Section 3:

Consumer Theory

Fall 2021 Final Exam Question 2
Consumer Theory

2. **[16 points]**

   Suppose a consumer consumes only two goods, x and y. In this case, if the consumer has lexicographic preferences, then the consumer will choose between bundles (x_1, y_1) and (x_2, y_2) in one of the following two ways. The first is that if $x_1 \neq x_2$, the consumer prefers the bundle whose x is larger; but if $x_1 = x_2$, the consumer prefers the bundle whose y is larger. The second is that if $y_1 \neq y_2$, the consumer prefers the bundle whose y is larger; but if $y_1 = y_2$, the consumer prefers the bundle whose x is larger.

   If a consumer has lexicographic preferences, there is no utility function which represents his or her preferences.

   Figure 1 is one author's attempt to describe a particular situation involving lexicographic preferences, with Y (measured on the vertical axis) representing the consumer's consumption of food and E (measured on the horizontal axis) representing the state of the natural environment. (I know that calling food "Y" is unusual, but the graph is not mine so I did not choose the notation.)

   (a) Why can the solid lines in Figure 1 not be indifference curves?

   (b) The author calls the solid lines in Figure 1 "quasi-indifference curves." Describe in words the preferences of the consumer depicted in Figure 1, making a plausible conjecture about the meaning of the quasi-indifference curves. (If your conjecture turns out to be wrong, you nevertheless may earn many points if your conjecture is reasonable and your interpretation of the graph is consistent with your conjecture.)

   (c) Sketch the income expansion path in Figure 1.

   (d) Show that there exists a (different) consumer whose preferences *can* be represented by a utility function who has the same income expansion path as the path you drew in part (c). Do this by sketching this consumer's indifference curves.

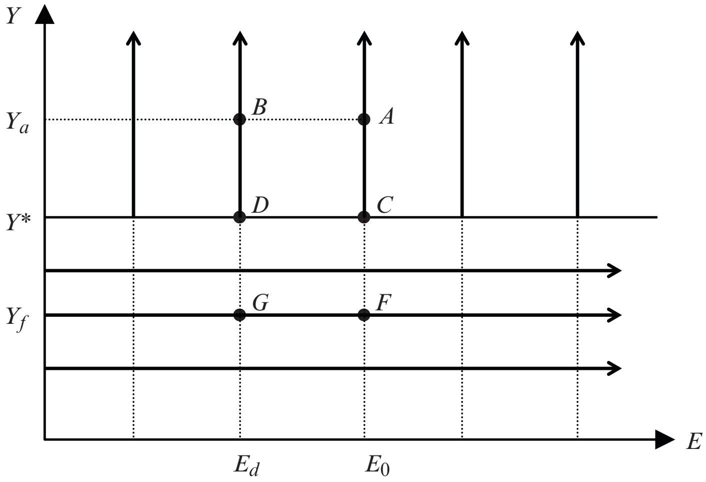


Figure 1.

Answer to Fall 2021 Econ. 7005 Final Exam, Question 2

a) Indifference curves are contour lines of the utility function. But there is no utility function which represents lexicographic preferences. (We showed in class that lexicographic preferences violate continuity of preferences.) So with lexicographic preferences, there are no indifference curves.

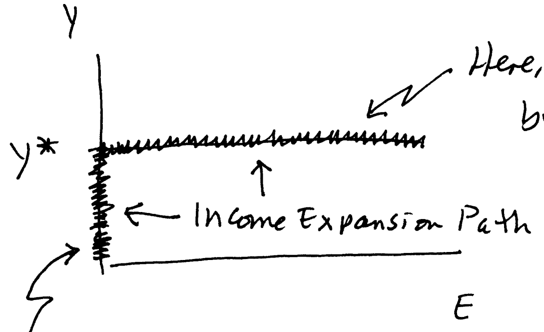
b) For $Y < Y^*$: prefer the bundle whose Y is larger. If they have the same Y, prefer the bundle whose E is larger.

For $Y > Y^*$: prefer the bundle whose E is larger. If they have the same E, prefer the bundle whose Y is larger.

If you have inadequate food, you only care about food, not environmental protection. But once you have adequate food, you only care about environmental protection, not food (unless there is a tie in food)

↑ unless there is a tie in food

c)

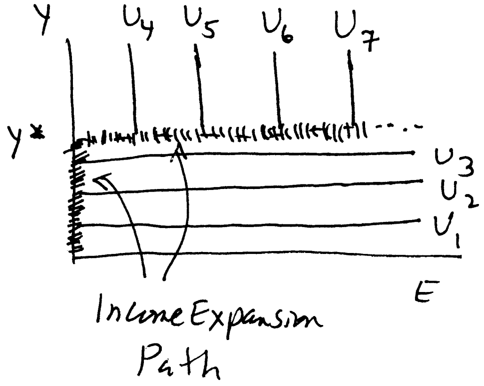


Here, all increases in income go to buying more environmental quality.

Here, all increases in income go to buying more food.

With lexicographic preferences, all income increases go to buying the good the consumer cares most about.

d)



$U_1 < U_2 < U_3 < U_4 < U_5 < U_6 < U_7$

For $Y < Y^*$, the consumer only cares about Y, not E. For $Y > Y^*$, the consumer only cares about E, not Y.

(For $Y < Y^*$, if two bundles have the same Y, the consumer is indifferent between them, even if one of them has more E than the other. This is unlike in parts (a)-(c). For $Y > Y^*$, analogous comments can be made.)

Optional: This suggests that lexicographic preferences are observationally equivalent to certain non-lexicographic preferences in a market environment.

Note: Figure from "Post Keynesian consumer choice theory and ecological economics," Marc Lavoie. In "Post Keynesian and Ecological Economics: Confronting Environmental Issues," edited by Richard P. F. Holt, Steven Pressman, & Clive L. Spash. Edward Elgar, 2009.

2018 Exam 1 Qu. 3

3. **[11 points]**

   Suppose a price-taking utility-maximizing consumer has a utility function of $u(\mathbf{x}) = \ln x_1 + \ln x_2$ and faces prices p_1 and p_2 for commodities x_1 and x_2, respectively; and has income m.

   (a) Find this consumer's optimal consumption for x_1 and x_2 (called x_1^* and x_2^*, respectively).

   (b) Find $\partial x_1^* / \partial p_1$ and $\partial x_2^* / \partial p_1$.

   (c) Find this consumer's income expansion path as a function of x_1^*. (Imagine a graph where x_1^* is on the horizontal axis). Hint: one way to proceed is to consider x_2^* / x_1^*.

   (d) How will the income expansion path shift when p_1 changes?

Answer to Question 3, Exam 1, Fall 2018, Econ. 7005

$u(\underset{\sim}{x}) = \ln x_1 + \ln x_2$

a) $\max_{\underset{\sim}{x}} u(\underset{\sim}{x})$ s.t. $p_1 x_1 + p_2 x_2 = m$. The Lagrangian:

$\mathcal{L} = \ln x_1 + \ln x_2 + \lambda(m - p_1 x_1 - p_2 x_2)$. First-order conditions:

(1) $0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2$ (or justify this as a repetition of the constraint rather than as $0 = \partial \mathcal{L}/\partial \lambda$)

$$\left.\begin{array}{ll} (2) & 0 = \frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1} - \lambda p_1 \\ (3) & 0 = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{x_2} - \lambda p_2 \end{array}\right\} \Rightarrow \lambda = \frac{1}{p_1 x_1} = \frac{1}{p_2 x_2}$$

$$\Rightarrow p_1 x_1 = p_2 x_2 \Rightarrow x_1 = \frac{p_2 x_2}{p_1}.$$

Substitute this into (1):

$$0 = m - p_2 x_2 - p_2 x_2 = m - 2 p_2 x_2 \Rightarrow 2 p_2 x_2 = m$$

$$\boxed{x_2 = \frac{m}{2p_2}}$$

and $\boxed{x_1 = \frac{p_2}{p_1} \frac{m}{2p_2} = \frac{m}{2p_1}}$.

Optional: $\lambda = \frac{1}{p_1 x_1} = \frac{1}{p_1} \frac{2p_1}{m} = \frac{2}{m}$.

b) _The easy way_ (since one is able, in part (a), to solve explicitly for x_1^* and x_2^*) :

$$\frac{\partial x_1^*}{\partial p_1} = \frac{\partial}{\partial p_1} \frac{m}{2p_1} \text{ from part (a)}$$

$$= \frac{-m}{2p_1^2}.$$

$$\frac{\partial x_2^*}{\partial p_1} = \frac{\partial}{\partial p_1} \frac{m}{2p_2} \text{ from part (a)}$$

$$= 0.$$

The hard way (which works regardless of whether explicit solutions for x_1^* and x_2^* were obtainable) : This is a comparative statics problem with endogenous variables λ, x_1, and x_2, and exogenous variables m, p_1, and p_2 ; but here the only exogenous variable that changes is p_1, so $dm = 0$ and $dp_2 = 0$.

The differentials of (1), (2), and (3) are, if $dm = 0$ and $dp_2 = 0$:

$$\begin{array}{lllll}
 & d\lambda & dx_1 & dx_2 & \\
0 = & & -p_1\,dx_1 & -p_2\,dx_2 & -x_1\,dp_1 \\
0 = & -p_1\,d\lambda & -\frac{1}{x_1^2}\,dx_1 & & -\lambda\,dp_1 \\
0 = & -p_2\,d\lambda & & -\frac{1}{x_2^2}\,dx_2 &
\end{array}$$

$$\underset{\sim}{0} = \begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & -1/x_1^2 & 0 \\ -p_2 & 0 & -1/x_2^2 \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} + \begin{bmatrix} -x_1 \\ -\lambda \\ 0 \end{bmatrix} dp_1$$

. Using Cramer's Rule after

rewriting as

$$\begin{bmatrix} x_1 \\ \lambda \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & -1/x_1^2 & 0 \\ -p_2 & 0 & -1/x_2^2 \end{bmatrix} \begin{bmatrix} d\lambda/dp_1 \\ dx_1/dp_1 \\ dx_2/dp_1 \end{bmatrix}$$

yields

$$\frac{dx_1}{dp_1} = \frac{\begin{vmatrix} 0 & x_1 & -p_2 \\ -p_1 & \lambda & 0 \\ -p_2 & 0 & -1/x_2^2 \end{vmatrix}}{\begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & -1/x_1^2 & 0 \\ -p_2 & 0 & -1/x_2^2 \end{vmatrix}}$$

Numerator: solve by cofactor expansion, here along the first row:

$$x_1(-1)^{1+2} \begin{vmatrix} -p_1 & 0 \\ -p_2 & -1/x_2^2 \end{vmatrix} - p_2(-1)^{1+3} \begin{vmatrix} -p_1 & \lambda \\ -p_2 & 0 \end{vmatrix}$$

$$= -x_1\left[\frac{p_1}{x_2^2}\right] - p_2\left[+p_2\lambda\right]$$

$$= -\frac{x_1 p_1}{x_2^2} - \lambda p_2^2$$

Denominator: also expand along the first row:

$$-p_1(-1)^{1+2} \begin{vmatrix} -p_1 & 0 \\ -p_2 & -1/x_2^2 \end{vmatrix} - p_2(-1)^{1+3} \begin{vmatrix} -p_1 & -1/x_1^2 \\ -p_2 & 0 \end{vmatrix}$$

$$= p_1\left(\frac{p_1}{x_2^2}\right) - p_2\left(-\frac{p_2}{x_1^2}\right) = \frac{p_1^2}{x_2^2} + \frac{p_2^2}{x_1^2}.$$

So

$$\frac{\partial x_1}{\partial p_1} = \frac{-\frac{p_1 x_1}{x_2^2} - \lambda p_2^2}{\frac{p_1^2}{x_2^2} + \frac{p_2^2}{x_1^2}}.$$

Also,

$$\frac{\partial x_2}{\partial p_1} = \frac{\begin{vmatrix} 0 & -p_1 & x_1 \\ -p_1 & -1/x_1^2 & \lambda \\ -p_2 & 0 & 0 \end{vmatrix}}{\text{same denominator as } \partial x_1/\partial p_1}$$

. The numerator is, expanding along its first row, $-p_1(-1)^{1+2}\begin{vmatrix} -p_1 & \lambda \\ -p_2 & 0 \end{vmatrix} + x_1(-1)^{1+3}\begin{vmatrix} -p_1 & -1/x_1^2 \\ -p_2 & 0 \end{vmatrix}$

$= p_1(\lambda p_2) + x_1\left(\frac{-p_2}{x_1^2}\right) = \lambda p_1 p_2 - p_2/x_1$. Thus

$$\frac{\partial x_2}{\partial p_1} = \frac{\lambda p_1 p_2 - p_2/x_1}{\frac{p_1^2}{x_2^2} + \frac{p_2^2}{x_1^2}} .$$

Optional:

As in "the easy way," we could at this point avail ourselves of the fact that we were able to solve for x_1^*, x_2^*, and λ^* in part (a) by substituting $x_1^* = m/(2p_1)$, $x_2^* = m/(2p_2)$, and $\lambda^* = 2/m$ into the expressions we derived for $\partial x_1/\partial p_1$ and $\partial x_2/\partial p_1$:

$$\frac{\partial x_1}{\partial p_1} = \frac{\frac{-p_1 x_1}{x_2^2} - \lambda p_2^2}{\frac{p_1^2}{x_2^2} + \frac{p_2^2}{x_1^2}} = \frac{\frac{-\enclose{circle}{p_1}\enclose{circle}{m}}{2\enclose{circle}{p_1}}\frac{4p_2^2}{m^{\enclose{circle}{2}}} - \frac{2}{m}\frac{p_2^2}{1}}{\frac{p_1^2 4p_2^2}{m^2} + \frac{p_2^2 4p_1^2}{m^2}} = \frac{-\frac{2p_2^2}{m} - \frac{2p_2^2}{m}}{\frac{8p_1^2p_2^2}{m^2}}$$

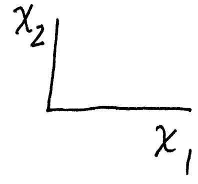
$$= \frac{-4\enclose{circle}{p_2^2}}{\enclose{circle}{m}} \cdot \frac{m^{\enclose{circle}{2}}}{8p_1^2\enclose{circle}{p_2^2}} = \frac{-m}{2p_1^2}$$

(which matches the answer obtained in "the easy way")

$$\frac{\partial x_2}{\partial p_1} = \frac{\lambda p_1 p_2 - p_2/x_1}{\frac{p_1^2}{x_2^2} + \frac{p_2^2}{x_1^2}}$$ has a numerator of $\frac{2}{m} p_1 p_2 - p_2 \frac{2p_1}{m}$

$$= \frac{2p_1p_2}{m} - \frac{2p_1p_2}{m} = 0$$

so $\partial x_2/\partial p_1 = 0$ (which matches the answer obtained in "the easy way").

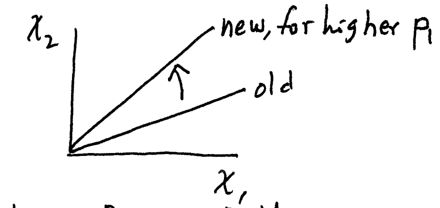
c) The income expansion path shows how the optimal consumption bundle in the  plane changes as income changes. From part (a),

$$\frac{x_2^*}{x_1^*} = \frac{\frac{m}{2p_2}}{\frac{m}{2p_1}} = \frac{m}{2p_2} \cdot \frac{2p_1}{m} = \frac{p_1}{p_2}$$ so $\boxed{x_2^* = \frac{p_1}{p_2} x_1^*}$ is

the income expansion path. It happens to be a straight line through the origin with slope p_1/p_2.

d) <u>The easy way</u>:

Nothing in part (c)'s calculations depends on a particular value of p_1, so the income expansion path is always described by $x_2^* = \frac{p_1}{p_2} x_1^*$, and a rise in p_1 will (since $\frac{x_1^*}{p_2} \geq 0$) raise x_2^* for any given $x_1^* > 0$ (although for $x_1^* = 0$, x_2^* will remain at zero).

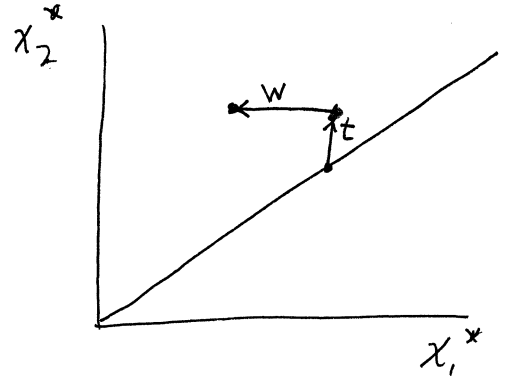


Income Expansion Path

The hard way:

$$\frac{\partial x_2^*}{\partial p_1} = \begin{cases} \bullet\ \frac{\partial}{\partial p_1}\frac{m}{2p_2} = 0 \text{ from part(a) ; or} \\[2ex] \bullet\ \dfrac{\lambda p_1 p_2 - p_2/x_1}{\dfrac{p_1^2}{x_2^2} + \dfrac{p_2^2}{x_1^2}} \text{ from part (b) ; or} \\[4ex] \bullet\ \frac{\partial}{\partial p_1}\left(\frac{p_1}{p_2} x_1^*\right) = \frac{x_1^*}{p_2} + \frac{p_1}{p_2}\frac{\partial x_1^*}{\partial p_1} \text{ from part (c)} \\[2ex] \qquad = \dfrac{x_1^*}{p_2} + \dfrac{p_1}{p_2}\,\dfrac{\dfrac{-p_1 x_1}{x_2^2} - \lambda p_2^2}{\dfrac{p_1^2}{x_2^2} + \dfrac{p_2^2}{x_1^2}} \text{ from part (b)} \end{cases}$$

$$\frac{\partial x_1^*}{\partial p_1} = \begin{cases} \bullet\ \frac{\partial}{\partial p_1}\frac{m}{2p_1} = \frac{-m}{2p_1^2} \text{ from part (a)} \\[3ex] \bullet\ \dfrac{\dfrac{-p_1 x_1}{x_2^2} - \lambda p_2^2}{\dfrac{p_1^2}{x_2^2} + \dfrac{p_2^2}{x_1^2}} \text{ from part (b)} \end{cases}$$



As p_1 increases, x_2^* shifts by $\frac{\partial x_2^*}{\partial p_1} dp_1$ (length "t") and x_1^* shifts by $\frac{\partial x_1^*}{\partial p_1} dp_1$ (length "w").

In our specific case we know that $\frac{\partial x_2^*}{\partial p_1} = 0$ so length "t" is zero, and $\frac{\partial x_1^*}{\partial p_1} = \frac{-m}{2p_1^2} < 0$ so x_1^* falls and the graph's "w" does go to the left as illustrated (although if $m = 0$, $x_1^* = 0$ and $\partial x_1^* / \partial p_1 = 0$, so at $x_1^* = 0$, length "w" is zero). It follows that the income expansion path pivots as shown under "the easy way."

Remark. The first bullet point for $\partial x_2^* / \partial p_1$ says it equals zero; the third says it equals $\frac{x_1^*}{p_2} + \frac{p_1}{p_2} \frac{\partial x_1^*}{\partial p_1}$. Are these compatible? Answer:

$$\frac{x_1^*}{p_2} + \frac{p_1}{p_2}\frac{\partial x_1^*}{\partial p_1} = \frac{(m/2p_1)}{p_2} + \frac{p_1}{p_2}\frac{\partial}{\partial p_1}\left(\frac{m}{2p_1}\right) = \frac{m}{2p_1p_2} + \frac{p_1}{p_2}\frac{-m}{2p_1^2}$$

$$= \frac{m}{2p_1p_2} - \frac{m}{2p_1p_2} = 0, \text{ yes.}$$

2. [14 points] Suppose a price-taking consumer has utility function

$$u(\mathbf{x}) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

for commodities x_1, x_2, …, x_n, where n is a positive integer greater than two. Find this consumer's demand for each of the commodities. (You need not check the second-order conditions.)

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②

$$\max \underbrace{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}_{\text{"}u\text{"}} \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = \underset{\substack{\nearrow \\ \text{income}}}{m}$$

(Formally: $\max \prod_{i=1}^{n} x_i^{\alpha_i}$ s.t. $\sum_{i=1}^{n} p_i x_i = m$.)

$$\mathcal{L} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} + \lambda\left[m - p_1x_1 - p_2x_2 - \cdots - p_nx_n\right].$$

$$0 = \partial\mathcal{L}/\partial\lambda = m - p_1x_1 - p_2x_2 - \cdots - p_nx_n$$

$$0 = \partial\mathcal{L}/\partial x_i = \underbrace{\alpha_i \frac{u}{x_i}} - \lambda p_i$$ or see alternative solution on next page →

This can also be written $\alpha_i x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_i^{\alpha_i - 1} \cdots x_n^{\alpha_n}$

$$= \alpha_i x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_i^{\alpha_i} x_i^{-1} \cdots x_n^{\alpha_n}$$

$$= \alpha_i u x_i^{-1}.$$

So

$$\lambda p_i = \frac{\alpha_i}{x_i} u$$

$$\Rightarrow x_i = \frac{u}{\lambda}\frac{\alpha_i}{p_i} \text{ and}$$

$$m = \sum_i p_i x_i = \sum_i p_i \frac{u}{\lambda}\frac{\alpha_i}{p_i} = \sum_i \frac{u}{\lambda}\alpha_i = \frac{u}{\lambda}\sum_i \alpha_i$$

$$\Rightarrow \frac{u}{\lambda} = \frac{m}{\sum_i \alpha_i}.$$

Hence $x_i^* = \frac{u}{\lambda}\frac{\alpha_i}{p_i} = \boxed{\frac{m}{\sum_i \alpha_i}\frac{\alpha_i}{p_i}}$. → (Note that often, $\sum_i \alpha_i = 1$, in which case $x_i^* = m\alpha_i/p_i$.)

Alternative solution: F.O.C.'s are

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 - \cdots - p_n x_n$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_i} = \alpha_i x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_i^{\alpha_i - 1} \cdots x_n^{\alpha_n} - \lambda p_i$$ and similarly

$$\underline{0 = \alpha_j x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_j^{\alpha_j - 1} \cdots x_n^{\alpha_n} - \lambda p_j}$$ so

$$\lambda = \frac{\alpha_i}{p_i} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_i^{\alpha_i - 1} \cdots x_n^{\alpha_n}$$

$$\lambda = \frac{\alpha_j}{p_j} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_j^{\alpha_j - 1} \cdots x_n^{\alpha_n} \quad \Rightarrow$$

$$\frac{\alpha_i}{p_i} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_i^{\alpha_i - 1} \cdots x_n^{\alpha_n} = \frac{\alpha_j}{p_j} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_j^{\alpha_j - 1} \cdots x_n^{\alpha_n}$$

$$\frac{\alpha_i}{p_i} x_i^{\alpha_i - 1} x_j^{\alpha_j} = \frac{\alpha_j}{p_j} x_i^{\alpha_i} x_j^{\alpha_j - 1}$$

$$\frac{\alpha_i}{p_i} x_i^{-1} = \frac{\alpha_j}{p_j} x_j^{-1}$$

$$\frac{\alpha_i}{p_i x_i} = \frac{\alpha_j}{p_j x_j}$$ or $\frac{p_i x_i}{\alpha_i}$ = a constant $\forall i$; call this constant "L".

Then $m = \sum_i p_i x_i = \sum_i \alpha_i \cdot L = L \sum_i \alpha_i$ and $L = m / \sum_i \alpha_i$.

From the definition of L,

$$x_i^* = \frac{\alpha_i}{p_i} L = \frac{\alpha_i}{p_i} \frac{m}{\sum_i \alpha_i}$$

as before.

2. **[12 points]**
Suppose the consumers in this problem are competitive. This is true for *both* parts (a) and (b) of this question! For simplicity, assume there is only one consumer (this just saves on notation). This consumer only consumes two goods, X and Y. Suppose this consumer's income is \$10, the price of good X is \$2/unit, and the price of good Y is \$1/unit. Suppose the consumer's utility function is $X^{1/2}Y^{1/2}$.

   (a) How much X and Y will this consumer buy? Be sure to verify the second-order conditions.

   (b) Suppose when this consumer goes to the store to buy X and Y, he can only find 4 units of Y in the store. (This should be less than the amount you calculated that he desired to buy in part (a).) What do you think will happen? Bidding the price of good Y up? In the end, how much X and Y will he end up with?

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Section 2 Question 2.

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(a) max $x^{1/2}y^{1/2}$ s.t. $2x+1y=10$

$$\mathcal{L} = x^{1/2}y^{1/2} + \lambda(10-2x-y)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = 10-2x-y$$

$$\left.\begin{array}{l} 0 = \frac{\partial \mathcal{L}}{\partial x} = \frac{1}{2}\,\frac{x^{1/2}y^{1/2}}{x} - 2\lambda \\ 0 = \frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2}\,\frac{x^{1/2}y^{1/2}}{y} - \lambda \end{array}\right\} \lambda = \frac{1}{4}\,\frac{x^{1/2}y^{1/2}}{x} = \frac{1}{2}\,\frac{x^{1/2}y^{1/2}}{y}$$

$$\Rightarrow \frac{1}{4x} = \frac{1}{2y} \Rightarrow y = 2x \text{ and } 10 = 2x+y$$

$$= 2x+2x = 4x \Rightarrow x^* = \frac{10}{4} = \frac{5}{2},$$

$$y^* = 2x = 5.$$

S.O.C. number of constraints $m=1$

number of variables $n=2$

$2m+1=3$

$m+n=3$

So we need D_3 of $\nabla^2\mathcal{L}$ to have the sign of $(-1)^{m+1} = 1 > 0$.

$$\nabla^2\mathcal{L} = \begin{bmatrix} \mathcal{L}''_{\lambda\lambda} & \mathcal{L}''_{\lambda x} & \mathcal{L}''_{\lambda y} \\ \mathcal{L}''_{x\lambda} & \mathcal{L}''_{xx} & \mathcal{L}''_{xy} \\ \mathcal{L}''_{y\lambda} & \mathcal{L}''_{yx} & \mathcal{L}''_{yy} \end{bmatrix} = \begin{bmatrix} 0 & -2 & -1 \\ -2 & -\frac{1}{4}x^{-3/2}y^{1/2} & \frac{1}{4}x^{-1/2}y^{-1/2} \\ -1 & \frac{1}{4}x^{-1/2}y^{-1/2} & -\frac{1}{4}x^{1/2}y^{-3/2} \end{bmatrix}$$

D_3 of $D^2\mathcal{L}$ is its determinant. Expanding along the first row,

$$|D^2\mathcal{L}| = (-1)^{1+2}(-2)\left[-2\cdot\frac{-1}{4}x^{1/2}y^{-3/2} - (-1)\frac{1}{4}x^{-1/2}y^{-1/2}\right]$$

$$+(-1)^{1+3}(-1)\left[-2\cdot\frac{1}{4}x^{-1/2}y^{-1/2} - (-1)\frac{-1}{4}x^{-3/2}y^{1/2}\right]$$

$$= 2\left[\frac{1}{2}x^{1/2}y^{-3/2} + \frac{1}{4}x^{-1/2}y^{-1/2}\right]$$

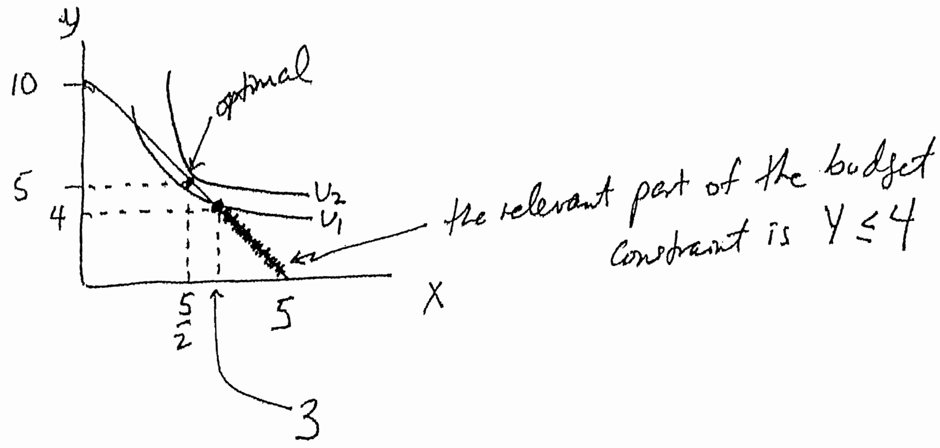
$$-\left[\frac{-1}{2}x^{-1/2}y^{-1/2} - \frac{1}{4}x^{-3/2}y^{1/2}\right]$$

$$= x^{1/2}y^{-3/2} + \frac{1}{2}x^{-1/2}y^{-1/2} + \frac{1}{2}x^{-1/2}y^{-1/2} + \frac{1}{4}x^{-3/2}y^{1/2}$$

$$= x^{1/2}y^{-3/2} + x^{-1/2}y^{-1/2} + \frac{1}{4}x^{-3/2}y^{1/2}$$

> 0. (Note that it's not even necessary to substitute in $x^* = \frac{5}{2}$ and $y^* = 5$ here.) (One could also instead prove that this utility function is quasiconcave, then appeal to the result that the S.O.C. for a max are satisfied if the objective function is quasiconcave and there is exactly one, linear constraint.)

