & \left.\lambda \min _{\sim}^{x} p_{i} \cdot \underset{\sim}{x} \text { (sit. } u(\underset{\sim}{x}) \geqslant u_{0}\right) \\
& \left.\quad+(1-\lambda) \min _{\sim}^{x} p_{2} \cdot x \quad \text { s.t. } u(x) \geqslant u_{0}\right) \\
= & \lambda e\left(p_{1}, u_{0}\right)+(1-\lambda) e\left(p_{2}, u_{0}\right) .
\end{aligned}
$$



Step 4 . Since 'e is concave in $p, \nabla_{p}^{2} e$ is negative semidefinite.
So it diagonal elements are $\leq 0$. From Step 2, the means that the diagonal elements of $\nabla_{\sim} \underset{\sim}{L}$ are $\leq 0$; therefore $\partial h_{i} / \partial p_{i} \leq 0$ for all i.

Qualifying Exam 1997
Answer 2 cont..

## 2015 Final Exam Qu. 2

2. [18 points] If $h_{i}$ denotes a competitive consumer's Hicksian demand curve for good $i$ and $p_{i}$ denotes the price of good $i$, prove that

$$
\begin{aligned}
& \partial h_{i} / \partial p_{i} \leq 0 \quad \text { and } \\
& \partial h_{i} / \partial p_{j}=\partial h_{j} / \partial p_{i} .
\end{aligned}
$$

If you use an "Envelope Theorem" result, prove it (by applying the Envelope Theorem). If you contend that a function is concave or convex, prove it.
a) $\underset{\sim}{h}=\nabla_{\sim}^{p}{ }_{\uparrow} e(p, \bar{u})$. $P_{w o f}: e(p, \bar{u})=\min _{\sim}^{x} \underset{\sim}{p} \underset{\sim}{x}$ s.t. $u(x)=\bar{u}$
the expaditure
function
so by the Envelope Theorem,

$$
\begin{aligned}
\frac{\partial e}{\partial p_{i}} & =\frac{\partial z^{*}}{\partial p_{i}} \\
& =\frac{\partial}{\partial p_{i}}[\underset{\sim}{p} \cdot \underset{\sim}{x}+\lambda(\bar{u}-u(x))]
\end{aligned}
$$

$=x_{i}$ or, instanderel motation, $h_{i}$,
the thicksian demend curre for food $i$.
(a) $\Rightarrow$

$$
\begin{equation*}
\nabla_{\sim}^{p} \underset{\sim}{h}=\nabla_{p}^{2} e(\underset{\sim}{p}, \bar{u}) . \tag{b}
\end{equation*}
$$

(c) $e(\underset{\sim}{p}, \bar{u})$ is concave in $p$. Poof:

$$
e\left(p_{a}\right)
$$


(Here I'm supressing the depanduce of $e$ on $\bar{u}$.)

$$
(1-\alpha) p_{a}+\alpha p_{b} \text { for } 0 \leq \alpha \leq 1
$$

$e\left((1-\alpha) p_{a}+\alpha p_{b}\right)=\min _{\sim}^{x}\left[(1-\alpha) p_{a}+\alpha p_{b}\right] \cdot \underset{\sim}{x}$ s.t.

$\geqslant \min _{\sim}^{x}(1-\alpha) p_{a} \cdot \underset{\sim}{x}+\operatorname{man}_{\sim}^{x} \alpha p_{b} \cdot \underset{\sim}{x}$ (because
$\min _{x}[f(x)+g(x)] \leq \operatorname{mix}_{x} f(x)+\min _{x} g(x)$ cs
shown in class)

$$
\begin{aligned}
& =(1-\alpha) \min _{x}^{x} p_{a} \cdot x+\alpha \max _{\sim}^{x} p_{b} \cdot x \\
& =(1-\alpha) e\left(p_{a}\right)+\alpha e\left(p_{b}\right)
\end{aligned}
$$

which proves conccurity.
(c) $\Rightarrow{\underset{\sim}{p}}^{\nabla_{p}} e(\underline{p}, \bar{u})$ is negative semidefinite, and hence is diagonal terms are $\leq 0$. (This was also shown in class: if $A$ is negative semidefinite, $x^{\top} A_{\sim}^{x} \leqslant 0 \forall x$; take $\underset{\sim}{x}=(1,0,0, \ldots, 0)$, then $x=(0,1,0,0, \ldots, 0)$, eta. $)$

Then from (b), the diagonal terms of ${\underset{\sim}{p}}_{p} h_{i}$ are $\leq 0 ;$ but theecterms are just $\partial h_{i} / \partial p_{i}$.
(b) $\Rightarrow \partial h_{i}\left|\partial p_{j}=\partial h_{j}\right| \partial p_{i}$ because $D^{2} e$ is symmetric (its $(i, j)^{n}$ clamant is $\partial^{2} e \mid \partial p_{i} \partial p_{j}$ and its $(j, i)^{\text {th }}$ element is $\partial^{2} e / \partial p_{j} \partial p_{i}$, which are equal).

Fall 2021 Exam 1 Question 3

## 3. [11 points]

Suppose a consumer has an $n \times n$ negative semidefinite symmetric Slutsky Substitution Matrix $\mathbf{S}$, but an economist has data only on $n^{\prime}<n$ of the commodities. Should the $n^{\prime} \times n^{\prime}$ matrix $\mathbf{S}^{\prime}$ (whose $i, j$ entry is $\partial h_{i} / \partial p_{j}$ ) which is formed by using data only from $n^{\prime}$ of the commodities be negative semidefinite and symmetric? (This question requires a proof, not just a "yes or no" answer.)

Answer to Question 3, Exam 1, Full 2021, Econ. 7005

Suppose $n=4, n^{\prime}=2$, and the commodities one has data on are 1 and 3 .

$$
\text { We know that }\left[\begin{array}{llll}
S_{11} & S_{12} & S_{13} & S_{14} \\
S_{21} & S_{22} & S_{23} & S_{24} \\
S_{31} & S_{32} & S_{33} & S_{34} \\
S_{41} & S_{42} & S_{43} & S_{44}
\end{array}\right]=S \text { is negative semiditimite and }
$$

symmetric. But the order of commodities is arbitrary. Put them in this order: $1,3,2,4$. (that is, put the commodities you hare data on first.) then

$$
\hat{S}=\left[\begin{array}{llll}
S_{11} & S_{13} & S_{12} & S_{14} \\
S_{31} & S_{33} & S_{32} & S_{34} \\
S_{21} & S_{23} & S_{22} & S_{24} \\
S_{41} & S_{43} & S_{42} & S_{44}
\end{array}\right]
$$

is also negative somidefirite and symmetric. Hence $S_{13}=S_{31}$. Also, being negative semidefinite implies that:
all the $\Delta_{1}$ of $\hat{S}$ are $\leq 0$
will the $\Delta_{2}$ of $\hat{S}$ are $\geqslant 0$. But $S^{\prime}=\left[\begin{array}{ll}S_{11} & S_{13} \\ S_{31} & S_{33}\end{array}\right]$ is the upper
left-hand corner of $\hat{S}$, so the abovecunditions on the $\Delta$ 's of $\hat{S}$ imply that all the $\Delta_{1}$ of $S^{\prime}$ are $\leq 0$
all the $\Delta_{2}$ of $S^{\prime}$ are $\geqslant 0$. Hence $S^{\prime}$ is negative semidefinite. Its symmetry was proven above when we observed that $S_{13}=S_{31}$.

Extending this proof to other values of $n$ and $n '$ is strajklfforwacl.
Note: there may be other ways to prove that the leading prixapal minors of a negative definite symmetric matrix are negative defiute symmetric.
3. [11 points] Starting from the identity

$$
h_{k}(\mathbf{p}, u) \equiv x_{k}(\mathbf{p}, e(\mathbf{p}, u))
$$

where $x_{k}$ is the Marshallian demand for good $k$ and $h_{k}$ is the Hicksian demand for good $k$, determine the conditions on $x$ under which

$$
\frac{\partial x_{j}(\mathbf{p}, m)}{\partial p_{i}}=\frac{\partial x_{i}(\mathbf{p}, m)}{\partial p_{j}}
$$

[Fall 2004 Exam 1.]
3)

$$
\begin{aligned}
& h_{k}(\underset{\sim}{p}, u) \equiv x_{k}(p, e(p, u)) \\
& \frac{\partial h_{k}}{\partial p_{e}}=\frac{\partial x_{k}}{\partial p_{e}}+\frac{\partial x_{k}}{\partial e} \frac{\partial e}{\partial p_{e}} \\
& =\frac{\partial x_{k}}{\partial p_{e}}+\frac{\partial x_{k}}{\partial m}\left(x_{l}\right) \text { actual } h e \begin{array}{l}
\text { but the quantities are } \\
\text { same the the } \\
\text { turctions dither }
\end{array} \\
& \Rightarrow \frac{\partial x_{k}}{\partial p_{e}}=\frac{\partial h_{k}}{\partial p_{e}}-x_{e} \frac{\partial x_{k}}{\partial m} \text { the Shutsky Equation. } \\
& \text { So } \quad \frac{\partial x_{j}}{\partial p_{i}} \stackrel{?}{=} \frac{\partial x_{i}}{\partial p_{j}} \Leftrightarrow \\
& \frac{\partial h_{j}}{\partial p_{i}}-x_{i} \frac{\partial x_{j}}{\partial m} \stackrel{?}{=} \frac{\partial h_{i}}{\partial p_{j}}-x_{j} \frac{\partial x_{i}}{\partial m}
\end{aligned}
$$

1 cancel because the slutsky substitution matrix is symmetric

$$
\Leftrightarrow \quad x_{i} \frac{\partial x_{j}}{\partial m}=x_{j} \frac{\partial x_{i}}{\partial m}
$$

or $\quad \frac{1}{x_{j}} \frac{\partial x_{j}}{\partial m}=\frac{1}{x_{i}} \frac{\partial x_{i}}{\partial m}$.
(or $\quad \frac{m}{x_{j}} \frac{\partial x_{j}}{\partial m}=\frac{m}{x_{i}} \frac{\partial x_{i}}{\partial m}$, in other words, $i$ and $j$ have the same income elasticity.)

2017 Final Exam Qu. 1; resembles 1997 Ex. 1 Qu. 1, and has the same answer as it (see next problem)

## 1. [16 points]

Prove that a consumer's expenditure function $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$. Fully explain your work.

$$
\begin{gathered}
\text { Exam 1 } \\
1997 \\
\text { Question 1 }
\end{gathered}
$$

## (1)

Answer all of the following three questions.

1. Prove that a consumer's expenditure function $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$. Fully explain your work.
Hint: Here is the beginning and middle of a bare, unexplained proof that a firm's cost function $c(\mathbf{w}, y)$ is concave in $\mathbf{w}$ :
To prove: $c\left(t \mathbf{w}_{a}+(1-t) \mathbf{w}_{b}, y\right) \geq t c\left(\mathbf{w}_{a}, y\right)+(1-t) c\left(\mathbf{w}_{b}, y\right)$ for $0 \leq t \leq$ 1.

Proof:

$$
\begin{aligned}
c\left(t \mathbf{w}_{a}+(1-t) \mathbf{w}_{b}, y\right) & =\min _{\mathbf{x} \in V(y)}\left[t \mathbf{w}_{a}+(1-t) \mathbf{w}_{b}\right] \cdot \mathbf{x} \\
& =\min _{\mathbf{x} \in V(y)}\left[t \mathbf{w}_{a} \cdot \mathbf{x}+(1-t) \mathbf{w}_{b} \cdot \mathbf{x}\right] \\
& \geq \min _{\mathbf{x} \in V(y)} t \mathbf{w}_{a} \cdot \mathbf{x}+\min _{\mathbf{x} \in V(y)}(1-t) \mathbf{w}_{b} \cdot \mathbf{x}
\end{aligned}
$$

(1)

$$
e\left(t{\underset{\sim}{p}}_{a}+(1-t) p_{b}, u\right)=\min _{\underset{\sim}{n}}\left[\underset{\sim}{p_{a}}+(1-t) p_{b}\right] \cdot \underset{\sim}{x} \quad \text { st. } u(\underset{\sim}{x})=u
$$

by the definition of the expenditure function, where $\underset{\sim}{x}$ is the commodity bundle purchased and $u(\underset{\sim}{x})$ is the utility function

$$
\begin{aligned}
& =\min _{\sim}^{x}\left[t{\underset{\sim}{p}} \cdot \underset{\sim}{x}+(1-t) p_{\dot{b}} \cdot \underset{\sim}{x}\right] \text { set. } u(\underset{\sim}{x})=u \\
& \geqslant \min _{\sim}^{x} t p_{a} \cdot \underset{\sim}{x}[s . t . u(\underset{\sim}{x})=u] \\
& +\min _{\sim}^{x}(1-t) p_{b} \cdot x[s, t . u(x)=u]^{\prime} \\
& \text { since } \min _{x}[f(x)+g(x)] \geqslant \min _{x} f(x)+\min _{x} g(x) \text {, as illustrated } \\
& \text { by }{ }^{1} \Omega^{f(x)} x \text { and } \left\lvert\, \frac{\Lambda^{2}}{\cdots(x)} x\right. \text { where } \min (f+g)=\min (0)=0 \\
& \text { but mun }+\min g=0+(-1)=-1 \text {. } \\
& =t \min _{\sim}^{x}{\underset{\sim}{r}}^{P_{a}} \cdot \underset{\sim}{x}[s \cdot t \cdot u(x)=u]+(1-t) \operatorname{mox}_{\sim} \underset{\sim}{p} P_{b} \cdot x[s \cdot t \cdot u(x)=u .] \\
& =t e\left(p_{a}, u\right)+(1-t) e\left(p_{b}, u\right)
\end{aligned}
$$

Exam 1

$$
1997
$$

Answer 1 cont...
and the fore that $e(p, i)$ ' is concave.

# Qualifying. Exam 

1997
Question 1
(1)

## 621 Section (must answer one)

Question 1. Suppose a price-taking consumer buys $n$ commodities $q$ which are indexed by $i$, as in: $q_{1}, q_{2}, \ldots, q_{i}, \ldots, q_{n}$. Let the price of commodity $i$ be $p_{i}$. Let the consumer's income be $m$. Let the consumer's budget share for item $i$ be

$$
\alpha_{i}=\frac{p_{i} q_{i}}{m}
$$

Let

$$
\epsilon_{i j}=\frac{\partial \ln q_{i}}{\partial \ln p_{j}}
$$

What is this?
Let

$$
\eta_{i}=\frac{\partial \ln q_{i}}{\partial \ln m}
$$

What is this?
a) Prove the so-called Cournot Aggregation Condition:

$$
\sum_{j} \alpha_{j} \epsilon_{j i}=-\alpha_{i}
$$

As a hint: differentiate the budget constraint with respect to $p_{i}$.
b) Prove the so-called Engel Aggregation Condition:

$$
\sum_{j} \alpha_{j} \ddot{\eta}_{j}=1
$$

As a hint: differentiate the budget constraint with respect to $m$.

Optional Question 1. $\varepsilon_{i j}$ is the elasticity of demand for yod $i$ with respect. to the price of yod $j$. $\eta_{i}$ is the income elastrity of good $i$. Also,

$$
\begin{aligned}
& \varepsilon_{i j}=\frac{\partial \ln q_{i}}{\partial \ln p_{j}}=\frac{p_{j}}{q_{i}} \frac{\partial q_{i}}{\partial p_{j}} \text { and } \eta_{i}=\frac{\partial \ln q_{i}}{\partial \ln m}=\frac{m}{q_{i}} \frac{\partial q_{i}}{\partial m} \text {. } \\
& \text { a) Differ }
\end{aligned}
$$

a) Differentiating $\sum_{j} p_{j} q_{j}=m$ with respect to $p_{i}$ yields

$$
\begin{gathered}
\sum_{j}\left(\frac{d p_{j}}{d p_{i}} q_{j}+p_{j} \frac{d q_{j}}{d p_{i}}\right)=\frac{d m}{d p_{i}} \\
L_{0 \text { except when }} \\
j=i, \text { in which case } \\
d p_{j} / d p_{i}=d p_{i} / d p_{i}=1 \\
\left(\text { and } q_{j}=q_{i} \operatorname{since} j=i\right)
\end{gathered}
$$

Qualifying Exam 1997
Answer 1

Therefore

$$
1 \cdot q_{i}+\sum_{j} p_{j} \frac{d q_{j}}{d p_{i}}=0 \text {. Multiply by } \frac{p_{i}}{m} \text { and multiply inside the }
$$ summation by $q_{j} / q_{j}$ :

$$
\begin{aligned}
& \sum_{j} \frac{p_{i}}{m} p_{j} \frac{q_{j}}{q_{j}} \frac{d q_{j}}{d p_{i}}=-q_{i} \frac{p_{i}}{m} \\
& \sum_{j} \frac{p_{j} q_{j}}{m} \cdot \frac{p_{i}}{q_{j}} \frac{d q_{j}}{d p_{i}}=-\frac{q_{i} p_{i}}{m} \Rightarrow \sum_{j} \alpha_{j} \varepsilon_{j i}=-\alpha_{i} .
\end{aligned}
$$

b) Differentiating $\sum_{j} p_{j} q_{j}=m$ with respect to $m$ yields

$$
\begin{aligned}
& \sum_{j} p_{j} \frac{d q_{j}}{d m}=1 \text {. Multiply the LHS by } \frac{q_{j}}{m} \frac{m}{q_{j}} \text { : } \\
& \sum_{j}\left[\frac{p_{j} q_{j}}{m} \cdot \frac{m}{j} \frac{d q_{j}}{d m}\right]=1 \\
& \Rightarrow \sum_{j} \alpha_{j} \eta_{j}=1 . \\
& \text { Qualifying Exam } \\
& 1997 \\
& \text { Answer } 1 \text { cont... }
\end{aligned}
$$

new: 2020 Exam 1, Qu. 1.

## 1. [11 points]

Suppose a consumer consumes two goods, " 1 " and " 2 ," and that if the consumer consumes $\mathbf{x}=\left(x_{1}, x_{2}\right)$ amounts of those two goods, his utility function is

$$
u(\mathbf{x})=x_{1} x_{2} .
$$

Assume

$$
\begin{aligned}
& x_{1} \geq 0 \\
& x_{2} \geq 0 .
\end{aligned}
$$

Consider another family of preferences

$$
\hat{u}_{\gamma}(\mathbf{x})=\left(x_{1} x_{2}\right)^{2}+\gamma
$$

where $\gamma \in \mathbf{R}^{1}=(-\infty, \infty)$.
(a) For what values of $\gamma$ do $\hat{u}_{\gamma}(\mathbf{x})$ and $u(\mathbf{x})$ represent the same preferences?
(b) State whether or not $u$ is homogeneous, and if it is, state its degree of homogeneity.
(c) State for what value or values of $\gamma$, if any, $\hat{u}_{\gamma}(\mathbf{x})$ is homogeneous and what the degree of homogeneity is (or the degrees of homogeneity are).
(d) State whether or not $u$ is homothetic. Hint: You could use the definition of homotheticity to prove this, but you need not go through that trouble here: there is a way to answer by using the answer to a previous part of this question and then just appealing to a result that I showed in class and that you do not have to prove here.
(e) State for what value or values of $\gamma$, if any, $\hat{u}_{\gamma}(\mathbf{x})$ is homothetic. Use the definition of homotheticity given in class as the basis for your answer.

## Answers to Exam 1, Econ. 7005, Fall 2020

1. (a) Observe that $\hat{u}$ is the following transformation $f$ of $u$ :

$$
\hat{u}_{\gamma}=f(u)=u^{2}+\gamma .
$$

The domain is $u \geq 0$ because the problem states that $x_{1} \geq 0$ and $x_{2} \geq 0$, making $u=x_{1} x_{2} \geq 0$. As the following graph shows, this $f(u)$ is increasing in $u$ over the entire domain $u \geq 0$ for all $\gamma$. (The value of $\gamma$ could be negative or positive or zero.)


This shows that " $\hat{u}_{\gamma}$ is an increasing transformation of $u$ " (which just means that $\hat{u}_{\gamma}$ is increasing in $u$ (that is, $d \hat{u}_{\gamma} / d u>0$ ), not that $\hat{u}_{\gamma}>u$; note that the latter is not true in the graph I drew because $\hat{u}_{\gamma}$ is sometimes negative while $u$ is never negative). Therefore, $\hat{u}_{\gamma}(\mathbf{x})$ and $u(\mathbf{x})$ represent the same preferences for all values of $\gamma$.
(b) A function $f(\mathbf{x})$ is by definition "homogeneous of degree $k$ " if $f(\lambda \mathbf{x})=\lambda^{k} f(\mathbf{x})$. Here, we have $u(\lambda \mathbf{x})=\lambda x_{1} \cdot \lambda x_{2}=\lambda^{2} x_{1} x_{2}$. So $u$ is homogeneous of degree 2.
(c) Here,

$$
\begin{aligned}
\hat{u}_{\gamma}(\lambda \mathbf{x}) & =\left(\lambda x_{1} \lambda x_{2}\right)^{2}+\gamma \\
& =\lambda^{4} x_{1}^{2} x_{2}^{2} \stackrel{?}{=} \lambda^{k}\left[x_{1}^{2} x_{2}^{2}+\gamma\right]=\lambda^{k} \hat{u}_{\gamma}(\mathbf{x})
\end{aligned}
$$

so $\hat{u}_{\gamma}(\mathbf{x})$ is homogeneous only if $k=4$ and $\gamma=0$.
(d) From part (b), $u$ is homogeneous. All homogeneous functions are homothetic, so $u$ is homothetic.
(e) If $\hat{u}_{\gamma}(\mathbf{x})$ is homothetic then, by definition,

$$
\begin{equation*}
\hat{u}_{\gamma}(\mathbf{x})=\hat{u}_{\gamma}(\mathbf{y}) \tag{1}
\end{equation*}
$$

would imply that

$$
\begin{equation*}
\hat{u}_{\gamma}(\lambda \mathbf{x})=\hat{u}_{\gamma}(\lambda \mathbf{y}) . \tag{2}
\end{equation*}
$$

(1) implies that

$$
\begin{align*}
x_{1}^{2} x_{2}^{2}+\gamma & =y_{1}^{2} y_{2}^{2}+\gamma \quad \Leftrightarrow \\
x_{1}^{2} x_{2}^{2} & =y_{1}^{2} y_{2}^{2} . \tag{3}
\end{align*}
$$

On the other hand, (2) would imply that

$$
\begin{align*}
\lambda^{2} x_{1}^{2} x_{2}^{2}+\gamma & =\lambda^{2} y_{1}^{2} y_{2}^{2}+\gamma \\
\lambda^{2} x_{1}^{2} x_{2}^{2} & =\lambda^{2} y_{1}^{2} y_{2}^{2} \\
x_{1}^{2} x_{2}^{2} & =y_{1}^{2} y_{2}^{2} . \tag{4}
\end{align*}
$$

Since (3) is the same as (4), $\hat{u}_{\gamma}(\mathbf{x})$ is homothetic.
Optional: Thus this problem shows three things:

- $u$ and $\hat{u}_{\gamma}$ represent the same preferences;
- $u$ and $\hat{u}_{\gamma}$ are not both homogeneous (in general); and
- $u$ and $\hat{u}_{\gamma}$ are both homothetic.

This confirms what we said in class about homotheticity rather than homogeneity being the important concept in consumer theory.

1. (a) Prove that Hicksian demand curves $h(\mathbf{p}, u)$ are homogeneous of degree zero in $\mathbf{p}$.
(b) As some of you may already know, Euler proved the following: if $f(\mathrm{x})$ is differentiable and is homogeneous of degree $k$, then

$$
\nabla f(\mathrm{x}) \cdot \mathbf{x}=k f(\mathbf{x})
$$

(Do not forget that the left-hand side has a "• $x$ " in it.) What property of Hicksian demand curves can you derive from this

2005 Qualifier

## See. 2

 result, given what you already know from part (a)?(c) Rewrite your answer to part (b) for the special case when the total number of commodities is exactly three.
(d) For any two commodities $j$ and $k$, here are two definitions:

$$
\begin{aligned}
\partial h_{j}(\mathbf{p}, u) / \partial p_{k} \geq 0 & \Longleftrightarrow j \text { and } k \text { are "substitutes" } \\
\partial h_{j}(\mathbf{p}, u) / \partial p_{k}<0 & \Longleftrightarrow j \text { and } k \text { are "complements." }
\end{aligned}
$$

[By the way, if instead of using the Hicksian demand curve $h_{j}(\mathbf{p}, u)$ on the left-hand side, we used Marshallian demand curves $x_{j}(\mathbf{p}, m)$ where $m$ is income, then we would use the terms "gross substitutes" and "gross complements" on the right-hand side (but this is not important for this exam).]
Use the previous parts of this question, and other information, to prove that if the total number of commodities is three, then every good has at least one substitute:-
(e) Prove that every good has at least one substitute (regardless of what the total number of commodities may be).

Section 2
(1) a) Hicksian demand curves solve

$$
\begin{equation*}
\min _{x} p \cdot \underset{\sim}{x} \text { s.t. } u(x)=\bar{u} . \tag{1}
\end{equation*}
$$

If $p$ changes to $\lambda p$, the problem becomes

$$
\begin{array}{ll} 
& \operatorname{mix}_{\sim}^{x} \lambda \\
\mu & p \cdot x  \tag{3}\\
\sim
\end{array} \text { s.t. } u(\underset{\sim}{x})=\bar{u} .
$$

(P3) Las the same optimal point ${\underset{\sim}{x}}^{*}$ (or " $\sim^{* \prime \prime}$ ) as (P1).
So $\underset{\sim}{h}(p, \bar{u})$ doesn't change when $p$ changes.
b) ${\underset{\sim}{p}}_{p}^{h_{i}}(\underset{\sim}{p}, u) \cdot \underset{\sim}{p}=k h_{i}(p, u)$ where $k$ is the degree of homogeneity of $h_{i}(p, u)$ in $p$, who is zero:

$$
\underset{\sim}{p} h_{i}(\underset{\sim}{p}, n) \cdot \underset{\sim}{p}=0 .
$$

c) $\frac{\partial h_{i}}{\partial p_{1}} p_{1}+\frac{\partial h_{i}}{\partial p_{2}} p_{2}+\frac{\partial h_{i}}{\partial p_{3}} p_{3}=0$ for $i \in\{1,2,3\}$.
d) If $i=1$ then $p \cot (c) \Rightarrow \frac{\partial h_{1}}{\partial p_{1}} p_{1}+\frac{\partial h_{1}}{\partial p_{2}} p_{2}+\frac{\partial h_{1}}{\partial p_{3}} p_{3}=0$.

Similar reasoning apples for $i=2$ and $i=3$

But this is $\theta$ (Hicksian own-demand arses always slope downwards). So one of the other terms has to be positive. (for all there to add uptozeros,
e) One of the terms in ${\underset{\sim}{p}} h_{i}$ is $\partial h_{i} / \partial p_{i}$, which is $<0$, so for $\nabla_{p} h \cdot p=0$, one of the other $\partial h_{i} l \partial p_{j}$ has to be positive.

# Qualifying Elam <br> 2000 <br> Question 1 

## Short Section

## 6710 Section (must answer one)

Question 1. On p. 147 of Varian's Microeconomic Analysis, he writes:
We saw in our discussion of production theory that if a production function was homogeneous of degree 1 , then the cost function could be written as $c(\mathbf{w}, y)=c(\mathbf{w}) y$. It follows from this observation that if the utility function is homogeneous of degree 1 , then the expenditure function can be written as $e(\mathbf{p}, u)={ }^{\prime} e(\mathbf{p}) u$.

You may use this information (without proving that it is true) in the questons below.
a) Prove that if the utility function is homogeneous of degree 1 , then the indirect utility function can be written as

$$
v(\mathbf{p}, m)=v(\mathbf{p}) m
$$

Hint:' You may use $e(\mathbf{p}, v(\mathbf{p}, m))=m$ without proving it.
b) Prove that if the indirect utility function can be written as $v(\mathbf{p}, \dot{m})=$ $v(\mathbf{p}) m$, then the demand functions can be written as

$$
x_{i}(\mathbf{p}, m)=x_{i}(\mathbf{p}) m
$$

-ie., they are linear functions of income. Hint: You need not prove Roy's Identity.
c) Prove that if the demand functions can be written as $x_{i}(\mathbf{p}, m)=x_{i}(\mathbf{p}) m$ -ie., they are linear functions of income-then

$$
\frac{\partial x_{i}(\mathbf{p}, m)}{\partial p_{j}}=\frac{\partial x_{j}(\mathbf{p}, m)}{\partial p_{i}}
$$

Hint: If you use the result that $\mathbf{h}=\nabla_{\mathbf{p}} e$ here, then you should prove that $\mathbf{h}=\nabla_{\mathbf{p}} e$. If you use the Slutsky equation here, then you should prove the Slutsky equation. If you use $h_{i}(\mathbf{p}, u)=x_{i}(\mathbf{p}, e(\mathbf{p}, u))$ here, you do not need to prove it.

Short Section.
Question 1.
utility fiction is hoongenerers of degree $1 \Rightarrow \underset{\sim}{e}(p, u)=e(p) u . \quad$ (1)
a)

$$
\left.\begin{array}{rl}
e(p, v(p, m)) & =m . \text { From }(1), \\
e(p) v(p, m) & =m \\
\underset{\sim}{p} & v(p, m)
\end{array}\right) \frac{1}{e(p)} \cdot m . \text { So it we let } v(p) \text { equal } \frac{1}{e(p)}, ~ l
$$

the claim is proven.
b). $v(p, m)=v(p) m$.

Roy's identity is $\underset{\sim}{x_{i}(p, m)}=-\frac{\partial v(p, m) / \partial p_{i}}{\partial v(p, m) / \partial m}$; in this case;,

$$
\begin{aligned}
& =-\frac{\frac{\partial}{\partial p_{i}}\left[v(\rho)_{m}\right]}{\frac{\partial}{\partial m}\left[v(\rho)_{m}\right]}=-\frac{\partial v(\rho) / \partial p_{i} \cdot m}{v(p)} \\
& =\left[\frac{-\partial v(p) / \partial p_{i}}{v(p)}\right] \cdot m .
\end{aligned}
$$

So if we let this $\uparrow$ (in brackets) be $x_{i}(p)$, the claim is proven.
c) $\quad h_{i}\left(p_{i n}, u\right)=x_{i}(p, e(p, u))$

$$
\begin{align*}
& \frac{\partial h_{i}}{\partial p_{j}}=\frac{\partial x_{i}}{\partial p_{j}}+\frac{\partial x_{i}}{\partial m} \frac{\partial e}{\partial p_{j}} \Rightarrow \\
& \frac{\partial x_{i}}{\partial p_{j}}=\frac{\partial h_{i}}{\partial p_{j}}-\underbrace{\frac{\partial e}{\partial p_{j}}} \frac{\partial x_{i}}{\partial m} \text {. }  \tag{1}\\
& =h_{j} \text { (sec proof below), } \\
& \text { or, as oVarian write, } x_{j} \text {, } \\
& \text { understood notes a function } \\
& \text { but as a single quantity. } \\
& =\frac{\partial}{\partial m} x_{i}(p, m) \text {; here, } \\
& =\frac{\partial}{\partial m}\left[x_{i}(p) \cdot m\right] \\
& =x_{i}(p) \text {. }
\end{align*}
$$

So (1) $\Rightarrow$

$$
\begin{equation*}
\frac{\partial x_{i}^{\prime}}{\partial p_{j}}=\frac{\partial h_{i}}{\partial p_{j}}-x_{j} x_{i} . \tag{2}
\end{equation*}
$$

By writing " $j$ "for " $i$ " and " $i$ " for " $j$ "in the above proof, one derives

$$
\begin{equation*}
\frac{\partial x_{j}}{\partial p_{i}}=\frac{\partial h_{j}}{\partial p_{i}}-x_{i} x_{j} \tag{3}
\end{equation*}
$$

We wish to prove that the LHS of (2) equals the LH'S of (3). Looking at their RHS's, all we need to show is that $\partial h_{i} / \partial p_{j}=\partial h_{j} / \partial p_{i}$. This is straightforward: since $h_{\sim}^{h}={\underset{\sim}{D}}_{p} e$ (see proof below),
$\nabla_{P} h_{n}=\nabla_{P}^{2} e$. The latter is symmetric he cause Af is a Hessian, so ${\underset{p}{p}}^{h}$ is symmetric, which is what needed to be shown.

Qualilefing Exam 2000
Answer 1 cont.
Above, we used $\partial e / \partial p_{j}=h_{j}$ and $\nabla_{p} e=h$ which are the same result: shepherd's Lemma applied to consumer theory. It is proven by posing the consumers problem

$$
\begin{gathered}
e(p, \bar{u})=\operatorname{mix}_{\sim} \underset{\sim}{p} \cdot \underset{\sim}{x} \text { s.t. } u(\underset{\sim}{x}) \geqslant \bar{u} \Rightarrow \\
\mathscr{L}=\underset{\sim}{p} \cdot \underset{\sim}{x}+\lambda(u(\underset{\sim}{x})-\bar{u}),
\end{gathered}
$$

then applying the eurclope theorem:

$$
\begin{aligned}
\frac{\partial e\left(p_{,} \bar{u}\right)}{\partial p_{i}} & =\frac{\partial \mathcal{L}^{x}}{\partial p_{i}} \\
& =x_{i}, \text { which is the Hticksian demand awe here }
\end{aligned}
$$

(ivoully denoted hi), not the Mashatlian demand cure, because this ss an expenditure' - minimization problem, and this " $x_{i}$ " depends on pard $\bar{a}$, not pond m.


## Answer all of the following three questions.

1. If $\mathbf{h}(\mathbf{p}, u)$ are the Hicksian demand curves for a consumer, prove that $d \mathbf{p} \cdot d \mathbf{h} \leq 0$.
Also: very briefly discuss why this result is or is not surprising.
Hint: use the analogue for consumers of Shepherd's Lemma; find a total differential. You do not have to prove Shepherd's Lemma, nor do you have to prove the convexity or concavity of a function, although if this were the qualifying exam then I would want you to prove these things.

Answers to Exam! Econ.621, Winter 1996
(1) $\quad \underset{\sim}{h}(p, u)=\nabla_{p} e(p ; u)$ is the analogue of Shepherd's Lemma, for consumers. Taking the differential of both sides,

$$
d \underline{\sim}(p, u)=\nabla_{\sim}^{2} e(\underset{\sim}{p}, u) \cdot d p+\left(\frac{\partial}{\partial u}{\underset{\sim}{p}}_{p} e(\underset{\sim}{p}, u)\right) d u .
$$

Exam 1 1996

But du $=0$ here, so
Answer 1
$d h_{\sim}=\nabla_{p}^{2} e(p, n) \cdot d p$ and left-multiplying by $d p$. (actually, by the trampose of (p) given
$\underset{\sim}{d p} \cdot \underset{\sim}{d h}=d p \cdot \nabla_{p}^{2} e(p, u) \cdot d p \leqslant 0$ because $\underset{\sim}{d} p \cdot \nabla_{\sim} \nabla_{p}^{2} e \cdot d p$ is ${ }_{\sim}^{\prime \prime a}$ quadratic form and $\nabla^{2} e$ is negative definite since $e(p, a)$ is concave in $p$. The revolt is ht supping be carse we already knew that $\partial h_{i} / \partial_{p} \leq 0$ (thitssian demand comes are down ward sloping): this result extends the simpler one to cases when move than one price or quantity demand changes. $\qquad$

Summer 2011 Qualifying Exam, Sec. 1 Qu. 2.
2. [13 points] On August 22, 2010, the Los Angeles Times published an opinion piece entitled "Disincentivizing Greed" written by Neal Gabler (who was then a public policy scholar at the Woodrow Wilson Center in Washington, DC). Here is an excerpt of the piece; it essentially argues that decreasing tax rates increases the amount of dishonest labor, which is an assertion about a comparative statics derivative.

To a surprising degree, economic misfortune has correlated with low top marginal tax rates. The top marginal tax rate at the time of the 1929 crash was $24 \%$. After his election, Roosevelt promptly raised it to $63 \%$ and then to $94 \%$, and one could easily make the case that it was this rise, rather than financial regulation, that played the primary... role in curbing abuses by attacking greed at its source, without, by the way, damaging the economy. Roosevelt essentially taxed away big money.

During the long postwar economic boom, the top marginal rates hovered at $91 \%$, removing a lot of the incentive to game the financial system. There was no point in scheming if you couldn't profit from it. Still, the country prospered. So did Wall Street.

Then came the greed deluge. ...[W]hen President Reagan cut the top marginal tax rate drastically from $70 \%$ to $50 \%$ in 1981 and then to $28 \%$ in 1988 (putting aside for the moment the cut in the capital gains tax and other investment incentives), that's when the troubles began-from the S\&L crisis right through to the fall of Lehman Bros. It wasn't enough for the rich to be rich. Human nature being what it is, they had to be super-rich. Or put another way, tax cuts, including the Bush tax cuts, fed some of the worst aspects of human nature and led to some of the worst excesses. It was just a matter of time before Wall Street went wild.

When the fire of greed is stoked this way, financial reforms cannot possibly bank it.... We now live in a country that seems to worship wealth, and we may just have to live with the consequences - a Bernie Madoff, an Enron, a Lehman Bros., and a steep recession when the super-rich overplay
their hand. The alternative is regulation that goes to the source by raising those marginal tax rates (and capital gains taxes) and forcing the super-rich to merely be rich again....
(a) Argue that a reasonable way - certainly not the only way, but a reasonable way - to model the (indirect) utility that the "rich" or "super-rich" people described in this article get from their pretax income is

$$
\text { "honest income" }+\sqrt{\text { "dishonest income" }} .
$$

(This is not a standard way of modeling indirect utility, of course.)
(b) If the tax rate is $t$, interpret

$$
\begin{aligned}
& \text { "honest income" }+\sqrt{\text { "dishonest income" }} \\
& \quad-t \cdot(\text { "honest income" }+ \text { "dishonest income" }) .
\end{aligned}
$$

(c) Modelling "income" as a wage rate (consider an "honest wage" and a "dishonest wage") times a number of hours worked (consider "honest labor time" and "dishonest labor time") and imposing some constraint on the number of hours humans work, discuss whether or not the expression in part (b) supports Gabler's hypothesis by calculating an appropriate comparative statics derivative. Does the appropriate second-order condition hold?
Hint: If you substitute the constraint on working hours into the objective function, the new problem has only one endogenous variable, which is much easier to work with. You may ignore leisure (and hence any leisure-labor tradeoff) in your answer.

Section 1 Question 2.
a) Both kinds of income increase (indirect) utility. This captures the reason some super-nich people did unethical things. However, the square root function means that $\underbrace{\text { whethically-earned income contributes less to }}_{\text {d } 1 \text { of }}$ क1 of
(indirect) utility than $\$ 1$ of ethically-earned income: so the super-nich do have some moral misgivings about unethically-earned income.
b) The utility gained from pre-tax income, minus taxes. An objective function for the super-rizh.
c) $W_{h}$ wage rate of honest work $w_{d}$ " "dishonest"
$l_{h}$ hours worked doing honest labor ld " " " dishonest"

Working time constraint: $l_{h}+l_{d}=1$ (the "1" stands for "one working day"; instead of "1" you could use "18 hours" or "8 hours" or " 24 hover).
honest income $=W_{h} l_{h}$
dishonest income $=w_{d} l_{d}=w_{d}\left(1-l_{h}\right)$.

Objective: $\max w_{h} l_{h}+\sqrt{w_{d}\left(1-l_{h}\right)}-t\left[w_{k} l_{h}+w_{d}\left(1-l_{h}\right)\right]$
over $l_{h}:$

$$
\begin{equation*}
0=\frac{d \text { (objective) }}{d \ell_{h}}=w_{h}+\frac{1}{2} \frac{-w_{d}}{\sqrt{w_{d}\left(1-\ell_{h}\right)}}-t\left[w_{h}-w_{d}\right] \tag{1}
\end{equation*}
$$

Solution Method 1: No need to solve for $l_{h}$.

$$
\begin{aligned}
& 0=d t\left[-w_{h}+w_{a l}\right]+d l_{h}\left[-\frac{1}{4} \frac{-w_{d}}{\left(w_{d}\left(1-l_{h}\right)\right)^{+3 / 2}}\left(-w_{d}\right)\right] \\
&\left(w_{h}-w_{d}\right) d t=\left[-\frac{1}{4} w_{d}^{2} w_{d}^{-3 / 2} \frac{1}{\left(1-l_{h}\right)^{3 / 2}}\right] d l_{h} \\
&=\frac{-\sqrt{w_{d}}}{4\left(1-l_{h}\right)^{3 / 2}} d l_{h}
\end{aligned}
$$

$$
\Rightarrow \quad \frac{d l_{h}}{d t}=\frac{w_{h}-w_{d}}{-\sqrt{w_{d}}} 4\left(1-l_{h}\right)^{3 / 2}=\frac{w_{h}-w_{d}}{-\sqrt{w_{d}}} 4 l_{d}^{3 / 2}
$$

$=\frac{4 \ell_{d}^{3 / 2}}{\sqrt{w_{d}}}\left(w_{d}-w_{h}\right)$. So if dishonest labor pays more than honest labor lit if
didn't, then in this model no one would do dishonest labor, which contrachets the article's opinion), one has $w_{d l}-w_{h}>0$, so $d l_{n} \mid d t>0$, therefore (due to the working hour constraint) we
obtain $d l_{d} / d t<0$, supporting the author's hypothesis.

Solutionkethod 2: Solving for $l_{h}$.
From (1),

$$
\begin{aligned}
& t\left(w_{h}-w_{d}\right)=w_{h}-\frac{1}{2} \frac{w_{d}}{\sqrt{w_{d}} \sqrt{1-l_{h}}}=w_{h}-\frac{1}{2} \frac{\sqrt{w_{d}}}{\sqrt{1-l_{h}}} \Rightarrow \\
& \frac{1}{2} \frac{\sqrt{w_{d}}}{\sqrt{1-l_{h}}}=w_{h}-t\left(w_{h}-w_{d}\right) \\
& \frac{\sqrt{w_{d}}}{2\left[w_{h}-t\left(w_{h}-w_{d}\right)\right]}=\sqrt{1-l_{n}} \Rightarrow \\
& 1-\ell_{h}=\frac{w_{d}}{4\left[w_{h}-t\left(w_{h}-w_{d}\right)\right]^{2}} \text {. It would be frontal to solve this } \\
& \text { for } l_{h} \text {, but it's even easier } \\
& \text { to sole if for } 2 \\
& l_{d}=\frac{w_{d}}{4\left[w_{h}-t\left(w_{h}-w_{d}\right)\right]^{2}} \text { and then calculate } \\
& \frac{d \ell_{d}}{d t}=\frac{w_{d}}{4} \frac{-2}{\left[w_{h}-t\left(w_{h}-w_{d}\right)\right]^{3}}\left[-\left(w_{h}-w_{d}\right)\right] \\
& =\frac{w_{d}}{4} \frac{1}{\left[w_{h}-t\left(w_{h}-w_{d}\right)\right]^{2}} \frac{-2}{\left[w_{h}-t\left(w_{h}-w_{d}\right)\right]}\left[-\left(w_{h}-w_{c l}\right)\right] \\
& =\ell_{d} \frac{-2}{w_{h}+\left(w_{h}-w_{d 1}\right)}\left[-\left(w_{h}-w_{c 1}\right)\right]=l_{d} \frac{2\left(w_{h}-w_{d}\right)}{w_{h}-t\left(w_{h}-w_{c 1}\right)}
\end{aligned}
$$

$$
=\frac{2 l_{d}\left(w_{h}-w_{d}\right)}{w_{h}+t\left(w_{d}-w_{h}\right)}=\frac{-2 l_{d}\left(w_{c l}-w_{h}\right)}{w_{h}+t\left(w_{d}-w_{h}\right)}<0
$$

since $W_{d}-W_{h}>0$. This is the same sign obtained using Method 1 .