(b)



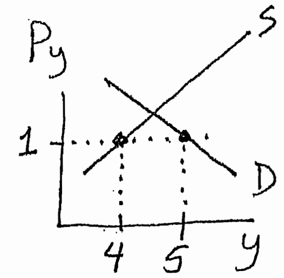
The consumer is competitive and so can't bid the price up.

So he can't get to the optimal point, $(\frac{5}{2}, 5)$, and indifference curve U_2.

He'd have to settle for the point $(3, 4)$, with indifference curve U_1,

which maximizes his utility given the additional $y \leq 4$ constraint.

Optional :



The idea here is that in a situation of excess demand, a competitive consumer can't bid the price up, so he will just take what he can get and spend more money than he'd like on the other good.

2019 Qualifying Exam Sec. 3 Qu. 2

2. **[16 points]**

   [Completely optional introduction: This will show that it is incorrect to use the change in "consumer surplus" as a measure of welfare change in the general case of goods having arbitrary income effects.]

   (a) Suppose a consumer has a utility function $u = x_1^{1/2} x_2^{1/2}$ and income $m = 2$ and takes the prices p_1 and p_2 as given. If x_1 is "cheese," find the consumer's (Marshallian) demand curve for cheese.

   (b) Make a rough, somewhat large sketch of this consumer's demand curve for cheese for $0 < x_1 = 1$ and identify the quantity demanded of cheese for prices p_1 of 1, 2, 3, and 4 dollars per pound ("\$/lb") of cheese.

   (c) Consider the following explanation of consumer surplus, which resembles what one might find in an undergraduate microeconomics textbook.

   > Consumer surplus, which is the area under the demand curve, measures how much a consumer would be willing and able to spend to buy cheese. To illustrate this, consider how much money the consumer whose demand curve you drew in part (b) would be willing and able to spend to buy a certain total amount of cheese. If the price of cheese were \$4/lb, he would be willing to buy *[fill in this blank, which is part (i) of this sub-part]* pounds of cheese, and so would spend the amount of money shown by area *[fill in this blank, which is part (ii) of this sub-part]* in the diagram. *[Designate areas in your graph by giving labels such as A, B, C, etc. to the vertices of those geometric areas, rather than say by shading the areas, because shading may make part (d) harder to superimpose onto this graph.]*
   >
   > If after making this transaction the price of cheese were to fall to \$3/lb, he would be willing to buy more cheese, raising his total cheese purchases to *[fill in this blank, which is part (iii) of this sub-part]* pounds of cheese, and so would in total spend the amount of money shown by area *[fill in this blank, which is part (iv) of this sub-part]* in the diagram.
   >
   > If after making this transaction the price of cheese were to fall further, to \$2/lb, he would be willing to buy more

cheese, raising his total cheese purchases to *[fill in this blank, which is part (v) of this sub-part]* pounds of cheese, and so would in total spend the amount of money shown by area *[fill in this blank, which is part (vi) of this sub-part]* in the diagram.

If, finally, after making this transaction the price of cheese were to fall even further, to \$1/lb, he would be willing to buy more cheese, raising his total cheese purchases to *[fill in this blank, which is part (vii) of this sub-part]* pounds of cheese, and so would in total spend the amount of money shown by area *[fill in this blank, which is part (viii) of this sub-part]* in the diagram. This amount of money is approximately equal to consumer surplus and thus shows that *[fill in this blank with the conclusion of this argument, which is part (ix) of this sub-part].*

(d) In this part you have to show that the explanation in part (c) is wrong. To do this, suppose the consumer has already spent the money to purchase, at a price of \$4/lb, the amount of cheese you answered in sub-part (i) of part (c). Suppose the consumer has taken ownership of this amount of cheese but has not eaten it yet. Before eating this cheese and before buying any x_2, the consumer gets the opportunity to buy more cheese at a price of \$3/lb.

  i. Show that he will not buy the total amount of cheese given in sub-part (iii) of part (c) by showing that the total amount of cheese he will actually buy is $7/24 \approx 0.29$ (where "$\approx$" means "is approximately equal to"). Hint: first calculate how much *extra* cheese he will buy.

  ii. Superimpose onto your prior graph this consumer's new demand curve for cheese for prices of 3, 2, and 1 dollars per pound, giving a numerical value for the amount of cheese demanded at each of these prices.

  iii. Construct an argument that the consumer surplus described in part (c) is not actually "how much a consumer would be willing and able to spend to buy cheese." Include a conceptual explanation of why the the demand curve you derived in part (a) generated a misleading answer to part (c).

Answer to 2019 Micro Qualifying Exam, Section 3 Question 2

a) $u = x_1^{1/2} x_2^{1/2}$ Budget constraint $m = p_1 x_1 + p_2 x_2$

$$\mathcal{L} = x_1^{1/2} x_2^{1/2} + \lambda(m - p_1 x_1 - p_2 x_2)$$

First-order conditions :

$$\left.\begin{aligned} 0 = \mathcal{L}_1' &= \frac{1}{2} \frac{x_1^{1/2} x_2^{1/2}}{x_1} - \lambda p_1 \Rightarrow \lambda = \frac{1}{2p_1} \frac{x_1^{1/2} x_2^{1/2}}{x_1} \\ 0 = \mathcal{L}_2' &= \frac{1}{2} \frac{x_1^{1/2} x_2^{1/2}}{x_2} - \lambda p_2 \Rightarrow \lambda = \frac{1}{2p_2} \frac{x_1^{1/2} x_2^{1/2}}{x_2} \end{aligned}\right\}$$

$$0 = \mathcal{L}_\lambda' = m - p_1 x_1 - p_2 x_2$$

$$\lambda = \frac{1}{2p_1} \frac{x_1^{1/2} x_2^{1/2}}{x_1} = \frac{1}{2p_2} \frac{x_1^{1/2} x_2^{1/2}}{x_2}$$

$$\frac{1}{p_1 x_1} = \frac{1}{p_2 x_2}$$

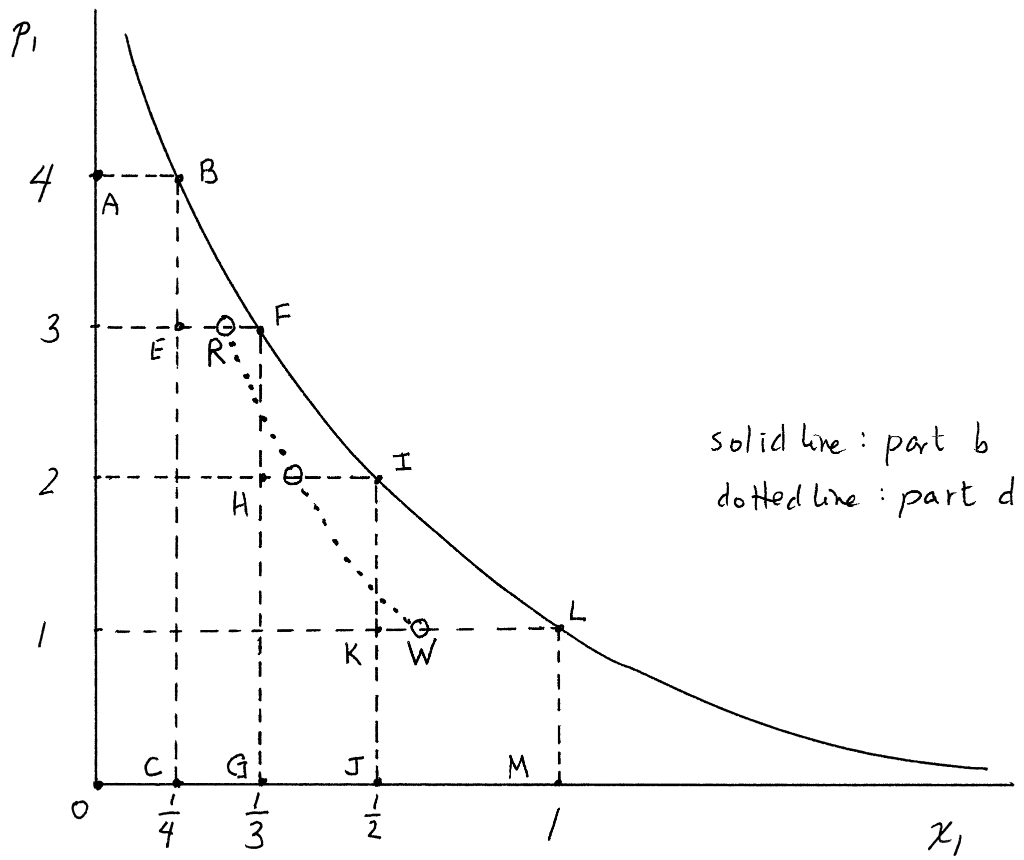
$$x_2 = \frac{p_1 x_1}{p_2} \Rightarrow$$

$$m = p_1 x_1 + p_2 x_2 = p_1 x_1 + p_2 \frac{p_1 x_1}{p_2} = 2 p_1 x_1$$

$\Rightarrow x_1 = \frac{m}{2p_1} = \frac{2}{2p_1} = \frac{1}{p_1}$ and

$$x_2 = \frac{p_1 x_1}{p_2} = \frac{p_1}{p_2} \frac{1}{p_1} = \frac{1}{p_2} .$$

b) over →



c) (i) 1/4 lb.

(ii) OABC

(iii) 1/3 lb.

(iv) OABC + CEFG = OABEFG

(v) 1/2 lb

(vi) answer to (iv) plus GHIJ = OABEFHIJ

(vii) 1 lb

(viii) answer to (vi) plus JKLM = OABEFHIKLM

(ix) consumer surplus (the area under the demand curve) is approximately equal to the amount of money this consumer would be willing and able to spend to buy 1 lb. of cheese.

d) (i) He has already bought 1/4 lb. of cheese. Let t be the amount of extra cheese he buys at \$3/lb. Then his utility is

$$u = \left(\frac{1}{4}+t\right)^{1/2} x_2^{1/2}.$$

He has already spent \$1 = area OABC in the diagram. From part (a), he started with m = \$2. So he has \$1 left now.

Thus his problem is to maximize $\left(\frac{1}{4}+t\right)^{1/2} x_2^{1/2}$ s.t. $\$1 = \underset{\substack{\nearrow \\ \text{price of} \\ \text{extra cheese}}}{3}\;\underset{\substack{\uparrow \\ \text{amount of} \\ \text{extra cheese}}}{t} + p_2 x_2$.

$$\mathcal{L} = \left(\frac{1}{4}+t\right)^{1/2} x_2^{1/2} + \lambda\left[1 - 3t - p_2 x_2\right]$$

$$0 = \mathcal{L}'_t = \frac{1}{2}\left(\frac{1}{4}+t\right)^{-1/2} x_2^{1/2} - 3\lambda \Rightarrow \lambda = \frac{1}{6}\left(\frac{1}{4}+t\right)^{-1/2} x_2^{1/2}$$

$$0 = \mathcal{L}'_2 = \frac{1}{2}\left(\frac{1}{4}+t\right)^{1/2} x_2^{-1/2} - \lambda p_2 \Rightarrow \lambda = \frac{1}{2p_2}\left(\frac{1}{4}+t\right)^{1/2} x_2^{-1/2}$$

$$\lambda = \frac{1}{6}\left(\frac{1}{4}+t\right)^{-1/2} x_2^{1/2} = \frac{1}{2p_2}\left(\frac{1}{4}+t\right)^{1/2} x_2^{-1/2}$$

$$x_2 = \frac{3}{p_2}\left(\frac{1}{4}+t\right).$$

Substituting into the budget constraint,

$$1 = 3t + p_2 \cdot \frac{3}{p_2}\left(\frac{1}{4}+t\right) = 3t + \frac{3}{4} + 3t = 6t + \frac{3}{4}$$

$$\frac{1}{4} = 6t \Rightarrow t = \frac{1}{24}.$$

This is the amount of extra cheese he will buy. The amount of total cheese he will buy is thus $\frac{1}{4} + \frac{1}{24} = \frac{6+1}{24} = \frac{7}{24} \approx 0.29.$

(ii) Re-working the optimization problem in part (i) with "p_1" replacing "3," the first-order conditions would become

$$\left.\begin{aligned} 0 = \mathcal{L}'_t &= \tfrac{1}{2}\left(\tfrac{1}{4}+t\right)^{-1/2} x_2^{1/2} - p_1\lambda \Rightarrow \lambda = \tfrac{1}{2p_1}\left(\tfrac{1}{4}+t\right)^{-1/2} x_2^{1/2} \\ 0 = \mathcal{L}'_2 &= \tfrac{1}{2}\left(\tfrac{1}{4}+t\right)^{1/2} x_2^{-1/2} - p_2\lambda \Rightarrow \lambda = \tfrac{1}{2p_2}\left(\tfrac{1}{4}+t\right)^{1/2} x_2^{-1/2} \end{aligned}\right\}$$

$$\frac{1}{2p_1}\left(\frac{1}{4}+t\right)^{-1/2} x_2^{1/2} = \frac{1}{2p_2}\left(\frac{1}{4}+t\right)^{1/2} x_2^{-1/2}$$

$$p_2 x_2 = p_1 \left(\frac{1}{4}+t\right)$$

$$x_2 = \frac{p_1}{p_2}\left(\frac{1}{4}+t\right).$$

Substituting into the budget constraint,

$$1 = p_1 t + p_2 \cdot \frac{p_1}{p_2}\left(\frac{1}{4}+t\right) = p_1 t + p_1\left(\frac{1}{4}+t\right) = p_1 t + \frac{1}{4}p_1 + p_1 t$$

$$= 2p_1 t + \frac{1}{4}p_1$$

$$1 - \frac{1}{4}p_1 = 2p_1 t \Rightarrow t = \frac{1-\frac{1}{4}p_1}{2p_1} = \frac{1}{2p_1} - \frac{1}{8}$$

For confirmation: $p_1 = 3 \Rightarrow t = \frac{1}{2\cdot 3} - \frac{1}{8} = \frac{1}{6} - \frac{1}{8} = \frac{4-3}{24} = \frac{1}{24}$

as before.

$p_1 = 2 \Rightarrow t = \frac{1}{2\cdot 2} - \frac{1}{8} = \frac{1}{4} - \frac{1}{8} = \frac{2-1}{8} = \frac{1}{8}$

Total cheese is $\frac{1}{8} + \frac{1}{4} = \frac{1}{8} + \frac{2}{8} = \frac{3}{8} = 0.375$

$p_1 = 1 \Rightarrow t = \frac{1}{2\cdot 1} - \frac{1}{8} = \frac{1}{2} - \frac{1}{8} = \frac{4-1}{8} = \frac{3}{8}$

Total cheese is $\frac{3}{8} + \frac{1}{4} = \frac{3+2}{8} = \frac{5}{8} = 0.625$

These points generate the dotted line and the small circles in the graph drawn a few pages ago.

(iii) It is true that at a price of \$4/lb. the consumer would be willing and able to spend the area under AB to buy the 1/4 lb. of cheese. However if after doing that the price fell to \$3/lb., the demand curve will shift down to RW, and the extra amount of money spent on cheese will be the area under ER, not EF.

The demand curve derived in part (a) assumed the consumer paid one price for all units of cheese. If that assumption is violated then that demand curve is invalid.

The new demand curve is lower than the old one because the consumer had to spend, in order to buy the first 1/4 lb. of cheese, more than $\$3/\text{lb} \cdot \frac{1}{4}\text{lb.} = \$3/4$, meaning his remaining income after buying the first 1/4 lb. of cheese is lower than it would otherwise have been, and this lower remaining income has an <u>income effect</u> which reduces the demand for cheese, meaning that cheese is a normal good for this consumer, and that the demand curve shifts down. It would shift down again if after buying ER the price fell further, to \$2/lb; then shift down yet again if the price fell to \$1/lb.

over →

We are of course used to demand curves not shifting when prices change, but that assumes a uniform price for all units of cheese and a demand curve obtained by assuming a uniform price for all units of cheese. If there is a non-uniform price for all units of cheese, a demand curve drawn assuming a uniform price for all units of cheese will not behave as expected.

Exam 1
1994
Question 1

①

Answer all of the following five questions.

1. Suppose a consumer's utility function u is given by $u(\mathbf{x}) = x_1\, x_2^2$ where x_1 and x_2 are amounts of two commodities consumed.
   (a) Is this utility function concave?
   (b) Find the utility-maximizing demand for x_1 and for x_2.
   (c) Are the second-order sufficient conditions for a maximum satisfied in part (b)?

Exam 1
1994
Answer 1

Answers to Exam 1, Econ. 621, Winter 1994

1. a. $u(\underset{\sim}{x}) = x_1 \, x_2^2$.

$$\frac{\partial u}{\partial x_1} = x_2^2 \qquad \frac{\partial u}{\partial x_2} = 2x_1x_2$$

$$\frac{\partial^2 u}{\partial x_1^2} = 0 \qquad \frac{\partial^2 u}{\partial x_2^2} = 2x_1$$

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = 2x_2$$

So the Hessian $\nabla^2 u(\underset{\sim}{x}) = \begin{bmatrix} 0 & 2x_2 \\ 2x_2 & 2x_1 \end{bmatrix}$

Newer notation:

$\tilde{D}_1 = 0$ and $2x_1 > 0$ so not concave $= \Delta_1$

$\tilde{D}_2 = -4x_2^2 < 0$ (also not convex) $= \Delta_2$

(The fact that $\tilde{D}_2 < 0$ is enough to conclude that u is not concave.)

b. $\max \; x_1 \, x_2^2$ s.t. $p_1x_1 + p_2x_2 = m$

$$\mathcal{L} = x_1 x_2^2 + \lambda(m - p_1x_1 - p_2x_2)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 = x_2^2 - \lambda p_1$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 = 2x_1x_2 - \lambda p_2$$

$$\Rightarrow \frac{x_2^2}{2x_1x_2} = \frac{p_1}{p_2} \text{ so } \frac{x_2}{2x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{p_1}{p_2}2x_1$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 = m - p_1x_1 - p_2x_2$$

and therefore $m = p_1x_1 + p_2\left(\frac{p_1}{p_2}2x_1\right)$

$$= p_1x_1 + 2p_1x_1 = 3p_1x_1$$

$\boxed{\frac{m}{3p_1} = x_1}$ and $x_2 = \frac{p_1}{p_2} 2 \left[\frac{m}{3p_1}\right]$

$$\Rightarrow \boxed{x_2 = \frac{2m}{3p_2}}.$$

Remark. Demand is unchanged by positive monotonic transformations of the utility function. That is why these demand curves are the same as those from the standard Cobb-Douglas example $\hat{u} = x_1^{1/3} x_2^{2/3}$ (since u here is simply $(\hat{u})^3$).

Exam 1
1994
Answer 1 cont.

$$\mathcal{L}_{\lambda\lambda} = 0 \quad \mathcal{L}_{\lambda 1} = -p_1 \quad \mathcal{L}_{\lambda 2} = -p_2$$

$$\mathcal{L}_{11} = 0 \quad \mathcal{L}_{12} = 2x_2$$

$$\mathcal{L}_{22} = 2x_1$$

$$\nabla^2\mathcal{L} = \begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & 0 & 2x_2 \\ -p_2 & 2x_2 & 2x_1 \end{bmatrix}$$

$m = 1$ (# of constraints)

$2m+1 = 3$. So check $|\nabla^2\mathcal{L}|$.

For a maximum, this should have sign $(-1)^{m+1} = (-1)^2 > 0$.

$$|\nabla^2\mathcal{L}| = +p_1 \begin{vmatrix} -p_1 & 2x_2 \\ -p_2 & 2x_1 \end{vmatrix} - p_2 \begin{vmatrix} -p_1 & 0 \\ -p_2 & 2x_2 \end{vmatrix}$$

$$= p_1\left[-2p_1x_1 + 2p_2x_2\right] - p_2\left(-2p_1x_2\right)$$

$= -2p_1^2x_1 + 2p_1p_2x_2 + 2p_1p_2x_2$ and substituting the optimal values,

(It would have been better to combine the last two terms into 4 p_1 p_2 x_2.)

$$= -2p_1^2\left(\frac{m}{3p_1}\right) + 2p_1p_2\left(\frac{2m}{3p_2}\right) + 2p_1p_2\left(\frac{2m}{3p_2}\right)$$

$$= \frac{-2}{3}p_1m + \frac{4}{3}p_1m + \frac{4}{3}p_1m$$

$= 2p_1m > 0$ so the 2^{ND} order condition for a maximum is fulfilled.

Final Exam
2000 ①
Question 5

5. If a consumer has a standard budget constraint and a utility function $u(\mathrm{x}) = x_1^\alpha + x_2^\alpha$, what conditions on α have to be satisfied if the consumer is to be able to maximize utility at an interior point? (Hint: second-order conditions.)

5) $\max\ u(\underset{\sim}{x}) = x_1^\alpha + x_2^\alpha$ s.t. $p_1 x_1 + p_2 x_2 = m$.

$$\mathcal{L} = x_1^\alpha + x_2^\alpha + \lambda\,(m - p_1 x_1 - p_2 x_2)$$

$$\text{F.O.C.} \begin{cases} 0 = \partial\mathcal{L}/\partial\lambda = m - p_1 x_1 - p_2 x_2 \\ 0 = \dfrac{\partial \mathcal{L}}{\partial x_1} = \alpha x_1^{\alpha-1} - p_1 \lambda \\ 0 = \dfrac{\partial \mathcal{L}}{\partial x_2} = \alpha x_2^{\alpha-1} - p_2 \lambda \end{cases}$$

$$\nabla^2\mathcal{L} = \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda x_1} & \mathcal{L}_{\lambda x_2} \\ \mathcal{L}_{x_1\lambda} & \mathcal{L}_{x_1 x_1} & \mathcal{L}_{x_1 x_2} \\ \mathcal{L}_{x_2\lambda} & \mathcal{L}_{x_2 x_1} & \mathcal{L}_{x_2 x_2} \end{bmatrix} = \begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & \alpha(\alpha-1)x_1^{\alpha-2} & 0 \\ -p_2 & 0 & \alpha(\alpha-1)x_2^{\alpha-2} \end{bmatrix}$$

S.O.C.:

$D_{2m+1}, \dots, D_{m+n}$ of $\nabla^2\mathcal{L}$ should alternate in sign beginning with $(-1)^{m+1}$.

$\downarrow \qquad \downarrow \qquad \downarrow$

$D_{2(1)+1} = D_3 \qquad D_{1+2} = D_3 \qquad (-1)^{1+1} = (-1)^2 = +1$

So we want $|\nabla^2\mathcal{L}| > 0$.

$$|\nabla^2\mathcal{L}| = -p_1^2\,\alpha(\alpha-1)\,x_2^{\alpha-2} - p_2^2\,\alpha(\alpha-1)\,x_1^{\alpha-2}$$ expanding along the first column

$$= \underset{\overset{\uparrow}{+}}{\alpha}\,(1-\alpha)\,\underbrace{\left[p_1^2 x_2^{\alpha-2} + p_2^2 x_1^{\alpha-2}\right]}_{+}$$

Therefore, $|\nabla^2\mathcal{L}|$ will satisfy the S.O.C. when $\alpha \in (0,1)$ because then $\alpha(1-\alpha)$ will be positive.

2. I suggest you read both parts of this question before you begin to work on the first part.

   (a) Suppose a consumer has a quasiconcave utility function $u(\mathbf{x})$ where $\mathbf{x} \in \mathbf{R}^n$. Prove that *if*

      $\mathbf{x}^*$ satisfies *the first-order conditions* for the problem

$$\max_{\mathbf{x}} u(\mathbf{x}) \quad \text{s.t. } \mathbf{p} \cdot \mathbf{x} = m \tag{1}$$

      where $\mathbf{p}$ are the prices (which the consumer takes as given) and m is income,

   then

      $\mathbf{x}^*$ must actually solve (1).

   (b) Show that if the consumer does not take $\mathbf{p}$ as given, but rather has some influence over it—say he receives lower prices for a particular commodity if he buys a great deal of it—then it is no longer necessarily true that "if $\mathbf{x}^*$ satisfies the first-order conditions for (1) then $\mathbf{x}^*$ must actually solve (1)."

2005 Qualifier, Section 2

② a) The Lagrangian is $\mathcal{L} = u(\underset{\sim}{x}) + \lambda(m - \underset{\sim}{p}\cdot\underset{\sim}{x})$. Although the question only asks for $\underset{\sim}{x} \in \mathbb{R}^3$, I'll give the general answer, that is for $\underset{\sim}{x} \in \mathbb{R}^n$.

F.O.C.:
$$0 = \partial\mathcal{L}/\partial\lambda = m - \underset{\sim}{p}\cdot\underset{\sim}{x}$$
$$0 = \partial\mathcal{L}/\partial x_1 = u_1' - \lambda p_1$$
$$\vdots$$
$$0 = \partial\mathcal{L}/\partial x_n = u_n' - \lambda p_n$$

An $\underset{\sim}{x}^*$ satisfying the F.O.C. will actually solve the optimization problem if it satisfies the second-order sufficient conditions for an optimum:

$$\nabla^2\mathcal{L} = \begin{bmatrix} \mathcal{L}''_{\lambda\lambda} & \mathcal{L}''_{\lambda x_1} & \cdots & \mathcal{L}''_{\lambda x_n} \\ \mathcal{L}''_{x_1\lambda} & \mathcal{L}''_{x_1x_1} & \cdots & \mathcal{L}''_{x_1x_n} \\ \vdots & & & \\ \mathcal{L}''_{x_n\lambda} & \mathcal{L}''_{x_nx_1} & \cdots & \mathcal{L}''_{x_nx_n} \end{bmatrix} = \begin{bmatrix} 0 & -p_1 & \cdots & -p_n \\ -p_1 & u''_{11} & \cdots & u''_{1n} \\ \vdots & \vdots & & \vdots \\ -p_n & u''_{n1} & \cdots & u''_{nn} \end{bmatrix}$$

but since from the F.O.C., $p_i = u_i'/\lambda$, we have

$$\nabla^2\mathcal{L} = \begin{bmatrix} 0 & -u_1'/\lambda & \cdots & u_n'/\lambda \\ -u_1'/\lambda & u''_{11} & \cdots & u''_{1n} \\ \vdots & \vdots & & \vdots \\ -u_n'/\lambda & u''_{n1} & & u''_{nn} \end{bmatrix}.$$

The S.O.C. are (with $m = 1$): D_{2m+1} of $\nabla^2\mathcal{L}$ has the sign of $(-1)^{m+1}$ $\Leftrightarrow$

D_3 of $\nabla^2\mathcal{L}$ " " " " ⊕

then $D_4 < 0$, $D_5 > 0$, etc.:

$$0 < D_3 \text{ of } \nabla^2\mathcal{L} = \begin{vmatrix} 0 & -u_1'/\lambda & -u_2'/\lambda \\ -u_1'/\lambda & u''_{11} & u''_{12} \\ -u_2'/\lambda & u''_{21} & u''_{22} \end{vmatrix} = \underbrace{\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)}_{\text{positive}} \begin{vmatrix} 0 & u_1' & u_2' \\ u_1' & u''_{11} & u''_{12} \\ u_2' & u''_{21} & u''_{22} \end{vmatrix}$$

$$0 > D_4 \text{ of } \nabla^2 \mathcal{L} = \begin{vmatrix} 0 & -u_1'/\lambda & -u_2'/\lambda & -u_3'/\lambda \\ -u_1'/\lambda & u_{11}'' & u_{12}'' & u_{13}'' \\ -u_2'/\lambda & u_{21}'' & u_{22}'' & u_{23}'' \\ -u_3'/\lambda & u_{31}'' & u_{32}'' & u_{33}'' \end{vmatrix}.$$

$$= \underbrace{\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)}_{\text{positive}} \begin{vmatrix} 0 & u_1' & u_2' & u_3' \\ u_1' & u_{11}'' & u_{12}'' & u_{13}'' \\ u_2' & u_{21}'' & u_{22}'' & u_{23}'' \\ u_3' & u_{31}'' & u_{32}'' & u_{33}'' \end{vmatrix}$$

etc.