## Section 1.

## Answer all of the following three questions.

## 1. [12 points]

There is an old saying,
"Idle hands are the devil's workshop."
In other words, excessive idleness ("leisure") is a bad thing for people. Suppose a price-taking consumer's utility depends on his purchases of a good $x \geq 0$ and his consumption of leisure $z$ (the notation " $z$ " recalls the sound at the beginning of the second syllable of "leisure").
(a) Argue that postulating a utility function of

$$
u=\ln x+\left[-(z-1)^{2}+1\right] \quad \text { with } 0 \leq z \leq 2
$$

is a reasonable way of modeling the idea that excessive idleness is bad.
(b) Suppose the wage rate (that is, the payment for the opposite of leisure) is $w$. Assume that all of the consumer's income comes from selling his non-leisure time. Suppose the price of $x$ is $p$. What is the consumer's budget constraint?
(c) From the first-order conditions, argue that in order for the optimall $x$ to be positive, the optimal $z$ must be in $[0,1)$.
(d) Find the optimal $x$ and $z$ explicitly in terms of exogenous variables.
(e) What does this consumer's labor supply curve look like?
(f) Check the second-order conditions for the optimization problem.
(g) What is this consumer's indirect utility function?

Answas to 7005 section of $2008 M_{\text {re no economics }}$
Qualifying Exam
(1) a) The function $-z^{2}$ looks like


The function - $(z-1)^{2}$ thus looks like


$$
\text { and }-(z-1)^{2}+1 \text { looks like }
$$


$z=0$ is no leisure; $z=2$ is $100 \%$ leisure. For $z \in[0,1]$, increasing leisure increases utility. So if you have rather little leis re, monde leisure is better. But once you have the "best possible "amount of leisure, $z=1$, getting any more leisure decreases utility; hence the negative slope for $z>1$.

The last, " +1 " term of " $-(z-1)^{2}+1$ " does nothing mathematically and could be left ont. I just put it in because positive values for utility are slightly easier to understand.
Optional: We can predict that $z^{*}$ will not be greater than 1, because mencasing $z$ beyond 1 decreases utility and decreases nome. See also part (c).
b) in come $=$ expenditure


Amount of work; $z=0 \Rightarrow$ mo leisure $\Rightarrow$ all time is spent on work, $z=2 \Rightarrow$ no work $\Rightarrow$ no income.
c)

$$
\left.\begin{array}{rl}
u & =\ln x+\left[-\left(z^{2}-2 z+1\right)+1\right] \\
& =\ln x+\left[-z^{2}+2 z-1+1\right] \\
& =\ln x-z^{2}+2 z \\
\varphi & =\ln x-z^{2}+2 z+\lambda[\omega(2-z)-p \pi]
\end{array}\right\}
$$

$$
\left.\begin{array}{l}
0=\mathscr{L}_{\lambda}^{\prime}=w(2-z)-p x \\
0=\mathscr{L}_{x}^{\prime}=\frac{1}{x}-\lambda p \Rightarrow \lambda=\frac{1}{x p} \\
0=\mathscr{L}_{z}^{\prime}=-2 z+z-\lambda w \Rightarrow \lambda=\frac{-2 z+2}{w}
\end{array}\right\} \begin{aligned}
& \frac{1}{x p}=\frac{2-2 z}{w} \\
& x p
\end{aligned}=\frac{w}{2-2 z} . \quad \begin{aligned}
x & =\frac{1}{2-2 z} \frac{w}{p} \\
& =\frac{1}{2} \frac{1}{1-z} \frac{w}{p} \quad \text { over } \rightarrow
\end{aligned}
$$

Recall that the range of $z$ is $[0,2]$. If $z>1$, then $x=\frac{1}{2} \frac{1}{1-z} \frac{w}{p}$ would be negative. If $z=1, x$ wold be $\infty$. So to make sense, $z$ needs to be in $[0,1)$.
d) $w(2-z)=p x$ from (b) or the first F.O.C.

$$
\begin{aligned}
& =p \cdot \frac{1}{2} \frac{1}{1-z} \frac{w}{p} \\
& =\frac{1}{2} \frac{1}{1-z} w
\end{aligned}
$$

$$
\begin{gathered}
2 w(2-z)(1-z)=w \\
2(2-z)(1-z)=1 \\
(z-2)(z-1)=\frac{1}{2} \\
z^{2}-3 z+2=\frac{1}{2} \\
z^{2}-3 z+\frac{3}{2}=0 \\
z=\frac{3 \pm \sqrt{9-4 \cdot \frac{3}{2}}}{2}=\frac{3 \pm \sqrt{9-6}}{2}=\frac{3 \pm \sqrt{3}}{2}
\end{gathered}
$$

Choosing the " + "sign would yield $z=\frac{3+\sqrt{3}}{2}>\frac{3+0}{2}=1.5$ violating part (c)'s
conclusion that $z$ has to be between. 0 and 1.
So $z^{*}=\frac{3-\sqrt{3}}{2}$. (This is about 0.6.)

$$
\begin{aligned}
x^{*} & =\frac{1}{2} \frac{1}{1-z^{*}} \frac{w}{p}=\frac{\omega}{2 p} \frac{1}{1-\frac{3-\sqrt{3}}{2}}=\frac{\omega}{2 p} \frac{2}{2-(3-\sqrt{3})}=\frac{\omega}{2 p} \frac{2}{2-3+\sqrt{3}} \\
& =\frac{1}{p} \frac{1}{\sqrt{3}-1} .\left(T_{\text {us }} \text { is }>0 \text { since } \sqrt{3}>\sqrt{1}=1 .\right)
\end{aligned}
$$

e) Since $z^{*}=\frac{3-\sqrt{3}}{2}$ is a constant, $z^{*}$ does not depend on w. Work hours (2- $z^{*}$ ) hence do not depend on w either. So the labor supply care is vertical: $w \frac{\sum_{2-2}^{\text {Labor }}}{5}$
f)

$$
\nabla^{2} \mathcal{L}=\left[\begin{array}{ccc}
\mathcal{L}_{\lambda \lambda}^{\prime \prime} & \mathcal{L}_{\lambda x}^{\prime \prime} & \mathcal{L}_{\lambda z}^{\prime \prime} \\
\mathscr{L}_{x \gamma_{2}}^{\prime \prime} & \mathcal{L}_{x x}^{\prime \prime} & \mathcal{L}_{x z}^{\prime \prime} \\
\mathcal{L}_{z \lambda \prime}^{\prime \prime} & \mathcal{L}_{z x}^{\prime \prime} & \mathcal{L}_{z z}^{\prime \prime}
\end{array}\right]=\left[\begin{array}{ccc}
0 & -p & -w \\
-p & -1 x^{2} & 0 \\
-w & 0 & -2
\end{array}\right]
$$

S.O.C. for a maximum: $D_{2 m+1}, \cdots, D_{m+n}$ alternate a sign beginning with $(-1)^{m+1}$.

$$
\begin{aligned}
n & =2 \quad \text { \#vanables } \\
m & =1 \quad \text { \# constraints } \\
2 m+1 & =3 \\
m+n & =3
\end{aligned}
$$

S. $D_{3}$ should hare the sign of $(-1)^{2}>0$.
$D_{3}$ is $(-1)^{2+1}(-p)[2 p-0]+(-1)^{3+1}(-w)\left[0-\frac{w}{x^{2}}\right]$

$$
=p(2 p)-w\left(\frac{-w}{x^{2}}\right)=2 p^{2}+\frac{w^{2}}{x^{2}} \text { which, be cause all the }
$$

variables are squared, is clearly positive even without substituting the optimal $x$ of part (d) in to this expression.
g)

$$
\begin{aligned}
v= & \ln x^{*}+\left[-\left(z^{*}-1\right)^{2}+1\right] \\
= & \ln \left[\frac{w}{p} \frac{1}{\sqrt{3}-1}\right]+\left[-\left(\frac{3-\sqrt{3}}{2}-1\right)^{2}+1\right] \\
= & +\left[-\left(\frac{3-\sqrt{3}}{2}-\frac{2}{2}\right)^{2}+1\right] \\
& +\left[-\left(\frac{1-\sqrt{3}}{2}\right)^{2}+1\right] \\
& +\left[-\left(\frac{1-2 \sqrt{3}+3}{4}\right)+1\right] \\
= & +\left[-\left(\frac{4-2 \sqrt{3}}{4}\right)+1\right] \\
= & +\left[-\left(1-\frac{\sqrt{3}}{2}\right)+1\right] \\
= & +\left[-1+\frac{\sqrt{3}}{2}+1\right] \\
= & +\frac{\sqrt{3}}{2}
\end{aligned}
$$

# This is the end of the Consumer Theory positive questions, and the beginning of the Consumer Theory normative questions. 

The normative questions are not covered on the midterm exam. So if you are studying for the midterm exam, you need not go beyond this point.
5. [11 points] Suppose a consumer buys only two commodities, called $x$ and $y$, and suppose this consumer's preferences for $x$ and $y$ are strongly monotonic. Suppose the government is contemplating giving (for free) the consumer a certain amount more of commodity $x$. By drawing one graph with $x$ on the horizontal axis and $y$ on the vertical axis, illustrate the typical result that for this contemplated change, "willingness to pay" is less than "willingness to accept." Also, describe how your graph would have to change in order to obtain the atypical result that WTP is equal to WTA.
(5)


A: original point
$B$ : kew point if given $x$ for free
$B C$ : WTP to set the new amount of $x$ li.e., how much y you'd be willing to jive up)

AD: WTA compensation (ii., more y) in return for not being given the extra $x$

If the maifference corves were parallel straight lines, them WT P $=$ WT:
$y$


1. [17 points] Using $\mathbf{p}$ to denote prices, $m$ to denote income, $e$ to denote the expenditure function, and $v$ to denote the indirect utility function, consider the expression

$$
e\left(\cdot, v\left(\mathbf{p}^{\prime}, m^{\prime}\right)\right)-e\left(\cdot, v\left(\mathbf{p}_{0}, m_{0}\right)\right)
$$

if the dot (".") is replaced by $\mathbf{p}_{0}$, this expression measures "Equivalent Variation," and if it is replaced by $\mathbf{p}^{\prime}$, it measures "Compensating Variation."

If a consumer buys two goods, $a$ and $b$, at prices $p_{a}$ and $p_{b}$, and has utility function $u=a \times b$, then find the Compensating Variation (not the Equivalent Variation) for this consumer.

Fall 2010, Final Exam, Qu. 1

Answer to Question 1 of Econ 7005 Final Exam, Fall 2010
(1)

$$
u=a b
$$

To find $v$ :
$\max u$ s.t. $p_{a} a+p_{b} b=m$

$$
\begin{aligned}
& \mathscr{L}=a b+\lambda\left[m-p_{a} a-p_{b} b\right] \\
& 0=\frac{\partial \mathscr{L}}{\partial \lambda}=m \cdot p_{a} a-p_{b} b \\
& 0=\frac{\partial \mathscr{Z}}{\partial a}=b+\lambda\left[-p_{a}\right] \Rightarrow \lambda p_{a}=b \Rightarrow \lambda=\frac{b}{p_{a}} \\
& 0=\frac{\partial \mathscr{L}}{\partial b}=a+\lambda\left[-p_{b}\right] \Rightarrow \lambda p_{b}=a
\end{aligned}
$$

$$
\frac{b}{P_{a}} P_{b}=a \text {; substizte into }
$$

the budget constraint:

$$
\begin{aligned}
& m=p_{a} a+p_{b} b=p_{a}\left(\frac{p_{b}}{p_{a}} b\right)+p_{b} b=2 p_{b} b \Rightarrow \\
& b^{*}=\frac{m}{2 p_{b}} \text { and } a^{*}=\frac{p_{b}}{p_{a}} b^{*}=\frac{p_{b}}{p_{a}} \frac{m}{2 p_{b}}=\frac{m}{2 p_{a}} .
\end{aligned}
$$

So $u^{*}=a^{*} b^{*}=\frac{m}{2 p_{a}} \frac{m}{2 p_{b}}=\frac{m^{2}}{4 p_{a} p_{b}}=v(p, m)$.
To fond $e$ : Either use $v(\underset{\sim}{p}, e(\underset{p}{p}, \hat{u}))=\hat{u} \Rightarrow \frac{e^{2}}{4 p_{a} p_{b}}=\hat{u} \Rightarrow e=2 \sqrt{p_{a} P_{b} \hat{u}}$ or: min $p_{a} a+p_{b} b$ s.t. $u=\hat{u}$
$\mathscr{L}=p_{a} a+p_{b} b+\mu[\hat{u}-a b]$ where I use $\mu$ instead of $l$ for the Lagrange mulaptier just to aroid contusion with the " $\lambda$ " of the previous part.

$$
\begin{aligned}
& D=\frac{\partial \mathcal{Z}}{\partial \mu}=\hat{u}-a b \\
& \left.\begin{array}{l}
0=\frac{\partial \mathscr{Z}}{\partial a}=p_{a}-\mu b \\
0=\frac{\partial \mathscr{Z}}{\partial b}=p_{b}-\mu a
\end{array}\right\} \Rightarrow \frac{p_{a}}{p_{b}}=\frac{b}{a} \Rightarrow b=\frac{p_{a}}{p_{b}} a \Rightarrow \\
& \hat{u}=a b=a\left(\frac{P_{a}}{P_{b}} a\right)=\frac{p_{a}}{p_{b}} a^{2} \Rightarrow \\
& a^{*}=\sqrt{p_{b} \hat{u} / p_{a}} \text { and } \\
& b^{*}=\frac{P_{a}}{P_{b}} a^{*}=\sqrt{\frac{P_{a}^{2}}{P_{b}^{2}} \frac{P_{b} \hat{u}}{P_{a}}}=\sqrt{P_{a} \hat{u} / P_{b}} .
\end{aligned}
$$

So $e\left(p_{\sim}, \hat{u}\right)=p_{a} a^{*}+p_{b} b^{*}=p_{a} \sqrt{\frac{p_{b} \hat{u}}{p_{a}}}+p_{b} \sqrt{\frac{p_{a} \hat{u}}{p_{b}}}$

To find CV:

$$
=\sqrt{p_{a} p_{b} \hat{u}}+\sqrt{P_{a} P_{b} \hat{u}}=2 \sqrt{P_{a} P_{b} \hat{u}} \text {. }
$$

$$
\begin{aligned}
C V & =e \underset{\sim}{p}\left(\underset{\sim}{p}, v\left(p_{1}^{\prime} m^{\prime}\right)\right)-e \underset{\sim}{p} \\
& \left.=2 \sqrt{p_{a}^{\prime}, v\left(p_{0}, m_{0}\right)}\right) \\
& =2 \sqrt{p_{a}^{\prime} v\left(p_{b}^{\prime}, m^{\prime}\right)}-2 \sqrt{p_{a}^{\prime} p_{b}^{\prime} v\left(p_{0}, m_{0}\right)} \\
& =m^{\prime}-m_{0} \sqrt{\frac{\left(m^{\prime}\right)^{2}}{p_{a}^{\prime} p_{b}^{\prime}}}-2 \sqrt{p_{a}^{\prime} p_{b}^{\prime} p_{b}^{\prime}}
\end{aligned} .
$$

* Remark: If you found $e(p, \hat{u})$ first, you can obtain $v(\underset{\sim}{p}, m)$ via: $m=e(p, v(p, m))=$

$$
2 \sqrt{p_{a} p_{b} v} \Rightarrow m^{2}=4 p_{a} p_{b} v \Rightarrow v=m^{2} /\left(4 p_{a} p_{b}\right)
$$

# Qualfyngog Exam <br> 2004 <br> Question 1 

6 ?

## Section 2.

Answer two of the following three questions.

1. Attached to this exam is an excerpt from pages 167 and 168 of Varian's textbook. This excerpt ends with Varian stating that

$$
p^{0}>p^{1} \Rightarrow E V>C V .
$$

(a) If $p^{0}<p^{1}$, is $E V$ greater than or less than $C V$ ?
(b) Related to compensating and equivalent variation are:

- "willingness to pay" an amount of money in order to avoid suffering the price increase from $p^{0}$ to $p^{1}$; and
- "willingness to accept" an amount of money in order to accept the price increase from $p^{0}$ to $p^{1}$.
Is "willingness to pay" equal to CV or to EV? Is "willingness to accept" equal to CV or to EV? Why?

Section 2 \#1.

$$
\begin{aligned}
& E V=\int_{p^{\prime}}^{p^{0}} h\left(p, u^{\prime}\right) d \rho \\
& C V=\int_{p^{\prime}}^{p^{0}} h\left(p, u^{0}\right) d \rho
\end{aligned}
$$

Qualifying Exam

$$
2004
$$

Answer 1
a) $p^{0}<p^{\prime} \Rightarrow u^{0}>u^{\prime} \Rightarrow h\left(p, u^{0}\right)>h\left(p, u^{\prime}\right)$


$$
|E V|=a b f g \quad|C V|=a c e g \quad>a b f g=|E V|
$$

$E V=\Theta$ because $h>0 \quad C V=\Theta$ for smile reasons and the integral's low r limit
( $p^{\prime}$ ) is larger then is upper
limit ( $p^{0}$ )
So $|E V|<|C V|, E V<0, C V<0$. This implies that $C V<E V<0$.
Hence $C V<E V$ regardless of whether $P^{\circ}$ is $>o r<P^{\prime}$ :
b) From the second paragraph of the excerpt f om p. 167 of Varian,

EV uses base year pries and
$C V$ uses time year prices.
The morefrom $p^{\circ}$ to $p^{\prime}$ is a loss to the consumer. In this case of a
loss, WTP to avoid the loss must be measured using base year pries (since the los is avoided, the base prices orevelevant). So $E V=W T P$ to wood the loss. Similarly, $C V$, since in uses final year prices, goes with the loss having already occurred. So $C V=W T A$ compensation for the loss.
optional: For a jain. the situation differs.
WTA to togo a jain involves base year prices, so it's EV.
WTP for the jain it you jet the jain involves final yea prizes, so it's $C V$.

Summary:

|  | Loss | Gain |
| :--- | :--- | :--- |
| $E V$ | FTP | WTA |
| $C V$ | WTA | FTP |

Qualifying Exam
2004
Answer' 1 cont...
new: 2019 Final Exam, Qu. 2.

## 2. [17 points]

Suppose a consumer consumes apples " $a$ " and bananas " $b$ " and has utility function $u=\ln a+3 \ln b$. Suppose this consumer takes the price of apples $p_{a}$ and the price of bananas $p_{b}$ as given.
(a) Find this consumer's expenditure function.
(b) Find this consumer's indirect utility function " $v$ " by using the result of part (a), not by solving an optimization problem. (If you were not able to solve part (a), then make up a hypothetical solution for it in order to work this part of the problem.)
(c) One way of expressing the welfare change experienced by a consumer who faced an initial price of apples $p_{0 a}$ and an initial price of bananas $p_{0 b}$ and had an initial income $m_{0}$ and then is placed in a new economic environment in which his utility becomes $u_{1}$ is the "equivalent variation," denoted $E V$, which can be expressed in several ways, one of which is

$$
v\left(p_{0 a}, p_{0 b}, m_{0}+E V\right)=u_{1} .
$$

Find an expression for the $E V$ of this consumer.

Answer to Econ. 7005 Fall 2019 Final Exam, Qu. 2

$$
u=\ln a+3 \ln b
$$

a) $\min p_{a} a+p_{b} b$ s.t. $\ln a+3 \ln b=\bar{u}$

$$
\begin{aligned}
& \mathcal{L}=p_{a} a+p_{b} b+\lambda[\bar{u}-\ln a-3 \ln b] \\
& \left.\begin{array}{rl}
0=\mathscr{L}_{a}^{\prime}=p_{a}-\frac{\lambda}{a} \\
0=\mathscr{L}_{b}^{\prime}=p_{b}-\frac{3 \lambda}{b}
\end{array}\right\} \Rightarrow \lambda=a p_{a}=\frac{b p_{b}}{3} \Rightarrow b=\frac{3 a p_{a}}{p_{b}} \text { and } \\
& \begin{aligned}
0=\mathscr{L}_{\lambda}^{\prime}=\bar{u}-\ln a-3 \ln b \Rightarrow \bar{u} & =\ln a+3 \ln \frac{3 a p_{a}}{p_{b}} \\
& =\ln a+\ln \frac{3^{3} a^{3} p_{a}^{3}}{p_{b}^{3}}=\ln \frac{3^{3} a^{4} p_{a}^{3}}{p_{b}^{3}} \\
E^{\bar{u}} & =3^{3} a^{4} p_{a}^{3} / p_{b}^{3}
\end{aligned}
\end{aligned}
$$

I'lluse "E "rather then " $e$ " to represent the number while is the base of natural loganthms, because this problem asks for the expaditure function, whose traditional notation is "e."

$$
\begin{aligned}
\Rightarrow a=\left(\frac{p_{b}^{3} E^{\bar{u}}}{3^{3} p_{a}^{3}}\right)^{1 / 4} \text { and } b & =\frac{3 p_{a}}{p_{b}}\left(\frac{p_{b}^{3} E^{\bar{u}}}{3^{3} p_{a}^{3}}\right)^{1 / 4}=\left[\frac{3^{4} p_{a}^{4} p_{b}^{3} E^{\bar{u}}}{p_{b}^{4} 3^{3} p_{a}^{3}}\right]^{1 / 4} \\
& =\left(\frac{3 p_{a} E^{\bar{u}}}{p_{b}}\right)^{1 / 4} .
\end{aligned}
$$

Then expenditure is

$$
\begin{aligned}
e & =p_{a} a+p_{b} b=\left(p_{a}^{4} \frac{p_{b}^{3} E^{\bar{u}}}{3^{3} p_{a}^{3}}\right)^{1 / 4}+\left(p_{b}^{4} \frac{3 p_{a} E^{\bar{u}}}{p_{b}}\right)^{1 / 4} \\
& =\left(\frac{p_{a} p_{b}^{3} E^{\bar{u}}}{3^{3}}\right)^{1 / 4}+\left(3 p_{a} p_{b}^{3} E^{\bar{u}}\right)^{1 / 4}=\left(\frac{1}{3^{3 / 4}+3^{1 / 4}}\left(p_{a} p_{b}^{3} E^{\bar{u}}\right)^{1 / 4}\right.
\end{aligned}
$$

Note that $\frac{1}{3^{3 / 4}}+3^{1 / 4}=\frac{1+3^{\frac{1}{4}} 3^{\frac{3}{4}}}{3^{3 / 4}}=\frac{1+3}{3^{3 / 4}}=\frac{4}{3^{3 / 4}}$ so alternatively

$$
e(p, \bar{u})=4 \cdot 3^{-3 / 4}\left(p_{a} p_{b}^{3} E^{\bar{u}}\right)^{1 / 4} .
$$

b)

$$
\begin{gathered}
e(p, v(p, m)) \equiv m \text { so from part }(a), \\
4 \cdot 3^{-3 / 4}\left(p_{a} p_{b}^{3} E^{v}\right)^{1 / 4}=m \text {. Solving for } v: \\
4^{4} 3^{-3} p_{a} P_{b}^{3} E^{v}=m^{4} \\
E^{v}=\frac{3^{3} m^{4}}{4^{4} P_{a} p_{b}^{3}} \\
v=\ln \frac{3^{3} m^{4}}{4^{4} p_{a} P_{b}^{3}} .
\end{gathered}
$$

c) Set $u_{i}$ equal to $v\left(P_{O a}, P_{O b}, m_{0}+E V\right)=\ln \frac{3^{3}\left(m_{0}+E V\right)^{4}}{4^{4} P_{O a} P_{0 b}^{3}}$

$$
\begin{aligned}
E^{u_{1}} & =\frac{3^{3}\left(m_{0}+E V\right)^{4}}{4^{4} P_{0 a} P_{0 b}^{3}} \\
\left(m_{0}+E V\right)^{4} & =4^{4} P_{0 a} P_{0 b}^{3} E^{u_{1}} / 3^{3} \\
m_{0}+E V & =4\left(P_{0 a} P_{0 b}^{3} E^{u_{1}} / 3^{3}\right)^{1 / 4} \\
E V & =4\left(P_{0 a} P_{O b}^{3} E^{u_{1}} / 3^{3}\right)^{1 / 4}-m_{0}
\end{aligned}
$$

## Fall 2021 Final Exam Question 1

## Ch. 10 Partial Equilibrium

## 1. [17 points]

Suppose a consumer takes prices as given and consumes two goods, $x$ and $y$, which generate utility $u(x, y)=x y$. Suppose this consumer faces an "original" situation with prices $\mathbf{p}^{0}=\left(p_{x}^{0}, p_{y}^{0}\right)$ and income $m^{0}$, or a "new" situation with prices $\mathbf{p}^{\prime}=\left(p_{x}^{\prime}, p_{y}^{\prime}\right)$ and income $m^{\prime}$. One definition of compensating variation is

$$
C V=e\left(\mathbf{p}^{\prime}, v\left(\mathbf{p}^{\prime}, m^{\prime}\right)\right)-e\left(\mathbf{p}^{\prime}, v\left(\mathbf{p}^{0}, m^{0}\right)\right)
$$

and one definition of equivalent compensating variation is

$$
E V=e\left(\mathbf{p}^{0}, v\left(\mathbf{p}^{\prime}, m^{\prime}\right)\right)-e\left(\mathbf{p}^{0}, v\left(\mathbf{p}^{0}, m^{0}\right)\right) .
$$

(a) Find $C V$ and $E V$ if $p_{x}^{\prime}=p_{x}^{0}+\epsilon, p_{y}^{\prime}=p_{y}^{0}$, and $m^{\prime}=m^{0}$.
(b) If $\epsilon>0$, what is the sign of $C V$ and what is the sign of $E V$ ?
(c) In the case when $\epsilon>0$, argue that $E V$ is "willingness and ability to pay," and show mathematically that the absolute value of its formula in part (a) is always less than income. In the case when $\epsilon>0$, argue that $C V$ is "willingness to accept," and show mathematically that the absolute value of its formula in part (a) is not always less than income.

Answer to Fall 2021 Econ. F005 Final Exame, Question 1
a)

$$
\begin{aligned}
& C V=e\left(\underset{\sim}{p}, v\left(p^{\prime}, m^{\prime}\right)\right)-e\left({\underset{\sim}{p}}^{\prime}, v\left(p^{p}, m^{0}\right)\right) \\
& =m^{\prime}-e\left({\underset{\sim}{p}}^{\prime}, v\left(p_{\sim}^{0}, m^{0}\right)\right) . \text { Similaly, } \\
& E V=e\left(p^{p}, v\left(p_{\sim}^{\prime}, m^{\prime}\right)\right)-e\left(p^{0}, v\left({\underset{\sim}{p}}^{0}, m^{0}\right)\right) \\
& =e\left({\underset{\sim}{p}}^{0}, v\left({\underset{\sim}{p}}^{\prime}, m^{\prime}\right)\right)-m^{0} \text {. I vese these simplifita tions be low. }
\end{aligned}
$$

Method 1:
Find e.
min $p_{x} x+p_{y} y$ s.t. $x y=u(x, y)=\bar{u}$

$$
\begin{aligned}
& \mathscr{L}=p_{x} x+p_{y} y+\lambda(\bar{u}-x y) \\
& \text { F.O.C. }
\end{aligned}
$$

$$
\left\{\begin{array}{l}
0=\mathscr{L}_{x}^{\prime}=p_{x}-\lambda y \\
0=\mathscr{L}_{y}^{\prime}=p_{y}-\lambda x \\
0=\bar{u}-x y=\mathscr{L}_{\lambda}^{\prime}
\end{array}\right] \begin{aligned}
& \lambda=\frac{p_{x}}{y}=\frac{p_{y}}{x} \\
& \Rightarrow x=\frac{y p_{y}}{p_{x}} \text { and } \\
& \bar{u}=x y=\frac{y p_{y}}{p_{x}} y=\frac{p_{y}}{p_{x}} y^{2}
\end{aligned}
$$

Method2:
Find v.

$$
\begin{aligned}
& \max \quad u(x, y) \text { s.t. } p_{x} x+p_{y} y=m \\
& \hat{r}_{x y} \\
& \mathcal{L}=x y+\lambda\left(m-p_{x} x-p_{y} y\right)
\end{aligned}
$$

F.O.C.

$$
\left.\begin{array}{l}
0=\mathscr{L}_{x}^{\prime}=y-\lambda p_{x} \\
0=\mathscr{L}_{y}^{\prime}=x-\lambda p_{y} \\
0=\mathscr{L}_{\lambda}^{\prime}=m-p_{x} x-p_{y} y
\end{array}\right\} \begin{aligned}
& \lambda=\frac{y}{p_{x}}=\frac{x}{p_{y}} \\
& \Rightarrow x=\frac{p_{y}}{p_{x}} y \text { and } \\
& m=p_{x} x+p_{y} y=p_{x} \frac{p_{y}}{p_{x}} y+p_{y} y \\
& =2 p_{y} y
\end{aligned}
$$

Method 1, continued

$$
\begin{aligned}
\text { So } y & =\sqrt{\frac{P_{x}}{P_{y}} \bar{u}} \\
\text { and } x & =\frac{y P_{y}}{P_{x}}=\sqrt{\frac{P_{x}}{P_{y}} \bar{u} \frac{P_{y}^{2}}{P_{x}^{2}}} \\
& =\sqrt{\frac{P_{y}}{P_{x}} \bar{u}}
\end{aligned}
$$

making

$$
\begin{aligned}
e & =p_{x} x^{*}+p_{y} y^{x} \\
& =p_{x} \sqrt{\frac{p_{y}}{p_{x}} \bar{u}}+p_{y} \sqrt{\frac{p_{x}}{p_{y}} \bar{u}} \\
& =\sqrt{p_{x} P_{y} \bar{u}}+\sqrt{p_{x} p_{y} \bar{u}} \\
& =2 \sqrt{p_{x} P_{y} \bar{u}}
\end{aligned}
$$

Knowing $e$, we can find $v$ using

$$
\begin{aligned}
e\left(p_{\sim}^{p}, v\left(p_{\sim}, m\right)\right. & =m \\
2 \sqrt{p_{x} p_{y} v} & =m \\
p_{x} p_{y} v & =(m / 2)^{2} \\
v & =\frac{m^{2}}{4 p_{x} p_{y}}
\end{aligned}
$$

Method 2, continued
So $y=\frac{m}{2 p_{y}}$

$$
\text { and } \begin{aligned}
x & =\frac{p_{y}}{p_{x}} y=\frac{p_{y}}{p_{x}} \frac{m}{2 p_{y}} \\
& =\frac{m}{2 p_{x}},
\end{aligned}
$$

making

$$
v=x^{*} y^{*}=\frac{m}{2 p_{x}} \frac{m}{2 p_{y}}=\frac{m^{2}}{4 p_{x} p_{y}}
$$

Knowing $v$, we can find $e$ using

$$
\begin{aligned}
v\left(\underset{\sim}{p}, e\left(p_{1}, u\right)\right) & =u . \\
\frac{e^{2}}{4 p_{x} p_{y}} & =u \\
e^{2} & =4 p_{x} p_{y} u \\
e\left(p_{x}, p_{y}, \bar{u}\right) & =2 \sqrt{p_{x} p_{y} \bar{u}} .
\end{aligned}
$$

Clearly the two methods result
in the same $e$ and $v$ functions.

The publemsays $P_{x}{ }^{\prime}=p_{x}{ }^{0}+\varepsilon$

$$
\begin{aligned}
& P_{y}{ }^{\prime}=P_{y}^{0} \\
& m^{\prime}=m^{0}
\end{aligned}
$$

so

$$
\begin{aligned}
c V & =m^{\prime}-e\left(p_{\sim}^{\prime}, v\left(\underset{\sim}{p}, m^{0}\right)\right) \\
& =m^{\prime}-2 \sqrt{p_{x}^{\prime} p_{y}^{\prime} \frac{\left(m^{0}\right)^{2}}{4 p_{x}^{0} P_{y}^{0}}} \\
& =m^{0}-2 \sqrt{p_{x}^{\prime} p_{y}^{\prime} v\left(p_{\sim}^{0}, m 0\right)} \\
\left(p_{x}^{0}+\varepsilon\right) p_{y}^{0} \frac{\left(m^{0}\right)^{2}}{4 p_{x}^{0} p_{y}^{0}} & m^{0}-m^{0} \sqrt{\frac{P_{x}^{0}+\varepsilon}{P_{x}^{0}}} \\
& =m^{0}-m^{0} \sqrt{1+\frac{\varepsilon}{P_{x}^{0}}}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E V & =e\left(\underset{\sim}{P^{0}}, v\left(P^{\prime}, m^{\prime}\right)\right)-m^{0}=2 \sqrt{P_{x}^{0} P_{y}^{0} v\left(P^{\prime}, m^{\prime}\right)}-m \\
& =2 \sqrt{P_{x}^{0} P_{y}^{0} \frac{\left(m^{\prime}\right)^{2}}{4 P_{x}^{\prime} P_{y}^{\prime}}}-m^{0}=m^{\prime} \sqrt{P_{x}^{0} P_{y}^{0} \frac{1}{\left(P_{x}^{0}+\varepsilon\right) P_{y}^{0}}}-m^{0} \\
& =m^{0} \sqrt{\frac{P_{x}^{0}}{P_{x}^{0}+\varepsilon}}-m^{0}=m^{0} \sqrt{\frac{1}{1+\varepsilon / p_{x}^{0}}}-m^{0}
\end{aligned}
$$

b) (If $\varepsilon>0, p_{x}^{\prime}>p_{x}^{0}$, and since $p_{y}$ and $m$ stay the same, the consume must be worse off.) $C V=m^{0}-m^{0} \sqrt{1+\frac{\varepsilon}{P_{x}^{0}}}$ and $\sqrt{1+\frac{\varepsilon}{P_{x} 0}}>1$ so $C V<0$.

$$
\begin{aligned}
E V=m^{0} \sqrt{\frac{1}{1+\varepsilon / p_{1}}}-m^{0} . \text { With } \varepsilon>0,1+\frac{\varepsilon}{p_{0}} & >1 \\
\frac{1}{1+\frac{2}{p_{0}}} & <1 \\
\sqrt{\frac{1}{1+\varepsilon} p_{0}} & <1 \\
\text { so } E V & <0 .
\end{aligned}
$$ the WATP in gratitude for not increasing the price of $x$.

From part (a) $E V=m^{0}\left[\sqrt{\frac{1}{1+\varepsilon / p_{0}}}-1\right]$. From pat (b), $E V<O$;
So $|E V|=-E V=m^{0}\left[1-\sqrt{\frac{1}{1+\varepsilon / p_{0}}}\right]$.
It $\varepsilon=0$, the is 0 .
As $s \rightarrow \infty$, this is $m^{0}$.
For $\&$ between $O$ and $\infty$, this is between $O$ and $m{ }^{\circ}$.
From pat (a), $C V=m^{0}-m^{0} \sqrt{1+\frac{\varepsilon}{P_{x}^{0}}}$. From pat (b), $C V<0$;

$$
\text { so }|c V|=-C V=m^{0} \sqrt{1+\frac{\varepsilon}{P_{x}}}-m^{0} \text {. At } \varepsilon=0 \text {, this is } 0 \text {. }
$$

As $\varepsilon \rightarrow \infty$, this is $\infty$. So it is not bounded by income.
With $\varepsilon>0, C V$ is, if the poling is carried out (the pries increased - so, using new prices) the WTA compensation for the $\uparrow p_{x}$.

2021 Qualifying Exam Sec. 3 Qu. 1

1. [10 points] Let $p^{0}$ be an "initial" price vector, $p^{\prime}$ be a "final" price vector, $m^{0}$ be "initial" income, and $m^{\prime}$ be "final" income. One definition of equivalent variation, $E V$, and compensating variation, $C V$, is

$$
\begin{aligned}
& E V=e\left(p^{0}, v\left(p^{\prime}, m^{\prime}\right)\right)-m^{0} \\
& C V=m^{\prime}-e\left(p^{\prime}, v\left(p^{0}, m^{0}\right)\right.
\end{aligned}
$$

where $e(\cdot)$ denotes the expenditure function of a consumer and $v(\cdot)$ denotes the indirect utility function of that consumer. Let $u^{0}=v\left(p^{0}, m^{0}\right)$ and $u^{\prime}=v\left(p^{\prime}, m^{\prime}\right)$. Under the assumption that $m^{0}=m^{\prime}$, Varian writes on his page 167 that:

$$
\begin{aligned}
& E V=e\left(p^{0}, u^{\prime}\right)-e\left(p^{\prime}, u^{\prime}\right) \\
& C V=e\left(p^{0}, u^{0}\right)-e\left(p^{\prime}, u^{0}\right) .
\end{aligned}
$$

Finally, using the fact that the Hicksian demand function is the derivative of the expenditure function, so that $h(p, u) \equiv \partial e / \partial p$, we can write these expressions as

$$
\begin{align*}
& E V=e\left(p^{0}, u^{\prime}\right)-e\left(p^{\prime}, u^{\prime}\right)=\int_{p^{\prime}}^{p^{0}} h\left(p, u^{\prime}\right) d p \\
& C V=e\left(p^{0}, u^{0}\right)-e\left(p^{\prime}, u^{0}\right)=\int_{p^{\prime}}^{p^{0}} h\left(p, u^{0}\right) d p \tag{10.2}
\end{align*}
$$

It follows from these expressions that the compensating variation is the integral of the Hicksian demand curve associated with the initial level of

Show that if $m^{0} \neq m^{\prime}$, Varian's (10.2) have to be modified to

$$
\begin{aligned}
& E V=\int_{p^{\prime}}^{p^{0}} h\left(p, u^{\prime}\right) d p+\text { something more } \\
& C V=\int_{p^{\prime}}^{p^{0}} h\left(p, u^{0}\right) d p+\text { something more } .
\end{aligned}
$$

by finding what the "something more" parts are.

## Answer to Summer 2021 Qualifying Exam, Section 3 Question 1

Let $u^{\prime}=v\left(p^{\prime}, m^{\prime}\right)$. Then

$$
\begin{equation*}
E V=e\left(p^{0}, v\left(p^{\prime}, m^{\prime}\right)\right)-m^{0}=e\left(p^{0}, u^{\prime}\right)-m^{0} . \tag{1}
\end{equation*}
$$

Next, notice that the second equality in the first equation of Varian's (10.2) is still true when $m^{0} \neq m^{\prime}$ :

$$
\begin{equation*}
e\left(p^{0}, u^{\prime}\right)-e\left(p^{\prime}, u^{\prime}\right)=\int_{p^{\prime}}^{p^{0}} \frac{\partial e\left(p, u^{\prime}\right)}{\partial p} d p=\int_{p^{\prime}}^{p^{0}} h\left(p, u^{\prime}\right) d p \tag{2}
\end{equation*}
$$

(the first equality here is the Fundamental Theorem of Calculus, that $e=$ $\int(d e / d p) d p$, and the second is, as Varian points out, $\left.h(p, u) \equiv \partial e(p, u) / \partial p\right)$. So, if we can get the right-hand side of (1) to look more like the left-hand side of (2), we should be able to express $E V$ in terms of the right-hand side of (2). To do this, add and subtract $e\left(p^{\prime}, u^{\prime}\right)$ from (1):

$$
\begin{aligned}
E V & =e\left(p^{0}, u^{\prime}\right)-m^{0}+\left[e\left(p^{\prime}, u^{\prime}\right)-e\left(p^{\prime}, u^{\prime}\right)\right] \\
& =e\left(p^{0}, u^{\prime}\right)-e\left(p^{\prime}, u^{\prime}\right)-m^{0}+e\left(p^{\prime}, u^{\prime}\right)
\end{aligned}
$$

and from (2),

$$
\begin{equation*}
=\int_{p^{\prime}}^{p^{0}} h\left(p, u^{\prime}\right) d p-m^{0}+e\left(p^{\prime}, u^{\prime}\right) . \tag{3}
\end{equation*}
$$

This is good enough for an answer. However, it's easier to interpret if you use the fact that $m^{\prime}=e\left(p^{\prime}, u^{\prime}\right)$, which is true because one of the four basic identities on Varian's p. 106 in Ch. 7 says $m=e(p, v(p, u))$ and because at the beginning of this answer I set $u^{\prime}=v\left(p^{\prime}, m^{\prime}\right)$. This results in

$$
\begin{equation*}
E V=\int_{p^{\prime}}^{p^{0}} h\left(p, u^{\prime}\right) d p-m^{0}+m^{\prime} \tag{4}
\end{equation*}
$$

showing that the "something more" asked for in the question is just the change in income, $m^{\prime}-m^{0}$.

The $C V$ proof is completely analogous. Let $u^{0}=v\left(p^{0}, m^{0}\right)$. Then

$$
\begin{equation*}
C V=m^{\prime}-e\left(p^{\prime}, v\left(p^{0}, m^{0}\right)\right)=m^{\prime}-e\left(p^{\prime}, u^{0}\right) . \tag{5}
\end{equation*}
$$

The second equality in the second equation of Varian's (10.2) is still true when $m^{0} \neq m^{\prime}$, for the same reason as given for the second equality in the first equation of Varian's (10.2) being still true when $m^{0} \neq m^{\prime}$, so

$$
\begin{equation*}
e\left(p^{0}, u^{0}\right)-e\left(p^{\prime}, u^{0}\right)=\int_{p^{\prime}}^{p^{0}} \frac{\partial e\left(p, u^{0}\right)}{\partial p} d p=\int_{p^{\prime}}^{p^{0}} h\left(p, u^{0}\right) d p . \tag{6}
\end{equation*}
$$

Add and subtract $e\left(p^{0}, u^{0}\right)$ from (5):

$$
\begin{aligned}
C V & =m^{\prime}-e\left(p^{\prime}, u^{0}\right)+\left[e\left(p^{0}, u^{0}\right)-e\left(p^{0}, u^{0}\right)\right] \\
& =e\left(p^{0}, u^{0}\right)-e\left(p^{\prime}, u^{0}\right)+m^{\prime}-e\left(p^{0}, u^{0}\right)
\end{aligned}
$$

and from (6),

$$
\begin{equation*}
=\int_{p^{\prime}}^{p^{0}} h\left(p, u^{\prime}\right) d p+m^{\prime}-e\left(p^{0}, u^{0}\right) . \tag{7}
\end{equation*}
$$

This is good enough for an answer. However, it's easier to interpret if you use $m^{0}=e\left(p^{0}, u^{0}\right)$ (setting $u^{0}=v\left(p^{0}, m^{0}\right)$ ), resulting in

$$
\begin{equation*}
C V=\int_{p^{\prime}}^{p^{0}} h\left(p, u^{\prime}\right) d p+m^{\prime}-m^{0} \tag{8}
\end{equation*}
$$

showing that the "something more" asked for in the question is again just the change in income, $m^{\prime}-m^{0}$.