But since $\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right) > 0$, these are just the conditions for u to be quasiconcave, which was already assumed to be true in the question.

b) The F.O.C. are

$$0 = \partial\mathcal{L}/\partial\lambda = m - \underset{\sim}{p} \cdot \underset{\sim}{x} \quad \text{as before, but}$$

$$0 = \partial\mathcal{L}/\partial x_i = u_i' - \lambda p_i - \lambda \frac{\partial p_i}{\partial x_i} x_i$$

$$= u_i' - \lambda p_i - \lambda p_i' x_i \quad \text{which differs from before.}$$

$$\nabla^2\mathcal{L} = \begin{bmatrix} 0 & -p_1 - p_1' x_1 & \cdots \\ -p_1 - p_1' x_1 & u_i'' - \lambda p_i' - \lambda p_i'' x_i - \lambda p_i' & \cdots \\ \vdots & & \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -u_1'/\lambda & \cdots \\ -u_1'/\lambda & u_1'' - 2\lambda p_1' - \lambda p_1'' x_1 & \cdots \\ \vdots & \vdots & \end{bmatrix}.$$

Unlike before, this has no simple relation to the conditions for u to be quasiconcave, so assuming u to be quasiconcave does not assure that $\underset{\sim}{x}^*$ satisfies the S.O.C.

2. **[14 points]** Suppose a person consumes two goods, x and y. The price of x is \$0.50 per unit (that is, \$1/2 per unit). The price of y reflects a volume discount, and is

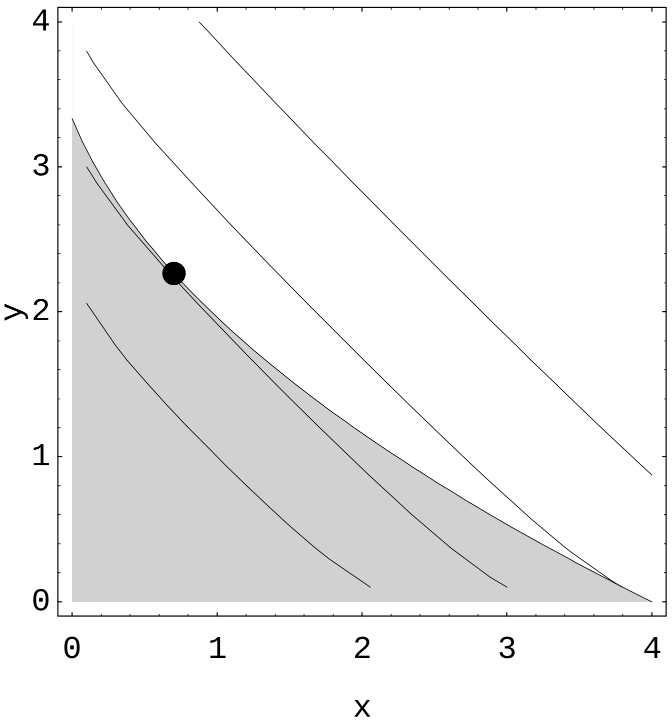
$$1 - 0.12y$$

as long as that is positive. The consumer's income is \$2. This consumer's affordable set is the shaded area in the graph below.

Suppose this consumer's utility function is

$$x + y + \tfrac{1}{10}\ln x + \tfrac{1}{10}\ln y\,.$$

Some of this consumer's indifference curves are shown in the graph below.



(a) Show that the utility function is strictly concave.

(b) Show that any (x^*, y^*, λ^*) satisfying

$$\begin{aligned}
2 &= -\frac{1}{10} \cdot \frac{2.4y^2 - 10y}{4.8y^2 - 10y + 1} + y - 0.12y^2 \\
x &= -\frac{1}{5} \cdot \frac{2.4y^2 - 10y}{4.8y^2 - 10y + 1} \\
\lambda &= 2 + \frac{1}{5x}
\end{aligned}$$

(or an equivalent set of equations) satisfies the first-order conditions for utility maximization. (These equations should have been written as functions of x^*, y^*, and λ^*, but I omitted the asterisks for enhanced legibility.) Do not try to solve the system for y^*, x^*, or λ^*.

(c) What sufficient condition would ensure that a vector (x^*, y^*, λ^*) satisfying the conditions of part (b) actually is a maximum? Your answer should be a function of x^*, y^*, and λ^*, but you can omit the asterisks for enhanced legibility.

(d) It can be shown that $(x^*, y^*, \lambda^*) = (0.704259, 2.26171, 2.28399)$ satisfies the conditions of part (b). This point is marked as a dot on the graph. However, it violates the condition of part (c) (do not prove this; take my word for it). What is the implication of this violation? Could you have predicted this violation?

(e) What do you guess the consumer's utility-maximizing bundle is? Why? (I am asking for a *guess* here, not a mathematical investigation.)

Summer 2012, Qualifying Exam, Section 1 Qu. 2

②

$P_x = \frac{1}{2}$

$P_y = 1 - 0.12y$ as long as this is positive

$u(x,y) = x + y + \frac{1}{10}\ln x + \frac{1}{10}\ln y$

a) $u'_x = 1 + \frac{1}{10x}$ $\qquad u'_y = 1 + \frac{1}{10y}$

$u''_{xx} = \frac{-1}{10x^2}$ $\qquad u''_{yy} = \frac{-1}{10y^2}$

$u''_{xy} = 0$

$$\nabla^2 u(x,y) = \begin{bmatrix} -1/(10x^2) & 0 \\ 0 & -1/(10y^2) \end{bmatrix}$$

A sufficient condition for u to be strictly concave is that

D_1 of $\nabla^2 u$ be < 0 and

D_2 of $\nabla^2 u$ be > 0.

Here D_1 of $\nabla^2 u$ is $\frac{-1}{10x^2} < 0$ and

D_2 of $\nabla^2 u$ is $\frac{-1}{10x^2} \cdot \frac{-1}{10y^2} - 0 = \frac{1}{100x^2y^2} > 0$

So u is strictly concave. Optional: so it is also concave, and quasiconcave.

b) The consumer's problem is to

$$\max u(x,y) \text{ s.t. } 2 = p_x x + p_y y$$
$$= \frac{1}{2}x + (1-0.12y)y$$
$$= \frac{1}{2}x + y - 0.12y^2.$$

$$\mathcal{L} = x + y + \frac{1}{10}\ln x + \frac{1}{10}\ln y + \lambda\left[2 - \frac{1}{2}x - y + 0.12y^2\right]$$

F.O.C.

$$0 = \partial\mathcal{L}/\partial\lambda = 2 - \frac{1}{2}x - y + 0.12y^2 \qquad (1)$$

$$0 = \partial\mathcal{L}/\partial x = 1 + \frac{1}{10x} + \lambda\left[\frac{-1}{2}\right] \qquad (2)$$

$$0 = \partial\mathcal{L}/\partial y = 1 + \frac{1}{10y} + \lambda\left[-1 + 0.24y\right] \qquad (3)$$

(2) only involves λ and x, like the third equation given in the question, so let's see if we can obtain it:

$$(2) \Rightarrow \quad \frac{\lambda}{2} = 1 + \frac{1}{10x}$$

$$(4) \qquad \lambda = 2 + \frac{1}{5x}.$$ Yes, this confirms the third equation.

The next-simplest equation is (3), because unlike (1), (3) does not contain any squared terms. So we'll work on it next, substituting in λ from (4):

$$0 = 1 + \frac{1}{10y} + \left(2 + \frac{1}{5x}\right)\left(-1 + 0.24y\right).$$

It should be simple to solve this for x in terms of y, obtaining the

Second equation of the answer:

$$0 = \frac{10y}{10y} + \frac{1}{10y} + \left(2 + \frac{1}{5x}\right)(0.24y - 1)$$

$$-\frac{1+10y}{10y} = \left(2 + \frac{1}{5x}\right)(0.24y - 1)$$

$$\frac{-1}{10y}\ \frac{1+10y}{0.24y-1} = 2 + \frac{1}{5x}$$

$$\frac{-1}{5x} = 2 + \frac{1}{10y}\ \frac{1+10y}{0.24y-1} = 2 + \frac{1+10y}{2.4y^2 - 10y}$$

$$= \underbrace{\frac{4.8y^2 - 20y}{2.4y^2 - 10y}}_{\text{this is "2"}} + \frac{1+10y}{2.4y^2-10y} = \frac{4.8y^2 - 10y + 1}{2.4y^2 - 10y}$$

$$\Rightarrow x = \frac{-1}{5}\ \frac{2.4y^2 - 10y}{4.8y^2 - 10y + 1}, \qquad (5)$$

confirming the second equation.

The remaining equation must come from (1) since we have not used it yet. (1) implies

$$2 = \frac{1}{2}x + y - 0.12y^2 \text{ ; from (5), this is}$$

$$2 = \frac{-1}{10}\ \frac{2.4y^2 - 10y}{4.8y^2 - 10y + 1} + y - 0.12y^2,$$

confirming the question's remaining equation.

c) the signs of D_{2m+1} of $\nabla^2\mathcal{L}, \dots, D_{m+n}$ of $\nabla^2\mathcal{L}$ should alternate, starting with the sign of $(-1)^{m+1}$. Here $n = 2$ and $m = 1$, so the sign of D_3 of ∇^2 should be $(-1)^{1+1} > 0$.

From (1), (2), and (3):

$$\mathcal{L}''_{\lambda\lambda} = 0$$

$$\mathcal{L}''_{x\lambda} = \frac{-1}{2} \qquad \mathcal{L}''_{xx} = \frac{-1}{10x^2}$$

$$\mathcal{L}''_{y\lambda} = -1 + 0.24y \qquad \mathcal{L}''_{yx} = 0 \qquad \mathcal{L}''_{yy} = \frac{-1}{10y^2} + 0.24\lambda$$

$$\Rightarrow \nabla^2\mathcal{L} = \begin{bmatrix} 0 & -1/2 & 0.24y-1 \\ \frac{-1}{2} & \frac{-1}{10x^2} & 0 \\ 0.24y-1 & 0 & \frac{-1}{10y^2} + 0.24\lambda \end{bmatrix}$$

D_3 of $\nabla^2\mathcal{L} = |\nabla^2\mathcal{L}| =$ (expanding by the first row)

$$(-1)^{1+2}\left(\frac{-1}{2}\right)\left[\frac{-1}{2}\left(\frac{-1}{10y^2} + 0.24\lambda\right) - 0\right]$$

$$+ (-1)^{1+3}(0.24y-1)\left[0 - \frac{-1}{10x^2}(0.24y-1)\right]$$

$$= \frac{1}{2}\cdot\frac{-1}{2}\left(0.24\lambda - \frac{1}{10y^2}\right) + (0.24y-1)\frac{1}{10x^2}(0.24y-1)$$

$$= \frac{1}{4}\left(\frac{1}{10y^2} - 0.24\lambda\right) + \frac{1}{10x^2}(0.24y-1)^2.$$

This should be positive for the candidate (x, y, λ) to be a maximum.

d) Optional: confirmation that this point satisfies the First Order Conditions (part b) is given in Out [6], Out [7], and Out [8] of the accompanying Mathematica program. Confirmation that it violates the Second Order Conditions (part c) is given in Out [10] and Out [11] of that program.

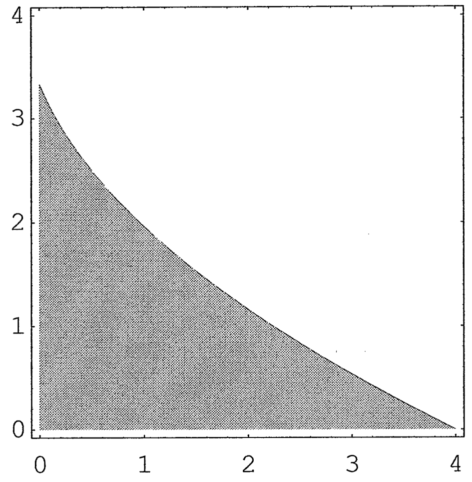
Not optional: The implication is that this point is a local utility minimum. This could have been predicted just by looking at the graph, where motion along the budget constraint in any direction away from the black circle leads to a higher indifference curve.

e) From the graph, it looks like it is a corner solution at the lower right-hand corner. ("Out [37]" of the Mathematica program shows this point is (4,0).)

Optional: The Mathematica program shows that appearances are somewhat deceiving. $u(4,0) = -\infty$ ("Out [38]"). The graph on p. 6 of the Mathematica printout shows three relevant points satisfying the FOC's (the fourth point has $x < 0$); I labeled them A, B, and C. A and C satisfy the S.O.C. (Out [29] and Out [31]), so are local maxima. Point C is a global max because it has the highest utility level (Out [33] exceeds Out [35]). It is at (3.80, 0.099651), so close to (4,0) but not at (4,0).

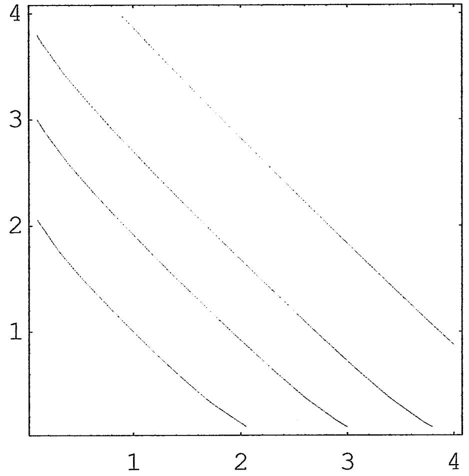
In[1]:=

```
BudgetConstraintGraph=
ContourPlot[(1/2) x + (1 - 0.12 y) y - 2, {x,0,4},{y,0,4},Contours->{0},
  ColorFunction->(GrayLevel[(#+.6)^(0.2)/1.1]&)(*,DisplayFunction->Identity*),PlotPoints->30];
```



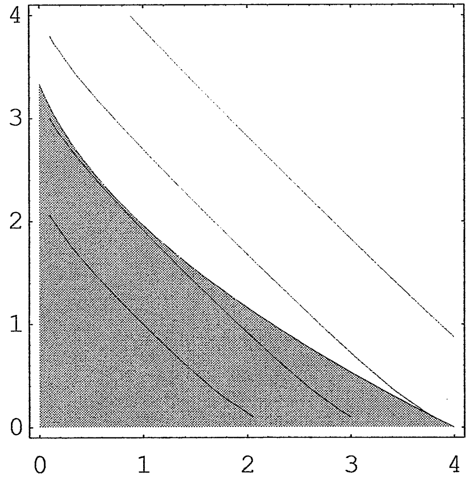
In[2]:=

```
IndifferenceCurveGraph=
ContourPlot[Log[x]/10 + Log[y]/10 + x+y,{x,0.1,4},{y,0.1,4},(*DisplayFunction->Identity,*)
    Contours->{2,2.98,3.8,5},ContourShading->False,PlotPoints->15];
```



In[3]:=

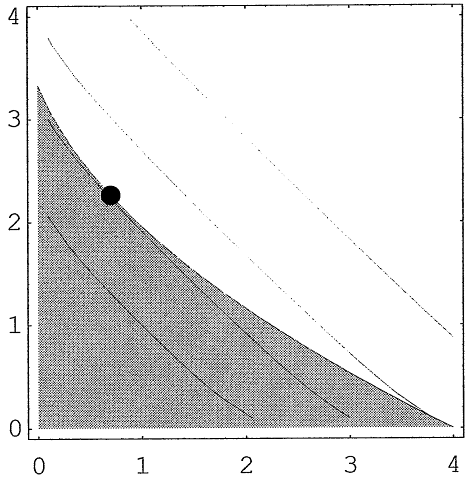
```
Show[BudgetConstraintGraph,IndifferenceCurveGraph];
```



```
In[4]:=
  Clear[y];
  NSolve[2==(-1/10)(  (2.4 y^2 - 10y) / (4.8 y^2 - 10 y + 1) ) + y - 0.12 y^2,
  y]
  y=y/.%[[2]]
Out[5]=
  {{y -> 0.0996514}, {y -> 2.26171}, {y -> 3.12428}, {y -> 4.93102}}
Out[6]=
  2.26171
In[7]:=
  x=(-1/5) ( (2.4 y^2 - 10y) / (4.8 y^2 - 10 y + 1) )
  lambda = 2 + 1/(5 x)
  {2 - (1/2) x - y + 0.12 y^2,
  1 + (1/(10 x)) + lambda (-1/2),
  1 + (1/(10 y)) + lambda (-1 + 2*0.12 y)}
Out[7]=
  0.704259
Out[8]=
  2.28399
Out[9]=
               -15                 -16
  {-4.88498 10    , 0., -2.22045 10   }
In[10]:=
  (1/4)( (1/(10 y^2)) - 0.24 lambda) + (1/(10 x^2)) (0.24 y -1)^2
Out[10]=
  -0.0900087
In[11]:=
  Det[{   {0,-1/2, -1+0.24y},   {-1/2, -1/(10 x^2), 0},
       {-1+0.24y, 0, (-1/(10 y^2)) + 0.24 lambda}}]
Out[11]=
  -0.0900087
```

In[12]:=

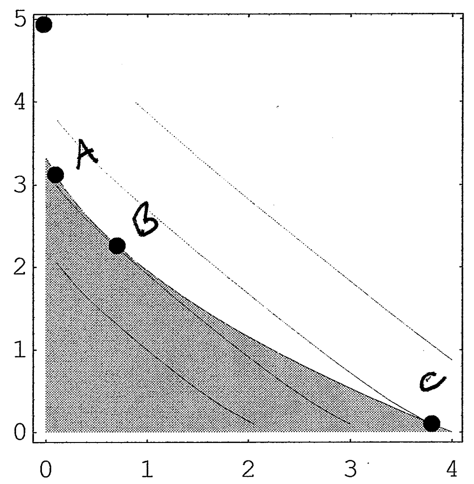
```
Show[Graphics[{AbsolutePointSize[15],Point[{x,y}]}],DisplayFunction->Identity];
Show[BudgetConstraintGraph,IndifferenceCurveGraph,%];
```



In[14]:=

```
Clear[y];
NSolve[2==(-1/10)(  (2.4 y^2 - 10y) / (4.8 y^2 - 10 y + 1) ) + y - 0.12 y^2,
y];
firsty=y/.%[[1]];
secondy=y/.%%[[2]];
thirdy=y/.%%%[[3]];
fourthy=y/.%%%%[[4]];
Clear[x];
x[y_] := (-1/5) ( (2.4 y^2 - 10y) / (4.8 y^2 - 10 y + 1) )
Clear[lambda];
lambda[x_] := 2 + 1/(5 x)
firstx=x[firsty]; secondx=x[secondy]; thirdx=x[thirdy];
fourthx=x[fourthy];
Show[Graphics[{PointSize[0.08],
{Point[{firstx,firsty}], Point[{secondx,secondy}], Point[{thirdx,thirdy}],
 Point[{fourthx,fourthy}] }
}],DisplayFunction->Identity];
Show[BudgetConstraintGraph,IndifferenceCurveGraph,%];
{{firstx,firsty}, {secondx,secondy}, {thirdx,thirdy},
 {fourthx,fourthy}}
```

```
General::spell1:
   Possible spelling error: new symbol name "firstx"
     is similar to existing symbol "firsty".
General::spell1:
   Possible spelling error: new symbol name "secondx"
     is similar to existing symbol "secondy".
General::spell1:
   Possible spelling error: new symbol name "thirdx"
     is similar to existing symbol "thirdy".
General::stop:
   Further output of General::spell1
     will be suppressed during this calculation.
General::spell1:
   Possible spelling error: new symbol name "fourthx"
     is similar to existing symbol "fourthy".
```



Out[28]=

```
{{3.80308, 0.0996514}, {0.704259, 2.26171}, {0.0941096, 3.12428},
  {-0.0264489, 4.93102}}
```

In[29]:=

```
Det[{   {0,-1/2, -1+0.24y},   {-1/2, -1/(10 x^2), 0},
     {-1+0.24y, 0, (-1/(10 y^2)) + 0.24 lambda}}]/.{x->firstx,y->firsty,lambda->lambda[firstx]}
Det[{   {0,-1/2, -1+0.24y},   {-1/2, -1/(10 x^2), 0},
     {-1+0.24y, 0, (-1/(10 y^2)) + 0.24 lambda}}]/.{x->secondx,y->secondy,lambda->lambda[secondx]]
Det[{   {0,-1/2, -1+0.24y},   {-1/2, -1/(10 x^2), 0},
     {-1+0.24y, 0, (-1/(10 y^2)) + 0.24 lambda}}]/.{x->thirdx,y->thirdy,lambda->lambda[thirdx]}
```

Out[29]=

```
2.40096
```

Out[30]=

```
-0.0900087
```

Out[31]=

```
0.461712
```

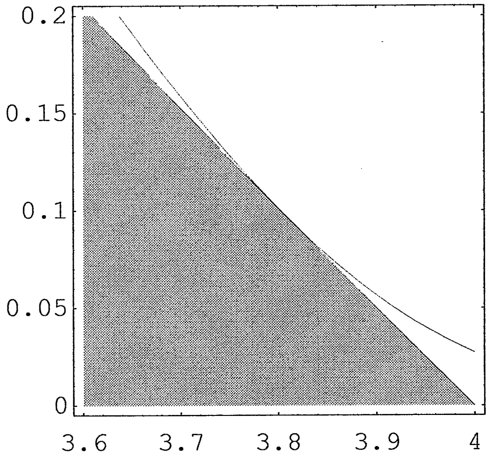
```
In[29]:=
```

```
In[32]:=
   u[x_,y_] := Log[x]/10 + Log[y]/10 + x+y
   u[firstx,firsty]
   u[secondx,secondy]
   u[thirdx,thirdy]
Out[33]=
   3.80571
Out[34]=
   3.01252
Out[35]=
   3.09598
In[36]:=
   (* At what x level is y=0? *)
   NSolve[0== (1/2) x + (1 - 0.12 0) 0 - 2,x]
Out[37]=
   {{x -> 4.}}
In[38]:=
   u[4,0]
Out[38]=
   -Infinity
```

In[39]:=

```
BCpartial=
ContourPlot[(1/2) x + (1 - 0.12 y) y - 2, {x,3.6,4},{y,0,0.2},Contours->{0},
  ColorFunction->(GrayLevel[(#+.6)^(0.2)/1.1]&),DisplayFunction->Identity,PlotPoints->30];
ICpartial=
ContourPlot[Log[x]/10 + Log[y]/10 + x+y,{x,3.6,4},{y,0.001,0.2},DisplayFunction->Identity,
     Contours->{3.80571},ContourShading->False,PlotPoints->15];
Show[BCpartial,ICpartial,DisplayFunction->$DisplayFunction];
```

```
General::spell1:
   Possible spelling error: new symbol name "ICpartial"
    is similar to existing symbol "BCpartial".
```



2. [11 points] Suppose a utility-maximizing consumer does not take prices as given. You should represent the prices of the two commodities x_1 and x_2 he may consume by $p_1(x_1)$ and $p_2(x_2)$, respectively.

   (a) What are the sufficient conditions for a utility maximum? Your answer may include λ^*, x_1^*, and x_2^*.

   (b) Presuming the conditions in (a) are satisfied, what else needs to be true in order for x_1 to be a normal good?

Fall 2010 Ex. 1 Qu. 2

②

The budget constraint becomes $\underset{\substack{\uparrow \\ \text{income}}}{m} = p_1(x_1)\, x_1 + p_2(x_2)\, x_2$,

and the Lagrangian is

$$\mathcal{L} = u(x_1, x_2) + \lambda \left[m - p_1(x_1)\, x_1 - p_2(x_2)\, x_2 \right].$$

F.O.C.

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1(x_1)\, x_1 - p_2(x_2)\, x_2$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_1} = u_1' - \lambda \frac{dp_1}{dx_1} x_1 - \lambda p_1(x_1) \text{ where } u_1' = \frac{\partial u}{\partial x_1};$$

and abbreviating further,

$$= u_1' - \lambda p_1' x_1 - \lambda p_1 = u_1' - \lambda(p_1' x_1 + p_1)$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_2} = u_2' - \lambda p_2' x_2 - \lambda p_2 = u_2' - \lambda(p_2' x_2 + p_2)$$

Then

$$\nabla^2 \mathcal{L} = \begin{bmatrix} \mathcal{L}''_{\lambda\lambda} & \mathcal{L}''_{\lambda x_1} & \mathcal{L}''_{\lambda x_2} \\ \mathcal{L}''_{x_1 \lambda} & \mathcal{L}''_{x_1 x_1} & \mathcal{L}''_{x_1 x_2} \\ \mathcal{L}''_{x_2 \lambda} & \mathcal{L}''_{x_2 x_1} & \mathcal{L}''_{x_2 x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -(p_1' x_1 + p_1) & -(p_2' x_2 + p_2) \\ -(p_1' x_1 + p_1) & u''_{11} - \lambda(p_1'' x_1 + p_1' + p_1') & u''_{12} \\ -(p_2' x_2 + p_2) & u''_{21} & u''_{22} - \lambda(p_2'' x_2 + p_2' + p_2') \end{bmatrix}$$

a) $n = 2$ # variables
$m = 1$ # constraints

$D_{2m+1}, \ldots, D_{m+n}$ of $\nabla^2 \mathcal{L}$ need to alternate in sign beginning with $(-1)^{m+1}$ to satisfy the sufficient condition for a maximum.

So we need D_3 $(= D_{2m+1} = D_{m+n})$ to have the sign of $(-1)^{1+1} > 0$.

For example, expanding along the first row, we want

$$0 < (-1)^{1+2}(-1)(p_1'x_1+p_1)\Big[-(p_1'x_1+p_1)[u''_{22} - \lambda(p_2''x_2+2p_2')] + u''_{12}(p_2'x_2+p_2)\Big]$$
$$+ (-1)^{1+3}(-1)(p_2'x_2+p_2)\Big[-(p_1'x_1+p_1)u''_{21} + (p_2'x_2+p_2)[u''_{11} - \lambda(p_1''x_1+2p_1')]\Big]$$

$$= (p_1'x_1+p_1)\{-(p_1'x_1+p_1)[u''_{22} - \lambda(p_2''x_2+2p_2')] + u''_{12}(p_2'x_2+p_2)\}$$
$$-(p_2'x_2+p_2)\{-(p_1'x_1+p_1)u''_{21} + (p_2'x_2+p_2)\{u''_{11} - \lambda(p_1''x_1+2p_1')\}\}.$$

b) Take the differential of the F.O.C.s (m is the only exogenous variable and is the variable of interest):

$$0 = \mathcal{L}''_{\lambda\lambda}\, d\lambda + \mathcal{L}''_{\lambda x_1}\, dx_1 + \mathcal{L}''_{\lambda x_2}\, dx_2 + 1\, dm$$

$$0 = \mathcal{L}''_{x_1\lambda}\, d\lambda + \mathcal{L}''_{x_1x_1}\, dx_1 + \mathcal{L}''_{x_1x_2}\, dx_2 + 0\, dm$$

$$0 = \mathcal{L}''_{x_2\lambda}\, d\lambda + \mathcal{L}''_{x_2x_1}\, dx_1 + \mathcal{L}''_{x_2x_2}\, dx_2 + 0\, dm$$

$$\Rightarrow \quad \nabla^2\mathcal{L} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} -dm \\ 0 \\ 0 \end{bmatrix} \quad \text{or}$$

$$\nabla^2\mathcal{L} \begin{bmatrix} d\lambda/dm \\ dx_1/dm \\ dx_2/dm \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Solve for $\partial x_1/\partial m$ (by Cramer's Rule or any other method):

$$\frac{\partial x_1}{\partial m} = \frac{\begin{vmatrix} 0 & -1 & -(p_2' x_2 + p_2) \\ -(p_1' x_1 + p_1) & 0 & u''_{12} \\ -(p_2' x_2 + p_2) & 0 & u''_{22} - \lambda(p_2'' x_2 + 2p_2') \end{vmatrix}}{|\nabla^2\mathcal{L}|}$$

$$= \frac{(-1)^{1+2}(-1)\left[-(p_1' x_1 + p_1)[u''_{22} - \lambda(p_2'' x_2 + 2p_2')] + (p_2' x_2 + p_2)\, u''_{12}\right]}{|\nabla^2\mathcal{L}|}$$

$$= \frac{-(p_1' x_1 + p_1)[u''_{22} - \lambda(p_2'' x_2 + 2p_2')] - (p_2' x_2 + p_2)\, u''_{12}}{|\nabla^2\mathcal{L}|}.$$

The denominator is positive from part (a); if the numerator is positive also, then $\partial x_1/\partial m > 0$ and the good is normal.