## 2023 Qualifying Exam Sec. 3 Qu. 1

1. [12 points] Suppose a consumer gets utility from consumption of apples " $a$ " and bananas " $b$ " according to a strictly quasiconcave utility function $u(a, b)$. Suppose the consumer's initial consumption bundle is $\left(a_{0}, b_{0}\right)$, and let $U_{0}$ be $u\left(a_{0}, b_{0}\right)$ where $u$ is increasing in $a$ and $b$. The purpose of this question is to investigate one way to define the "value" to this consumer of giving this consumer more apples, moving his consumption bundle to ( $a_{1}, b_{0}$ ), where $a_{1}>a_{0}$.
(a) (2 points) Let compensating variation " $C V$ " for this environment which lacks prices and incomes be implicitly defined by

$$
u\left(a_{0}, b_{0}\right)=u\left(a_{1}, b_{0}-C V\right)
$$

(In microeconomic theory textbooks, $C V$ is only defined in environments with prices and incomes.) How could $C V$, defined in this way, be interpreted as a measure of the value of moving from $a_{0}$ to $a_{1}$ ?
(b) (2 points) Sketch a graph with $a$ on the horizontal axis and $b$ on the vertical axis, illustrating $C V$. Hint: begin by drawing the indifference curve which $u\left(a_{0}, b_{0}\right)$ lies on.
(c) (3 points) By calculating the appropriate (total) differential, show that

$$
\frac{\partial C V}{\partial a_{1}}=\frac{\partial u\left(a_{1}, b_{0}-C V\right) / \partial a_{1}}{\partial u\left(a_{1}, b_{0}-C V\right) / \partial\left(b_{0}-C V\right)}
$$

which is an abbreviated notation for

$$
\begin{equation*}
\frac{\partial C V}{\partial a_{1}}=\left.\frac{\left.\frac{\partial u(a, b)}{\partial a}\right|_{\left(a_{1}, b_{0}-C V\right)}}{\frac{\partial u(a, b)}{\partial b}}\right|_{\left(a_{1}, b_{0}-C V\right)} \tag{1}
\end{equation*}
$$

(d) (3 points) As $a_{1}$ increases, does the strict quasiconcavity of $u$ imply that the right-hand side of (1) increases, decreases, or remains constant? Why? (If you make an assertion about how the strict quasiconcavity of $u$ affects the indifference curves, you do not have to prove that assertion.) Hint: Rather than calculate
any derivatives, think about what happens to the Marginal Rate of Substitution of $a$ for $b$. It is completely acceptable for you to simply assert, rather than rigorously prove, the connection between the Marginal Rate of Substitution and this problem, because that proof is an undergraduate-level exercise.
(e) (2 points) Sketch a rough graph of $C V$ versus $a_{1}$. Make sure the first derivative of your sketch of $C V\left(a_{1}\right)$ is consistent with (1) and the second derivative of your sketch of $C V\left(a_{1}\right)$ is consistent with your answer to part (d).


Figure 1. Two non-price- nor income-based measures of the value of a policy of moving from $a_{0}$ to $a_{1}>a_{0}$. Only the compensating variation ( $C V$ ) version of this measure of value was asked for in this question.

## Answer to Summer 2023 Qualifying Exam, Section 3 Question 1

(a) When apples increase from $a_{0}$ to $a_{1}$, utility goes up. What is the maximum number of bananas the consumer is willing to give up for this increase in apples? The answer is $C V$ as defined in this equation, because if the consumer had to give up more than this number of bananas, the consumer would be worse off than if he had not gotten the increased number of apples, whereas if the consumer only had to give up less than this amount of bananas, he would be strictly better off than he was originally, so he would not have given up a maximum number of bananas.
(b) The graph is Figure 1, where the initial position is ( $a_{0}, b_{0}$ ), receiving the extra apples moves the consumer to the open circle at $\left(a_{1}, b_{0}\right)$, and if the consumer then gives up $C V$ bananas, the consumer ends up at the solid circle at $\left(a_{1}, b_{0}-C V\right)$, which is on the same $U_{0}$ indifference curve as when the process began.
The graph has extra information that was not asked for in this question, and it is not expected that you included this extra information in your graph. The extra, not needed information is the "willingness
and ability to pay" (WATP), the "willingness to accept" (WTA), the "equivalent variation" $(E V)$, and the "new" indifference curve $U_{1}$.
(c) To take the differential of the equation defining $C V$, it is convenient to rewrite it as being equal to zero:

$$
u\left(a_{1}, b_{0}-C V\right)-u\left(a_{0}, b_{0}\right)=0
$$

In taking the differential, $a_{0}$ and $b_{0}$ are fixed, while $a_{1}$ and $C V$ vary:

$$
\frac{\partial u\left(a_{1}, b_{0}-C V\right)}{\partial a_{1}} d a_{1}+\frac{\partial u\left(a_{1}, b_{0}-C V\right)}{\partial\left(b_{0}-C V\right)} \frac{\partial\left(b_{0}-C V\right)}{\partial C V} d C V=0 .
$$

Since $\partial\left(b_{0}-C V\right) / \partial C V=-1$, this leads to

$$
\frac{\partial u\left(a_{1}, b_{0}-C V\right)}{\partial a_{1}} d a_{1}=\frac{\partial u\left(a_{1}, b_{0}-C V\right)}{\partial\left(b_{0}-C V\right)} d C V
$$

and the expression given in the exam follows.
(d) The key insight here is that the right-hand side of (1) is the marginal rate of substitution of $a$ for $b$, as is shown in undergraduate Intermediate Microeconomics textbooks (and in Varian's §7.1). (Note that the right-hand side of (1) is positive.)
Next, the marginal rate of substitution is -1 times the slope of the indifference curve, as is also shown in those texts.
Next, a quasiconcave function is defined as a function having convex upper level sets. Since the utility function in this problem was specified as being quasiconcave, its upper levels sets-and in particular, its upper level set for $U_{0}$-is a convex set. This means that the indifference curve $U_{0}$, thought of as a function of $a$, is a convex function. Since the utility function in this problem was specified as being strictly quasiconcave, the indifference curve $U_{0}$, thought of as a function of $a$, is a strictly convex function, as drawn in Figure 1. Hence its slope gets closer to zero as $a_{1}$ increases. But this slope is just -1 times (the right-hand side of) (1); hence (1) also gets closer to zero as $a_{1}$ increases.
It follows that $\partial C V / \partial a_{1}$ gets closer to zero as $a_{1}$ increases, so $\partial^{2} C V / \partial a_{1}^{2}<$ 0 .
(e) From the previous part, $C V$ is a positive, increasing, concave function of $a_{1}$ for $a_{1}>a_{0}$. ("Positive" is implied by parts (a) and (b); "increasing" is implied by (1) because its right-hand side is positive; and "concave" is implied by part (d).) The rough graph should have these properties. (Optional: at $a_{1}=a_{0}, C V$ is zero.)

## Completely Optional Remarks

The problem has shown that the value of apples (in terms of bananas, measured using $C V$ ) (I would call this the "use value" of apples, as opposed to their "exchange value," which is their price; see https: //en.wikipedia.org/wiki/Use_value) is an increasing, concave, cardinal function of apple consumption. In the late nineteenth and early twentieth centuries, an individual's utility was seen by the Utilitarians (or at least by Marshall and Pigou) as being an increasing, concave, cardinal function of wealth. If in addition one assumes either that everyone has the same utility function, or that the social planner wishes to act as if everyone has the same utility function when it comes to decisions on distribution, then maximizing social welfare implies giving everyone the same amount of each commodity. The rise of ordinalism destroyed this egalitarian argument. This problem's definition of use value resurrects it.
Historical comments: In his 1973 book "On Economic Inequality," Nobel laureate Amartya Sen calls cardinal "utilitarianism, the dominant faith of 'old' welfare economics" (p. 23). He explains how cardinal Utilitarianism came to have an egalitarian reputation (pp. 15-17):

Once the information content of individual preferences has been broadened to include interpersonally comparable cardinal welfare functions, many methods of social judgement become available. The most widely used approach is that of utilitarianism in which the sum of the individual utilities is taken as the measure of social welfare, and alternative social states are ordered in terms of the value of the sum of individual utilities. Pioneered by Bentham (1789), this approach has been widely used in economics for social judgements, notably by Marshall (1890), Pigou (1920), and Robertson (1952). In the context of the measurement of inequality of income distribution, and in
that of judging alternative distributions of income, it has been used by Dalton (1920), Lange (1938), Lerner (1944), Aigner and Heins (1967), and Tinbergen (1970), among others.

The trouble with this approach is that maximizing the sum of individual utilities is supremely unconcerned with the interpersonal distribution of that sum. This should make it a particularly unsuitable approach to use for measuring or judging inequality. Interestingly enough, however, not only has utilitarianism been fairly widely used for distributional judgements, it has-somewhat amazingly-even developed the reputation of being an egalitarian criterion. This seems to have come about through a peculiar dialectical process whereby such adherents of utilitarianism as Marshall and Pigou were attacked by Robbins and others for their supposedly egalitarian use of the utilitarian framework. This gave utilitarianism a ready-made reputation for being equality-conscious.

The whole thing arises from a very special coincidence under some extremely simple assumptions. The maximization of the sum of individual utilities through the distribution of a given total of income between different persons requires equating the marginal utilities from income of different persons, and if the special assumption is made that everyone has the same utility function, then equating marginal utilities amounts to equating total utilities as well. Marshall and others noted this particular aspect of utilitarianism, though they were in no particular hurry to draw any radical distributive policy prescription out of this. But when the attack on utilitarianism came, this particular aspect of it was singled out for an especially stern rebuke.

While this dialectical process gave utilitarianism its ill-deserved egalitarian reputation, the true character of that approach can be seen quite easily by considering a case where one person $A$ derives exactly twice as much utility as person $B \ldots$

Sen really did not think utilitarianism deserved its egalitarian reputation, continuing (p. 18):

It seems fairly clear that fundamentally utilitarianism is very far from an egalitarian approach. It is, therefore, odd that virtually all attempts at measuring inequality from a welfare point of view, or exercises in deriving optimal distributional rules, have concentrated on the utilitarian approach. It might be thought
that this criticism would not apply at all if utilitarianism were combined with the assumption that everyone has the same utility function. But this is not quite the case. The distribution of welfare between persons is a relevant aspect of any problem of income distribution, and our evaluation of inequality will obviously depend on whether we are concerned only with the loss of the sum of individual utilities through a bad distribution of income, or also with the inequality of welfare levels of different individuals. Its lack of concern with the latter tends to make utilitarianism a blunt approach to measuring and judging different extents of inequality even if the assumption is made that everyone has the same utility function. As a framework of judging inequality, utilitarianism is indeed a non-starter, despite the spell that this approach seems to have cast on this branch of normative economics.

However, this criticism by Sen of the consequences of "the assumption that everyone has the same utility function" seems unconvincing.

2022 Qualifying Exam Sec. 3 Qu. 1

1. [12 points] Suppose a consumer gets utility from two commodities, apples $a$ and bananas $b$, according to the utility function

$$
u(a, b)=a b^{2} .
$$

Suppose the consumer takes as given the price of apples, $p_{a}$, and the price of bananas, $p_{b}$, and that the consumer has income denoted by $m$.
(a) Find this consumer's demand for apples and demand for bananas. Do not bother checking the second-order conditions.
(b) For the rest of this problem, suppose that the price of apples is $p_{a}=1 / 3$ and the price of bananas is $p_{b}=2 / 3$.
If this consumer's initial income is $m_{0}=3$, determine his initial consumption of apples and of bananas.
(c) If this consumer's income changes to $m^{\prime}=4$, determine his new consumption of apples and of bananas.
(d) [Optional motivation for the rest of the problem: You might think that all "valuation" methods for the $\$ 1$ in extra income the consumer gets when going from $m_{0}=3$ to $m^{\prime}=4$ would assign this change a value of exactly $\$ 1$, but do they?]
Make a sketch of this consumer's indifference curve through his initial consumption point (the one for $m_{0}=3$ ) and this consumer's indifference curve through his new consumption point (the one for $m^{\prime}=4$ ), graphing apples on the horizontal axis and bananas on the vertical axis. You do not draw this graph very precisely. In particular, you do not need to use the mathematical form of this consumer's actual utility function, $u=a b^{2}$, in making your graph; it is fine to draw your graph in a generic way, using the sort of indifference curves discussed in undergraduate textbooks.
(e) On the graph you just drew, indicate this consumer's "Willingness (and Ability) to Pay," measured in terms of apples, to move from his initial consumption point to his new consumption point. This is equal to this consumer's compensating variation, measured in terms of apples.
(f) Numerically calculate this consumer's Willingness (and Ability) to Pay, measured in terms of apples, to move from his initial consumption point to his new consumption point. To do this,
you need to use the fact that the consumer's utility function is $u=a b^{2}$, and you need to use the coordinates in the (apples, bananas) plane of the initial and final consumption points.
Since you do not have calculators, you do not need to simplify expressions that are purely numerical. For example, you would not need to simplify $4-27 / 16$.
(g) In the previous part of this problem, you calculated this consumer's Willingness (and Ability) to Pay, measured in terms of apples, to move from his initial consumption point to his new consumption point. What is the dollar value of this WATP number of apples, using the prevailing price of apples? (Again, you do not need to simplify expressions that are purely numerical.)
(h) On the graph you drew above, indicate this consumer's "Willingness (and Ability) to Pay," measured in terms of bananas, to move from his initial consumption point to his new consumption point. This is equal to this consumer's compensating variation, measured in terms of bananas.
(i) Numerically calculate this consumer's Willingness (and Ability) to Pay, measured in terms of bananas, to move from his initial consumption point to his new consumption point. To do this, you need to use the fact that the consumer's utility function is $u=a b^{2}$, and you need to use the coordinates in the (apples, bananas) plane of the initial and final consumption points.
You do not need to simplify expressions that are purely numerical.
(j) In the previous part of this problem, you calculated this consumer's Willingness (and Ability) to Pay, measured in terms of bananas, to move from his initial consumption point to his new consumption point. What is the dollar value of this WATP number of bananas, using the prevailing price of bananas? (Again, you do not need to simplify expressions that are purely numerical.)

## Answer to Summer 2021 Qualifying Exam, Section 3 Question 1

(a) The Lagrangian is

$$
\mathscr{L}=a b^{2}+\lambda\left(m-p_{a} a-p_{b} b\right) .
$$

So

$$
\begin{aligned}
& 0=\mathscr{L}_{a}^{\prime}=b^{2}-\lambda p_{a} \quad \text { and } \\
& 0=\mathscr{L}_{b}^{\prime}=2 a b-\lambda p_{b} .
\end{aligned}
$$

From the first and second equations, respectively, it follows that

$$
\lambda=\frac{b^{2}}{p_{a}}=\frac{2 a b}{p_{b}}
$$

so $b=2 a p_{a} / p_{b}$ and $m=p_{a} a+p_{b} b=p_{a} a+p_{b}\left(2 a p_{a} / p_{b}\right)=3 a p_{a}$ and the demand for apples is

$$
a^{D}=\frac{m}{3 p_{a}} .
$$

Then the demand for bananas is

$$
b^{D}=\frac{2 p_{a} a}{p_{b}}=\frac{2 p_{a}}{p_{b}} \frac{m}{3 p_{a}}=\frac{2 m}{3 p_{b}} .
$$

(b) With $p_{a}=1 / 3$ and $p_{b}=2 / 3$, and $m_{0}=3$, the demand curves from part (a) result in $a=3$ and $b=3$.
(c) With income changing to $m^{\prime}=4$, the demand curves from part (a) result in $a=4$ and $b=4$.
(d) See Figure 1. The relevant characteristics of that figure for this question are that the original indifference curve $U_{0}$ passes through $(3,3)$ and that the new indifference curve $U^{\prime}$ passes through $(4,4)$.
(e) See the $W_{A T P}^{a}$ of Figure 1. This answers the question "if the consumer were able to move to the new point $(4,4)$, how many apples would he then be willing to give up," because if he had to give up any more apples than this, his utility would fall below $U_{0}$, which he would not willingly do.
(Optional: The question says this is compensating variation because it assumes the consumer first moves to the new point.)


Figure 1. With an initial income of $\$ 3$, a consumer with utility function $a b^{2}$ (where $a$ is apples and $b$ is bananas) facing prices $p_{a}=1 / 3$ and $p_{b}=2 / 3$ consumes at $(3,3)$, where budget constraint $B C_{0}$ is tangent to indifference curve $U_{0}$. With a new income of $\$ 4$, that consumer consumes at $(4,4)$, where budget constraint $B C^{\prime}$ is tangent to indifference curve $U^{\prime}$. Four measures of the consumer's value of this $\$ 1$ increase in income are shown. The WATP measures are for the exam's Section 3 Question 1, and the WTA measures are for the exam's Section 3 Question 2. Optional: The horizontal distance between $B C_{0}$ and $B C^{\prime}$ is one dollar's worth of apples. Since this graph has been drawn precisely, this means that $W A T P_{a}$ is less than $\$ 1$ (to see this, compare $W A T P_{a}$ to the gap between the dotted lines along the dashed line at $b=4$ ), and $W T A_{a}$ is more than $\$ 1$ (to see this, compare $W T A_{a}$ to the gap between the dotted lines along the dashed line at $b=3$ ). Similarly, the vertical distance between $B C_{0}$ and $B C^{\prime}$ is one dollar's worth of bananas. Hence $W A T P_{b}$ is less than $\$ 1$ (to see this, compare $W A T P_{b}$ to the gap between the dotted lines along the dashed line at $a=4$ ), and $W T A_{b}$ is more than $\$ 1$ (to see this, compare $W T A_{b}$ to the gap between the dotted lines along the dashed line at $a=3$ ).
(f) Starting from the new point $(4,4)$, he would be willing to give up apples up to, but not beyond, the point where his utility after giving up those apples, $u\left(4-W_{A T P}^{a}, 4\right)$, was equal to his original utility, $u(3,3)$. So we have

$$
\begin{aligned}
u(3,3) & =u\left(4-W A T P_{a}, 4\right) \\
3 \cdot 3^{2} & =\left(4-W A T P_{a}\right) \cdot 4^{2} \\
27 / 16 & =4-W A T P_{a} \\
W A T P_{a} & =4-27 / 16=2 \frac{5}{16} \approx 2.3125 .
\end{aligned}
$$

Throughout this problem, you were told that you did not have to simplify purely numerical expressions.
(g) It is $p_{a}$ times $W A T P_{a}$, so $1 / 3$ times $2 \frac{5}{16}$, which is $\$ 37 / 48 \approx \$ 0.77<$ $\$ 1$. The "less than $\$ 1$ " part is optional; it is interesting that "willingness and ability to pay," when measured in apples, is less than one dollar's worth of apples.
(h) See the $W A T P_{b}$ of Figure 1. This answers the question "if the consumer were able to move to the new point $(4,4)$, how many bananas would he be willing to give up," because if he had to give up any more bananas than this, his utility would fall below $U_{0}$, which he would not willingly do.
(Optional: The question says this is compensating variation because it assumes the consumer moves first to the new point.)
(i) Starting from the new point $(4,4)$, he would be willing to give up bananas up to, but not beyond, the point where his utility after giving up those bananas, $u\left(4,4-W A T P_{b}\right)$, was equal to his original utility, $u(3,3)$. So we have

$$
\begin{aligned}
u(3,3) & =u\left(4,4-W A T P_{b}\right) \\
3 \cdot 3^{2} & =4 \cdot\left(4-W A T P_{a}\right)^{2} \\
\frac{\sqrt{3} \cdot 3}{2} & =4-W A T P_{b} \\
W A T P_{b} & =4-\frac{\sqrt{3} \cdot 3}{2}=\frac{8-3 \sqrt{3}}{2} \approx 1.40 .
\end{aligned}
$$

(j) It is $p_{b}$ times $W A T P_{b}$, so $2 / 3$ times $\frac{8-3 \sqrt{3}}{2}$, which is $\$ \frac{8-3 \sqrt{3}}{3} \approx \$ 0.93<$ $\$ 1$. The "less than $\$ 1$ " part is optional; it is interesting that "willingness and ability to pay," when measured in bananas, is less than one dollar's worth of bananas.

Also optional: the dollar value of "WATP $P_{a}$ for $\$ 1$ in extra income" is not the same as the dollar value of " $W A T P_{b}$ for $\$ 1$ in extra income."