3. Suppose a consumer buys only two goods, x_1 and x_2, from his fixed income m. The prices for the two goods are p_1 and p_2, respectively. Under what conditions will a rise in p_2 cause the consumer's purchases of x_1 to fall? (Your answer may contain λ^*, x_1^*, and x_2^*.)

1999, Exam 1, Qu. 3

(3)

$$\max_{x_1, x_2} u(x_1, x_2) \quad s.t. \quad p_1 x_1 + p_2 x_2 = m$$

Exam 1
1999
Answer 3

$$\mathcal{L} = u(x_1, x_2) + \lambda(m - p_1 x_1 - p_2 x_2)$$

$$0 = \mathcal{L}'_\lambda = m - p_1 x_1 - p_2 x_2 \qquad (1)$$

$$0 = \mathcal{L}'_{x_1} = u'_1(x_1, x_2) - \lambda p_1 \qquad (2)$$

$$0 = \mathcal{L}'_{x_2} = u'_2(x_1, x_2) - \lambda p_2 \qquad (3)$$

endogenous : λ, x_1, x_2

exogenous : p_1, p_2, m

$\uparrow \qquad \uparrow$ unchanging, so set $dp_1 = 0$ and $dm = 0$

Take total differential :

$$\begin{array}{lccccc}
 & d\lambda & dx_1 & dx_2 & dp_2 & \\
(1) \Rightarrow & 0\, d\lambda & -p_1 dx_1 & -p_2 dx_2 & -x_2 dp_2 & = 0 \\
(2) \Rightarrow & -p_1 d\lambda & +u''_{11} dx_1 & +u''_{12} dx_2 & +0\, dp_2 & = 0 \\
(3) \Rightarrow & -p_2 d\lambda & +u''_{21} dx_1 & +u''_{22} dx_2 & -\lambda\, dp_2 & = 0 .
\end{array}$$

Exam 1
1999
Answer 3 cont...

$$\begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{11} & u''_{12} \\ -p_2 & u''_{21} & u''_{22} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \\ \lambda \end{bmatrix} dp_2$$

$$\begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{11} & u''_{12} \\ -p_2 & u''_{21} & u''_{22} \end{bmatrix} \begin{bmatrix} d\lambda / dp_2 \\ dx_1 / dp_2 \\ dx_2 / dp_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \\ \lambda \end{bmatrix}$$. By Cramer's Rule,

$$\frac{dx_1}{dp_2} = \frac{\begin{vmatrix} 0 & x_2 & -p_2 \\ -p_1 & 0 & u''_{12} \\ -p_2 & \lambda & u''_{22} \end{vmatrix}}{\begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{11} & u''_{12} \\ -p_2 & u''_{21} & u''_{22} \end{vmatrix}}$$

$$(-1)^{1+2}(x_2)(-p_1 u''_{22} + p_2 u''_{12})$$
$$+ (-1)^{1+3}(-p_2)(-p_1 \lambda - 0)$$
$$= -x_2(p_2 u''_{12} - p_1 u''_{22}) + p_1 p_2 \lambda$$
$$= \underbrace{-x_2 p_2 u''_{12}}_{\textcircled{?}} + \underbrace{x_2 p_1 u''_{22}}_{\ominus} + \underbrace{p_1 p_2 \lambda}_{\oplus}$$

if u is strictly concave, which it might not be

Note that the S.O.C. for a maximum, since $2m+1 = 3$, are:

D_3 of $\nabla^2 \mathcal{L}$ has the sign of $(-1)^{m+1} = \oplus$ (sufficient condition);

$\hat{\Delta}_3$ of $\nabla^2 \mathcal{L}$ ——— " ——— (necessary condition).

Since D_3 of $\nabla^2 \mathcal{L} = \hat{\Delta}_3$ of $\nabla^2 \mathcal{L} = |\nabla^2 \mathcal{L}|$ = the denominator of dx_1/dp_2, this denominator is $\oplus$. (Only when $\nabla^2 \mathcal{L}$ is a 3×3 matrix is D_3 of $\nabla^2 \mathcal{L} = \hat{\Delta}_3$ of $\nabla^2 \mathcal{L} = |\nabla^2 \mathcal{L}|$.)

The exam asks when dx_1/dp_2 will be negative. Given what we now know about the numerator and denominator of dx_1/dp_2, this will be true when

$$-x_2 p_2 u''_{12} + x_2 p_1 u''_{22} + p_1 p_2 \lambda < 0.$$

Exam 1
1999
Answer 3 cont...

Fall
2006
Final

1. [11 points] Suppose a price-taking consumer receives utility from two goods, x_1 and x_2. How does the consumer's demand for good 1 change when the consumer's income changes infinitesimally?

Answers to Econ. 7005 Final Exam, Fall 2006

①

$$\max u(x_1,x_2) \text{ s.t. } p_1x_1+p_2x_2=m$$

$$\mathcal{L}=u(x_1,x_2)+\lambda(m-p_1x_1-p_2x_2)$$

$$\left.\begin{aligned}0&=\frac{\partial\mathcal{L}}{\partial\lambda}=m-p_1x_1-p_2x_2\\0&=\frac{\partial\mathcal{L}}{\partial x_1}=u_1'-\lambda p_1\\0&=\frac{\partial\mathcal{L}}{\partial x_2}=u_2'-\lambda p_2\end{aligned}\right\}\underset{\Longrightarrow}{\text{take differentials}}\left\{\begin{aligned}0&=\overset{0\,d\lambda}{\swarrow}-p_1dx_1-p_2dx_2+dm\leftarrow\text{if } dp_1=dp_2=0\\0&=-p_1d\lambda+u_{11}''dx_1+u_{12}''dx_2+0\,dm\\0&=-p_2d\lambda+u_{21}''dx_1+u_{22}''dx_2+0\,dm\end{aligned}\right.$$

$$\underset{\sim}{0}=\begin{bmatrix}0&-p_1&-p_2\\-p_1&u_{11}''&u_{12}''\\-p_2&u_{21}''&u_{22}''\end{bmatrix}\begin{bmatrix}d\lambda\\dx_1\\dx_2\end{bmatrix}+\begin{bmatrix}dm\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&-p_1&-p_2\\-p_1&u_{11}''&u_{12}''\\-p_2&u_{21}''&u_{22}''\end{bmatrix}\begin{bmatrix}d\lambda/dm\\dx_1/dm\\dx_2/dm\end{bmatrix}=\begin{bmatrix}-1\\0\\0\end{bmatrix}$$

. Use Cramer's Rule:

$$\frac{dx_1}{dm}=\frac{\begin{vmatrix}0&-1&-p_2\\-p_1&0&u_{12}''\\-p_2&0&u_{22}''\end{vmatrix}}{\begin{vmatrix}0&-p_1&-p_2\\-p_1&u_{11}''&u_{12}''\\-p_2&u_{21}''&u_{22}''\end{vmatrix}}$$

$\leftarrow (-1)^{1+2}(-1)(-p_1u_{22}''+p_2u_{12}'')$

$= -p_1u_{22}''+p_2u_{12}''$

Optional

$\leftarrow$ from 2^{ND} Order Conditions, # variables = 2, # constraints = 1, D_3 of $\nabla^2\mathcal{L}$ should, for a maximum, be $(-1)^{1+1}=(-1)^2=+1$, positive. ($D_{2m+1}=D_3$ and $D_{m+n}=D_3$.)

Optional: From the F.O.C., $p_1 = u_1'/\lambda$ and $p_2 = u_2'/\lambda$, so the numerator of the answer equals $(-u_1' u_{22}'' + u_2' u_{12}'')/\lambda$. Presumably $u_1' > 0$ and $u_2' > 0$. Also, from the Envelope Theorem, $\frac{\partial v}{\partial m} = \frac{\partial \mathcal{L}^*}{\partial m} = \lambda^*$, and since as $m\uparrow$, indirect utility $v\uparrow$, we infer that $\lambda^* > 0$. However, u_{22}'' has an unknown sign (it'd be negative if u were concave but we usually only assume quasiconcavity), and u_{12}'' is certainly of unknown sign. $\left(\frac{\partial x_1}{\partial m} \gtrless 0 \text{ as } x_1 \text{ is } \begin{cases} \text{normal} \\ \text{inferior}. \end{cases} \right)$

Also from the F.O.C., the denominator of dx_1/dm is

$$\begin{vmatrix} 0 & -u_1'/\lambda & -u_2'/\lambda \\ -u_1'/\lambda & u_{11}'' & u_{12}'' \\ -u_2'/\lambda & u_{21}'' & u_{22}'' \end{vmatrix} = \underbrace{\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)}_{\oplus} \underbrace{\begin{vmatrix} 0 & u_1' & u_2' \\ u_1' & u_{11}'' & u_{12}'' \\ u_2' & u_{21}'' & u_{22}'' \end{vmatrix}}_{\oplus \text{ if } u \text{ is quasiconcave}} > 0 \text{ if } u \text{ is quasiconcave.}$$

Denominator. Calculating the determinant:

$$(-1)^{2+1}(-p_1)(-p_1 u_{22}'' + p_2 u_{21}'') + (-1)^{3+1}(-p_2)(-p_1 u_{12}'' + p_2 u_{11}'')$$

$$= p_1(-p_1 u_{22}'' + p_2 u_{21}'') - p_2(-p_1 u_{12}'' + p_2 u_{11}'')$$

$$= -p_1^2 u_{22}'' + p_1 p_2 u_{21}'' + p_1 p_2 u_{12}'' - p_2^2 u_{11}''$$

$$= -p_1^2 u_{22}'' + 2 p_1 p_2 u_{12}'' - p_2^2 u_{11}''.$$

Winter 1994 Exam 1 Qu. 5

5. Suppose a price-taking utility-maximizing consumer receives utility from two goods x_1 and x_2.
   - (a) How does this consumer's demand for good 1 change when p_1 changes infinitesimally?
   - (b) How does this consumer's demand for good 1 change when p_2 changes infinitesimally?
   - (c) Express, as a function of the change in p_1, how this consumer's demand for good 1 changes when: "p_1 and p_2 change simultaneously in such a way that $p_1 + p_2$ is unchanged."

Exam I
1994
Answers

5. $\mathcal{L} = u(x_1, x_2) + \lambda(m - p_1 x_1 - p_2 x_2)$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 = \frac{\partial u}{\partial x_1} - \lambda p_1$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 = \frac{\partial u}{\partial x_2} - \lambda p_2$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 = m - p_1 x_1 - p_2 x_2$$

Short way: $$\left[\nabla^2 \mathcal{L}\right] \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} = - \underset{p_1, p_2, m}{d} \begin{bmatrix} m - p_1 x_1 - p_2 x_2 \\ u_1 - \lambda p_1 \\ u_2 - \lambda p_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda x_1} & \mathcal{L}_{\lambda x_2} \\ \mathcal{L}_{x_1 \lambda} & \mathcal{L}_{x_1 x_1} & \mathcal{L}_{x_1 x_2} \\ \mathcal{L}_{x_2 \lambda} & \mathcal{L}_{x_2 x_1} & \mathcal{L}_{x_2 x_2} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} = - \begin{bmatrix} -x_1 \\ -\lambda \\ 0 \end{bmatrix} dp_1 - \begin{bmatrix} -x_2 \\ 0 \\ -\lambda \end{bmatrix} dp_2$$

$$- \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T dm \Rightarrow$$

$$\begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \lambda \\ 0 \end{bmatrix} dp_1 + \begin{bmatrix} x_2 \\ 0 \\ \lambda \end{bmatrix} dp_2 + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} dm. \quad (1)$$

Long way: Take the total differential of each F.O.C.

$$0 = m - p_1 x_1 - p_2 x_2 \Rightarrow \quad 0 = -p_1\, dx_1 - p_2\, dx_2 + dm - x_1\, dp_1 - x_2\, dp_2$$

$$0 = u_1 - \lambda p_1 \Rightarrow \quad 0 = -p_1\, d\lambda + u_{11} dx_1 + u_{12} dx_2 - \lambda\, dp$$

$$0 = u_2 - \lambda p_2 \Rightarrow \quad 0 = -p_2\, d\lambda + u_{12}\, dx_1 + u_{22} dx_2 - \lambda\, dp_2$$

Rearranging gives (1).

a. $$\begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \lambda \\ 0 \end{bmatrix} dp_1$$

$$\frac{dx_1}{dp_1} = \frac{\begin{vmatrix} 0 & x_1 & -p_2 \\ -p_1 & \lambda & u_{12} \\ -p_2 & 0 & u_{22} \end{vmatrix}}{\begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{21} & u_{22} \end{vmatrix}} = \frac{+p_1\, x_1 u_{22} - p_2 \left(x_1 u_{12} + \lambda p_2\right)}{+p_1\left(-p_1 u_{22} + p_2 u_{12}\right) - p_2\left(-p_1 u_{12} + p_2 u_{11}\right)}$$

$$= \frac{p_1\, x_1 u_{22} - p_2\, x_1 u_{12} - \lambda p_2^2}{-p_1^2 u_{22} + 2 p_1 p_2 u_{12} - p_2^2 u_{11}}$$

Note: Second Order conditions for a maximum are: $m = 1$, prin. minor of $\nabla^2 \mathcal{L}$ of order $2m+1 = 3$ has sign $(-1)^{m+1} > 0$. So this determinant is > 0.

b. $$\begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} d\lambda / dp_2 \\ dx_1 / dp_2 \\ dx_2 / dp_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \\ \lambda \end{bmatrix}$$

$$\frac{dx_1}{dp_2} = \frac{\begin{vmatrix} 0 & x_2 & -p_2 \\ -p_1 & 0 & u_{12} \\ -p_2 & \lambda & u_{22} \end{vmatrix}}{} = \frac{-x_2\left(-p_1 u_{22} + p_2 u_{12}\right) + p_2 p_1 \lambda}{\text{same as in (a)}}$$

Exam 1
1994
Answer 5 cont.

$$= \frac{p_1 x_2 u_{22} - p_2 x_2 u_{12} - p_1 p_2 \lambda}{-p_1^2 u_{22} + 2 p_1 p_2 u_{12} - p_2^2 u_{11}} .$$

A is the answer to part (a), namely dx_1/dp_1 when $dp_2 = dm = 0$.
B is the answer to part (b), namely dx_1/dp_2 when $dp_1 = dm = 0$.

c. Let $\frac{dx_1}{dp_1} = A$ and $\frac{dx_1}{dp_2} = B$. Here, $p_1 + p_2 = \text{const.}$ so $dp_1 + dp_2 = 0$ and $dp_2 = -dp_1$. Hence

$$\begin{aligned} dx_1 &= A\,dp_1 + B\,dp_2 \\ &= A\,dp_1 + B(-dp_1) \\ &= (A - B)\,dp_1 . \end{aligned}$$

So $\frac{dx_1}{dp_1} = A - B$ when $dp_2 = -dp_1$.

2. [10 points] Suppose a consumer has a standard budget constraint and has a utility function of the form

$$u(x_1, x_2) = x_1^{\beta\alpha} x_2^{\beta(1-\alpha)}$$

where $\alpha \in (0, 1)$ and $\beta > 0$.

(a) Find the Marshallian demand for x_1.

(b) How does the Marshallian demand for x_1 change as β changes? Why?

(c) Find the Hicksian demand for x_1.

(d) How does the Hicksian demand for x_1 change as β changes? Why?

Exam 1

2000

Question 2

①

Exam 1
2000
Answer 2

② a) $\mathcal{L} = x_1^{\beta\alpha} x_2^{\beta(1-\alpha)} + \lambda(m - p_1 x_1 - p_2 x_2)$ for $\max_{\underset{\sim}{x}} u(\underset{\sim}{x})$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = m$.

$$\text{F.O.C.: } 0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 \qquad (1)$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_1} = \beta\alpha \frac{x_1^{\beta\alpha} x_2^{\beta(1-\alpha)}}{x_1} - \lambda p_1 \qquad (2)$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_2} = \beta(1-\alpha) \frac{x_1^{\beta\alpha} x_2^{\beta(1-\alpha)}}{x_2} - \lambda p_2 \qquad (3)$$

Divide (2) & (3) ⇒ $$\frac{\beta\alpha}{\beta(1-\alpha)} \frac{x_1^{\beta\alpha} x_2^{\beta(1-\alpha)}}{x_1} \frac{x_2}{x_1^{\beta\alpha} x_2^{\beta(1-\alpha)}} = \frac{\lambda p_1}{\lambda p_2}$$

$$\frac{\alpha}{1-\alpha} \frac{x_2}{x_1} = \frac{p_1}{p_2}$$

$x_2^* = \frac{p_1}{p_2} \frac{1-\alpha}{\alpha} x_1^*$. Substitute into budget constraint:

$$m = p_1 x_1^* + p_2 \left(\frac{p_1}{p_2} \frac{1-\alpha}{\alpha} x_1^* \right)$$

$$= p_1 x_1^* + p_1 \frac{1-\alpha}{\alpha} x_1^* = p_1 \left(1 + \frac{1-\alpha}{\alpha}\right) x_1^* = p_1 \left(\frac{\alpha}{\alpha} + \frac{1-\alpha}{\alpha}\right) x_1^* = \frac{p_1}{\alpha} x_1^*$$

$$\Rightarrow \boxed{x_1^* = \frac{\alpha m}{p_1}}$$

b) x_1^* is unaffected by β. The utility function is $\left[x_1^{\alpha} x_2^{1-\alpha}\right]^{\beta}$, which is a monotonic transformation of $x_1^{\alpha} x_2^{1-\alpha}$, so they have the same demands.

Exam 1
2000
Answer 2 cont...

c) $\min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x}$ s.t. $u(\underset{\sim}{x}) = u_0 \Rightarrow$

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \text{ s.t. } x_1^{\beta\alpha} x_2^{\beta(1-\alpha)} = u_0 .$$

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda \left(x_1^{\beta\alpha} x_2^{\beta(1-\alpha)} - u_0 \right)$$

F.O.C. $0 = \frac{\partial \mathcal{L}}{\partial \lambda} = x_1^{\beta\alpha} x_2^{\beta(1-\alpha)} - u_0$

$$\left.\begin{aligned} 0 = \frac{\partial \mathcal{L}}{\partial x_1} &= p_1 + \lambda \beta\alpha \frac{x_1^{\beta\alpha} x_2^{\beta(1-\alpha)}}{x_1} = p_1 + \lambda\beta\alpha \frac{u}{x_1} \\ 0 = \frac{\partial \mathcal{L}}{\partial x_2} &= p_2 + \lambda \beta(1-\alpha) \frac{x_1^{\beta\alpha} x_2^{\beta(1-\alpha)}}{x_2} = p_2 + \lambda\beta(1-\alpha) \frac{u}{x_2} \end{aligned}\right\}$$

$$\frac{p_1}{p_2} = \frac{\lambda\beta\alpha\, u/x_1}{\lambda\beta(1-\alpha)\, u/x_2} = \frac{\alpha x_2}{(1-\alpha) x_1}$$

$\Rightarrow x_2^* = \frac{p_1}{p_2} \frac{1-\alpha}{\alpha} x_1^*$. Substitute into the constraint:

$$u_0 = x_1^{\beta\alpha} \left[\frac{p_1}{p_2} \frac{1-\alpha}{\alpha} x_1 \right]^{\beta(1-\alpha)}$$

$$= \left(\frac{p_1}{p_2} \frac{1-\alpha}{\alpha} \right)^{\beta(1-\alpha)} x_1^{\beta\alpha} x_1^{\beta(1-\alpha)} = \left(\frac{p_1}{p_2} \frac{1-\alpha}{\alpha} \right)^{\beta(1-\alpha)} x_1^{\beta}$$

$$u_0 \left(\frac{p_2}{p_1} \frac{\alpha}{1-\alpha} \right)^{\beta(1-\alpha)} = x_1^{\beta}$$

$$\boxed{u_0^{1/\beta} \left(\frac{p_2}{p_1} \frac{\alpha}{1-\alpha} \right)^{1-\alpha} = x_1^* .}$$

d) β only affects the cardinal magnitude of the utility target level.

Final Exam
2004
Question 2

①

✓ 2. [16 points] Suppose a consumer's utility function takes the form $u(x, y)$ and is quasiconcave. Suppose the consumer's income is fixed at m. Under what conditions on u will x be an inferior good?

To receive full credit, your answer should involve only u or its derivatives.

(You lose five points if you have to ask me to give you the definition of an "inferior good.")

② $\max u(x,y)$ s.t. $p_x x + p_y y = m$

$\mathcal{L} = u(x,y) + \lambda(m - p_x x - p_y y)$

$0 = \partial\mathcal{L}/\partial\lambda = m - p_x x - p_y y$

$0 = \partial\mathcal{L}/\partial x = u'_x - \lambda p_x$

$0 = \partial\mathcal{L}/\partial y = u'_y - \lambda p_y$

x an inferior good $\Leftrightarrow \partial x/\partial m < 0$
so this is a comparative
statics problem

Take differentials of the FOC's: $dp_x = dp_y = 0$

$d\lambda \quad dx \quad dy \quad dm$

$0 d\lambda \; - p_x dx \; - p_y dy \; + dm = 0$

$-p_x d\lambda \; + u''_{xx} dx + u''_{xy} dy + 0 dm = 0$

$-p_y d\lambda \; + u''_{yx} dx + u''_{yy} dy + 0 dm = 0$

Final Exam
2004
Answer 2

$$\begin{bmatrix} 0 & -p_x & -p_y \\ -p_x & u''_{xx} & u''_{xy} \\ -p_y & u''_{yx} & u''_{yy} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx \\ dy \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} dm = \underset{\sim}{0} \, .$$

$$\Rightarrow \text{either} \begin{bmatrix} 0 & -p_x & -p_y \\ -p_x & u''_{xx} & u''_{xy} \\ -p_y & u''_{yx} & u''_{yy} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx \\ dy \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} dm \text{ or } \underbrace{\begin{bmatrix} 0 & p_x & p_y \\ p_x & u''_{xx} & u''_{xy} \\ p_y & u''_{yx} & u''_{yy} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx \\ dy \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

this case is easier; I'll omit its continuation, since it's just like the other case without all the negative signs to worry about.

$$\begin{bmatrix} 0 & -p_x & -p_y \\ -p_x & u''_{xx} & u''_{xy} \\ -p_y & u''_{yx} & u''_{yy} \end{bmatrix} \begin{bmatrix} d\lambda/dm \\ dx/dm \\ dy/dm \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Cramer's Rule:

$$\frac{\partial x}{\partial m} = \frac{\begin{vmatrix} 0 & -1 & -p_y \\ -p_x & 0 & u''_{xy} \\ -p_y & 0 & u''_{yy} \end{vmatrix}}{\begin{vmatrix} 0 & -p_x & -p_y \\ -p_x & u''_{xx} & u''_{xy} \\ -p_y & u''_{yx} & u''_{yy} \end{vmatrix}}$$

Final Exam
2004
Answer 2 cont...

From FOC: $-p_x = -u'_x/\lambda$

$-p_y = -u'_y/\lambda$

So the numerator of $\partial x/\partial m$ is

$$\begin{vmatrix} 0 & -1 & -u'_y/\lambda \\ -u'_x/\lambda & 0 & u''_{xy} \\ -u'_y/\lambda & 0 & u''_{yy} \end{vmatrix} = (-1)^{1+2}(-1)\left[-\frac{u'_x}{\lambda}u''_{yy} + \frac{u'_y}{\lambda}u''_{xy}\right] \text{ by cofactor expansion along second column}$$

$$= \frac{1}{\lambda}\left[u'_y u''_{xy} - u'_x u''_{yy}\right].$$

The denominator of $\partial x/\partial m$ is $\begin{vmatrix} 0 & -u'_x/\lambda & -u'_y/\lambda \\ -u'_x/\lambda & u''_{xx} & u''_{xy} \\ -u'_y/\lambda & u''_{yx} & u''_{yy} \end{vmatrix} =$

$$\underbrace{\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)}_{\substack{\oplus \\ \text{since this} \\ \text{is } (-1/\lambda)^2}} \underbrace{\begin{vmatrix} 0 & u'_x & u'_y \\ u'_x & u''_{xx} & u''_{xy} \\ u'_y & u''_{yx} & u''_{yy} \end{vmatrix}}$$

Final Exam
2004
Answer 2 cont...

a bordered Hessian determinant, called δ_2 in my handout;
u quasiconcave $\Rightarrow \delta_2 > 0$.

So the denominator of $\partial x/\partial m$ is positive.

Furthermore, λ is positive since it's $\partial u^*/\partial m$ (more money "in" $\Rightarrow$ higher u^*).

So for $\partial x/\partial m < 0$ we'd need

$$u'_y u''_{xy} - u'_x u''_{yy} < 0.$$

2016 Exam 1 Qu. 3

3. [11 points] Suppose a consumer's utility function takes the form $u(x, y)$ and is quasiconcave. Suppose the consumer's income is fixed at m. Under what conditions on u will x be a Giffen good?

   To receive full credit, your answer should involve only u or its derivatives.

   (You lose three points if you have to ask me to give you the definition of a ”Giffen good.”)

③

$\max u(x,y)$ s.t. $p_x x + p_y y = m$.

"x is a Giffen good" $\iff \dfrac{\partial x}{\partial p_x} > 0$

so this is a comparative statics problem

$$\mathcal{L} = u(x,y) + \lambda(m - p_x x - p_y y)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - p_x x - p_y y$$

$$0 = \frac{\partial \mathcal{L}}{\partial x} = u'_x - \lambda p_x$$

$$0 = \frac{\partial \mathcal{L}}{\partial y} = u'_y - \lambda p_y$$

Take differentials of the F.O.C.'s with $dm = 0$ and $dp_y = 0$:

$$\begin{array}{ccccc}
d\lambda & dx & dy & dp_x & \\
0\,d\lambda & -p_x\,dx & -p_y\,dy & -x\,dp_x & = 0 \\
-p_x\,d\lambda & +u''_{xx}\,dx & +u''_{xy}\,dy & -\lambda\,dp_x & = 0 \\
-p_y\,d\lambda & +u''_{yx}\,dx & +u''_{yy}\,dy & +0\,dp_x & = 0
\end{array}$$

$$\begin{bmatrix} 0 & -p_x & -p_y \\ -p_x & u''_{xx} & u''_{xy} \\ -p_y & u''_{yx} & u''_{yy} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx \\ dy \end{bmatrix} + \begin{bmatrix} -x \\ -\lambda \\ 0 \end{bmatrix} dp_x = \underset{\sim}{0}$$

$$\begin{bmatrix} 0 & -p_x & -p_y \\ -p_x & u''_{xx} & u''_{xy} \\ -p_y & u''_{yx} & u''_{yy} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx \\ dy \end{bmatrix} = \begin{bmatrix} x \\ \lambda \\ 0 \end{bmatrix} dp_x$$

$$\begin{bmatrix} 0 & -p_x & -p_y \\ -p_x & u''_{xx} & u''_{xy} \\ -p_y & u''_{yx} & u''_{yy} \end{bmatrix} \begin{bmatrix} d\lambda / dp_x \\ dx / dp_x \\ dy / dp_x \end{bmatrix} = \begin{bmatrix} x \\ \lambda \\ 0 \end{bmatrix}.$$ Use Cramer's Rule to

Solve for dx/dp_x :

$$\frac{dx}{dp_x} = \frac{\begin{vmatrix} 0 & x & -p_y \\ -p_x & \lambda & u''_{xy} \\ -p_y & 0 & u''_{yy} \end{vmatrix}}{\begin{vmatrix} 0 & -p_x & -p_y \\ -p_x & u''_{xx} & u''_{xy} \\ -p_y & u''_{yx} & u''_{yy} \end{vmatrix}} .$$

Using the F.O.C.'s, the denominator is $\begin{vmatrix} 0 & -u'_x/\lambda & -u'_y/\lambda \\ -u'_x/\lambda & u''_{xx} & u''_{xy} \\ -u'_y/\lambda & u''_{yx} & u''_{yy} \end{vmatrix}$

$$= \underbrace{\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)}_{\frac{1}{\lambda^2} > 0} \underbrace{\begin{vmatrix} 0 & u'_x & u'_y \\ u'_x & u''_{xx} & u''_{xy} \\ u'_y & u''_{yx} & u''_{yy} \end{vmatrix}}_{}$$

a bordered Hessian determinant, called δ_2 in my handout;

u quasiconcave $\Rightarrow \delta_2 > 0$.

So the denominator of dx/dp_x is positive. Its numerator is

$$\left|\begin{array}{ccc|cc} 0 & x & -p_y & 0 & x \\ -p_x & \lambda & u''_{xy} & -p_x & \lambda \\ -p_y & 0 & u''_{yy} & -p_y & 0 \end{array}\right.$$

$$\text{determinant} = x\, u''_{xy} (-p_y) - x(-p_x) u''_{yy} - (-p_y)\lambda(-p_y)$$

$$= -x\, u''_{xy}\, p_y + x\, p_x\, u''_{yy} - p_y^2 \lambda .$$

From the F.O.C.'s, we could use either $\lambda = \frac{u'_x}{p_x}$ or $\lambda = \frac{u'_y}{p_y}$:

Numerator of $dx/dp_x = -x u''_{xy} p_y + x p_x u''_{yy} - p_y^2 \left(\frac{u'_y}{p_y} \right)$

$$= x\left(p_x u''_{yy} - p_y u''_{xy} \right) - p_y u'_y .$$

In order for x to be Giffen, this has to be positive:

$$x\left(p_x u''_{yy} - p_y u''_{xy} \right) - p_y u'_y > 0$$

$$\underbrace{\underset{\oplus}{x}\;\underset{\oplus}{p_x}\;\underset{\substack{\uparrow \\ \text{often } \ominus}}{u''_{yy}} \;-\; \underset{\oplus}{p_y}\;\underset{\substack{\uparrow \\ \oplus}}{u'_y}}_{\text{often } \ominus} > p_y u''_{xy}$$

So u''_{xy} would have to be very negative.

<u>Alternatively</u>: for x to be Giffen,

$$x \frac{u'_x}{\lambda} u''_{yy} - \frac{u'_y}{\lambda} u'_y > \frac{u'_y}{\lambda} u''_{xy} .$$

$\lambda > 0$ because using the envelope theorem, $\frac{\partial v}{\partial m} = \lambda$ and $\frac{\partial v}{\partial m} > 0$ (more income ⇒ greater utility).

So the condition becomes

$$\underset{\oplus}{x}\,\underset{\oplus}{u'_x}\,\underset{\substack{\uparrow \\ \text{often} \\ \ominus}}{u''_{yy}} - \underset{\oplus}{(u'_y)^2} > \underset{\oplus}{u'_y}\, u''_{xy}$$

↑ needs to be very negative, as before.

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5. [14 points] Consider a consumer who has a quasi-concave utility function defined over two goods. Determine, if possible, the sign of the slope of this consumer's Hicksian demand curve for good 1 if the consumer *does not* take the price of good 2 as given, but rather considers the price of good 2 to be a function of how much of good 2 he consumes.

⑤

The expenditure-minimization problem is to

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$$\min \; p_1 x_1 + p_2(x_2)\, x_2 \quad s.t. \quad u(x_1, x_2) = \bar{u}$$

$$\mathcal{L} = p_1 x_1 + p_2(x_2)\, x_2 + \lambda\,(\bar{u} - u(x_1, x_2))$$

F.O.C.'s: $0 = \partial\mathcal{L}/\partial\lambda = \bar{u} - u(x_1, x_2)$

$0 = \partial\mathcal{L}/\partial x_1 = p_1 - \lambda u_1'$

$0 = \partial\mathcal{L}/\partial x_2 = p_2' x_2 + p_2 - \lambda u_2'$ where p_2' is dp_2/dx_2.

Endogenous variables: λ, x_1, x_2

Exogenous variables: p_1 is the only one which changes (the others are p_2 and $\bar{u}$).

Find the differential of each F.O.C.:

$$\begin{array}{lllllll} & d\lambda & dx_1 & dx_2 & dp_1 & dp_2 = d\bar{u} = 0 \\ 0 = & 0 & -u_1' dx_1 & -u_2' dx_2 & +0 dp_1 & \\ 0 = & -u_1' d & -\lambda u_{11}'' dx_1 & -\lambda u_{12}'' dx_2 & +dp_1 & \\ 0 = & -u_2' d\lambda & -\lambda u_{21}'' dx_1 & +(p_2'' x_2 + p_2' + p_2' - \lambda u_{22}'')\, dx_2 & +0\, dp_1 & \end{array}$$

$$\underset{\sim}{0} = \begin{bmatrix} 0 & -u_1' & -u_2' \\ -u_1' & -\lambda u_{11}'' & -\lambda u_{12}'' \\ -u_2' & -\lambda u_{21}'' & p_2'' x_2 + 2p_2' - \lambda u_{22}'' \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} + \begin{bmatrix} 0 \\ dp_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -u_1' & -u_2' \\ -u_1' & -\lambda u_{11}'' & -\lambda u_{12}'' \\ -u_2' & -\lambda u_{21}'' & -\lambda u_{22}'' + p_2'' x_2 + 2 p_2' \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -dp_1 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & -u_1' & -u_2' \\ -u_1' & -\lambda u_{11}'' & -\lambda u_{12}'' \\ -u_2' & -\lambda u_{21}'' & -\lambda u_{22}'' + p_2'' x_2 + 2 p_2' \end{bmatrix} \begin{bmatrix} d\lambda / dp_1 \\ dx_1 / dp_1 \\ dx_2 / dp_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Cramer's Rule ⇒

$$\frac{dx_1}{dp_1} = \frac{\begin{vmatrix} 0 & 0 & -u_2' \\ -u_1' & -1 & -\lambda u_{12}'' \\ -u_2' & 0 & -\lambda u_{22}'' + p_2'' x_2 + 2 p_2' \end{vmatrix}}{\begin{vmatrix} 0 & -u_1' & -u_2' \\ -u_1' & -\lambda u_{11}'' & -\lambda u_{12}'' \\ -u_2' & -\lambda u_{21}'' & -\lambda u_{22}'' + p_2' x_2 + 2 p_2' \end{vmatrix}}$$

Numerator: expand along second column to get

$$(-1)^{2+2}(-1)(0 - (-u_2')(-u_2')) = -1\left[-u_2' u_2'\right]$$

$$= (u_2')^2 > 0$$

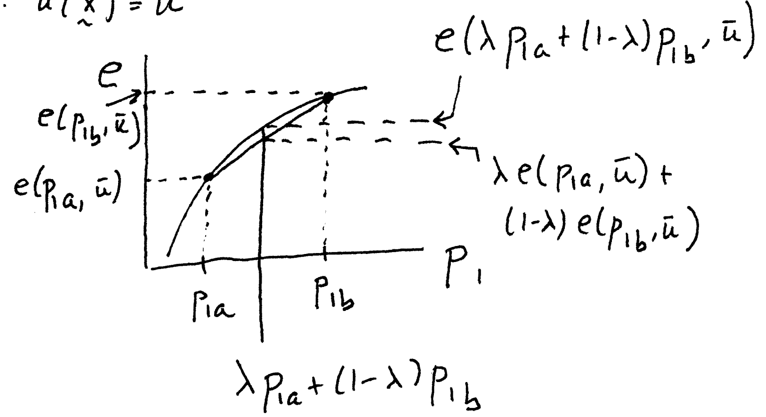
Denominator: This is $|\nabla^2 \mathcal{L}|$, which for a minimum should have the sign of $(-1)^m = -1^1 = -1$, negative. ↑ assume it is a minimum

So $dx_1/dp_1 < 0$, the usual down-ward sloping Hicksian demand curve.

Alternative Answer to Econ. 7005 Fall 2004 Final Exam's Question 5

$$e(p_1, \bar{u}) = \min_{x_1, x_2} \; p_1 x_1 + p_2(x_2)\cdot x_2 \quad \text{s.t. } u(\underset{\sim}{x}) = \bar{u}$$

To prove concavity of $e(p_1, \bar{u})$ in p_1 :



$$e(\lambda p_{1a} + (1-\lambda) p_{1b}, \bar{u}) = \min_{x_1, x_2} \; (\lambda p_{1a} + (1-\lambda) p_{1b}) x_1 + p_2(x_2) x_2 \text{ s.t. } u(\underset{\sim}{x}) = \bar{u}$$

$$= \min_{x_1, x_2} \; \lambda p_{1a} x_1 + (1-\lambda) p_{1b} x_1 + \underbrace{[\lambda + (1-\lambda)]}_{=1} p_2(x_2) x_2 \quad \text{s.t. } u(\underset{\sim}{x}) = \bar{u}$$

$$= \min_{x_1, x_2} \; \lambda p_{1a} x_1 + \lambda p_2(x_2) x_2 + (1-\lambda) p_{1b} x_1 + (1-\lambda) p_2(x_2) x_2 \text{ s.t. } u(\underset{\sim}{x}) = \bar{u}$$

$$= \min_{x_1, x_2} \; \lambda (p_{1a} x_1 + p_2(x_2) x_2) + (1-\lambda)(p_{1b} x_1 + p_2(x_2) x_2) \quad \text{s.t. } u(\underset{\sim}{x}) = \bar{u}$$

$$\geq \min_{x_1, x_2} \; \lambda (p_{1a} x_1 + p_2(x_2) x_2) \text{ s.t. } u(\underset{\sim}{x}) = \bar{u}$$
$$+ \min_{x_1, x_2} \; (1-\lambda)(p_{1b} x_1 + p_2(x_2) x_2) \quad \text{s.t. } u(\underset{\sim}{x}) = \bar{u}$$

$$= \lambda \min_{x_1, x_2} \; p_{1a} x_1 + p_2(x_2) x_2 \text{ s.t. } u(\underset{\sim}{x}) = \bar{u}$$
$$+ (1-\lambda) \min_{x_1, x_2} \; p_{1b} + p_2(x_2) x_2 \quad \text{s.t. } u(\underset{\sim}{x}) = \bar{u}$$

$$= \lambda \, e(p_{1a}, \bar{u}) + (1-\lambda) \, e(p_{1b}, \bar{u}).$$ Then $0 \geq \frac{\partial^2 e}{\partial p_1^2} = \frac{\partial}{\partial p_1} \frac{\partial e}{\partial p_1}$

$= \frac{\partial}{\partial p_1} h_1$, so h_1 is downward-sloping.

Answer all of the following three questions.

Fall 2006 Ex. 1

1. [11 points] Suppose a consumer's utility function is given by a quasi-concave function $u(x_1, x_2)$.
   (a) Suppose the consumer takes the price of x_1 as given and the price of x_2 as given. Call these prices p_1 and p_2. Let income be m. Implicitly find the consumer's demand for x_1 and x_2 and verify that these demands actually do maximize the consumer's utility.
   (b) Suppose the consumer takes the price of x_1 as given, but the consumer faces a price of x_2 which declines the more of x_2 the consumer buys. Implicitly find the consumer's demand for x_1 and x_2 and try to verify that these demands actually do maximize the consumer's utility.

Proposition 2. **[Test of Pseudoconvexity.]** *Let f be a C^2 function defined in an open, convex set S in R^n. Define the "bordered Hessian" determinants $\delta_r(\mathrm{x})$, $r = 1, \ldots, n$ by*

$$\delta_r(\mathrm{x}) = \begin{vmatrix} 0 & f_1'(\mathrm{x}) & \cdots & f_r'(\mathrm{x}) \\ f_1'(\mathrm{x}) & f_{11}''(\mathrm{x}) & \cdots & f_{1r}''(\mathrm{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_r'(\mathrm{x}) & f_{r1}''(\mathrm{x}) & \cdots & f_{rr}''(\mathrm{x}) \end{vmatrix}.$$

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A sufficient condition for f to be pseudoconvex is that $\delta_r(\mathrm{x}) < 0$ for $r = 2, \ldots, n$, and all $\mathrm{x} \in S$.

[Proposition 2′: Similarly, a sufficient condition for f to be pseudoconcave is that $\delta_r(\mathrm{x})$ alternate in sign beginning with > 0 for $r = 2, \ldots, n$, and all $\mathrm{x} \in S$.]

Answers to Econ 7005 Midterm, Fall 2006

① a) max $u(x_1, x_2)$ s.t. $p_1 x_1 + p_2 x_2 = m$

$$\mathcal{L} = u(x_1, x_2) + \lambda(m - p_1 x_1 - p_2 x_2)$$

F.O.C.

$$0 = \partial\mathcal{L}/\partial\lambda = m - p_1 x_1 - p_2 x_2 \quad (1)$$

$$0 = \partial\mathcal{L}/\partial x_1 = u_1' - \lambda p_1 \quad (2)$$

$$0 = \partial\mathcal{L}/\partial x_2 = u_2' - \lambda p_2 \quad (3)$$

These implicitly define x_1^* and x_2^* as a function of p_1, p_2, and m.

S.O.C.

$$\nabla^2\mathcal{L} = \begin{bmatrix} \mathcal{L}''_{\lambda\lambda} & \mathcal{L}''_{\lambda x_1} & \mathcal{L}''_{\lambda x_2} \\ \mathcal{L}''_{x_1\lambda} & \mathcal{L}''_{x_1 x_1} & \mathcal{L}''_{x_1 x_2} \\ \mathcal{L}''_{x_2\lambda} & \mathcal{L}''_{x_2 x_1} & \mathcal{L}''_{x_2 x_2} \end{bmatrix} = \begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{11} & u''_{12} \\ -p_2 & u''_{21} & u''_{22} \end{bmatrix}$$

but from (2), $-p_1 = -u_1'/\lambda$; from (3), $-p_2 = -u_2'/\lambda$.

Substituting,

$$\nabla^2\mathcal{L} = \begin{bmatrix} 0 & -u_1'/\lambda & -u_2'/\lambda \\ -u_1'/\lambda & u''_{11} & u''_{12} \\ -u_2'/\lambda & u''_{21} & u''_{22} \end{bmatrix}.$$

The S.O.C. for a maximum are that $D_{2m+1} \cdots D_{m+n}$ of $\nabla^2\mathcal{L}$ alternate in sign starting with $(-1)^{m+1}$. Here $m = 1$ (constraint) and $n = 2$ (variables), so $2m+1 = 3$ and, $m+n = 3$ and $m+1 = 2$, so we need D_3 of $\nabla^2\mathcal{L}$ to be the same sign as $(-1)^2 > 0$.

$$D_3 \text{ of } \nabla^2\mathcal{L} = |\nabla^2\mathcal{L}| = \left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right) \begin{vmatrix} 0 & u_1' & u_2' \\ u_1' & u''_{11} & u''_{12} \\ u_2' & u''_{21} & u''_{22} \end{vmatrix}.$$

Obviously $\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right) = \left(\frac{-1}{\lambda}\right)^2 > 0$. In addition, $\begin{vmatrix} 0 & u_1' & u_2' \\ u_1' & u_{11}'' & u_{12}'' \\ u_2' & u_{21}'' & u_{22}'' \end{vmatrix} = \delta_2$, the "bordered Hessian" of u. Since u is quasiconcave, we know that $\delta_2 > 0$. Thus D_3 of $\nabla^2\mathcal{L}$ is > 0, fulfilling the S.O.C.

b) $\mathcal{L}$ is the same as in (a), but now p_2 is a function of x_2:

F.O.C. $0 = \partial\mathcal{L}/\partial\lambda = m - p_1x_1 - p_2x_2$ (1')

$0 = \partial\mathcal{L}/\partial x_1 = u_1' - \lambda p_1$ (2')

$0 = \partial\mathcal{L}/\partial x_2 = u_2' - \lambda p_2 - \lambda p_2' x_2$ with $p_2' = \frac{dp_2}{dx_2}$. (3')

$$\nabla^2\mathcal{L} = \begin{bmatrix} 0 & -p_1 & -p_2 - p_2'x_2 \\ -p_1 & u_{11}'' & u_{12}'' \\ -p_2 - p_2'x_2 & u_{21}'' & u_{22}'' - \lambda p_2' - \lambda p_2''x_2 - \lambda p_2' \end{bmatrix} \text{; (2') and (3')} \Rightarrow$$

$$= \begin{bmatrix} 0 & -u_1'/\lambda & -u_2'/\lambda \\ -u_1'/\lambda & u_{11}'' & u_{12}'' \\ -u_2'/\lambda & u_{21}'' & u_{22}'' - 2\lambda p_2' - \lambda p_2''x_2 \end{bmatrix}$$

and as before,

$$|\nabla^2\mathcal{L}| = \left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right) \begin{vmatrix} 0 & u_1' & u_2' \\ u_1' & u_{11}'' & u_{12}'' \\ u_2' & u_{22}'' & u_{22}'' - 2\lambda p_2' - \lambda p_2''x_2 \end{vmatrix}.$$

This is not a bordered Hessian, and while $p_2' < 0$ and λ is (typically) positive, not knowing p_2'' makes it clear we will be unable to sign $|\nabla^2\mathcal{L}|$. So we don't know if the x_1^* and x_2^* implicitly defined in (1')-(3') satisfy the S.O.C.

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4. [11 points] Compose a problem of economic importance which involves quasiconcavity. Then work the problem you composed and demonstrate the importance of quasiconcavity in your mathematical working-out of that problem.

Answers will vary.

2. [11 points] Suppose a price-taking consumer has income m and utility function $u = \ln y + \ln z$ where y and z are the two goods which the consumer buys. For this consumer, calculate separately the left-hand side and right-hand side of the Slutsky Equation

$$\frac{\partial x_y(\mathbf{p}, m)}{\partial p_z} = \frac{\partial h_y(\mathbf{p}, v(\mathbf{p}, m))}{\partial p_z} - \frac{\partial x_y(\mathbf{p}, m)}{\partial m} x_z(\mathbf{p}, m)$$

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(where x denotes the Marshallian demand curve and h the Hicksian demand curve), and show that the left-hand side is equal to the right-hand side. (In the process, it helps to calculate the consumer's indirect utility function.)

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②

max $\ln y + \ln z$ s.t. $p_y y + p_z z = m$ → Or could start by taking a monotonic transformation of the utility function; one such transformation is $e^u = e^{(\ln y + \ln z)} = e^{\ln(yz)} = yz$.

$$\mathcal{L} = \ln y + \ln z + \lambda(m - p_y y - p_z z)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - p_z z - p_y y$$

$$\left.\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial y} = \frac{1}{y} - \lambda p_y \Rightarrow \lambda = \frac{1}{y p_y} \\ 0 &= \frac{\partial \mathcal{L}}{\partial z} = \frac{1}{z} - \lambda p_z \Rightarrow \lambda = \frac{1}{z p_z} \end{aligned}\right\} \quad \frac{1}{y p_y} = \frac{1}{z p_z}$$

$z = \frac{y p_y}{p_z}$ and

$$0 = m - p_z\left(\frac{y p_y}{p_z}\right) - p_y y$$

$$= m - y p_y - y p_y = m - 2 y p_y$$

$$2 y p_y = m$$

$$y = \frac{m}{2 p_y} \text{ and } z = \frac{p_y}{p_z} \cdot \frac{m}{2 p_y} = \frac{m}{2 p_z}.$$

Marshallian Demand Curves

$$v = \ln y^* + \ln z^* = \ln \frac{m}{2 p_y} + \ln \frac{m}{2 p_z}$$

$$= \ln \frac{m^2}{4 p_y p_z} = v(\underset{\sim}{p}, m).$$

$v(\underset{\sim}{p}, e(\underset{\sim}{p}, u)) \equiv u$ so from above,

$$\ln \frac{e(\underset{\sim}{p}, u)^2}{4 p_y p_z} = u$$

$$\frac{e(\underset{\sim}{p}, u)2}{4 p_y p_z} = e^u$$

$$e(\underset{\sim}{p}, u)^2 = 4 p_y p_z e^u$$

$$e(\underset{\sim}{p}, u) = 2\sqrt{p_y p_z}\, e^{u/2}$$

$$h_y = \frac{\partial e}{\partial p_y} = \sqrt{\frac{p_z}{p_y}}\, e^{u/2}$$

(or find h_y directly by solving the expenditure-minimization problem)

or: $h_y(\underset{\sim}{p}, u) = x_y(\underset{\sim}{p}, e(\underset{\sim}{p}, u)) = \frac{e(\underset{\sim}{p}, u)}{2 p_y} = \frac{2\sqrt{p_y p_z}\, e^{u/2}}{2 p_y} = \sqrt{\frac{p_z}{p_y}}\, e^{u/2}$.

$$\frac{\partial h_y}{\partial p_z} = \frac{1}{2}\sqrt{\frac{1}{p_y p_z}}\, e^{u/2}$$

$$\frac{\partial h_y(\underset{\sim}{p}, v(\underset{\sim}{p}, m))}{\partial p_z} = \frac{1}{2}\sqrt{\frac{1}{p_y p_z}}\, e^{v(\underset{\sim}{p}, m)/2}$$

$$= \frac{1}{2}\sqrt{\frac{1}{p_y p_z}}\, e^{\frac{1}{2}\ln \frac{m^2}{4 p_y p_z}}$$

$$= \frac{1}{2}\sqrt{\frac{1}{p_y p_z}}\, e^{\ln\left(\frac{m^2}{4 p_y p_z}\right)^{1/2}} = \frac{1}{2}\sqrt{\frac{1}{p_y p_z}}\, e^{\ln \frac{m}{2\sqrt{p_y p_z}}}$$

$$= \frac{1}{2}\sqrt{\frac{1}{p_y p_z}}\, \frac{m}{2\sqrt{p_y p_z}} = \frac{m}{4 p_y p_z}$$

$$\frac{\partial x_y}{\partial m} = \frac{\partial}{\partial m}\, \frac{m}{2p_y} = \frac{1}{2p_y}$$

$$\frac{\partial x_y}{\partial m} x_z = \frac{1}{2p_y}\, \frac{m}{2p_z} = \frac{m}{4 p_y p_z}$$

RHS of Slutsky Equation : $\frac{\partial h_y(\underset{\sim}{p}, V)}{\partial p_z} - \frac{\partial x_y}{\partial m} x_z = \frac{m}{4p_y p_z} - \frac{m}{4 p_y p_z} = 0$

LHS of Slutsky Equation: $\frac{\partial x_y}{\partial p_z} = \frac{\partial}{\partial p_z}\, \frac{m}{2p_y} = 0.$

Since $0 = 0$, verification is complete.

3. [11 points] Suppose a consumer consumes two goods, x_1 and x_2, and has a utility function of

$$u(\mathbf{x}) = x_1^{1/2} x_2^{1/2}.$$

This consumer takes the prices of the goods p_1 and p_2 as given, and has a fixed income m.

(a) Find the consumer's Hicksian demand curve for good 1, $h_1(\mathbf{p}, \hat{u})$, without explicitly solving a utility-maximization problem.

(b) Find this consumer's expenditure function. Hint: one way of writing this consumer's expenditure function is

$$\left(\frac{p_2}{p_1}\right)^{1/2} p_1 \, \hat{u} + \left(\frac{p_1}{p_2}\right)^{1/2} p_2 \, \hat{u} \, .$$

(c) Find this consumer's indirect utility function from this consumer's expenditure function. Hint: one way of writing this consumer's indirect utility function is

$$m p_1^{-1/2} p_2^{-1/2}.$$

(d) Derive this consumer's Marshallian demand function for good 1 from the consumer's indirect utility function.

(e) Using your answers to parts (a) and (d), verify the Slutsky equation

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial h_1}{\partial p_1} - x_1 \frac{\partial x_1}{\partial m}$$

for this consumer. (In other words, calculate each side of this equation separately, then show that they are equal to each other.) If you need this consumer's Marshallian demand function for good 2 in order to solve this problem, you may deduce its form by symmetry from the answer to part (d) instead of deriving it.

Fall 2012 Exam 1

Answer to Question 3 of Exam 1, Fall 2012, Econ. 7005

a) $\min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x}$ s.t. $u(\underset{\sim}{x}) = \hat{u}$

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda \left[\hat{u} - x_1^{1/2} x_2^{1/2} \right]$$

$$0 = \partial \mathcal{L} / \partial \lambda = \hat{u} - x_1^{1/2} x_2^{1/2} \qquad (1)$$

$$0 = \partial \mathcal{L} / \partial x_1 = p_1 - \lambda \left(\tfrac{1}{2}\right) x_1^{-1/2} x_2^{1/2} \qquad (2)$$

$$0 = \partial \mathcal{L} / \partial x_2 = p_2 - \lambda \left(\tfrac{1}{2}\right) x_1^{1/2} x_2^{-1/2}. \qquad (3)$$

(2) $\Rightarrow \lambda = 2 p_1 x_1^{1/2} x_2^{-1/2}$; substituting into (3),

$$p_2 = 2 p_1 x_1^{1/2} x_2^{-1/2} \cdot \left(\tfrac{1}{2}\right) x_1^{1/2} x_2^{-1/2}$$

$= p_1 x_1 x_2^{-1} \Rightarrow x_1 = (p_2/p_1) x_2$. Substitute into (1):

$\hat{u} = \left(\frac{p_2}{p_1}\right)^{1/2} x_2^{1/2} \cdot x_2^{1/2} = x_2 \sqrt{\frac{p_2}{p_1}} \Rightarrow x_2 = \hat{u} \sqrt{\frac{p_1}{p_2}}$. Then from the sentence before last, $x_1 = \left(\frac{p_2}{p_1}\right) x_2 = \left(\frac{p_2}{p_1}\right) \hat{u} \sqrt{\frac{p_1}{p_2}} = \hat{u} \sqrt{\frac{p_2}{p_1}}$.

Hence

$$h_1(\underset{\sim}{p}, \hat{u}) = \hat{u} \sqrt{\frac{p_2}{p_1}}.$$

b) $e(\underset{\sim}{p}, \hat{u}) = p_1 h_1^* + p_2 h_2^* = p_1 \hat{u} \sqrt{\frac{p_2}{p_1}} + p_2 \hat{u} \sqrt{\frac{p_1}{p_2}}$; simplifying,

$$= \hat{u} \sqrt{p_1 p_2} + \hat{u} \sqrt{p_1 p_2} = 2 \hat{u} \sqrt{p_1 p_2}.$$

c) $m = e(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) = e(\underset{\sim}{p}, v) = 2 v \sqrt{p_1 p_2}$ from (b);

so $v = \dfrac{m}{2\sqrt{p_1 p_2}} = \dfrac{m}{2} p_1^{-1/2} p_2^{-1/2}$.

d) $$x_1 = \frac{-\partial v/\partial p_1}{\partial v/\partial m} = - \frac{\frac{m}{2} \cdot \frac{-1}{2} p_1^{-3/2} p_2^{-1/2}}{\frac{1}{2} p_1^{-1/2} p_2^{-1/2}} = \frac{m}{2} p_1^{-1} = \frac{m}{2p_1}.$$

e) $\dfrac{\partial x_1}{\partial p_1} = \dfrac{\partial}{\partial p_1} \dfrac{m}{2} p_1^{-1} = \dfrac{-m}{2p_1^2}$. This is the left-hand side.

$$\frac{\partial h_1}{\partial p_1} = \frac{\partial}{\partial p_1} \hat{u} \sqrt{\frac{p_2}{p_1}} = \frac{\partial}{\partial p_1} \hat{u}\, p_2^{1/2} p_1^{-1/2} = \frac{-1}{2} \hat{u}\, p_2^{1/2} p_1^{-3/2}.$$

$\dfrac{\partial x_1}{\partial m} = \dfrac{\partial}{\partial m} \dfrac{m}{2p_1} = \dfrac{1}{2p_1}$. So the right-hand side is

$$\frac{-1}{2} \hat{u}\, p_1^{-3/2} p_2^{1/2} - \frac{m}{2p_1} \frac{1}{2p_1} = \frac{-1}{2} \underbrace{\left(x_1^{1/2} x_2^{1/2} \right)} p_1^{-3/2} p_2^{1/2} - \frac{m}{4p_1^2}$$

or instead from (c) use $\hat{u} = m/(2\sqrt{p_1 p_2})$.

$$= \frac{-1}{2} \left[\left(\frac{m}{2p_1}\right)^{1/2} \underset{\uparrow\ x_2 \text{ by symmetry}}{\left(\frac{m}{2p_2}\right)^{1/2}} \right] p_1^{-3/2} p_2^{1/2} - \frac{m}{4p_1^2}$$

$$= \frac{-1}{2} \frac{m}{2\sqrt{p_1 p_2}} p_1^{-3/2} p_2^{1/2} - \frac{m}{4p_1^2} = \frac{-m}{2} \cdot \frac{1}{2} p_1^{-\frac{3}{2} - \frac{1}{2}} p_2^{-\frac{1}{2} + \frac{1}{2}} - \frac{m}{4p_1^2}$$

$$= \frac{-m}{4p_1^2} - \frac{m}{4p_1^2} = \frac{-m}{2p_1^2},$$ which agrees with the left-hand side.