2022 Qualifying Exam Sec. 3 Qu. 2
2. [12 points] Suppose a consumer gets utility from two commodities, apples $a$ and bananas $b$, according to the utility function

$$
u(a, b)=a b^{2} .
$$

Suppose the consumer takes as given the price of apples, $p_{a}$, and the price of bananas, $p_{b}$, and that the consumer has income denoted by $m$.
(a) Find this consumer's demand for apples and demand for bananas. Do not bother checking the second-order conditions.
(b) For the rest of this problem, suppose that the price of apples is $p_{a}=1 / 3$ and the price of bananas is $p_{b}=2 / 3$.
If this consumer's initial income is $m_{0}=3$, determine his initial consumption of apples and of bananas.
(c) If this consumer's income changes to $m^{\prime}=4$, determine his new consumption of apples and bananas.
(d) [Optional motivation for the rest of the problem: You might think that all "valuation" methods for the $\$ 1$ in extra income the consumer gets when going from $m_{0}=3$ to $m^{\prime}=4$ would assign this change a value of exactly $\$ 1$, but do they?]
Make a sketch of this consumer's indifference curve through his initial consumption point (the one for $m_{0}=3$ ) and this consumer's indifference curve through his new consumption point (the one for $m^{\prime}=4$ ), graphing apples on the horizontal axis and bananas on the vertical axis. You do not draw this graph very precisely. In particular, you do not need to use the mathematical form of this consumer's actual utility function, $u=a b^{2}$, in making your graph; it is fine to draw your graph in a generic way, using the sort of indifference curves discussed in undergraduate textbooks.
(e) On the graph you just drew, indicate this consumer's "Willingness to Accept" compensation, measured in terms of apples, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. This is equal to this consumer's equivalent variation, measured in terms of apples.
(f) Numerically calculate this consumer's Willingness to Accept compensation, measured in terms of apples, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. To do this, you need to use the fact that the consumer's utility function is $u=a b^{2}$, and you need to use the coordinates in the (apples, bananas) plane of the initial and final consumption points.
Since you do not have calculators, you do not need to simplify expressions that are purely numerical. For example, you would not need to simplify $4-27 / 16$.
(g) In the previous part of this problem, you calculated this consumer's Willingness to Accept compensation, measured in terms of apples, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. What is the dollar value of this WTA number of apples, using the prevailing price of apples? (Again, you do not need to simplify expressions that are purely numerical.)
(h) On the graph you drew above, indicate this consumer's "Willingness to Accept" compensation, measured in terms of bananas, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. This is equal to this consumer's equivalent variation, measured in terms of bananas.
(i) Numerically calculate this consumer's Willingness to Accept compensation, measured in terms of bananas, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. To do this, you need to use the fact that the consumer's utility function is $u=a b^{2}$, and you need to use the coordinates in the (apples, bananas) plane of the initial and final consumption points.
You do not need to simplify expressions that are purely numerical.
(j) In the previous part of this problem, you calculated this consumer's Willingness to Accept compensation, measured in terms of bananas, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. What is the dollar value of this WTA number of bananas, using the prevailing price of bananas? (Again, you do not need to simplify expressions that are purely numerical.)

## Answer to Summer 2022 Qualifying Exam, Section 3 Question 2

The answers of parts (a)-(d) are the same as for Summer 2022 Qualifying Exam Section 3 Question 1.
(e) See the $W T A_{a}$ of Figure 1. This answers the question "if the consumer were not able to move to the new point $(4,4)$, how many apples would he require in compensation," because if he is given this many apples, his utility would be the same, $U^{\prime}$, as it would have been if he had been able to move to the new point.
(Optional: The question says this is equivalent variation because it assumes the consumer is not allowed to move to the new point.)
(f) Starting from the old point $(3,3)$, he would need to be given apples up to the point where his utility after being given those apples, $u\left(3+W T A_{a}, 3\right)$, was equal to the utility he would have if he had been allowed to move to the new point, $u(4,4)$. So we have

$$
\begin{aligned}
u\left(3+W T A_{a}, 3\right) & =u(4,4) \\
\left(3+W T A_{a}\right) \cdot 3^{2} & =4 \cdot 4^{2} \\
64 / 9 & =3+W T A_{a} \\
W T A_{a} & =64 / 9-3=37 / 9=4 \frac{1}{9} \approx 4.11
\end{aligned}
$$

Throughout this problem, you were told that you did not have to simplify purely numerical expressions.
(g) It is $p_{a}$ times $W T A_{a}$, so $1 / 3$ times $37 / 9$, which is $\$ 37 / 27 \approx \$ 1.37>$ $\$ 1$. The "greater than $\$ 1$ " part is optional; it is interesting that "willingness to accept," when measured in apples, is greater than one dollar's worth of apples.
(h) See the $W T A_{b}$ of Figure 1. This answers the question "if the consumer were not able to move to the new point $(4,4)$, how many bananas would he require in compensation," because if he is given this many bananas, his utility would be the same, $U^{\prime}$, as it would have been if he had been able to move to the new point.
(Optional: The question says this is equivalent variation because it assumes the consumer is not allowed to move to the new point.)
(i) Starting from the old point $(3,3)$, he would need to be given bananas up to the point where his utility after being given those bananas,

For Figure 1 see the previous question.
$u\left(3,3+W T A_{b}\right)$, was equal to the utility he would have if he had been allowed to move to the new point, $u(4,4)$. So we have

$$
\begin{aligned}
u\left(3,3+W T A_{b}\right) & =u(4,4) \\
3 \cdot\left(3+W T A_{b}\right)^{2} & =4 \cdot 4^{2} \\
\left(3+W T A_{b}\right)^{2} & =4^{3} / 3 \\
3+W T A_{b} & =4 \sqrt{4 / 3}=8 / \sqrt{3} \\
W T A_{b} & =8 / \sqrt{3}-3=(8 \sqrt{3}-9) / 3 \approx 1.62 .
\end{aligned}
$$

(j) It is $p_{b}$ times $W T A_{b}$, so $2 / 3$ times $(8 \sqrt{3}-9) / 3$, which is $2(8 \sqrt{3}-$ $9) / 9 \approx \$ 1.08>\$ 1$. The "greater than $\$ 1$ " part is optional; it is interesting that "willingness to accept," when measured in apples, is greater than one dollar's worth of apples.
Also optional: the dollar value of "WTA $a_{a}$ for $\$ 1$ in extra income" is not the same as the dollar value of " $W T A_{b}$ for $\$ 1$ in extra income."

Optional: Note that in Figure 1, the Compensating Variation measures, $W A T P_{a}$ and $W A T P_{b}$, are measured from the new point $(4,4)$, whereas the Equivalent Variation measures, $W T A_{a}$ and $W T A_{b}$, are measured from the old point, $(3,3)$. We can summarize the answers to this question and the previous question as:

- The change in income is $\$ 1$, which could buy, among other possibilities, " 3 more apples and no more bananas," or "no more apples and 1.5 more bananas," or " 1 more apple and 1 more banana." The consumer in this problem does the last of these.
- $W A T P_{a} \approx 2.31$ apples, worth approximately $\$ 0.77$
- $W A T P_{b} \approx 1.40$ bananas, worth approximately $\$ 0.93$
- $W T A_{a} \approx 4.11$ apples, worth approximately $\$ 1.37$
- $W T A_{b} \approx 1.62$ bananas, worth approximately $\$ 1.08$.

These results are related to the "lump sum" principle taught in Intermediate Microeconomics, which is the superiority of lump sum taxes or subsidies over taxes or subsidies on just one good. In this problem, receiving \$1 worth of apples is worth less than receiving $\$ 1$ in cash, so if you received the cash (compensating variation), you would be WATP less than $\$ 1$ worth of apples in return. The same is true for bananas. On the other hand, if you did not receive the $\$ 1$ in cash (equivalent variation), you would need more than $\$ 1$ worth of apples or bananas (WTA) to compensate you for not receiving the $\$ 1$ cash.

2020 Qualifying Exam Sec. 3 Qu. 1 [Open-book exam due to the pandemic]

1. [16 points] Suppose a consumer consumes two goods, $x$ and $y$, has income $m$, and has the "Constant Elasticity of Substitution" utility function $u(x, y)=\left(x^{1 / 2}+y^{1 / 2}\right)^{2}$. Suppose

- the price of $y$, denoted $p_{y}$, is always equal to one.

You may use without proof the facts that Varian's book states on its p. 112, namely that:

- if a consumer has the CES utility function $u\left(x_{1}, x_{2}\right)=\left(x_{1}^{\rho}+x_{2}^{\rho}\right)^{1 / \rho}$
- and if we define $r=\frac{\rho}{\rho-1}$
- then the consumer's expenditure function is $e(\mathbf{p}, u)=\left(p_{1}^{r}+\right.$ $\left.p_{2}^{r}\right)^{1 / r} u$,
- their indirect utility function is $v(\mathbf{p}, m)=\left(p_{1}^{r}+p_{2}^{r}\right)^{-1 / r} m$,
- their demand for $x_{1}$ is $x_{1}(\mathbf{p}, m)=\frac{m p_{1}^{r-1}}{p_{1}^{r}+p_{2}^{r}}$,
- and their money metric utility function is $\mu(\mathbf{p} ; \mathbf{q}, m)=\left(p_{1}^{r}+\right.$ $\left.p_{2}^{r}\right)^{1 / r}\left(q_{1}^{r}+q_{2}^{r}\right)^{-1 / r} m$.

On p. 161 of Varian's book, upon defining

$$
\mu(\mathbf{q} ; \mathbf{p}, m)=e(\mathbf{q}, v(\mathbf{p}, m)),
$$

Varian gives the equivalent and compensating variation as, respectively,

$$
\begin{aligned}
& E V=\mu\left(\mathbf{p}^{0} ; \mathbf{p}^{\prime}, m^{\prime}\right)-m^{0} \quad \text { and } \\
& C V=m^{\prime}-\mu\left(\mathbf{p}^{\prime} ; \mathbf{p}^{0}, m^{0}\right) .
\end{aligned}
$$

(a) The consumer's "expenditure on $x$ " is the price of $x$ times the quantity of $x$ which he buys, in other words, $x p_{x}$. Assuming that $p_{x}=3$, find an expression for this consumer's expenditures on $x$. (This expression will depend on $m$.)
(b) Assuming that $p_{x}=3$, find an expression for this person's consumer surplus generated from $x$. (This expression will depend on $m$.) You may use without proof the following result:

$$
\int \frac{p^{-2}}{p^{-1}+1} d p=\int \frac{d p}{p+p^{2}}=\int \frac{1+p-p}{p(1+p)} d p
$$

$$
\begin{aligned}
& =\int\left[\frac{1+p}{p(1+p)}-\frac{p}{p(1+p)}\right] d p=\int\left(\frac{1}{p}-\frac{1}{1+p}\right) d p \\
& =\ln p-\ln (1+p)
\end{aligned}
$$

(c) Express $E V$ and $C V$ in terms of the expenditure function [better: in terms of income and the old and new prices] and describe the relationship between $E V$ and $C V$, on the one hand, and "willingness to pay" ("WTP") and "willingness to accept" ("WTA"), on the other hand.
(d) Find this consumer's WTP for a decrease in the price of $x$ from infinity to 3 . Assume as before that $p_{y}=1$. (Your expression for WTP will depend on $m$.)
(e) Find this consumer's WTA for an increase in the price of $x$ from 3 to infinity. Assume as before that $p_{y}=1$. (Your expression for WTA will depend on $m$.)
(f) Sketch a graph showing how this person's expenditures on $x$, consumer surplus generated from $x$, WTP, and WTA all depend on $m$. Use your results from (a), (b), (d), and (e) to do this.

Answer to Summer 2020 Qualifying Exam, Section 3 Question 1

$$
u(x, y)=\left(x^{1 / 2}+y^{1 / 2}\right)^{2} \quad p_{y}=1
$$

a) The translation between Varian's notation and the notation of this 3 problem is

| $\frac{\text { this problem }}{x}$ | $\frac{V_{\text {arian }}}{x}$ |
| :---: | :---: |
| $y$ | $x_{2}$ |
| $\frac{1}{2}$ | $\rho$ |
| -1 | $r=\frac{\rho}{\rho-1}=\frac{1 / 2}{1 / 2-1}=\frac{1 / 2}{-1 / 2}=-1$ |
| $P_{x}$ | $P_{1}$ |
| $P_{y}$ | $P_{2}$ |
| $m$ | $m$ |

So Varian's $x_{1}=\frac{m p_{1}^{r-1}}{p_{1}^{r}+p_{2}^{r}}$ becomes $x=\frac{m p_{x}^{-1-1}}{p_{x}^{-1}+p_{y}^{-1}}=\frac{m p_{x}^{-2}}{p_{x}^{-1}+1}=\frac{m}{p_{x}+p_{x}^{2}}$.
Expenditure on $x$ is $x p_{x}=\frac{m p_{x}}{p_{x}+p_{x}^{2}}=\frac{m}{1+p_{x}}$.
Expenditure on $x$ with $p_{x}=3$ is $\frac{m}{1+3}=\frac{m}{4}$.
b) Consumer surplus is this area:


This is $\int_{P_{x_{0}}}^{\infty} D(p) d p$. Here $P_{x_{0}}=3$ and $D(p)=\frac{m}{P_{x}+P_{x}^{2}}$ so consumer

$$
\begin{aligned}
\text { surplus is } \int_{3}^{\infty} \frac{m d p}{P_{x}+P_{x}^{2}} & =\left.m[\ln p-\ln (p+1)]\right|_{3} ^{\infty} \\
\text { from the formula given } & =\left.m \ln \frac{p}{p+1}\right|_{3} ^{\infty}
\end{aligned}=m\left(\ln 1-\ln \frac{3}{4}\right) .
$$ in the question

c)

$$
\begin{aligned}
& E V=\mu\left(p_{\sim}^{0} ;{\underset{\sim}{p}}^{1}, m^{1}\right)-m^{0} \\
& C V=m^{\prime}-\mu\left({\underset{\sim}{p}}^{\prime} ;{\underset{\sim}{p}}^{0}, m^{0}\right)
\end{aligned}
$$

with Variant's $\mu(\underset{\sim}{p}=\underset{\sim}{q}, m)=\left(p_{1}^{r}+p_{2}^{r}\right)^{1 / r}\left(q_{1}^{r}+q_{2}^{r}\right)^{-1 / r} m$ imply (moving the "o"superscripts down to subscinpts to aroid the exponents):

$$
\begin{aligned}
& E V=\left(p_{01}^{r}+p_{02}^{r}\right)^{1 / r}\left(p_{1}^{\prime r}+p_{2}^{\prime r}\right)^{-1 / r} m^{\prime}-m_{0} \\
& C V=m^{\prime}-\left(p_{1}^{\prime r}+p_{2}^{\prime r}\right)^{1 / r}\left(p_{01}^{r}+p_{02}^{r}\right)^{-1 / r} m_{0}
\end{aligned}
$$

So in our problem

$$
\begin{aligned}
& E V=\left(p_{x}^{-1}+p_{y_{0}}^{-1}\right)^{-1}\left(p_{x}^{(-1)}+p_{y}^{(-1)}\right)^{\frac{-1}{-1}} m-m \\
& =\left(p_{x_{0}}^{-1}+1\right)^{-1}\left(p_{x}^{\prime(-1)}+1\right) m-m \\
& =m \frac{1+1 / p_{x}^{\prime}}{1+1 / p_{x_{0}}}-m=m \frac{1+\frac{1}{p_{x}^{\prime}}-1-\frac{1}{p_{x_{0}}}}{1+1 / p_{x_{0}}} \\
& =\left(\frac{1}{p_{x}^{\prime}}-\frac{1}{p_{x_{0}}}\right) \frac{m}{1+1 / p_{x 0}} \text { and } \\
& C V=m-\left(p_{x}^{\prime(-1)}+p_{y}^{\prime(-1)}\right)^{\frac{1}{-1}}\left(p_{x 0}^{-1}+p_{y_{0}}^{-1}\right)^{\frac{-1}{-1}} m \\
& =m-\left(p_{x}^{(-1)}+1\right)^{-1}\left(p_{x_{0}}^{-1}+1\right) m \quad \leftarrow \text { oxtoleave } m \text { this } \\
& =m\left[1-\frac{1+1 / p_{x_{0}}}{1+1 / p_{x}^{\prime}}\right]=\frac{m}{1+1 / p_{x}^{\prime}}\left[1+\frac{1}{p_{x}^{\prime}}-1-\frac{1}{p_{x}}\right] \\
& =\left(\frac{1}{p_{x}^{\prime}}-\frac{1}{p_{x_{0}}}\right) \frac{m}{1+1 / p_{x}^{\prime}} . \\
& \leftarrow \text { or to leave in third } \\
& \text { unsmpatired form } \\
& \text { unsimplitied form }
\end{aligned}
$$

On this page, this level of detail is not needed; the main point is being able to get from WTP/WTA to EV/CV or vice versa.

Also,

$$
\begin{aligned}
& E V=\mu\left({\underset{\sim}{p}}^{0} ;{\underset{\sim}{p}}^{\prime}, m^{\prime}\right)-m^{0}=e\left({\underset{\sim}{p}}^{0}, v\left(\underset{\sim}{p^{\prime}}, m^{\prime}\right)\right)-m^{0} \\
& C V=m^{\prime}-\mu\left({\underset{\sim}{p}}^{\prime}{\underset{\sim}{p}}^{0}, m^{0}\right)=m^{\prime}-e\left({\underset{\sim}{p}}^{\prime}, v\left({\underset{\sim}{p}}^{0}, m^{0}\right)\right) .
\end{aligned}
$$

Thus EV uses base year prizes, asking:
if we did not do this, what would you have to $\left\{\begin{array}{l}\text { be paid (if "this" is a gain) } \\ \text { pay (if "thus" B a loss) }\end{array}\right\}$
to make you as well off as if we had done it?
CV uses new year prices, asking:

$$
\text { if we did this, what would you have to }\left\{\begin{array}{l}
\text { pay (if "this" Ba gain) } \\
\text { be paid (if 'this" Ba loss) }
\end{array}\right\}
$$

to make you as well off as if we had not done if?

| Therefore: | Contemplated <br> Gain | Contemplated <br> Loss |
| ---: | :--- | :--- |
| if we didn't dothis: EV | "be paid": WTA | "pay": WTP |
| if we did this: CV | "Pay": WTP | "be paid": WTA |

WTP: witloyness to pay
WTA: willingness to accept
(A betterterm than WTP
Conclusion:
In both parts (b) and (c), we are asking
what happens if a policy is undertaken, so we will use $(V$ not $E V$. would be "willingness and ability to pay," but that would be nonstandard.) $\uparrow \uparrow$ (optional)
d)

$$
\begin{aligned}
& \text { old } p_{x}: \infty \\
& \text { New } p_{x}: 3
\end{aligned}
$$

This is a gain, hence the question asks WT P and CV.
$\tau_{\text {which actually happens, not which }}$
we retrain from carrying out
Fuomabone,

$$
\begin{aligned}
C V & =\left(\frac{1}{p_{x}^{\prime}}-\frac{1}{p_{x_{0}}}\right) \frac{m}{1+1 / p_{x}^{\prime}}=\left(\frac{1}{3}-\frac{1}{\infty}\right) \frac{m}{1+\frac{1}{3}} \\
& =m \frac{1 / 3}{1+1 / 3}=m \frac{1 / 3}{4 / 3}=\frac{m}{4} .
\end{aligned}
$$

e)

$$
\begin{aligned}
& \text { old } p_{x}: 3 \\
& \text { New } p_{x}: \infty
\end{aligned}
$$

Thus is an actual loss (not a loss we netrain from carry ing out), hence the question asks WT $A$ and $C V$.
From above,

$$
\begin{aligned}
C V & =\left(\frac{1}{p_{x}^{\prime}}-\frac{1}{P_{x_{0}}}\right) \frac{m}{1+1 / p_{x}^{\prime}}=\left(\frac{1}{\infty}-\frac{1}{3}\right) \frac{m}{1+\frac{1}{\infty}} \\
& =\frac{-1}{3} \frac{m}{1}=\frac{-m}{3} .
\end{aligned}
$$

- of you graph
f)
$W T A, W T P, C S$

optional
These are three ways of measuring the consumer's value of a " $p_{x}=3$ " policy. If society adopts this policy valuation, richer people's value has greater weight than poorer people's value.

THIS PAGE COMPLETELY OPTIONAL!

To obtain the figure on the next page, which illustrates these results when $m={ }^{*} 100$ and $P_{x}=3$, obtain the Marshallian demand curve from part (a) above:

$$
x=\frac{m}{p_{x}+p_{x}^{2}}=\frac{100}{3+9}=\frac{100}{12}=\frac{25}{3} .
$$

Frompart (b), CS $\approx m \ln \frac{4}{3} \approx 29$.
To obtain the Hicksian demand curves,

$$
\begin{aligned}
h_{x} & =\frac{\partial e}{\partial p_{x}}=\frac{\partial}{\partial p_{x}}\left(p_{x}^{-1}+1\right)^{-1} u=-\left(p_{x}^{-1}+1\right)^{-2}\left(-1 p_{x}^{-2}\right) u \\
& =\frac{+u}{\left[\left(p_{x}^{-1}+1\right) p_{x}\right]^{2}}=\frac{u}{\left(1+p_{x}\right)^{2}} .
\end{aligned}
$$

- u when $p_{x}=3$ : From the Marshallian demand curve, $x=\frac{25}{3}$, and expenditures on $x$ are $P_{x} \cdot x=3 \cdot \frac{25}{3}=25$. Since $m=100$, expenditures on $y$ must be $100-25=$ 75. Since $p_{y}=1$, this means that $y=75$. Then $u=\left(\sqrt{\frac{25}{3}}+\sqrt{75}\right)^{2}=$

$$
\begin{aligned}
& \left(\frac{5}{\sqrt{3}}+\sqrt{25 \cdot 3}\right)^{2}=\left(\frac{5}{\sqrt{3}}+5 \sqrt{3}\right)^{2}=\frac{25}{3}+2 \cdot \frac{5}{\sqrt{3}} \cdot 5 \sqrt{3}+25 \cdot 3=\frac{25}{3}+50+75 \\
& =8 \frac{1}{3}+125=133 \frac{1}{3} . \text { Hence } h_{x}=\frac{133 \frac{1}{3}}{\left(1+p_{x}\right)^{2}} .
\end{aligned}
$$

- u when $P_{x}=\infty$. From the Marshallian demand curve, $x=0$, so expenditures on $x$ are zero, so the entire income of $m=10010$ spent on $y$, meaning, since $p_{y}=1$, that $y=100$. Then $u=(\sqrt{0}+\sqrt{100})^{2}=100$. Hence $h_{x}=\frac{100}{\left(1+P_{x}\right)^{2}}$.