Exam 1
1997
Question 2

①

2. Suppose a consumer's expenditure function $e(\mathbf{p}, u)$ is equal to $p_1^a p_2^{1-a} u$, where the p's denote prices and u denotes a utility level.
   (a) Find the consumer's Hicksian demand curve for good 2, $h_2(\mathbf{p}, u)$.
   (b) Find the consumer's Marshallian demand curve for good 2, $x_2(\mathbf{p}, m)$, where m is income.
   (c) Verify the following Slutsky equation for this consumer:

$$\frac{\partial x_2(\mathbf{p}, m)}{\partial p_2} = \frac{\partial h_2(\mathbf{p}, u)}{\partial p_2} - x_2(\mathbf{p}, m)\frac{\partial x_2(\mathbf{p}, m)}{\partial m}.$$

Exam 1
1997
Answer 2

② $e(\underset{\sim}{p}, u) = p_1^{a} p_2^{1-a} u$

a) $h_2 = \frac{\partial e}{\partial p_2} = (1-a) p_1^{a} p_2^{-a} u$

b) Since $e(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) = m$ where m is income,

$$p_1^{a} p_2^{1-a} v(\underset{\sim}{p}, m) = m \Rightarrow$$

$$v(\underset{\sim}{p}, m) = m p_1^{-a} p_2^{a-1}.$$

Then by Roy's Identity,

$$x_2 = -\frac{\partial v/\partial p_2}{\partial v/\partial m} = -\frac{(a-1) m p_1^{-a} p_2^{a-2}}{p_1^{-a} p_2^{a-1}} = (1-a) m p_2^{-1};$$

alternatively,

$$x_2(\underset{\sim}{p}, m) = h_2(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) = (1-a) p_1^{a} p_2^{-a} (m p_1^{-a} p_2^{a-1}) = (1-a) m p_2^{-1}.$$

Exam 1
1997
Answer 2 cont...

c) The left-hand side is

$$\frac{\partial x_2}{\partial p_2} = \frac{\partial}{\partial p_2} (1-a) m\, p_2^{-1} = (a-1)\, m\, p_2^{-2}.$$

The right-hand side is

$$\frac{\partial h_2}{\partial p_2} - x_2 \frac{\partial x_2}{\partial m} = \frac{\partial \left[(1-a) p_1^{a} p_2^{-a} u\right]}{\partial p_2} - \frac{(1-a)m}{p_2} \frac{\partial}{\partial m} \frac{(1-a)m}{p_2}$$

$$= -a(1-a) p_1^{a} p_2^{-a-1} u - \frac{(1-a)m}{p_2} \frac{(1-a)}{p_2}$$; since $u = v(\underset{\sim}{p}, m)$,

$$= -a(1-a) \underline{p_1}^{a} \underline{\underline{p_2}}^{-a-1} m\, \underline{p_1}^{-a} \underline{\underline{p_2}}^{a-1} - \frac{(1-a)^2 m}{p_2^2}$$

$$= \frac{-a(1-a)m}{p_2^2} - \frac{(1-a)^2 m}{p_2^2} = \frac{-(1-a)m}{p_2^2} \left[a + 1 - a\right]$$

$= (a-1)\, m\, p_2^{-2}$, which is the same as the left-hand side.

Final Exam

2000 ①

Question 4

4. Suppose a consumer has a standard budget constraint and a utility function $u(\mathbf{x}) = x_1 + \frac{1}{2}x_2$. Find this consumer's indirect utility function.

Final Exam
2000
Answer 4

④ $u(\underset{\sim}{x}) = x_1 + \frac{1}{2}x_2$. The budget constraint is $p_1 x_1 + p_2 x_2 = m$.

$$\mathcal{L} = x_1 + \tfrac{1}{2}x_2 + \lambda\,(m - p_1 x_1 - p_2 x_2)$$

$$\left.\begin{array}{l} 0 = \dfrac{\partial \mathcal{L}}{\partial x_1} = 1 - \lambda p_1 \Rightarrow \lambda = \dfrac{1}{p_1} \\[2ex] 0 = \dfrac{\partial \mathcal{L}}{\partial x_2} = \dfrac{1}{2} - \lambda p_2 \Rightarrow \lambda = \dfrac{1}{2p_2} \end{array}\right\} \quad \frac{1}{p_1} = \frac{1}{2p_2} \Rightarrow p_1 = 2p_2 \text{ for interior solutions.}$$

<u>Suppose $p_1 = 2p_2$</u>. Then let $x_1 = \alpha$. We would have $x_2 = \dfrac{m - p_1 x_1}{p_2} = \dfrac{m - p_1\alpha}{p_2}$

and utility would be $\alpha + \frac{1}{2}\dfrac{m - p_1\alpha}{p_2} = \alpha + \dfrac{m}{2p_2} - \dfrac{p_1\alpha}{2p_2} = \dfrac{m}{2p_2} + \dfrac{2p_2\alpha - p_1\alpha}{2p_2}$

$= \dfrac{m + (2p_2 - p_1)\alpha}{2p_2} = \dfrac{m}{2p_2}$ since $2p_2 - p_1 = 0$

$= m/p_1$.

or, note that $\frac{p_1}{2p_2} = 1$, so the third term is -1 times the first term

<u>If $p_1 \neq 2p_2$</u>, there is a corner solution, not an interior solution. (The indifference curves in this problem are linear.) If $p_1 < 2p_2$, the consumer spends all his income on good 1, and utility is m/p_1 (since $x_1 = m/p_1$). If $p_1 > 2p_2$, the consumer spends all his income on good 2, and utility is $\frac{1}{2}\frac{m}{p_2}$ (since $x_2 = \frac{m}{p_2}$).

So to summarize,

$$v(p_1, p_2, m) = \begin{cases} m/p_1 & \text{if } p_1 < 2p_2 \\ \dfrac{m}{p_1} = \dfrac{m}{2p_2} & \text{if } p_1 = 2p_2 \\ \dfrac{m}{2p_2} & \text{if } p_1 > 2p_2 \,. \end{cases}$$

This can also be written $v(p_1, p_2, m) = \max\left\{\dfrac{m}{p_1}, \dfrac{m}{2p_2}\right\}$.

1/2

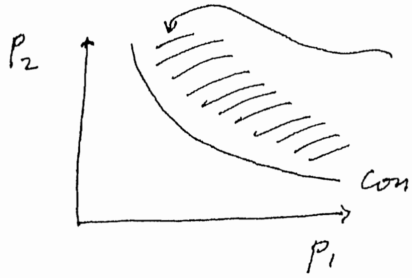
Answer all of the following five questions.

Fall 2005 Final

1. [8 points] Give a two-dimensional graphical interpretation of the result that the indirect utility function is quasiconvex in prices.

Answers to Final Exam, Econ. 7005, Fall 2005

① "quasiconvex" means "convex lower level sets"



lower level set for v_0 (since $\uparrow p \Rightarrow \downarrow v$)

contour line of constant v (constant utility); call this level "v_0"

In the diagram, the lower level set needs to be a convex set since v is quasiconvex. So the indicated contour line, if thought of as being a function of p_1, needs to be convex.

A "convex set" has the property that a straight line drawn between any two members of the set stays within the set.

Fall 2004
Ex. 1

1. [11 points] Suppose a consumer receives utility from consumption of two goods, x_1 and x_2, according to the utility function $x_1^a x_2^{(1-a)}$, where $a \in (0, 1)$.

   (a) Find this consumer's indirect utility function.

   (b) Verify that this consumer's indirect utility function is quasiconvex in $\mathbf{p}$. Directly check for quasiconvexity; do not use the result that a convex function is quasiconvex, and do not use the result that a strictly convex function is quasiconvex. (If you could not solve part (a), then work part (b) with the indirect utility function $\ln m - a \ln p_1 - (1-a) \ln p_2$, which may or may not be the correct answer to part (a).)

Answers to Exam 1, Econ 7005, Fall 2004

① See Varian p. 111.

a) max u $\Leftrightarrow$ max $\ln u$ since $\ln u$ is an increasing function of u.

$$\therefore \max x_1^a x_2^{(1-a)} \Leftrightarrow \max \ln x_1^a x_2^{1-a}$$

$$\Updownarrow$$

$$\max (\ln x_1^a + \ln x_2^{1-a})$$

$$\Updownarrow$$

$$\max [a \ln x_1 + (1-a)\ln x_2]$$ which is easier to work with

$\max [a \ln x_1 + (1-a) \ln x_2]$ s.t. $p_1 x_1 + p_2 x_2 = m$

$$\mathcal{L} = a \ln x_1 + (1-a) \ln x_2 + \lambda [m - p_1 x_1 - p_2 x_2]$$

$$0 = \partial \mathcal{L}/\partial \lambda = m - p_1 x_1 - p_2 x_2$$

$$\left.\begin{aligned} 0 &= \partial \mathcal{L}/\partial x_1 = \frac{a}{x_1} - \lambda p_1 \\ 0 &= \partial \mathcal{L}/\partial x_2 = \frac{1-a}{x_2} - \lambda p_2 \end{aligned}\right\} \lambda = \frac{a}{p_1 x_1} = \frac{1-a}{p_2 x_2}$$

$\Rightarrow x_1 = \dfrac{a p_2 x_2}{(1-a) p_1}$. Substitute into the

constraint: $m = p_1 \dfrac{a p_2 x_2}{(1-a) p_1} + p_2 x_2$

$$= \frac{a}{1-a} p_2 x_2 + p_2 x_2 = p_2 x_2 \left[\frac{a}{1-a} + 1\right] = p_2 x_2 \frac{a + 1 - a}{1-a}$$

$$= \frac{p_2}{1-a} x_2 \Rightarrow x_2^* = (1-a)\frac{m}{p_2}$$ and

$$x_1^* = \frac{a p_2}{(1-a) p_1} \cdot (1-a) \frac{m}{p_2} = a \frac{m}{p_1}.$$

$$\text{So } u^* = a \ln \frac{am}{p_1} + (1-a) \ln \frac{(1-a)m}{p_2}$$

$$= a \ln a + a \ln \frac{m}{p_1} + (1-a) \ln (1-a) + (1-a) \ln \frac{m}{p_2}$$

but since constants are not important, the first and third terms are unimportant, and we can write

$$v(\underset{\sim}{p}, m) = a \ln \frac{m}{p_1} + (1-a) \ln \frac{m}{p_2}$$

$$= a \ln m \;\; - a \ln p_1 + (1-a) \ln m \;\; - (1-a) \ln p_2$$

$$= \ln m - a \ln p_1 - (1-a) \ln p_2 .$$

b) The bordered Hessian is

$$\begin{bmatrix} 0 & v_1' & v_2' \\ v_1' & v_{11}'' & v_{12}'' \\ v_2' & v_{21}'' & v_{22}'' \end{bmatrix} = \begin{bmatrix} 0 & -a/p_1 & \frac{a-1}{p_2} \\ -a/p_1 & +a/p_1^2 & 0 \\ \frac{a-1}{p_2} & 0 & \frac{1-a}{p_2^2} \end{bmatrix}$$

For quasiconvexity we want $\delta_2 < 0$. In this case, δ_2 is the determinant of the entire bordered Hessian. This is (expanding along the first column):

$$(-1)^{2+1} \left(\frac{-a}{p_1}\right)\left(\frac{-a}{p_1} \cdot \frac{1-a}{p_2^2}\right) + (-1)^{3+1} \frac{a-1}{p_2} \left(- \frac{a}{p_1^2} \cdot \frac{a-1}{p_2}\right)$$

$$= \frac{a}{p_1} \frac{a}{p_1} \frac{a-1}{p_2^2} + \frac{a-1}{p_2} \frac{a}{p_1^2} \frac{1-a}{p_2} = \frac{a}{p_1^2 p_2^2} \left(a^2 - a + (a-1)(1-a)\right)$$

$$= \frac{a}{p_1^2 p_2^2} \left[a^2 - a + a - a^2 - 1 + a\right] = \frac{a\,(a-1)}{p_1^2\, p_2^2} < 0.$$

2017 Exam 1 Qu. 1

1. [11 points]

   (a) Suppose a consumer has income m and a standard budget constraint and a utility function $u(x, y) = \alpha \ln x + \beta \ln y$ with $\alpha > 0$ and $\beta > 0$. Find this consumer's indirect utility function v.

   (b) From your answer to part (a), find $\partial v/\partial m$ and $\partial^2 v/\partial m^2$. Interpret these in terms of "the marginal utility of money." What is the sign of $\partial v/\partial m$ and $\partial^2 v/\partial m^2$? What do these signs imply for the shape of a graph of v versus m?

   (c) Find $\partial v/\partial m$ by using the Envelope Theorem and verify that you get the same answer you got in part (b). Explain why the Envelope Theorem is relevant to the problem.

   (d) Suppose another consumer has income m and a standard budget constraint and a utility function $\hat{u}(x, y) = x^{\alpha} y^{\beta}$ with $\alpha > 0$ and $\beta > 0$. Without solving an optimization problem, explain why this consumer has the same demand curves for x and y as the consumer in the earlier parts of this question.

   (e) Find this second consumer's indirect utility function $\hat{v}$ and find $\partial \hat{v}/\partial m$ and $\partial^2 \hat{v}/\partial m^2$.

   (f) Is the sign of $\partial \hat{v}/\partial m$ the same as the sign of $\partial v/\partial m$? Why or why not?

   (g) Is the sign of $\partial^2 \hat{v}/\partial m^2$ the same as the sign of $\partial^2 v/\partial m^2$? Why or why not?

Answer to Qu. 1, Midterm Exam, Fall 2017 (Econ. 7005)

a) $\mathcal{L} = \alpha \ln x + \beta \ln y + \lambda (m - p_x x - p_y y)$ for max $u(x,y)$ s.t. $p_x x + p_y y = m$.

F.O.C.

$$0 = \mathcal{L}'_x = \frac{\alpha}{x} - \lambda p_x \Rightarrow \lambda p_x = \frac{\alpha}{x} \Rightarrow \lambda = \frac{\alpha}{p_x x}$$

$$0 = \mathcal{L}'_y = \frac{\beta}{y} - \lambda p_y \Rightarrow \lambda p_y = \frac{\beta}{y} \Rightarrow \lambda = \frac{\beta}{p_y y}$$

$$\Rightarrow \frac{\alpha}{p_x x} = \frac{\beta}{p_y y}$$

$$0 = m - p_x x - p_y y$$

$$\Downarrow$$

$$x = \frac{\alpha p_y y}{\beta p_x}$$

$$m = p_x \left(\frac{\alpha p_y y}{\beta p_x}\right) + p_y y$$

$$= \left(\frac{\alpha}{\beta} + 1\right) p_y y = \left(\frac{\alpha}{\beta} + \frac{\beta}{\beta}\right) p_y y = \frac{\alpha+\beta}{\beta} p_y y$$

$$\Rightarrow y^* = \frac{\beta}{\alpha+\beta} \frac{m}{p_y}.$$

Then from above, $x^* = \frac{\alpha p_y y}{\beta p_x} = \frac{\alpha p_y}{\beta p_x} \frac{\beta}{\alpha+\beta} \frac{m}{p_y} = \frac{\alpha}{\alpha+\beta} \frac{m}{p_x}$.

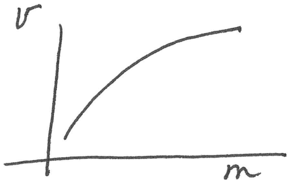
Then $v = \alpha \ln x^* + \beta \ln y^* = \alpha \ln \frac{\alpha m}{(\alpha+\beta) p_x} + \beta \ln \frac{\beta m}{(\alpha+\beta) p_y}$.

b) $$\frac{\partial v}{\partial m} = \alpha \frac{(\alpha+\beta) p_x}{\alpha m} \frac{\alpha}{(\alpha+\beta) p_x} + \beta \frac{(\alpha+\beta) p_y}{\beta m} \frac{\beta}{(\alpha+\beta) p_y}$$

$$= \frac{\alpha}{m} + \frac{\beta}{m} = \frac{\alpha+\beta}{m} > 0.$$

$$\frac{\partial^2 v}{\partial m^2} = -\frac{\alpha+\beta}{m^2} < 0.$$

Note: $\alpha > 0, \beta > 0, m > 0$.



v versus m is increasing and concave. It may be in Quadrant IV (v may be negative). (The sign of v is unimportant.)

$\partial v/\partial m$ is the "marginal utility of money." Here, $\partial v/\partial m$ is decreasing in m since the graph is concave.

c) $v = \max_{x,y} u(x,y)$ s.t. $p_x x + p_y y = m$.

By the Envelope Theorem, $\frac{\partial v}{\partial m} = \frac{\partial \mathcal{L}^*}{\partial m}$, and using $\mathcal{L}$ from the first line of part (a),

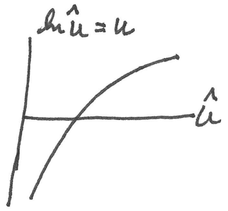
$$= \lambda^*.$$

From part (a), $\lambda^* = \frac{\alpha}{p_x x^*} = \frac{\alpha}{p_x} \frac{(\alpha+\beta) p_x}{\alpha m} = \frac{\alpha+\beta}{m}$ or

$$\lambda^* = \frac{\beta}{p_y y^*} = \frac{\beta}{p_y} \frac{(\alpha+\beta) p_y}{\beta m} = \frac{\alpha+\beta}{m}.$$

So $\partial v / \partial m = \frac{\alpha+\beta}{m}$, as in part (b).

d) Notice that $\ln \hat{u} = \ln(x^\alpha y^\beta) = \ln x^\alpha + \ln y^\beta = \alpha \ln x + \beta \ln y = u(x,y)$.

Also, 

, that is, "ln" is an increasing function. So $u(x,y)$ is an increasing function of $\hat{u}(x,y)$, and hence u and $\hat{u}$ represent the same preferences.

e) From (d), we know that (a)'s answers for x^* and y^* are true for $\hat{u}$. So

$$\hat{v} = (x^*)^\alpha (y^*)^\beta = \left(\frac{\alpha m}{(\alpha+\beta) p_x}\right)^\alpha \left(\frac{\beta m}{(\alpha+\beta) p_y}\right)^\beta = \frac{\alpha^\alpha \beta^\beta m^{\alpha+\beta}}{(\alpha+\beta)^{\alpha+\beta} p_x^\alpha p_y^\beta}.$$

$$\frac{\partial \hat{v}}{\partial m} = (\alpha+\beta) \frac{\alpha^\alpha \beta^\beta}{(\alpha+\beta)^{\alpha+\beta} p_x^\alpha p_y^\beta} m^{\alpha+\beta-1} = \frac{\alpha^\alpha \beta^\beta}{(\alpha+\beta)^{\alpha+\beta-1} p_x^\alpha p_y^\beta} m^{\alpha+\beta-1} > 0$$

and

$$\frac{\partial^2 \hat{v}}{\partial m^2} = (\alpha+\beta-1) \frac{\alpha^\alpha \beta^\beta}{(\alpha+\beta)^{\alpha+\beta-1} p_x^\alpha p_y^\beta} m^{\alpha+\beta-2}.$$

f) Yes, both $\partial v/\partial m$ and $\partial \hat{v}/\partial m$ are positive. (See part (e).)

g) From (b), $\partial^2 v/\partial m^2 < 0$.

However from (e), $\partial^2 \hat{v}/\partial m^2$ has the same sign as $\alpha+\beta-1$, which is positive if $\alpha+\beta > 1$ and negative if $\alpha+\beta < 1$ (and zero if $\alpha+\beta=1$).

For the case of $\alpha+\beta > 1$, the graph of $\hat{v}$ versus m would look like

.

Part (b)'s graph, with a falling marginal utility of money, reflects the conventional idea that a marginal dollar is worth more to a poor person than to a rich person. The graph of $\hat{v}$ versus m shows the opposite behavior. However, we have seen that v and $\hat{v}$ represent the same underlying preferences. This means that the "marginal utility of money" is a cardinal, not ordinal, idea, and hence that the notion of a "diminishing marginal utility of money" is meaningless in the context of ordinal utility theory.

2. [11 points] Suppose a price-taking consumer has a utility function

$$u(\mathbf{x}) = 2\ln x_1 + \ln x_2$$

over two goods x_1 and x_2.

(a) Show that this consumer's indirect utility function is

$$v(\mathbf{p}, m) = 2\ln\frac{2m}{3p_1} + \ln\frac{m}{3p_2}$$

where p_1 is the price of the first good, p_2 is the price of the second good, and m is income.

(b) If the price of the first good rises, what change in income would leave utility unchanged? [Hint: It is possible to use the indirect utility function from part (a) in answering part (b).]

Fall 2005 Ex. 1

8
2/3

②

a) $u = 2\ln x_1 + \ln x_2$

$\mathcal{L} = 2\ln x_1 + \ln x_2 + \lambda(m - p_1 x_1 - p_2 x_2)$

$0 = \frac{\partial \mathcal{L}}{\partial x_1} = \frac{2}{x_1} - \lambda p_1$

$0 = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{x_2} - \lambda p_2$

$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2$

$\frac{2}{x_1} \div \frac{1}{x_2} = \frac{p_1}{p_2}$

$\frac{2}{x_1}\frac{x_2}{1} = \frac{p_1}{p_2}$

$x_2 = \frac{p_1}{p_2}\frac{x_1}{2}$; into budget constraint $\Rightarrow$

$$m = p_1 x_1 + p_2\left(\frac{p_1}{p_2}\frac{x_1}{2}\right)$$
$$= p_1 x_1 + \tfrac{1}{2} p_1 x_1$$
$$= \tfrac{3}{2} p_1 x_1 \Rightarrow x_1^* = \frac{2}{3}\frac{m}{p_1} \quad \text{and}$$

$$x_2^* = \frac{\cancel{p_1}}{p_2}\frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}\frac{m}{\cancel{p_1}}\right) = \frac{m}{3p_2}.$$

So $v = u^* = 2\ln x_1^* + \ln x_2^*$

$$= 2\ln\frac{2m}{3p_1} + \ln\frac{m}{3p_2}.$$

b) We want utility to be unchanged, so we want $dv = 0$. But $dv = \frac{\partial v}{\partial p_1}dp_1 + \frac{\partial v}{\partial p_2}dp_2 + \frac{\partial v}{\partial m}dm$. With $dp_2 = 0$, this is

$$= 2\left(\frac{3p_1}{2m}\right)\frac{(-2m)}{3p_1^2}\,dp_1 + 0 + \left(2\,\frac{3p_1}{2m}\,\frac{2}{3p_1} + \frac{3p_2}{m}\,\frac{1}{3p_2}\right)dm$$

$$= 2\,\frac{-1}{p_1}\,dp_1 \qquad + \left(\frac{2}{m} + \frac{1}{m}\right)dm$$

$$= \frac{-2}{p_1} dp_1 + \frac{3}{m} dm .$$

In order for dv to be zero, one must have

$$0 = \frac{-2}{p_1} dp_1 + \frac{3}{m} dm$$

$$\frac{2}{p_1} dp_1 = \frac{3}{m} dm$$

$$dp_1 = \frac{3p_1}{2m} dm \Rightarrow \frac{dm}{dp_1} = \frac{2m}{3p_1} .$$

Optional: the elasticity required is $\frac{dm/m}{dp_1/p_1} = \frac{2}{3}$.

For every 1% ↑ in p_1, you'd need a $\frac{2}{3}$% ↑ in income.

A different approach: $v = \ln\left[\left(\frac{2m}{3p_1}\right)^2 \frac{m}{3p_2}\right]$. v = constant $\Rightarrow$

$$\text{constant} = \left(\frac{2m}{3p_1}\right)^2 \frac{m}{3p_2} = \frac{4m^3}{27p_1^2p_2} \qquad (*)$$

$$\Rightarrow m = \left[\frac{27}{4} \text{const.}\ p_1^2 p_2\right]^{1/3}$$

$$\frac{dm}{dp_1} = \frac{1}{3}\left[\frac{27}{4} \text{Const.}\ p_1^2 p_2\right]^{-2/3} \cdot \frac{27}{4} \text{Const.} (2p_1) p_2$$

$$= \frac{1}{3} \cdot \frac{1}{m^2} \cdot \frac{27}{4} \text{Const.} (2p_1) p_2$$

$$= \frac{27}{6} \text{Const.} \frac{p_1 p_2}{m^2} = \frac{27}{6} \frac{4m^3}{27p_1^2p_2} \frac{p_1p_2}{m^2}$$

$$= \frac{2}{3}\frac{m}{p_1} \text{ as before.}$$

A third approach: from (*), if m goes to $\hat{m}$ and p_1 to $\hat{p}_1$ we need

$$\frac{4m^3}{27p_1^2p_2} = \frac{4\hat{m}^3}{27\hat{p}_1^2 p_2} \Rightarrow \hat{m}^3 = \left(\frac{\hat{p}_1}{p_1}\right)^2 m^3 \text{ and}$$

$$\hat{m} = (\hat{p}_1/p_1)^{2/3} m .$$

A fourth approach:

$$v(\underset{\sim}{p}, e(\underset{\sim}{p}, u)) = u$$

From earlier, $v = 2 \ln \frac{2m}{3p_1} + \ln \frac{m}{3p_2} = \ln \left(\frac{2m}{3p_1}\right)^2 + \ln \frac{m}{3p_2}$

$$= \ln \frac{4m^2}{9p_1^2} + \ln \frac{m}{3p_2} = \ln \frac{4m^3}{27p_1^2 p_2} .$$

↑ like in the Second Approach

$$\text{So } v(\underset{\sim}{p}, e) = u = \ln \frac{4e^3}{27p_1^2 p_2} \Rightarrow$$

$$\exp(u) = \frac{4e^3}{27p_1^2 p_2}$$

"e" here is $e(\underset{\sim}{p}, u)$, not the number 2.718...

— Notation change —

$$e^u = \frac{4\, e(\underset{\sim}{p}, u)^3}{27 p_1^2 p_2}$$

$$\frac{27p_1^2 p_2}{4} e^u = e(\underset{\sim}{p}, u)^3 \Rightarrow$$

$$e(\underset{\sim}{p}, u) = 3 \left(p_1^2 p_2 / 4 \right)^{1/3} e^{u/3} \quad \text{and}$$

$$\left.\frac{\partial e(\underset{\sim}{p}, u)}{\partial p_1}\right|_u = \left[\frac{\partial}{\partial p_1} p_1^{2/3}\right] \cdot 3 \left(p_2/4\right)^{1/3} e^{u/3}$$

$$= \frac{2}{3} p_1^{-1/3} \cdot 3(p_2/4)^{1/3} e^{u/3} = \frac{2}{3} p_1^{2/3} p_1^{-1} \cdot 3(p_2/4)^{1/3} e^{u/3}$$

$$= \frac{2}{3p_1} \cdot 3\left(p_1^2 p_2/4\right)^{1/3} e^{u/3} = \frac{2}{3p_1} e(\underset{\sim}{p}, u) = \boxed{\frac{2m}{3p_1}} .$$

A fifth approach:

Compensating Variation $CV = e(\underset{\sim}{\hat{p}}, v(\underset{\sim}{\hat{p}}, \hat{m})) - e(\underset{\sim}{\hat{p}}, v(\underset{\sim}{p}, m))$

$= \hat{m} - e(\underset{\sim}{\hat{p}}, v(\underset{\sim}{p}, m))$. Using $e(\underset{\sim}{p}, u)$ from the Fourth Approach,

$= \hat{m} - 3(\hat{p}_1^2 \hat{p}_2 / 4)^{1/3} e^{v(\underset{\sim}{p}, m)/3}$; using $v(\underset{\sim}{p}, m)$ from either the Second Approach or the Fourth Approach,

$$= \hat{m} - 3(\hat{p}_1^2 \hat{p}_2 / 4)^{1/3} e^{\left(\ln \frac{4m^3}{27 p_1^2 p_2}\right)/3}$$

$$= \hat{m} - 3(\hat{p}_1^2 \hat{p}_2 / 4)^{1/3} e^{\frac{1}{3} \ln \frac{4m^3}{27 p_1^2 p_2}}$$

$$= \hat{m} - 3(\hat{p}_1^2 \hat{p}_2 / 4)^{1/3} e^{\ln \left[\frac{4m^3}{27 p_1^2 p_2}\right]^{1/3}}$$

$$= \hat{m} - 3 \left(\frac{\hat{p}_1^2 \hat{p}_2}{4}\right)^{1/3} \left(\frac{4m^3}{27 p_1^2 p_2}\right)^{1/3}$$

$$= \hat{m} - 3 \left[\frac{\hat{p}_1^2 \hat{p}_2}{p_1^2 p_2}\right]^{1/3} \frac{m}{3} = \hat{m} - \left[\frac{\hat{p}_1^2 \hat{p}_2}{p_1^2 p_2}\right]^{1/3} m.$$

We know that $\hat{p}_2 = p_2$. Find the $\hat{m}$ which makes $CV = 0$: *

$$0 = \hat{m} - \left(\frac{\hat{p}_1^2}{p_1^2}\right)^{1/3} m \Rightarrow \boxed{\hat{m} = (\hat{p}_1 / p_1)^{2/3} m.}$$

This is the same conclusion as the Third Approach.