## OPTIONAL PAGE



Figure 6. Demand curves for an income of $m=100$ and a utility function of $u(x, y)=\left(x^{1 / 2}+y^{1 / 2}\right)^{2}$ when $p_{y}=1$. Dotted curve: the Marshallian demand curve, $x=p_{x}^{-2} /\left(p_{x}^{-1}+1\right)=1 /\left(p_{x}+p_{x}^{2}\right)$. Consumer surplus when $p_{x}=3$ is the area left of this curve and above the line $p_{x}=3$; it is $m \ln (4 / 3) \approx \$ 29$. Solid curves: Hicksian demand curves, $u /\left(1+p_{x}\right)^{2}$. Right-most solid curve: Hicksian demand curve with utility fixed at its level when $p_{x}=3$, therefore $u(25 / 3,75)=\left((25 / 3)^{1 / 2}+75^{1 / 2}\right)^{2}=133 \frac{1}{3}$. WTA when $p_{x}=3$ is the area left of this curve and above the line $p_{x}=3$; it is $m / 3=\$ 33 \frac{1}{3}$. Left-most solid curve: Hicksian demand curve with utility fixed at its level when $p_{x}=\infty$, therefore $u(0,100)=\left(0^{1 / 2}+100^{1 / 2}\right)^{2}=100$. WTP when $p_{x}=3$ is the area left of this curve and above the line $p_{x}=3$; it is $m / 4=\$ 25$. When $p_{x}=3$, expenditure is $3 \cdot 25 / 3=\$ 25$. The value of " $p_{x}=3$ " or " $x=25 / 3$ " is the $\$ 25$ expenditure plus the measure of the surplus (either WTA or WTP or CS).
2. [16 points] Suppose a consumer consumes two goods, $x$ and $y$, has income $m$, and has the "Constant Elasticity of Substitution" utility function $u(x, y)=\left(x^{1 / 2}+y^{1 / 2}\right)^{2}$. Suppose

- the price of $y$, denoted $p_{y}$, is always equal to one;
- and the consumer's income $m=100$.

You may use without proof the facts that Varian's book states on its p. 112, namely that:

- if a consumer has the CES utility function $u\left(x_{1}, x_{2}\right)=\left(x_{1}^{\rho}+x_{2}^{\rho}\right)^{1 / \rho}$
- and if we define $r=\frac{\rho}{\rho-1}$
- then the consumer's expenditure function is $e(\mathbf{p}, u)=\left(p_{1}^{r}+\right.$ $\left.p_{2}^{r}\right)^{1 / r} u$,
- their indirect utility function is $v(\mathbf{p}, m)=\left(p_{1}^{r}+p_{2}^{r}\right)^{-1 / r} m$,
- their demand curve for $x_{1}$ is $x_{1}(\mathbf{p}, m)=\frac{m p_{1}^{r-1}}{p_{1}^{r}+p_{2}^{r}}$,
- and their money metric utility function is $\mu(\mathbf{p} ; \mathbf{q}, m)=\left(p_{1}^{r}+\right.$ $\left.p_{2}^{r}\right)^{1 / r}\left(q_{1}^{r}+q_{2}^{r}\right)^{-1 / r} m$.

On p. 161 of Varian's book, upon defining

$$
\mu(\mathbf{q} ; \mathbf{p}, m)=e(\mathbf{q}, v(\mathbf{p}, m)),
$$

Varian gives the equivalent and compensating variation as, respectively,

$$
\begin{aligned}
& E V=\mu\left(\mathbf{p}^{0} ; \mathbf{p}^{\prime}, m^{\prime}\right)-m^{0} \quad \text { and } \\
& C V=m^{\prime}-\mu\left(\mathbf{p}^{\prime} ; \mathbf{p}^{0}, m^{0}\right) .
\end{aligned}
$$

If there are other results which Varian proves which you want to use, simply cite the result and its page number; since this is an open-book exam, I think it's pointless to ask you to copy a proof from Varian's book straight onto your exam paper.
Please look at Figure 3.


Figure 3. Demand curves for an income of $m=100$ and a utility function of $u(x, y)=\left(x^{1 / 2}+y^{1 / 2}\right)^{2}$ when $p_{y}=1$.
(a) The " $W T A=33 \frac{1}{3}$ " label of Figure 3 denotes willingness to accept the price of $x$ increasing from three to infinity, and $33 \frac{1}{3}$ is the area to the left of the right-most curve and above the line $p_{x}=3$. Show that the equation of this right-most curve is

$$
\frac{u}{\left(1+p_{x}\right)^{2}}
$$

where

$$
u=\left(\sqrt{\frac{25}{3}}+\sqrt{75}\right)^{2}=133 \frac{1}{3} .
$$

(b) The " $W T P=25$ " label of Figure 3 denotes willingness to pay in return for the price of $x$ decreasing from infinity to three, and 25 is the area to the left of the left-most curve and above the line $p_{x}=3$. Show that the equation of this left-most curve is

$$
\frac{u}{\left(1+p_{x}\right)^{2}}
$$

where

$$
u=(\sqrt{0}+\sqrt{100})^{2}=100
$$

(c) The " $C S \approx 29$ " label of Figure 3 denotes consumer surplus, and $100 * \ln (4 / 3) \approx 29$ is the area to the left of the dotted curve and above the line $p_{x}=3$. What is the equation of this dotted curve? (You may express this either as a function of $p_{x}$ or as a function of $x$.)
(d) Consider the following two situations:
i. The situation depicted in Figure 3, with $p_{x}=3$ and $p_{y}=$ 1 and $m=100$ and consumer surplus from consumption of $x$ approximately equal to 29 and consumer surplus from consumption of $y$ (not illustrated); or
ii. A situation in which the consumer faces a price-discriminating seller who charges the consumer $\$ 25+\$ 29$ for consuming $x=25 / 3$ (which the consumer does consume and does pay) and the consumer faces a uniform price of $p_{y}=1$ for $y$ and the consumer is given an income of $\$ 100$ plus $\$ 34$.

In which situation does this consumer have the larger consumer surplus (considering both goods $x$ and $y$ )? There is no need to provide a numerically-calculated answer; one arrived at just by logical reasoning is what is being asked for. (This question may be too easy but it becomes interesting in light of the next question.)
(e) In which of the situations described in part (d) does this consumer have the larger utility? There is no need to provide a numerically-calculated answer; one arrived at just by logical reasoning is what is being asked for.

Answer to Summer 2020 Quality,

First we need to connect EV and CV with UTP and WTA. From the equations given in the problem, $E V=\mu\left(p^{0}: \sim_{\sim}^{\prime}, m^{\prime}\right)-m^{0}=e\left(p^{0}, v\left(p^{\prime}, m^{\prime}\right)\right)-m^{0}$

$$
C V=m^{\prime}-\mu\left(p^{\prime} ; p^{0}, m^{0}\right)=m^{\prime}-e\left(p^{\prime}, v\left(p^{0}, m^{0}\right)\right) .
$$

Thus EV uses base year prices, asking:
if we che not do this, what would you have to $\left\{\begin{array}{l}\text { be paid (if "tho") is a gain } \\ \text { pay (if "this") is a loss }\end{array}\right\}$ to make you as well off as if we had dore it?

CV uses new year prices, asking:
if we did this, what would you have to $\left\{\begin{array}{l}\text { pay (if "this "is a gain) } \\ \text { bepaid lift "this" s a loss) }\end{array}\right\}$ to make you as well off as if we had not done it?

Therefore:

| Contemplated | Con templated <br> Gain |
| :--- | :--- | :---: |
| if we didn't do this : EV | "be paid": WTA "pay": WTP |
| if we did this : CV | "pay": WTP "bepaid": WTA |

For the, wo ne, equation (10.2) on p. 167 of Varian five $E V=\int_{p,}^{p^{0}} h\left(p, u^{\prime}\right) d p$ and $C V=\int_{p,}^{p} \hbar\left(p, u^{0}\right) d p$. It follows that WTP and WTA are integrals of thicksian dem and curves. WTP and WT A would be areas under Hickian demand curves if those curveswere graphed with $p$ on the horizontal axis, but
with p graphed on the vertical axis as is traditional, WTP and WT A would be areas to the left of a Hick sian demand curve.
a) The problem gives the expenditure function as

$$
e(p, u)=\left(p_{1}^{r}+p_{2}^{r}\right)^{1 / r} u
$$

where $r=\frac{l}{e^{-1}}$ and $u=\left(x_{1}^{l}+x_{2}^{e}\right)^{1 / l}$. In our problem, $u(x y)-\left(x^{1 / 2}+y^{1 / 2}\right)^{2}$, so our $x$ and $y$ correspond to Varian's $x$, and $x_{2}$, our $\frac{1}{2}$ to Varian's $l$,

$$
\begin{aligned}
& \text { and } r=\frac{l}{\rho-1}= \frac{1 / 2}{1 / 2-1}=\frac{1 / 2}{-1 / 2}=-1 . S_{0} \\
& e(p, u)=\left(p_{x}^{-1}+p_{y}^{-1}\right)^{-1} u
\end{aligned}
$$

and the Hicksian demand curve

$$
\begin{aligned}
{\underset{\sim}{i}}^{h_{i}}(p, u) & =\frac{\partial \tilde{\sim}^{e(p, u)}}{\partial p_{i}} \\
h_{x}\left(p_{\sim}, u\right) & =\frac{\partial e}{\partial p_{x}}=-\left(p_{x}^{-1}+p_{y}^{-1}\right)^{-2} \cdot\left(-p_{x}^{-2}\right) u \\
& =\frac{u}{\left(p_{x}^{1}+p_{y}^{-1}\right)^{2} p_{x}^{2}}=\frac{u}{\left[\left(p_{x}^{-1}+p_{y}^{-1}\right) p_{x}\right]^{2}} \\
& =\frac{u}{\left(1+p_{x} / p_{y}\right)^{2}}=\frac{u}{\left(1+p_{x}\right)^{2}} \sin c e p_{y}=1 .
\end{aligned}
$$

To find u, note that the curve related to WT A goes through the point $\left(p_{x}=3\right.$, $x=25 / 3$ ). So at this point the consumer spent $3 \cdot 25 / 3=525$ on $x$ i since $m=\$ 100$, the consumer must spend ${ }^{5} 100-25=75$ on $y$, which, at a
price of $\$ 1$, means 75 units of $y$. So

$$
\begin{aligned}
u & =\left(x^{1 / 2}+y^{1 / 2}\right)^{2}=\left((25 / 3)^{1 / 2}+(75)^{1 / 2}\right)^{2}=\left(\frac{5}{\sqrt{3}}+5 \sqrt{3}\right)^{2} \\
& =\frac{25}{3}+2 \cdot \frac{5}{\sqrt{3}} \cdot 5 \sqrt{3}+25 \cdot 3=\frac{25}{3}+50+75 \\
& =8 \frac{1}{3}+125=133 \frac{1}{3} .
\end{aligned}
$$

Hence the relevant Hicksian demand curve throng h $\left(P_{x}=3, x=25 / 3\right)$ is

$$
h_{x}=\frac{133 \frac{1}{3}}{\left(1+p_{x}\right)^{2}} . \quad\left(\text { Note } \cdot p_{x}=3 \Rightarrow h_{x}=25 / 3 .\right)
$$

b) As in $(a), h_{x}=\frac{u}{\left(1+p_{x}\right)^{2}}$, so we only need to find $u$. This core corresponds with an initial situation of $P_{x}=\infty$, so $x=0$, so the consumer spencls all his income (of $\$ 100$ ) on $y$, whose prize is $p_{y}=1$; thus the consumer buys $y=100$. Utility then is $\left(x^{1 / 2}+y^{1 / 2}\right)^{2}=\left(0^{1 / 2}+100^{1 / 2}\right)^{2}=100$ and the relevant Hicksicen demand curve is

$$
h_{x}=\frac{100}{\left(1+p_{x}\right)^{2}}
$$

c) Consumer suplus is the area to the left of the Marshaltion demand curve, which isfiven by the question as $x_{1}=\frac{m p_{1}^{r-1}}{p_{1}^{r}+p_{2}^{r}}$, so in our problem it is

$$
\chi=\frac{m P_{x}^{-1-1}}{P_{x}^{-1}+P_{y}^{-1}}=\frac{100 \cdot P_{x}^{-2}}{P_{x}^{-1}+1}=\frac{100}{\left(P_{x}^{-1}+1\right) p_{x}^{2}}=\frac{100}{P_{x}+P_{x}^{2}}
$$

(Note: $P_{x}=3 \Rightarrow x=25 / 3$.)
d) In (i) the consumer has $\$ 29$ of consumer surplus from food $x$. In (ii) the consumer has no consumer surplus from $x$ be carse it all foes to the price-discriminating seller. However the consumer's income is increased enough ( $\$ 34>$ WTP or WTA or CS) to make the consumer a little bit better off in (ii); this must mean his consumption of y goes up a bit, since his consumption of $x$ is still 25/3. However, $y$ does not increase much ( $\$ 34$ is not much more th an WTP or WTA or CS), so any small increase in consumer surplus from will be overshadowed by the $\$ 29$ Loss in consumer surplus from $x$. So the consumer surplusislarger in (i).
e) In (ii), the consumer is over-compensated for having to face price discrimination, so if $x$ is still $25 / 3$, purchases of $y$ will go up; soutility is higher in (ii).

Completely Optional Remarks

Thequestion never asks you to conform that CS actually is approximately 29 , or that WTP actually is 25 , or that $W T A$ actually is $33 \frac{1}{3}$. There are two ways to do that.

Method 1. Use the rear its of the previous problem (not this problem). In that problem, with the same basic setup we found that with $p_{x}=3$ :

$$
\begin{array}{ll}
\text { expenditure }=m / 4 & \frac{\text { so if }}{25}=100 \longleftarrow \text { as in this problem } \\
C S=m \ln \frac{4}{3} & 100 \ln \frac{4}{3} \approx 29 \\
W T P=m / 4 & 25 \\
|W T A|=m / 3 & 33 \frac{1}{3}
\end{array}
$$

Method 2.
CS: the Marshallian demand woe is, from $p$ art (c), $\frac{100}{P_{x}+P_{x}^{2}} . S_{0}$

$$
C S=\int_{3}^{\infty} \frac{100}{P_{x}+P_{x}^{2}} d P_{x} \text {. Thusissolved in part (b) of the previous question not }
$$ this question).

WTP, WTA: The area "under" a Hicksian demand curve (to its left if price is cm the vertical axis) is, using parts (a) and (b),

$$
\int_{3}^{\infty} \frac{\bar{u}}{\left(1+p_{x}\right)^{2}} d p_{x}=\left.\frac{-\bar{u}}{1+p_{x}}\right|_{3} ^{\infty}=0+\frac{\bar{u}}{1+3}=\frac{\bar{u}}{4}
$$

In part (a), $\bar{u}=133 \frac{1}{3}$, so this is $133 \frac{1}{3} / 4=33 \frac{1}{3}$, the WTA.
In part (b), $\bar{u}=100$, so this is $100 / 4=25$, the $\omega T$ P.
Expenditure: As with C.S., the Marshallian demand cove is $\frac{100}{p_{x}+p_{x}^{2}}$. At $p_{x}=3$
this is $\frac{100}{3+9}=\frac{100}{12}=\frac{25}{3}=x$; then expenditure $x p_{x}=\frac{25}{3} \cdot 3=25$.

## OPTIONAL PAGE



Figure 6. Demand curves for an income of $m=100$ and a utility function of $u(x, y)=\left(x^{1 / 2}+y^{1 / 2}\right)^{2}$ when $p_{y}=1$. Dotted curve: the Marshallian demand curve, $x=p_{x}^{-2} /\left(p_{x}^{-1}+1\right)=1 /\left(p_{x}+p_{x}^{2}\right)$. Consumer surplus when $p_{x}=3$ is the area left of this curve and above the line $p_{x}=3$; it is $m \ln (4 / 3) \approx \$ 29$. Solid curves: Hicksian demand curves, $u /\left(1+p_{x}\right)^{2}$. Right-most solid curve: Hicksian demand curve with utility fixed at its level when $p_{x}=3$, therefore $u(25 / 3,75)=\left((25 / 3)^{1 / 2}+75^{1 / 2}\right)^{2}=133 \frac{1}{3}$. WTA when $p_{x}=3$ is the area left of this curve and above the line $p_{x}=3$; it is $m / 3=\$ 33 \frac{1}{3}$. Left-most solid curve: Hicksian demand curve with utility fixed at its level when $p_{x}=\infty$, therefore $u(0,100)=\left(0^{1 / 2}+100^{1 / 2}\right)^{2}=100$. WTP when $p_{x}=3$ is the area left of this curve and above the line $p_{x}=3$; it is $m / 4=\$ 25$. When $p_{x}=3$, expenditure is $3 \cdot 25 / 3=\$ 25$. The value of " $p_{x}=3$ " or " $x=25 / 3$ " is the $\$ 25$ expenditure plus the measure of the surplus (either WTA or WTP or CS).

## 2019 Qualifying Exam Sec. 3 Qu. 1

## 1. [16 points]

[Completely optional introduction: This is the beginning and the middle but not the end of a demonstration that George Stigler's "Coase Theorem" is false in the general case of goods having arbitrary income effects. (This would please Nobel Laureate Ronald Coase but it profoundly challenges followers of Stigler.)]
(a) Suppose a consumer's welfare depends on the number of apples $a$ which he consumes and on the amount of clean air in his environment. There is a polluting firm in the consumer's environment and the amount of air pollution it emits is proportional to the level of its output $Q$. For some fixed level of output $\bar{Q}>0$, argue that

$$
u(a, Q)=a \cdot(\bar{Q}-Q)
$$

is a reasonable specification for this consumer's utility function.
(b) Suppose this consumer sets out one day with $m$ dollars to visit the marketplace and buy some apples. Before he gets to the marketplace, he encounters the owner of the polluting firm. He may strike up a conversation with this owner in the hopes of affecting how much the firm pollutes. Perhaps he and the firm owner exchange money for a change in $Q$. Let $m_{a}$ denote the amount of money the consumer has when he takes leave of the firm owner and proceeds to the marketplace, at which time the amount of $Q$, and therefore air pollution, is irrevocably fixed (it will never change again).
Show that his utility at this point is destined to be

$$
\frac{m_{a}}{p_{a}}(\bar{Q}-Q)
$$

where $p_{a}$ is the price of apples.
(c) Suppose that in this country, firms have the right to emit pollution at will. (One could say that the firm has the "property right" to pollute.) Suppose that in the absence of any interaction or bargaining between the firm and the consumer,
the firm sees fit to produce $\bar{Q} / 2$ units of output.
Show that with that level of output, the (indirect) utility of the customer in this initial situation would be

$$
v_{0}=\frac{m \bar{Q}}{2 p_{a}} .
$$

(d) Upon meeting the firm owner, the consumer contemplates offering the firm owner money in return for a reduction of $Q$. If the consumer offered the firm owner $T$ dollars and in return the firm owner reduced output to $Q$, show that the consumer would, after making the bargain and then buying apples, have a utility level of

$$
v^{\prime}=\frac{m-T}{p_{a}}(\bar{Q}-Q)
$$

(e) Suppose that, for a given $Q$, the consumer is indifferent between paying $T(Q)$ in return for the firm producing only $Q$, on the one hand, and paying nothing and having the firm produce $\bar{Q} / 2$, on the other hand. Find $T$ as a function of $Q$.
Hint: I get

$$
T=m \frac{\bar{Q}-2 Q}{2 \bar{Q}-2 Q}>0 \quad \text { for } Q<\bar{Q} / 2
$$

(f) Show that

$$
\begin{aligned}
\frac{d T}{d Q} & =\frac{-m \bar{Q}}{2(Q-\bar{Q})^{2}}<0 \quad \text { for } Q<\bar{Q} / 2, \text { and that } \\
\frac{d^{2} T}{d Q^{2}} & =\frac{m \bar{Q}}{(Q-\bar{Q})^{3}}<0 \quad \text { for } Q<\bar{Q} / 2
\end{aligned}
$$

(g) Make a rough sketch of $T(Q)$, indicating the values of $T(0)$ and of $T(\bar{Q} / 2)$.
(h) If $E C$ denotes the "external cost" which pollution imposes on this consumer, argue that

$$
E C(Q)=T(0)-T(Q)
$$

(i) Show that the "marginal external cost"

$$
M E C=\frac{d E C}{d Q}=\frac{m \bar{Q}}{2(Q-\bar{Q})^{2}}>0
$$

(Prove the second equality.) Also show that

$$
\frac{d M E C}{d Q}=\frac{-m \bar{Q}}{(Q-\bar{Q})^{3}}=\frac{m \bar{Q}}{(\bar{Q}-Q)^{3}}>0
$$

(Prove at least one of the equalities and prove the inequality.)
(j) Now we contrast this situation to that under a different constitution in which consumers have the right to clean air and firms cannot pollute the air without obtaining permission from the consumer. (One could say that consumers have the "property right" to clean air.) Show that in the absence of any interaction or bargaining between the firm and the consumer, (indirect) utility of the customer in this initial situation would be

$$
v_{0}=\frac{m \bar{Q}}{p_{a}}
$$

(k) Upon meeting the firm owner, the consumer contemplates offering to allow the firm to increase output to $Q$ in return for the firm paying the consumer $\widehat{T}$ dollars. Show that the consumer would, after making the bargain and then buying apples, have a utility level of

$$
v^{\prime}=\frac{m+\widehat{T}}{p_{a}}(\bar{Q}-Q)
$$

(l) Suppose that, for a given $Q$, the consumer is indifferent between receiving $\widehat{T}(Q)$ in return for allowing the firm to increase its production to $Q$, on the one hand, and receiving nothing and making no bargain with the firm, on the other hand. Show that

$$
\widehat{T}=\frac{m Q}{\bar{Q}-Q}>0
$$

and show that

$$
\begin{aligned}
\frac{d \widehat{T}}{d Q} & =\frac{m \bar{Q}}{(\bar{Q}-Q)^{2}}>0 \quad \text { and that } \\
\frac{d^{2} \widehat{T}}{d Q^{2}} & =\frac{2 m \bar{Q}}{(\bar{Q}-Q)^{3}}>0 .
\end{aligned}
$$

(m) In this situation argue that external cost $\widehat{E C}(Q)=\widehat{T}(Q)$.
(n) Show that

$$
\widehat{M E C}=\frac{d \widehat{M E C}}{d Q}=2 M E C \quad \text { and } \quad \frac{d \widehat{M E C}}{d Q}=2 \frac{d M E C}{d Q}
$$

Answer to 2019 Miro Qualifying Exam, Section 3 Qu. 1
a) If $u=a(\bar{Q}-Q)$ then as $Q \uparrow$, the air be comes dirtier and utility falls. Formally, $\frac{\partial u}{\partial Q}=\frac{\partial}{\partial Q}[a \bar{Q}-a Q]=-a<O$. Optional: $\partial u / \partial a$ is also positive; the formulation is just Cobb-Douglas utility in apples and clean air.
b) By the time has come to purchase apples, $Q$ is exogenously fixed. Let ma be the amount of money the consumer has left at that time. He millspand all of ma buying apples, so $a^{*}=\frac{m_{a}}{P_{a}}$ and his utility will be $\frac{m_{a}}{p_{a}}(\bar{Q}-Q)$.
c) In this situation, $Q_{0}=\frac{1}{2} \bar{Q}$ is given. If the consumer does not bargain with the polluter, his utility will be

$$
\begin{aligned}
v_{0} & =\frac{m}{p_{a}}\left(\bar{Q}-\frac{1}{2} \bar{Q}\right) \quad \text { using the notation for indirect utility; } \\
& =\frac{m \bar{Q}}{2 p_{a}} .
\end{aligned}
$$

d) If the consumer does bargain with the polluter, suppose he pays the polluter $T$ dollars ("T" for "transfer"), and in return the polluter reduces output to $Q$. then

$$
v^{\prime}=\frac{m-T}{P_{a}}(\bar{Q}-Q) \quad \text { since } m-T=m_{a} \text {. }
$$

e) The maximum T which the consumer would be withing to pay for a given $Q$ would satisfy the property that $v^{\prime}=v_{0}$, be cause if $v^{\prime}$ were any lower than $v_{0}$, the consumer would peter to stay at $v_{0}$. IIngame theory, this is called the "participation constraint" or the "indindual rationality constraint."]