* Or calculate $-CV = \boxed{(\hat{p}_1 / p_1)^{2/3} m - m}$.

2015 Qualifying Exam Sec. 2 Qu. 2

2. **[8 points]**

   (a) If the price vector faced by a price-taking consumer changes from $\mathbf{p}$ to $\gamma\mathbf{p}$, what change in income would leave utility unchanged? Why?

   (b) Suppose a price-taking consumer has a utility function

$$u(\mathbf{x}) = 2\ln x_1 + \ln x_2$$

   over two goods x_1 and x_2. If the price of only the first good rises, what change in income would leave utility unchanged?

Section 2 Question 2

a) Before the change in prices, $v(\underset{\sim}{p}, m) = \max_{\underset{\sim}{x}} u(\underset{\sim}{x})$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = m$.

After the change in prices, suppose income changes from m to αm. Then

$$v(\gamma \underset{\sim}{p}, am) = \max_{\underset{\sim}{x}} u(\underset{\sim}{x}) \text{ s.t. } \gamma \underset{\sim}{p} \cdot \underset{\sim}{x} = \alpha m.$$

If $\alpha = \gamma$ then this constraint is $\gamma \underset{\sim}{p} \cdot \underset{\sim}{x} = \gamma m$

$$\Rightarrow \underset{\sim}{p} \cdot \underset{\sim}{x} = m$$

which is the same constraint as before. The objective function is also the same as before, so v will be the same as before. So the answer is that m also has to change by a factor of γ.

b) This combines (a) and (b) of the previous problem.

Exam 1
1994
Question 3

①

3. Suppose a consumer's utility function u is given by $u(\mathbf{x}) = x_1^{1/2} + 2x_2$ where x_1 and x_2 are amounts of two commodities consumed. Find this consumer's indirect utility function and expenditure function.

Exam 1
1994
Answer 3

3. $u(\underset{\sim}{x}) = x_1^{1/2} + 2x_2$

$$\mathcal{L} = x_1^{1/2} + 2x_2 + \lambda(m - p_1x_1 - p_2x_2)$$

$$\frac{\partial\mathcal{L}}{\partial x_1} = 0 = \frac{1}{2}x_1^{-1/2} - \lambda p_1 \Rightarrow \frac{2}{p_2}p_1 = \frac{1}{2}x_1^{-1/2} \Rightarrow x_1 = \left(\frac{1}{4}\frac{p_2}{p_1}\right)^2 \qquad (1)$$

$$\frac{\partial\mathcal{L}}{\partial x_2} = 0 = 2 - \lambda p_2 \Rightarrow 2 = \lambda p_2 \Rightarrow \lambda = 2/p_2$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} = 0 = m - p_1x_1 - p_2x_2 \Rightarrow m = p_1x_1 + p_2x_2$$

$$= p_1\left(\frac{1}{4}\frac{p_2}{p_1}\right)^2 + p_2x_2$$

$$= \frac{1}{16}\frac{p_2^2}{p_1} + p_2x_2$$

$$\frac{m}{p_2} - \frac{1}{16}\frac{p_2}{p_1} = x_2. \qquad (2)$$

Substitute (1) and (2) back into $u(\underset{\sim}{x})$ to obtain

$$v(\underset{\sim}{p}, m) = \frac{1}{4}\frac{p_2}{p_1} + 2\left[\frac{m}{p_2} - \frac{1}{16}\frac{p_2}{p_1}\right]$$

$$= \frac{1}{4}\frac{p_2}{p_1} + \frac{2m}{p_2} - \frac{1}{8}\frac{p_2}{p_1}$$

$$= \frac{2m}{p_2} + \frac{p_2}{8p_1}. \qquad (3)$$

Finally, $v(\underset{\sim}{p}, e(\underset{\sim}{p}, u)) \equiv u$ so from (3)

$$\frac{2\,e(\underset{\sim}{p}, u)}{p_2} + \frac{p_2}{8p_1} = u$$

$$2\,e(\underset{\sim}{p}, u) = u\,p_2 - \frac{p_2^2}{8p_1}$$

$$e(p, u) = \frac{1}{2}u\,p_2 - \frac{1}{16}p_2^2/p_1. \qquad (4).$$

Exam 1
1994
Answer 3 cont..

Optional Remark. Imposing $x_2 \geq 0$ in (2) implies that

$$\frac{m}{p_2} - \frac{1}{16}\frac{p_2}{p_1} \geq 0 \iff$$

$$m \geq \frac{1}{16}\frac{p_2^2}{p_1}. \qquad (5)$$

If (5) is violated then $x_2^* = 0$ and all the income goes to x_1, leading to $x_1^* = \frac{m}{p_1}$. In this case $\underbrace{v(\underset{\sim}{p}, m) = \sqrt{m/p_1}}_{(6)}$ and $\sqrt{e(\underset{\sim}{p},u)/p_1} \equiv u =$ $\underbrace{e(\underset{\sim}{p},u) = p_1 u^2}_{(7)}$.

To summarize, $v(\underset{\sim}{p}, m)$ is given by (6) if $m < \frac{1}{16}\frac{p_2^2}{p_1}$ and by (3) otherwise. This ought to be continuous at $m = \frac{1}{16}\frac{p_2^2}{p_1}$:

$$(3) \Rightarrow v = \frac{2}{p_2}\left[\frac{1}{16}\frac{p_2^2}{p_1}\right] + \frac{p_2}{8p_1} = \frac{p_2}{8p_1} + \frac{p_2}{8p_1} = \frac{p_2}{4p_1}$$

$$(6) \Rightarrow v = \sqrt{\frac{1}{16}\frac{p_2^2}{p_1}/p_1} = \frac{1}{4}\frac{p_2}{p_1} \longleftarrow \text{continuity OK!}$$

This shows that the utility level at the boundary between the two regimes is $\frac{p_2}{4p_1}$. For $u < \frac{p_2}{4p_1}$, $e(\underset{\sim}{p},u)$ is given by (7); otherwise it is given by (4). To check continuity of $e(\underset{\sim}{p},u)$ at the point $u = p_2/(4p_1)$:

$$(4) \Rightarrow e = \frac{1}{2}\left[\frac{p_2}{4p_1}\right]p_2 - \frac{p_2^2}{16p_1} = \frac{p_2^2}{8p_1} - \frac{p_2^2}{16p_1} = \frac{p_2^2}{16p_1}$$

$$(7) \Rightarrow e = p_1\left[\frac{p_2}{4p_1}\right]^2 = \frac{p_2^2}{16p_1} \longleftarrow \text{Continuity OK!}$$

Continuity of the demand for x_2 is obvious by construction (see (5)). To check continui of x_1, note that at $m = \frac{1}{16}\frac{p_2^2}{p_1}$, $x_1^* = \frac{m}{p_1} = \frac{1}{16}\frac{p_2^2}{p_1^2}$, which is the same as given by (1).

Exam 1

1996

Question 2

①

2. Suppose a consumer obtains utility from consumption of two goods, x_1 and x_2, according to the utility function $u(\mathbf{x}) = x_1 x_2$. This consumer's income is denoted by m. The consumer takes the price of the second good as a constant, p_2. However, the consumer's actions influence p_1, the price of the first good; p_1 increases with the consumer's purchases of x_1 according to the relationship $p_1 = x_1/3$.

   (a) Find this consumer's indirect utility function.

   (b) Find this consumer's expenditure function.

Exam 1
1996
Answer 2

②

$$\max u(\underset{\sim}{x}) \text{ s.t. } m = p_1 x_1 + p_2 x_2$$

$$= p_1(x_1)\, x_1 + p_2 x_2 = \frac{x_1}{3} x_1 + p_2 x_2 = \frac{1}{3} x_1^2 + p_2 x_2 .$$

$$\mathcal{L} = x_1 x_2 + \lambda \left[m - \frac{1}{3} x_1^2 - p_2 x_2 \right].$$ (the Lagrangian)

F.O.C.:

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - \frac{1}{3} x_1^2 - p_2 x_2 \qquad (a)$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_1} = x_2 - \frac{2}{3} \lambda x_1 \qquad (b)$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_2} = x_1 - \lambda p_2 \qquad (c).$$

Exam 1
1996
Answer 2 cont...

$$\left.\begin{array}{l} (b) \Rightarrow \; x_2 = \frac{2}{3}\lambda x_1 \\ (c) \Rightarrow \; x_1 = \lambda p_2 \end{array}\right\} \text{divide} \Rightarrow \frac{x_2}{x_1} = \frac{2}{3}\frac{x_1}{p_2} \Rightarrow p_2 x_2 = \frac{2}{3} x_1^2 .$$

Sub-stitute this into (a) to obtain

$$m = \frac{1}{3}x_1^2 + p_2 x_2$$

$$= \frac{1}{3}x_1^2 + \frac{2}{3}x_1^2 = x_1^2 \Rightarrow \underline{x_1^* = \sqrt{m}} .$$

Then $x_2^* = \frac{1}{p_2} \cdot \frac{2}{3}x_1^2 = \frac{1}{p_2} \cdot \frac{2}{3} \cdot m = \underline{\frac{2m}{3p_2}}$.

a) $v(\underset{\sim}{p}, m) = u(x_1^*(\underset{\sim}{p}, m), x_2^*(\underset{\sim}{p}, m)) = x_1^* \cdot x_2^* = \sqrt{m} \cdot \frac{2m}{3p_2}$

$$= \underline{\frac{2m^{3/2}}{3p_2}} .$$

b) $v(\underset{\sim}{p}, e(\underset{\sim}{p}, u)) = u$ is a basic identity. Using part a), this implies

$$\frac{2\,[e(\underset{\sim}{p},u)]^{3/2}}{3p_2} = u \Rightarrow [e(\underset{\sim}{p},u)]^{3/2} = \frac{3up_2}{2} \Rightarrow$$

$$\underline{e(\underset{\sim}{p},u) = \left[\frac{3}{2}up_2\right]^{2/3}} .$$

4. Suppose a consumer's expenditure function is

$$\left(\frac{p_1}{a}\right)^a \left(\frac{p_2}{1-a}\right)^{1-a} u^o .$$

Final Exam
1998
Question 4
①

Find the consumer's (direct) utility function.

Final Exam
1998
Answer 4

④ $e(\underset{\sim}{p}, u^0) = \left(\frac{p_1}{a}\right)^a \left(\frac{p_2}{1-a}\right)^{1-a} u^0$

$m = e(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) = \left(\frac{p_1}{a}\right)^a \left(\frac{p_2}{1-a}\right)^{1-a} v(\underset{\sim}{p}, m)$

$\Rightarrow v(\underset{\sim}{p}, m) = \left(\frac{a}{p_1}\right)^a \left(\frac{1-a}{p_2}\right)^{1-a} m.$

Then $u(\underset{\sim}{x}) = \min_{\underset{\sim}{p}} v(\underset{\sim}{p}, 1)$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = 1$. Therefore

$u(\underset{\sim}{x}) = \min_{p_1, p_2} \left(\frac{a}{p_1}\right)^a \left(\frac{1-a}{p_2}\right)^{1-a}$ s.t. $p_1 x_1 + p_2 x_2 = 1$.

$$\mathcal{L} = \left(\frac{a}{p_1}\right)^a \left(\frac{1-a}{p_2}\right)^{1-a} + \lambda(p_1 x_1 + p_2 x_2 - 1)$$

$$\overset{!}{=} a^a (1-a)^{1-a} p_1^{-a} p_2^{-(1-a)} + \lambda(p_1 x_1 + p_2 x_2 - 1)$$

$$0 = \frac{\partial \mathcal{L}}{\partial p_1} = a^a (1-a)^{1-a} (-a)\, p_1^{-a-1}\, p_2^{-(1-a)} + \lambda x_1 \qquad (1)$$

$$0 = \frac{\partial \mathcal{L}}{\partial p_2} = a^a (1-a)^{1-a}\, p_1^{-a} \left(-(1-a)\right) p_2^{-(1-a)-1} + \lambda x_2$$

$$= a^a (1-a)^{1-a}\, p_1^{-a} (a-1)\, p_2^{a-2} + \lambda x_2 \qquad (2)$$

(1) and (2) ⇒ $$\frac{a^a (1-a)^{1-a} (-a)\, p_1^{-a-1}\, p_2^{a-1}}{a^a (1-a)^{1-a} (a-1)\, p_1^{-a}\, p_2^{a-2}} = \frac{x_1}{x_2}$$

Final Exam
1998
Answer 4 cont...

$$\frac{a}{(1-a)} \frac{p_2}{p_1} = \frac{x_1}{x_2}$$

$$p_2 = \frac{1-a}{a} \frac{x_1}{x_2} p_1 .$$

Substitute this into the constraint, $0 = \frac{\partial \mathcal{L}}{\partial \lambda} = p_1 x_1 + p_2 x_2 - 1$:

$$1 = p_1 x_1 + \frac{1-a}{a} \frac{x_1}{x_2} p_1 x_2$$

$$= \left(x_1 + \frac{1-a}{a} \frac{x_1}{x_2} x_2\right) p_1 = \left(x_1 + \frac{1-a}{a} x_1\right) p_1 = \left(1 + \frac{1-a}{a}\right) p_1 x_1$$

$$= \left(\frac{a}{a} + \frac{1-a}{a}\right) p_1 x_1 = \frac{1}{a} p_1 x_1 \Rightarrow p_1^* = \frac{a}{x_1} .$$

Then $p_2^* = \frac{1-a}{a} \frac{x_1}{x_2} p_1^* = \frac{1-a}{a} \frac{x_1}{x_2} \frac{a}{x_1} = \frac{1-a}{x_2}$.

Finally, $u(\underline{x}) = \left(\frac{a}{p_1^*}\right)^a \left(\frac{1-a}{p_2^*}\right)^{1-a} = \left(\frac{a}{a/x_1}\right)^a \left(\frac{1-a}{(1-a)/x_2}\right)^{1-a} = \underline{x_1^a\, x_2^{1-a}}$.

3. [11 points] Suppose a consumer's indirect utility function is given by

$$v(\mathbf{p}, m) = \ln m - a \ln p_1 - (1 - a) \ln p_2 \,.$$

Find this consumer's:

(a) expenditure function;

(b) (direct) utility function.

2007
Midterm

③

$v = \ln m - a \ln p_1 - (1-a)\ln p_2$

a) $v(\underset{\sim}{p}, e(\underset{\sim}{p}, u)) = u$ so

$$\ln e(\underset{\sim}{p}, u) - a \ln p_1 - (1-a)\ln p_2 = u$$; solve for $e(\underset{\sim}{p}, u)$.

$$\ln e(\underset{\sim}{p}, u) = a \ln p_1 + (1-a)\ln p_2 + u$$
$$= \ln p_1^{a} + \ln p_2^{1-a} + u$$
$$= \ln p_1^{a} + \ln p_2^{1-a} + \underbrace{\ln e^{u}}$$

a trick ; "ln" and "e" are inverse functions, so $\ln e^u = u$

$$= \ln p_1^{a} p_2^{1-a} e^{u} \text{ so}$$
$$e(\underset{\sim}{p}, u) = p_1^{a} p_2^{1-a} e^{u}.$$

Note: Don't confuse this "e", which is the expenditure function, with this "e," which is the base of the natural logarithms, the number 2.71828....

b) $u(\underset{\sim}{x}) = \min_{\underset{\sim}{p}} v(\underset{\sim}{p}, 1)$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = 1$

$$\mathcal{L} = \underset{0}{\cancel{\ln 1}} - a \ln p_1 - (1-a)\ln p_2 + \lambda(p_1 x_1 + p_2 x_2 - 1)$$

F.O.C. $$\left.\begin{aligned} 0 &= \partial\mathcal{L}/\partial p_1 = \frac{-a}{p_1} + \lambda x_1 \\ 0 &= \partial\mathcal{L}/\partial p_2 = -\frac{1-a}{p_2} + \lambda x_2 \end{aligned}\right\} \lambda = \frac{a}{p_1 x_1} = \frac{1-a}{p_2 x_2}$$

$$\Rightarrow p_2 = \frac{1-a}{a} \frac{p_1 x_1}{x_2} \text{ and}$$

$$1 = p_1 x_1 + p_2 x_2 = p_1 x_1 + \frac{1-a}{a} p_1 x_1 = \left(\frac{a}{a} + \frac{1-a}{a}\right) p_1 x_1$$

$$= \frac{p_1 x_1}{a} \Rightarrow p_1^* = \frac{a}{x_1} \text{ and}$$

$$p_2^* = \frac{1-a}{a} \frac{a}{x_2} = \frac{1-a}{x_2}.$$ Then

$$\begin{aligned}
u &= \ln 1 - a \ln p_1^* - (1-a) \ln p_2^* \\
&= 0 - a \ln \frac{a}{x_1} - (1-a) \ln \frac{1-a}{x_2} \\
&= -a(\ln a - \ln x_1) - (1-a)(\ln(1-a) - \ln x_2) \\
&= -a \ln a + a \ln x_1 - (1-a) \ln(1-a) + (1-a) \ln x_2 \\
&= [-a \ln a - (1-a) \ln(1-a)] + a \ln x_1 + (1-a) \ln x_2 \\
&= [\qquad '' \qquad] + \ln x_1^a + \ln x_2^{1-a} \\
&= \underbrace{[\qquad '' \qquad]}_{\text{constants are irrelevant}} + \underset{\uparrow}{\ln} \underbrace{x_1^a x_2^{1-a}}_{\text{Cobb-Douglas utility}} .
\end{aligned}$$

a monotonically increasing transformation

optimal

Final Exam
1994
Question 2

①

2. Suppose the indirect utility function of a consumer is given by

$$v(\mathbf{p}, m) = \frac{4m^3}{27p_1\,p_2^2}.$$

(a) Find the (direct) utility function.

(b) Find the Hicksian demand curve for good 1.

Final Exam
1994
Answer 2

② $v(\underset{\sim}{p}, m) = \frac{4m^3}{27 p_1 p_2^2}$

a) $u(\underset{\sim}{x}) = \min\limits_{\underset{\sim}{p}} v(\underset{\sim}{p}, 1)$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = 1$ so set $m = 1$ and minimize v w.r.t. $\underset{\sim}{p}$:

$$\mathcal{L} = \frac{4}{27 p_1 p_2^2} + \lambda \left[1 - p_1 x_1 - p_2 x_2 \right]$$

Setup and FOC's: 8pts.

$$\left.\begin{aligned} 0 = \frac{\partial \mathcal{L}}{\partial p_1} &= \frac{-4}{27 p_1^2 p_2^2} - \lambda x_1 \\ 0 = \frac{\partial \mathcal{L}}{\partial p_2} &= \frac{-8}{27 p_1 p_2^3} - \lambda x_2 \end{aligned}\right\} \frac{x_1}{x_2} = \frac{-4}{27 p_1^2 p_2^2} \cdot \frac{27 p_1 p_2^3}{-8} = \frac{1}{2} \frac{p_2}{p_1}$$

$$\Rightarrow p_2 = 2p_1 \frac{x_1}{x_2}$$

and $1 = p_1 x_1 + p_2 x_2$

2pts

$$= p_1 x_1 + \left[2p_1 \frac{x_1}{x_2}\right] x_2 = p_1 x_1 + 2p_1 x_1 = 3p_1 x_1 \Rightarrow p_1 = \frac{1}{3x_1},$$

$$p_2 = 2\left[\frac{1}{3x_1}\right]\frac{x_1}{x_2} = \frac{2}{3x_2}.$$

2pts

So $v(\underset{\sim}{p}^*, 1) = \frac{4}{27}\,\frac{3x_1}{1}\,\frac{9x_2^2}{4} = \boxed{x_1 x_2^2 = u(\underset{\sim}{x})}$

2pts

b) $v(\underset{\sim}{p}, e(\underset{\sim}{p}, u)) \equiv u$ ← 2pts

$$\frac{4e^3}{27p_1p_2^2} = u \Rightarrow e^3 = \frac{27}{4} p_1 p_2^2 u,$$

$$e(\underset{\sim}{p}, u) = \frac{3}{\sqrt[3]{4}} p_1^{1/3} p_2^{2/3} u^{1/3}$$ ← 2pts

2pts

$$h_1(\underset{\sim}{p}, u) = \frac{\partial e}{\partial p_1} = \frac{1}{\sqrt[3]{4}} p_1^{-2/3} p_2^{2/3} u^{1/3} = \left(\frac{u p_2^2}{4p_1^2}\right)^{1/3}.$$

2. [11 points] Suppose a consumer with income m faces given prices p_1 and p_2 for commodities x_1 and x_2, respectively. Suppose this consumer's indirect utility function is

$$v(\mathbf{p}, m) = m(p_1^r + p_2^r)^{-1/r}$$

for some positive constant r.

(a) Find this consumer's Marshallian demand curves for x_1 and x_2.

(b) Prove either that this consumer's (direct) utility function $u(x_1, x_2)$ can be written as

$$\left\{ \frac{x_1^{\frac{r}{r-1}}}{\left[x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right]^r} + \frac{x_2^{\frac{r}{r-1}}}{\left[x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right]^r} \right\}^{-1/r}$$

or as

$$\left(x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}}$$

or, with $\rho = r/(1-r)$, as

$$(x_1^\rho + x_2^\rho)^{1/\rho}$$

(which is called the "Constant Elasticity of Substitution" ("CES") utility function). Hint: at some point you may find it helpful to multiply

$$\frac{1}{x_1 + x_1^{\frac{-1}{r-1}} x_2^{\frac{r}{r-1}}}$$

by one in the form of $x_1^{\frac{1}{r-1}}/x_1^{\frac{1}{r-1}}$ in order to obtain

$$\frac{x_1^{\frac{1}{r-1}}}{x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}}.$$

Fall 2013 Exam 1

Answer to Question 2, Exam 1,

Fall 2013

a) $v = m\,(p_1^r + p_2^r)^{-1/r}$. Use Roy's Identity:

$$x_1 = -\frac{\partial v/\partial p_1}{\partial v/\partial m} = -\frac{m\left(\frac{-1}{r}\right)(p_1^r+p_2^r)^{\frac{-1}{r}-1}\, r\, p_1^{r-1}}{(p_1^r+p_2^r)^{\frac{-1}{r}}} = \frac{+m}{r}(p_1^r+p_2^r)^{-1}\, r\, p_1^{r-1}$$

$$= \frac{p_1^{r-1}\, m}{p_1^r + p_2^r}.$$ Analogously, $$x_2 = \frac{p_2^{r-1}\, m}{p_1^r + p_2^r}.$$

b) $u(\underset{\sim}{x}) = \min\limits_{\underset{\sim}{p}} v(\underset{\sim}{p})$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = 1$ (so $m = 1$).

$$\mathcal{L} = v(\underset{\sim}{p}) + \lambda\,(p_1 x_1 + p_2 x_2 - 1)$$

F.O.C. $0 = p_1 x_1 + p_2 x_2 - 1$

$$0 = \partial\mathcal{L}/\partial p_1 = \partial v/\partial p_1 + \lambda x_1$$; from part (a)

$$= \frac{-m}{r}(p_1^r + p_2^r)^{\frac{-1}{r}-1}(r\, p_1^{r-1}) + \lambda x_1$$; using $m = 1$,

$$= -(p_1^r + p_2^r)^{\frac{-1}{r}-1}\, p_1^{r-1} + \lambda x_1$$

$$\Rightarrow \lambda = \frac{1}{x_1}(p_1^r + p_2^r)^{\frac{-1}{r}-1}\, p_1^{r-1}.$$

$$0 = \frac{\partial\mathcal{L}}{\partial p_2} = \frac{\partial v}{\partial p_2} + \lambda x_2 = \frac{-m}{r}(p_1^r + p_2^r)^{\frac{-1}{r}-1}(r p_2^{r-1}) + \lambda x_2 \Rightarrow$$

$$(p_1^r + p_2^r)^{\frac{-1}{r}-1}\, p_2^{r-1} = \lambda x_2 = \frac{x_2}{x_1}(p_1^r + p_2^r)^{\frac{-1}{r}-1}\, p_1^{r-1}$$

so $$p_2^{r-1} = \frac{x_2}{x_1}\, p_1^{r-1}.$$

Substitute into the constraint:

$$1 = p_1 x_1 + p_2 x_2 = p_1 x_1 + \overbrace{\left(\frac{x_2}{x_1}\right)^{\frac{1}{r-1}} p_1}^{\text{this is } p_2} \cdot x_2$$

$$= p_1 \left[x_1 + x_1^{\frac{-1}{r-1}} x_2^{\frac{1}{r-1}} x_2 \right] ; \quad \frac{1}{r-1} + 1 = \frac{1}{r-1} + \frac{r-1}{r-1} = \frac{r}{r-1} \Rightarrow$$

$$= p_1 \left[x_1 + x_1^{\frac{-1}{r-1}} x_2^{\frac{r}{r-1}} \right] \Rightarrow$$

$$p_1 = \frac{1}{x_1 + x_1^{\frac{-1}{r-1}} x_2^{\frac{r}{r-1}}} \cdot \frac{x_1^{\frac{1}{r-1}}}{x_1^{\frac{1}{r-1}}} = \frac{x_1^{\frac{1}{r-1}}}{x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}} ;$$

by analogy, $p_2 = \dfrac{x_2^{\frac{1}{r-1}}}{x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}}$.

So $v_{min} = 1\left((p_1^*)^r + (p_2^*)^r \right)^{-1/r}$

$$= \left[\frac{x_1^{\frac{r}{r-1}}}{\left(x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right)^r} + \frac{x_2^{\frac{r}{r-1}}}{\left(x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right)^r} \right]^{-1/r}$$

$$= \frac{\left(x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right)^{-1/r}}{\left(x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right)^{r \cdot \frac{-1}{r}}} ; \quad \frac{-1}{r} - \left(r \cdot \frac{-1}{r}\right) = \frac{-1}{r} + 1 =$$

$$\frac{-1}{r} + \frac{r}{r} = \frac{r-1}{r} \Rightarrow$$

$$= \left(x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}}.$$

Qualifying Exam
1996
Question 3

①

Question 3. If a consumer's expenditure function is $e(\mathbf{p}, u) = (p_1^r + p_2^r)^{1/r} u$, find the consumer's:

a) (direct) utility function;
b) Marshallian demand curve for good 2;
c) Hicksian demand curve for good 2.

If you decide to solve an optimization problem when you answer this question, you do not have to verify that the second-order conditions hold, but you do have to state the second-order conditions. In stating these second-order conditions, it is acceptable to leave derivatives unevaluated, as long as the only things you have left undone are simple differentiations.

Qualifying Exam
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Answer 3

<u>Optional Question #3.</u> See the last example in Ch. 7 of Varian.

$$e(\underset{\sim}{p}, u) = (p_1^r + p_2^r)^{1/r} u.$$

First I'll answer part (c): Use Shephard's Lemma.

$$h_2 = \frac{\partial e}{\partial p_2} = \underset{\uparrow}{\frac{1}{r}} \cdot (p_1^r + p_2^r)^{\frac{1}{r}-1} \underset{\uparrow}{r}\, p_2^{r-1} u = \boxed{(p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^{r-1} u}$$

is the Hicksian demand for good 2.

Now here's part (b). Obtain the indirect utility function by applying

$$e(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) = m \quad \Rightarrow \quad (p_1^r + p_2^r)^{1/r} v(\underset{\sim}{p}, m) = m \quad \Rightarrow$$

$v(\underset{\sim}{p}, m) = m(p_1^r + p_2^r)^{-1/r}$. From here there are two ways to proceed: either use $h(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) = x(\underset{\sim}{p}, m)$ with v and h_2 from above to obtain

$$x_2(\underset{\sim}{p}, m) = (p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^{r-1} \cdot m(p_1^r + p_2^r)^{\frac{-1}{r}}$$

$$= \boxed{m\, p_2^{r-1} / (p_1^r + p_2^r)}$$

or use Roy's Identity to obtain

$$x_2 = -\frac{\partial v / \partial p_2}{\partial v / \partial m} = \overset{\downarrow}{-} \frac{\overset{\downarrow\downarrow}{\left(\frac{-1}{r}\right)} m (p_1^r + p_2^r)^{\overset{\downarrow}{\frac{-1}{r}}-1} \overset{\downarrow}{r}\, p_2^{r-1}}{(p_1^r + p_2^r)^{\underset{\uparrow}{\frac{-1}{r}}}}$$

$$= \boxed{\frac{m\, p_2^{r-1}}{p_1^r + p_2^r}}$$

as before, for the Marshallian demand for good 2.