So $v_{0}=v^{\prime} \Rightarrow \frac{m \bar{Q}}{2 \mathrm{~Pa}}=\frac{m-T}{\mathrm{~Pa}}(\bar{Q}-Q)$ and solving for $T$ as a function of $Q$,

$$
\begin{aligned}
\frac{m}{2} & =(m-T)\left(1-\frac{Q}{\bar{Q}}\right) \\
\frac{m / 2}{1-\frac{Q}{\bar{Q}}} & =m-T \\
T & =m-\frac{m / 2}{1-\frac{Q}{\bar{Q}}}
\end{aligned}=m-\frac{\frac{m}{2}}{1-\frac{Q}{\bar{Q}}} \cdot \frac{2 \bar{Q}}{2 \bar{Q}} .
$$

This is valid for reductions in output below its initial level of $\frac{1}{2} \bar{Q}$, that is, for $Q \leqslant \frac{1}{2} \bar{Q}$. Note that this makes both $T^{\prime}$ 's numerator and its denominator have identical signs, so $T>0$.
f) For eve-larger decreases in $Q$, we expect the consumer to be willing to spend more $T$, so we expect $d T / d Q<0$.

$$
\begin{aligned}
\frac{d T}{d Q} & =m\left[\frac{2}{2 Q-2 \bar{Q}}-\frac{2}{(2 Q-2 \bar{Q})^{2}}(2 Q-\bar{Q})\right] \\
& =m\left[\frac{1}{Q-\bar{Q}}-\frac{2(2 Q-\bar{Q})}{2(Q-\bar{Q}) \cdot 2(Q-\bar{Q})}\right]=m\left[\frac{1}{Q-\bar{Q}}-\frac{2 Q-\bar{Q}}{2(Q-\bar{Q})^{2}}\right] \\
& =m \frac{2(Q-\bar{Q})-2 Q+\bar{Q}}{2(Q-\bar{Q})^{2}}=m \frac{2 Q-2 \bar{Q}-2 Q+\bar{Q}}{2(Q-\bar{Q})^{2}}=\frac{-m \bar{Q}}{2(Q-\bar{Q})^{2}}<0 .
\end{aligned}
$$

Also

$$
\begin{aligned}
\frac{d^{2} T}{d Q^{2}} & =\frac{-m \bar{Q}}{2} \frac{d}{d Q}(Q-\bar{Q})^{-2}=\frac{-m \bar{Q}}{2}(-2)(Q-\bar{Q})^{-3}(1) \\
& =\frac{m \bar{Q}}{(Q-\bar{Q})^{3}} \text { which is sega tire because } Q \leqslant \frac{1}{2} \bar{Q}<\bar{Q} .
\end{aligned}
$$

g)

n) It is worthwhile for the consumer to spend $T$ to reduce output to $Q$ because output imposes "extunal costs" "EC" on the consumer. We have

$$
\begin{equation*}
E C(Q)=T(0)-T(Q) \tag{1}
\end{equation*}
$$

because when $Q=0$ we should have $E C(0)=0$, which $(1)$ satisfies, and when $Q=\bar{Q} / 2$ we should have $E C(\bar{Q} / 2)=T(0)$, the consume 's entire willingness - to -pay to reduce output from $\bar{Q} / 2$ to zero, and (1) gives this also because it jives $E C(\bar{Q} / 2)=T(0)-T(\bar{Q} / 2)=T(0)-0=T(0)$.
i) Nee marginal external cost

$$
\begin{gathered}
M E C=\frac{d E C}{d Q}=-\frac{d T}{d Q}=\frac{m \bar{Q}}{2(Q-\bar{Q})^{2}}>0 \text { and } \\
\frac{d M E C}{d Q}=-\frac{d^{2} T}{d Q^{2}}=\frac{-m \bar{Q}}{(Q-\bar{Q})^{3}}>0 \text { because } Q \leqslant \frac{1}{2} \bar{Q}<\bar{Q} .
\end{gathered}
$$

It may be easier to express this as

$$
\begin{aligned}
& \frac{d M E C}{d Q}=\frac{-m \bar{Q}}{[(-1)(\bar{Q}-Q)]^{3}}=\frac{-m \bar{Q}}{(-1)^{3}(\bar{Q}-Q)^{3}}=\frac{m \bar{Q}}{(\bar{Q}-Q)^{3}}>0 . \\
& m>0, \bar{Q}>0, \text { and } \\
& \bar{Q}-Q>0 \text { since } Q<\bar{Q} / 2 .
\end{aligned}
$$

j) In this situation, $Q_{0}=0$ is given. If the consumer does not bar jain with the polluter, the consumer's utility will be

$$
v_{0}=\frac{m}{p_{a}}(\bar{Q}-0)=\frac{m}{P_{a}} \bar{Q}
$$

k) If the consumer does barguinmith the polluter, suppose that in return for allowing the polluter to increase output to $Q$, the consumer accepts a transfer of $\hat{T}$.
then

$$
v^{\prime}=\frac{m+\hat{T}}{P_{a}}(\bar{Q}-Q) .
$$

e) The minimum $\hat{T}$ which the consumer would be willing to accept for a given $Q$ would satisfy the property that $v^{\prime}=v_{0}$, because if $v^{\prime}$ were any lowe than $v_{0}$, the consumer would prefer to stag at $v_{0}$. So

$$
\begin{aligned}
& v_{0}=v^{\prime} \Rightarrow \frac{m \bar{Q}}{P a}=\frac{m+\hat{T}}{P a}(\bar{Q}-Q) \text { and solving for } \hat{T} \\
& \quad \text { as a function of } Q, \\
& \frac{m \bar{Q}}{}=(m+\hat{T})(\bar{Q}-Q) \\
& \frac{m \bar{Q}}{\bar{Q}-Q}=m+\hat{T}
\end{aligned}
$$

$$
\hat{T}=\frac{m \bar{Q}}{\bar{Q}-Q}-m=m\left[\frac{\bar{Q}}{\bar{Q}-Q}-\frac{\bar{Q}-Q}{\bar{Q}-Q}\right]=\frac{m Q}{\bar{Q}-Q}
$$

which is positive in the relevant range of $O \leq Q \leq \bar{Q} / 2$. If the polluter wants $Q$ to be larger, the consumer wi U demand that $\hat{T}$ be come larger, so we expect that $d \hat{T} / d Q>0$.

$$
\begin{aligned}
\frac{d \hat{T}}{d Q} & =\frac{m}{\bar{Q}-Q}-\frac{(-1)}{(\bar{Q}-Q)^{2}} m Q=m\left[\frac{1}{\bar{Q}-Q}+\frac{Q}{(\bar{Q}-Q)^{2}}\right] \\
& =m \frac{\bar{Q}-Q+Q}{(\bar{Q}-Q)^{2}}=\frac{m \bar{Q}}{(\bar{Q}-Q)^{2}}>0
\end{aligned}
$$

Also

$$
\frac{d^{2} \hat{T}}{d Q^{2}}=\frac{-2 m \bar{Q}}{(\bar{Q}-Q)^{3}}(-1)=\frac{2 m \bar{Q}}{(\bar{Q}-Q)^{3}}>0
$$

$m$ ) In this situation, $\hat{E C}(Q)=\hat{T}(Q)$ be case the reason the consumer is witliry to $\overbrace{\text { and not less }}^{\operatorname{accopt}} \overbrace{T}$ is because $Q$ causes $\hat{E C}(Q)$ indarmage.
n) Be cause of part $(\mathrm{m}), M \hat{E} C=d \hat{T} / d Q$.

In summary,

$$
\text { and from part }(\ell)
$$

$$
\begin{gathered}
M E C=\frac{m \bar{Q}}{2(Q-\bar{Q})^{2}} \quad M E C=\frac{m \bar{Q}}{(\bar{Q}-Q)^{2}}=2 M E C \\
\frac{d M E C}{d Q}=\frac{m \bar{Q}}{(\bar{Q}-Q)^{3}} \quad \frac{d M E C}{d Q}=\frac{2 m \bar{Q}}{(\bar{Q}-Q)^{3}}=2 \frac{d M E C}{d Q} .
\end{gathered}
$$

This Page Is Completely Optional!
A rough sketch would be the following, noting that the polluter's marginal profit $M \pi$ hits zero at $Q=\bar{Q} / 2$ because when the polluter has the property right, $Q=\bar{Q} / 2$ is in absence of bargaining.


So the assignment of the property right affects the location of the social optimum, which contradicts the so-called "Coase theorem," but is after all not supnising if one thanks of, in an Edgeworth Box, the effect of the endowment point on the allocations in the sore.

The only case in which the "Coase Theorem" would be true is when, using the language of this example, the utility function's guasilinear, so that the demand for dean air is completely unaffected by one's income. There are very few commodities consumed in equal amounts by the rich and the poor!


Figure 1. Willingness (and ability) to Pay, "WATP," is $m \bar{Q} /\left[2 \cdot(Q-\bar{Q})^{2}\right]$ from (i) of the 2019 exam, and $2 /(Q-2)^{2}$ substituting in the parameters here. Willingness to Accept, "WTA," is $\left.m \bar{Q} /(\bar{Q}-Q)^{2}\right)$ from (l) (that's the letter ' 1 ' not the number ' 1 ') of the 2019 exam, and $4 /(2-Q)^{2}$ substituting in the parameters here. Marginal Profit, " $M \Pi$," is assumed here to be $2-2 Q$.

## 2021 Qualifying Exam Sec. 3 Qu. 2

## 2. [10 points]

[Completely optional introduction: This is the end of a demonstration that George Stigler's "Coase Theorem" is false in the general case of goods having arbitrary income effects. (This would please Nobel Laureate Ronald Coase but it profoundly challenges followers of Stigler.)]
You have been given the question and answer to the 2019 Qualifying Exam's Section 3 Question 1. Adopt all the notation and situations
(see the previous question) described in that problem. In addition, in that problem, set $\bar{Q}=2$, $m=2$, and suppose the marginal profit of the firm is given by $M \Pi=$ $2-2 Q$. The figure on the last, "optional" page of the answer to the 2019 question then becomes Figure 1.
(a) Suppose, as in part (c) of the 2019 question, firms have the right to emit pollution at will. The maximum amount of money the
consumer is willing and able to pay for the output level " $Q$ " lying directly below points $c$ and $a$ is the area under the WATP curve, that is, under $a b$. However, in bargaining with the firm, the consumer might not have to pay this maximum amount of money. Explain briefly why the minimum amount of money the consumer would have to pay for the output to be reduced to that level of $Q$ is the area under the " $c d$ " segment of the $M \Pi$ curve.
(b) Show that the area under the $c d$ segment of the $M \Pi$ curve is $1-2 Q+Q^{2}$.
(c) From (c) of the 2019 exam, $v_{0}=m \bar{Q} /\left(2 p_{a}\right)$, which under our assumptions is equal to $2 / p_{a}$. From (d) of the 2019 exam, and with our assumptions,

$$
v^{\prime}=\frac{m-T}{p_{a}}(\bar{Q}-Q)=\frac{2-T}{p_{a}}(2-Q) .
$$

If $T$ corresponds not to WATP but to the minimum consumer payment, what value of $Q$ makes $v_{0}$ equal to $v^{\prime}$ ? It is sufficient to find an equation that defines $Q$ implicitly; you do not have to find $Q$ explicitly.
(d) Suppose, as in part (j) of the 2019 question, consumers have the right to clean air and firms cannot pollute the air without obtaining permission from the consumer. The minimum amount of money the consumer is willing to accept for the output level " $Q$ " lying directly below points $h$ and $f$ is the area under the WTA curve, that is, under ef. However, in bargaining with the firm, the consumer might not have to accept this minimum amount of money. The maximum amount of money firm would pay to the consumer for the output to be increased to that level of $Q$ is the area under the " $g h$ " segment of the $М \Pi$ curve. Show that that area under the $g h$ segment of the $M \Pi$ curve is $2 Q-Q^{2}$.
(e) From (j) of the 2019 exam, $v_{0}=m \bar{Q} / p_{a}$, which under our assumptions is equal to $4 / p_{a}$. From (k) of the 2019 exam, and with our assumptions,

$$
v^{\prime}=\frac{m+\widehat{T}}{p_{a}}(\bar{Q}-Q)=\frac{2+\widehat{T}}{p_{a}}(2-Q) .
$$

If $\widehat{T}$ corresponds not to WTA but to the maximum firm payment, what value of $Q$ makes $v_{0}$ equal to $v^{\prime}$ ? It is sufficient to find
an equation that defines $Q$ implicitly; you do not have to find $Q$ explicitly.

## Answer to Summer 2021 Qualifying Exam, Section 3 Question 2

(a) The area under $c d$ is the profit earned when the firm increases output from $Q$ to 1 . This is simply because total profit is the area under the marginal profit curve: $\int_{Q}^{1} M \Pi d Q=\int_{0}^{1}(d \pi / d Q) d Q=\pi(1)-\pi(Q)$ by the Fundamental Theorem of Calculus. The firm will refuse to reduce output to this $Q$ unless it receives payment equal to this lost profit.
(b)

$$
\begin{aligned}
\int_{Q}^{1}(2-2 \hat{Q}) d \hat{Q} & =\left.\left[2 \hat{Q}-\hat{Q}^{2}\right]\right|_{Q} ^{1} \\
& =(2-1)-\left(2 Q-Q^{2}\right)=1-2 Q+Q^{2}
\end{aligned}
$$

(This happens to be a perfect square, $(1-Q)^{2}$.)
(c) Substitute the answer to part (b) (which you know, even if you were not able to solve part (b), because its answer was given in the exam) for $T$ in the equation given in this part of the question for $v^{\prime}$, then set $v^{\prime}=v_{0}$ and recall that $\bar{Q}=2$ and that $m=2$ :

$$
\begin{align*}
\frac{2-\left(1-2 Q+Q^{2}\right)}{p_{a}}(2-Q) & =\frac{2 \bar{Q}}{2 p_{a}} \\
\left(1+2 Q-Q^{2}\right)(2-Q) & =\bar{Q}  \tag{9}\\
2+4 Q-2 Q^{2}-Q-2 Q^{2}+Q^{3} & =2 \\
3 Q-4 Q^{2}+Q^{3} & =0 .
\end{align*}
$$

Therefore one solution is $Q=0$. Other solutions are:

$$
\begin{aligned}
Q^{2}-4 Q+3 & =0 \\
(Q-3)(Q-1) & =0 .
\end{aligned}
$$

The $Q=3$ solution makes no sense in this context. The $Q=1$ solution would mean that less pollution reduction would be achieved when pollution victims had to pay little money to polluters than when they had to pay more money ( $Q=0.534$ ), which also makes no sense. Therefore, the correct answer is $Q=0$. (The question did not ask you to find such an explicit value for $Q$, only an implicit definition of it, so any of the displayed equations would be an adequate answer.)
Optional: If you have a computer, you can graph $v_{0}$ or, more straightforwardly from (9), $p_{a} v_{0}=\bar{Q}=2$, and also from (9), $p_{a} v^{\prime}=(1+2 Q-$ $\left.Q^{2}\right)(2-Q)$. This looks like Figure 2: starting from $Q=1$ (which is


Figure 2. Initial $\left(v_{0}\right)$ and final $\left(v^{\prime}\right)$ utility of the pollution victim (times the price of apples) if the firm has the property right to pollute and Coasian bargains are made at the minimum amount of money needed to induce the firm to lower production. If a quantity level has $p_{a} v^{\prime}>p_{a} v_{0}$, the consumer would strictly prefer being at that quantity level rather than remaining at $Q=1$. By construction, the firm is indifferent between $Q \in[0,1]$.
where $M \Pi=0$ ), the pollution victim gains from decreasing $Q$ and making the minimal payment to the polluter whenever $p_{a} v^{\prime}>p_{a} v_{0}$.
Optional: Figure 2 implies that any $Q \in[0,1)$ would be accepted by both parties as an alternative to $Q=1$. Bargaining starting from $Q=1$ and incrementally going left would stop at the maximum of $v^{\prime}$, which is at $(4-\sqrt{7}) / 3 \approx 0.451$.
(d) $\int_{0}^{Q}(2-2 \hat{Q}) d \hat{Q}=\left.\left[2 \hat{Q}-\hat{Q}^{2}\right]\right|_{0} ^{Q}=\left(2 Q-Q^{2}\right)-(0-0)=2 Q-Q^{2}$.
(e) Substitute the answer to part (d) (which you know, even if you were not able to solve part (d), because its answer was given in the exam) for $\widehat{T}$ in the equation given in this part of the question for $v^{\prime}$, then set $v^{\prime}=v_{0}$ and recall that $\bar{Q}=2$ and that $m=2$ :

$$
\begin{align*}
\frac{2+\widehat{T}}{p_{a}}(2-Q) & =\frac{4}{p_{a}} \\
{\left[2+\left(2 Q-Q^{2}\right)\right](2-Q) } & =4 \\
\left(4+4 Q-2 Q^{2}\right)-\left(2 Q+2 Q^{2}-Q^{3}\right) & =4 \\
Q^{3}-4 Q^{2}+2 Q+4 & =4  \tag{10}\\
Q^{3}-4 Q^{2}+2 Q & =0 .
\end{align*}
$$

Therefore one solution is $Q=0$. Other solutions solve $Q^{2}-4 Q+2=0$ so

$$
Q=\frac{4 \pm \sqrt{16-8}}{2}=\frac{4 \pm 2 \sqrt{2}}{2}=2 \pm \sqrt{2}
$$



Figure 3. Initial $\left(v_{0}\right)$ and final $\left(v^{\prime}\right)$ utility of the pollution victim (times the price of apples) if the pollution victim has the property right to clean air and Coasian bargains are made at the maximum amount of money the firms are willing to pay to increase production. If a quantity level has $p_{a} v^{\prime}>p_{a} v_{0}$, the consumer would strictly prefer being at that quantity level rather than remaining at $Q=0$. By construction, the firm is indifferent between $Q \in[0,1]$.
of which only $2-\sqrt{2} \approx 0.586$ makes sense in this context. (The question did not ask you to find such an explicit value for $Q$, only an implicit definition of it, so any of the displayed equations would be an adequate answer.)
Optional: If you have a computer, you can graph $v_{0}$ or, more straightforwardly from (10), $p_{a} v_{0}=4$, and also from (10), $p_{a} v^{\prime}=Q^{3}-4 Q^{2}+$ $2 Q+4$. This looks like Figure 3: starting from $Q=0$, the pollution victim gains from increasing $Q$ and receiving the maximal payment from the polluter whenever $p_{a} v^{\prime}>p_{a} v_{0}$.
Optional: Figure 3 implies that any $Q \in(0,0.586]$ would be accepted by both parties as an alternative to $Q=0$. Bargaining starting from $Q=0$ and incrementally going right would stop at the maximum of $v^{\prime}$, which is at $(4-\sqrt{10}) / 3 \approx 0.279$.

Optional: If the polluter has the property rights, we predict the outcome of Coasian bargaining will be between 0.534 (when pollution victims have to make maximal transfers to the firms, the area under WATP) and 0.451 (when pollution victims only have to make minimal transfers to the firms, the area under $M \Pi$ ). If the pollution victim has the property rights, we predict the outcome of Coasian bargaining will be between 0.279 (when firms have to make maximal transfers to the pollution victims, the area under $М П$ ) and 0.304 (when firms only have to make minimal transfers to the firms, the area under $W T A$ ).


Figure 4. The dark intervals on the $Q$ axis represent, respectively, the possible outcomes of Coasian bargaining when pollution victims have the property right to clean air (left) or when polluters have the property right to pollute (right).


[^0]:    2005 Qualifier, Section 2