Finally, part (a) $u(\underset{\sim}{x}) = \min_{\underset{\sim}{p}} v(\underset{\sim}{p})$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = 1$.

Using $v(\underset{\sim}{p}, m)$ from part (b) and setting $m=1$, the Lagrangian of this minimization problem is

$$\mathcal{L} = (p_1^r + p_2^r)^{-1/r} + \lambda (p_1 x_1 + p_2 x_2 - 1) \qquad (1)$$

First-order conditions : $0 = \frac{\partial \mathcal{L}}{\partial \lambda} = p_1 x_1 + p_2 x_2 - 1 \qquad (2)$

$$0 = \frac{\partial \mathcal{L}}{\partial p_1} = \frac{-1}{r}(p_1^r + p_2^r)^{-\frac{1}{r}-1} r p_1^{r-1} + \lambda x_1 \qquad (3)$$

$$0 = \frac{\partial \mathcal{L}}{\partial p_2} = \frac{-1}{r}(p_1^r + p_2^r)^{-\frac{1}{r}-1} r p_2^{r-1} + \lambda x_2 . \qquad (4)$$

(3)&(4) $\Rightarrow \dfrac{p_1^{r-1}}{p_2^{r-1}} = \dfrac{x_1}{x_2} \Rightarrow p_1^{r-1} = \dfrac{x_1}{x_2} p_2^{r-1} \Rightarrow p_1 = \left(\dfrac{x_1}{x_2}\right)^{\frac{1}{r-1}} p_2 .$ (5)

Substitute (5) into (1):

$$1 = p_1 x_1 + p_2 x_2 = \left(\frac{x_1}{x_2}\right)^{\frac{1}{r-1}} p_2 x_1 + p_2 x_2 = \left[\left(\frac{x_1}{x_2}\right)^{\frac{1}{r-1}} x_1 + x_2\right] p_2$$

$$= \left[\frac{x_1^{\frac{1}{r-1}} x_1}{x_2^{\frac{1}{r-1}}} + \frac{x_2}{1} \frac{x_2^{\frac{1}{r-1}}}{x_2^{\frac{1}{r-1}}}\right] p_2 = \frac{x_1^{\frac{1}{r-1}+1} + x_2^{\frac{1}{r-1}+1}}{x_2^{\frac{1}{r-1}}} p_2$$

and since $\frac{1}{r-1} + 1 = \frac{1}{r-1} + \frac{r-1}{r-1} = \frac{r}{r-1}$, this equals

$$\frac{x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}}{x_2^{\frac{1}{r-1}}}\, p_2 \Rightarrow p_2 = \frac{x_2^{\frac{1}{r-1}}}{x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}} . \qquad (6)$$

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Answer 3 cont...

Then from (5), $$p_1 = \frac{x_1^{\frac{1}{r-1}}}{x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}} . \qquad (7)$$

Substituting (6) and (7) into $u(\underset{\sim}{x}) = \min\limits_{\underset{\sim}{p}} (p_1^r + p_2^r)^{-1/r}$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = 1$ yields

$$u(\underset{\sim}{x}) = \left[\frac{x_1^{\frac{r}{r-1}}}{\left(x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right)^r} + \frac{x_2^{\frac{r}{r-1}}}{\left(x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right)^r} \right]^{-1/r}$$

$$= \left[\frac{x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}}{\left(x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right)^r} \right]^{-1/r} = \left[\left(x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right)^{1-r} \right]^{-1/r}$$

$$= \boxed{\left(x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}} .}$$ (If $\rho = \frac{r-1}{r}$ then $u(\underset{\sim}{x}) = (x_1^{\rho} + x_2^{\rho})^{1/\rho}$, the Constant Elasticity of Substitution utility function.)

The second-order sufficient conditions for a minimum are that the leading principal minors (the "D's") of the Hessian of the Lagrangian ($\nabla^2 \mathcal{L}$) have the same sign as $(-1)^m$, starting with D_{2m+1}. In this case, $m = 1$, so the condition is that D_3 (which equals $|\nabla^2 \mathcal{L}|$) should have the same sign as $(-1)^1 = -1 < 0$. Using subscripts to denote differentiation of v by p:

$$\nabla^2 \mathcal{L} = \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda 1} & \mathcal{L}_{\lambda 2} \\ \mathcal{L}_{1\lambda} & \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{2\lambda} & \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} = \begin{bmatrix} 0 & x_1 & x_2 \\ x_1 & v_{11} & v_{12} \\ x_2 & v_{21} & v_{22} \end{bmatrix}$$

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Answer 3 cont...

Expanding along the first column,

$$|\nabla^2 \mathcal{L}| = (-1)^{2+1} x_1 \left(x_1 v_{22} - x_2 v_{21} \right) + (-1)^{2+2} x_2 \left(x_1 v_{12} - x_2 v_{11} \right)$$

$$= -x_1 \left(x_1 v_{22} - x_2 v_{21} \right) + x_2 \left(x_1 v_{12} - x_2 v_{11} \right).$$

This should be negative for the second-order condition to hold.

new: 2019 Exam 1, Qu. 1.
Resembles 1997 Exam 1 Question 2 and 1996 Qualifying Exam Question 3

1. **[11 points]**

   If a consumer's expenditure function is $e(p, \bar{u}) = p_1^a p_2^{1-a} \bar{u}$, find the consumer's:

   (a) (direct) utility function;

   (b) Marshallian demand curve for good 2;

   (c) Hicksian demand curve for good 2.

   If you decide to solve an optimization problem when you answer this question, you do not have to verify that the second-order conditions hold, but you do have to state the second-order conditions. In stating these second-order conditions, it is acceptable to leave derivatives unevaluated, as long as the only things you have left undone are simple differentiations.

Answer to Exam 1, Fall 2019, Question 1

a) $e(p, \bar{u}) = p_1^{a} p_2^{1-a} \bar{u}$

To use $u = \min_{\underset{\sim}{p}} v(\underset{\sim}{p})$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = 1$, we need to find v :

$$e(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) = m \Rightarrow p_1^{a} p_2^{1-a} v(\underset{\smile}{p}, m) = m$$

$$\Rightarrow \quad v(\underset{\sim}{p}, m) = m\, p_1^{-a} p_2^{a-1} .$$

Now we solve $\min_{\underset{\sim}{p}} \underset{\uparrow_{m}}{(1)}\, p_1^{-a} p_2^{a-1}$ s.t. $p_1 x_1 + p_2 x_2 = 1$.

$$\mathcal{L} = p_1^{-a} p_2^{a-1} + \lambda [p_1 x_1 + p_2 x_2 - 1]$$

$$0 = \partial \mathcal{L} / \partial p_1 = -a\, p_1^{-a} p_1^{-1} p_2^{a-1} + \lambda x_1$$

$$0 = \partial \mathcal{L} / \partial p_2 = (a-1)\, p_1^{-a} p_2^{a} p_2^{-2} + \lambda x_2$$

$$\Rightarrow \lambda = a \underbrace{p_1^{-a}}_{\uparrow} p_1^{-1} \underbrace{p_2^{a-1}}_{\updownarrow} x_1^{-1} = (1-a) \underbrace{p_1^{-a}}_{\uparrow} \underbrace{p_2^{a} p_2^{-2}}_{\updownarrow} x_2^{-1}$$

$$a \quad p_1^{-1} \; p_2 \; x_1^{-1} = (1-a) \quad \cdot \quad x_2^{-1}$$

$$\frac{1}{a} p_1 p_2^{-1} x_1 = \frac{1}{1-a} x_2$$

$$\frac{1-a}{a} p_1 p_2^{-1} x_1 = x_2 \Rightarrow p_2 = \frac{1-a}{a} p_1 \frac{x_1}{x_2}$$ (remember it's the prices which are endogenous)

$$\Rightarrow 1 = p_1 x_1 + p_2 x_2 = p_1 x_1 + \frac{1-a}{a} p_1 x_1 = \left(\frac{a}{a} + \frac{1-a}{a}\right) p_1 x_1 = \frac{p_1 x_1}{a} \Rightarrow p_1 = \frac{a}{x_1} .$$

Then $p_2 = \frac{1-a}{a} \frac{a}{x_1} \frac{x_1}{x_2} = \frac{1-a}{x_2}$ and

$$u = v^* = \left(\frac{a}{x_1}\right)^{-a}\left(\frac{1-a}{x_2}\right)^{a-1} = \underline{x_1^{a}\,x_2^{1-a}} \cdot \underbrace{a^{-a}(1-a)^{a-1}}_{\text{a constant}}.$$ This is a Cobb-Douglas utility function.

To check second-order conditions:

$$\partial\mathcal{L}/\partial\lambda = p_1 x_1 + p_2 x_2 - 1$$

from above $\begin{cases} \partial\mathcal{L}/\partial p_1 = -a\,p_1^{-a-1}\,p_2^{a-1} + \lambda x_1 \\ \partial\mathcal{L}/\partial p_2 = (a-1)\,p_1^{-a}\,p_2^{a-2} + \lambda x_2 \end{cases}$

variables $n = 2$

constraints $m = 1$

$D_{2m+1} \dots D_{m+n}$ is $D_3 \dots D_3$.

For a minimum, we want D_3 of $\nabla^2\mathcal{L}$ to have the same sign as $(-1)^m = -1$.

$$\text{so } \nabla^2\mathcal{L} = \begin{bmatrix} \partial^2\mathcal{L}/\partial\lambda^2 & \partial^2\mathcal{L}/\partial\lambda\partial p_1 & \partial^2\mathcal{L}/\partial\lambda\partial p_2 \\ \partial^2\mathcal{L}/\partial p_1\partial\lambda & \partial^2\mathcal{L}/\partial p_1^2 & \partial^2\mathcal{L}/\partial p_1\partial p_2 \\ \partial^2\mathcal{L}/\partial p_2\partial\lambda & \partial^2\mathcal{L}/\partial p_2\partial p_1 & \partial^2\mathcal{L}/\partial p_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & x_1 & x_2 \\ x_1 & a(a+1)\,p_1^{-a-2}\,p_2^{a-1} & a(1-a)\,p_1^{-a-1}\,p_2^{a-2} \\ x_2 & a(1-a)\,p_1^{-a-1}\,p_2^{a-2} & (a-1)(a-2)\,p_1^{-a}\,p_2^{a-3} \end{bmatrix}. \xleftarrow{\boxed{\text{optional}}}$$

The second-order sufficient condition is that the determinant of this matrix be negative.

Optional: expanding along the first row, that condition is

$$0 > (-1)^{1+2}x_1\left[x_1(a-1)(a-2)\,p_1^{-a}\,p_2^{a-3} - x_2\,a(1-a)\,p_1^{-a-1}\,p_2^{a-2}\right]$$
$$+(-1)^{1+3}x_2\left[x_1 a(1-a)\,p_1^{-a-1}\,p_2^{a-2} - x_2\,a(a+1)\,p_1^{-a-2}\,p_2^{a-1}\right].$$

b) Use Roy's Identity $\qquad v(\underset{\sim}{p}, m) = m p_1^{-a} p_2^{a-1}$ from part (a)

$$X_2 = -\frac{\partial v/\partial p_2}{\partial v/\partial m} = -\frac{(a-1)\, m\, p_1^{-a} p_2^{a-2}}{p_1^{-a} p_2^{a-1}} = (1-a)\, m/p_2 .$$

c) Use Shephard's Lemma

$$h_2 = \frac{\partial e}{\partial p_2} = \frac{\partial}{\partial p_2} p_1^{a} p_2^{1-a} \bar{u} = (1-a)\, p_1^{a} p_2^{-a} \bar{u} .$$

Alternate answer for (c):

$h_2(\underset{\sim}{p}, \bar{u}) = x_2(\underset{\sim}{p}, e(\underset{\sim}{p}, m))$; from (b),

$= \frac{1-a}{p_2} e(\underset{\sim}{p}, m)$; from problem statement,

$= \frac{1-a}{p_2} p_1^{a} p_2^{1-a} \bar{u}$

$= (1-a)\, p_1^{a} p_2^{-a} \bar{u}$.

Alternate answer for (b):

$X_2(\underset{\sim}{p}, m) = h_2(\underset{\sim}{p}, v(\underset{\sim}{p}, m))$; from (c),

$= (1-a)\, p_1^{a} p_2^{-a}\, v(\underset{\sim}{p}, m)$; from v :

$= (1-a)\, p_1^{a} p_2^{-a}\, m\, p_1^{-a} p_2^{a-1}$

$= (1-a)\, p_2^{-1}\, m$.

1. [11 points] Suppose the expenditure function of a consumer is $e(\mathbf{p}, u) = (p_1^r + p_2^r)^{1/r} u$ where p_1 and p_2 are prices, $\mathbf{p}$ is the vector (p_1, p_2), and u is utility. Find this consumer's Marshallian demand curves.

Fall 2011, Exam 1 Qu 1

Answers to Exam 1, Econ. 7005, Fall 2011

①

$$e(\underset{\sim}{p}, u) = (p_1^{\,r} + p_2^{\,r})^{1/r}\, u$$

$$\left.\begin{array}{c} e(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) \equiv m \\ \parallel \\ (p_1^{\,r} + p_2^{\,r})^{1/r}\, v(\underset{\sim}{p}, m) \end{array}\right\} \Rightarrow (p_1^{\,r} + p_2^{\,r})^{1/r}\, v(\underset{\sim}{p}, m) = m$$

replacing u by $v(\underset{\sim}{p}, m)$

$$\Rightarrow v(\underset{\sim}{p}, m) = m\,(p_1^{\,r} + p_2^{\,r})^{-1/r}.$$

either

or

Roy's Identity

$$x_1(\underset{\sim}{p}, m) = -\frac{\partial v(\underset{\sim}{p}, m)/\partial p_1}{\partial v(\underset{\sim}{p}, m)/\partial m} = -\frac{m\left(\frac{-1}{r}\right)(p_1^{\,r} + p_2^{\,r})^{\frac{-1}{r}-1}\, r\, p_1^{\,r-1}}{(p_1^{\,r} + p_2^{\,r})^{-1/r}}$$

$$= \frac{m\, p_1^{\,r-1}}{p_1^{\,r} + p_2^{\,r}}$$

$$x_2(\underset{\sim}{p}, m) = -\frac{\partial v/\partial p_2}{\partial v/\partial m} = -\frac{m\left(\frac{-1}{r}\right)(p_1^{\,r} + p_2^{\,r})^{\frac{-1}{r}-1}\, r\, p_2^{\,r-1}}{(p_1^{\,r} + p_2^{\,r})^{-1/r}}$$

$$= \frac{m\, p_2^{\,r-1}}{p_1^{\,r} + p_2^{\,r}}$$

Alternatively, find h_1 and h_2 from e, then use

$$x_i(\underset{\sim}{p}, m) = \underset{\text{Hicksian demand curve}}{h_i}(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) \;:$$

Shephard's Lemma

$$h_1 = \frac{\partial e}{\partial p_1} = \frac{1}{r}\left(p_1^r + p_2^r\right)^{\frac{1}{r}-1} r\, p_1^{r-1}\, u = \left(p_1^r + p_2^r\right)^{\frac{1}{r}-1} p_1^{r-1}\, u$$

$$h_2 = \frac{\partial e}{\partial p_2} = \frac{1}{r}\left(p_1^r + p_2^r\right)^{\frac{1}{r}-1} r\, p_2^{r-1}\, u = \left(p_1^r + p_2^r\right)^{\frac{1}{r}-1} p_2^{r-1}\, u$$

and

$$x_1(\underset{\sim}{p}, m) = h_1(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) = \left(p_1^r + p_2^r\right)^{\frac{1}{r}-1} p_1^{r-1}\, v(\underset{\sim}{p}, m)$$

$$= \left(p_1^r + p_2^r\right)^{\frac{1}{r}-1} p_1^{r-1}\, m \left(p_1^r + p_2^r\right)^{\frac{-1}{r}} = \frac{m\, p_1^{r-1}}{p_1^r + p_2^r}$$

$$x_2(\underset{\sim}{p}, m) = h_2(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) = \left(p_1^r + p_2^r\right)^{\frac{1}{r}-1} p_2^{r-1}\, v(\underset{\sim}{p}, m)$$

$$= \left(p_1^r + p_2^r\right)^{\frac{1}{r}-1} p_2^{r-1}\, m \left(p_1^r + p_2^r\right)^{\frac{-1}{r}} = \frac{m\, p_2^{r-1}}{p_1^r + p_2^r}$$

as before.

Yet another method starting from $v(\underset{\sim}{p}, m)$ would be to obtain $u(\underset{\sim}{x})$ via $\min_{\underset{\sim}{p}} \underbrace{v(\underset{\sim}{p}, \underline{1})}_{\text{s.t. } \underset{\sim}{p} \cdot \underset{\sim}{x} = 1}$, then get x_1 and x_2 by $\max_{\underset{\sim}{x}} u(\underset{\sim}{x})$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = m$.

Exam 1

Spring 2004

Question 2

①

2. [11 points] Suppose a consumer has utility function $u(x_1, x_2) = \ln x_1 + \ln x_2$ where x_1 and x_2 are quantities of two goods. Suppose the consumer's income is m and let the prices of the goods be p_1 and p_2 respectively.

   (a) Find the consumer's Marshallian demand curves. (Hint: the answers are $x_1^* = m/(2p_1)$ and $x_2^* = m/(2p_2)$.)

   (b) State and verify the second-order conditions.

   (c) By using the Slutsky equation, find how the consumer's Hicksian (i.e., "compensated") demand curve for the first commodity varies as p_2 varies. (I want a formula for the appropriate derivative.) (Hint: the answer is $m/(4p_1p_2)$.)

   (d) Find the indirect utility function. (Hint: one way of writing the answer is $\ln\left[m^2/(4p_1p_2)\right]$.)

   (e) Find the expenditure function. (Hint: the answer is $2\sqrt{p_1p_2}\,e^{u/2}$ where "e" is the irrational number 2.718....)

   (f) From the expenditure function, derive the consumer's Hicksian demand curve for the first good and find how this demand changes with changes in p_2. (Hint: the answer is $e^{u/2}/(2\sqrt{p_1p_2})$.)

   (g) Verify that the answers to parts (c) and (f) are the same. You may want to use part (d) to help.

   (h) How does the consumer's Hicksian demand curve for the *second* commodity vary as p_1 varies?

Exam 1

2004

Answer 2

② a) $\max \ln x_1 + \ln x_2$ s.t. $p_1 x_1 + p_2 x_2 = m$

$$\mathcal{L} = \ln x_1 + \ln x_2 + \lambda(m - p_1 x_1 - p_2 x_2)$$

$$0 = \partial \mathcal{L} / \partial \lambda = m - p_1 x_1 - p_2 x_2$$

$$\left.\begin{array}{l} 0 = \frac{1}{x_1} - \lambda p_1 \\ 0 = \frac{1}{x_2} - \lambda p_2 \end{array}\right\} \lambda = \frac{1}{p_1 x_1} = \frac{1}{p_2 x_2} \Rightarrow x_2 = \frac{p_1 x_1}{p_2}$$

$$\Rightarrow m = p_1 x_1 + p_2 \frac{p_1 x_1}{p_2} \Rightarrow$$

$m = p_1 x_1 + p_1 x_1 = 2p_1 x_1 \Rightarrow x_1 = \frac{m}{2p_1}$ and $x_2 = \frac{m}{2p_2}$.

b) $$\nabla^2 \mathcal{L} = \begin{bmatrix} \mathcal{L}''_{\lambda\lambda} & \mathcal{L}''_{\lambda x_1} & \mathcal{L}''_{\lambda x_2} \\ \mathcal{L}''_{x_1\lambda} & \mathcal{L}''_{x_1 x_1} & \mathcal{L}''_{x_1 x_2} \\ \mathcal{L}''_{x_2\lambda} & \mathcal{L}''_{x_2 x_1} & \mathcal{L}''_{x_2 x_2} \end{bmatrix} = \begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & \frac{-1}{x_1^2} & 0 \\ -p_2 & 0 & \frac{-1}{x_2^2} \end{bmatrix}$$

So (for a max: $D_{2m+1} \cdots D_{n+m}$ alt. in sign starting with $(-1)^{m+1}$. $m=1$ $n=$

D_3 has sign of $(-1)^2 > 0$.

$$D_3 \text{ of } \nabla^2\mathcal{L} = |\nabla^2\mathcal{L}| = +p_1\left(p_1 \frac{1}{x_2^2} - 0\right) - p_2\left(-p_2 \frac{1}{x_1^2}\right)$$

$$= \frac{p_1^2}{x_2^2} + \frac{p_2^2}{x_1^2} > 0 \text{ OK}$$

c) $$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - \frac{\partial x_i}{\partial m} x_j$$

Exam 1
2004
Answer 2 cont...

$$\Rightarrow \frac{\partial h_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial m} x_j$$

$$\frac{\partial h_1}{\partial p_2} = \frac{\partial x_1}{\partial p_2} + \frac{\partial x_1}{\partial m} x_2 = 0 + \frac{1}{2p_1}\frac{m}{2p_2} = \frac{m}{4p_1 p_2}$$

d) $$v = \ln x_1^* + \ln x_2^* = \ln \frac{m}{2p_1} + \ln \frac{m}{2p_2} = \ln \frac{m^2}{4p_1 p_2}$$

e) $$v(\underset{\sim}{p}, \text{expend.}) \equiv u$$

$\uparrow e(\underset{\sim}{p}, u)$

$$\ln \frac{\text{expend.}^2}{4p_1 p_2} = u$$

$$\frac{\text{expend}^2}{4p_1 p_2} = e^u$$

$$\text{expend}^2 = 4p_1 p_2 e^u$$

$$\text{expenditure} = 2\sqrt{p_1 p_2}\, e^{u/2}$$

f) $h_1 = \frac{\partial e}{\partial p_1} = \sqrt{\frac{p_2}{p_1}}\; e^{u/2}$

$$\frac{\partial h_1}{\partial p_2} = \frac{1}{2}\,\frac{1}{\sqrt{p_1 p_2}}\; e^{u/2}$$

g) " $= \frac{1}{2}\,\frac{1}{\sqrt{p_1 p_2}}\; e^{\frac{1}{2}\ln\frac{m^2}{4p_1p_2}} = \frac{1}{2}\,\frac{1}{\sqrt{p_1 p_2}}\left(\frac{m^2}{4p_1 p_2}\right)^{1/2}$

$= \frac{1}{2}\,\frac{m}{2p_1p_2} = \frac{m}{4p_1p_2}$ ok

Exam 1
2004
Answer 2 cont...

h) $\frac{\partial h_2}{\partial p_1} = \frac{\partial h_1}{\partial p_2}$ by symmetry

↑ given in (f) and (c)

1. [**20 points**] Suppose a price-taking consumer consumes two commodities x and y and has a utility function of the form

$$u(x,y) = \alpha \ln x + \beta \ln y \,.$$

Suppose the price of x is $\bar{p}_x$ and the price of y is $\bar{p}_y$.

Fall
2008
Final

(a) Show that this consumer's indirect utility function can be written as

$$v(\bar{p}_x, \bar{p}_y, m) = \ln\left[\frac{\alpha^\alpha \beta^\beta m^{\alpha+\beta}}{\bar{p}_x^\alpha \bar{p}_y^\beta (\alpha+\beta)^{\alpha+\beta}}\right]$$

where m is the consumer's income.

(b) Show that this consumer's expenditure function is

$$e(\bar{p}_x, \bar{p}_y, u) = (\alpha+\beta)\left(\frac{\bar{p}_x^\alpha \bar{p}_y^\beta}{\alpha^\alpha \beta^\beta}\right)^{\frac{1}{\alpha+\beta}} \exp\left(\frac{u}{\alpha+\beta}\right) .$$

(c) Form this consumer's money metric indirect utility function,

$$\mu(\hat{p}_x, \hat{p}_y; \bar{p}_x, \bar{p}_y, m) \equiv e(\hat{p}_x, \hat{p}_y, v(\bar{p}_x, \bar{p}_y, m)) \,.$$

Your final expression should not explicitly involve the indirect utility function v.

(d) Briefly explain the economic interpretation of your answer to part (c).

Answers to Econ 7005 Final Exam, Fall 2008

① $u(x,y) = \alpha \ln x + \beta \ln y$

a) $\max u$ s.t. $m = \bar{p}_x x + \bar{p}_y y$

$$\mathcal{L} = \alpha \ln x + \beta \ln y + \lambda [m - \bar{p}_x x - \bar{p}_y y]$$

$$0 = \partial\mathcal{L}/\partial\lambda = m - \bar{p}_x x - \bar{p}_y y$$

$$\left.\begin{aligned} 0 &= \partial\mathcal{L}/\partial x = \frac{\alpha}{x} - \lambda \bar{p}_x \Rightarrow \lambda = \frac{\alpha}{\bar{p}_x x} \\ 0 &= \partial\mathcal{L}/\partial y = \frac{\beta}{y} - \lambda \bar{p}_y \Rightarrow \lambda = \frac{\beta}{\bar{p}_y y} \end{aligned}\right\} \frac{\alpha}{\bar{p}_x x} = \frac{\beta}{\bar{p}_y y} \Rightarrow$$

$$x = \frac{\alpha}{\beta} \frac{\bar{p}_y}{\bar{p}_x} y ;$$

substituting into the first F.O.C.:

$$m = \bar{p}_x \frac{\alpha}{\beta} \frac{\bar{p}_y}{\bar{p}_x} y + \bar{p}_y y = \frac{\alpha}{\beta} \bar{p}_y y + \bar{p}_y y = \left(\frac{\alpha}{\beta} + 1\right) \bar{p}_y y = \frac{\alpha+\beta}{\beta} \bar{p}_y y$$

$$\Rightarrow y^* = \frac{\beta}{\alpha+\beta} \frac{m}{\bar{p}_y}$$ and

$$x^* = \frac{\alpha}{\beta} \frac{\bar{p}_y}{\bar{p}_x} y = \frac{\alpha}{\beta} \frac{\bar{p}_y}{\bar{p}_x} \frac{\beta}{\alpha+\beta} \frac{m}{\bar{p}_y} = \frac{\alpha}{\alpha+\beta} \frac{m}{\bar{p}_x} .$$

Then

$$v = u(x^*, y^*) = \alpha \ln x^* + \beta \ln y^*$$

$$= \ln (x^*)^\alpha + \ln (y^*)^\beta = \ln (x^*)^\alpha (y^*)^\beta$$

$$= \ln \left(\frac{\alpha}{\alpha+\beta}\right)^\alpha \left(\frac{m}{\bar{p}_x}\right)^\alpha \left(\frac{\beta}{\alpha+\beta}\right)^\beta \left(\frac{m}{\bar{p}_y}\right)^\beta$$

over →

$$= \ln \frac{\alpha^{\alpha}\beta^{\beta} m^{\alpha+\beta}}{\bar{p}_x^{\alpha}\,\bar{p}_y^{\beta}\,(\alpha+\beta)^{\alpha+\beta}}$$ as was to be shown. (This "v" is $v(\bar{p}_x, \bar{p}_y, m)$.)

b) Since $u = v(\underset{\sim}{p}, e(\underset{\sim}{p}, u))$ and we know v from part (a):

$$u = v(\bar{p}_x, \bar{p}_y, e) = \ln \frac{\alpha^{\alpha}\beta^{\beta} e^{\alpha+\beta}}{\bar{p}_x^{\alpha}\,\bar{p}_y^{\beta}\,(\alpha+\beta)^{\alpha+\beta}}.$$

This "e" is the expenditure function, not the base of the natural logarithms. To avoid confusion – that is, to avoid having two different "e" symbols in this problem – I'll use exp (x) to denote raising the base of the natural logarithms to the "x" power.

Solve for the expenditure function, e :

$$\exp(u) = \frac{\alpha^{\alpha}\beta^{\beta} e^{\alpha+\beta}}{\bar{p}_x^{\alpha}\,\bar{p}_y^{\beta}(\alpha+\beta)^{\alpha+\beta}}$$

$$\frac{\bar{p}_x^{\alpha}\,\bar{p}_y^{\beta}}{\alpha^{\alpha}\beta^{\beta}}(\alpha+\beta)^{\alpha+\beta}\exp(u) = e^{\alpha+\beta} \quad \text{so}$$

$$e(\bar{p}_x, \bar{p}_y, u) = \left(\frac{\bar{p}_x^{\alpha}\,\bar{p}_y^{\beta}}{\alpha^{\alpha}\beta^{\beta}}\right)^{\frac{1}{\alpha+\beta}}(\alpha+\beta)\exp\left(\frac{u}{\alpha+\beta}\right)$$

as was to be shown.

c) Using the answers to (a) and (b),

$$\mu(\underset{\sim}{\hat{p}}; \underset{\sim}{\bar{p}}, m) \equiv e(\underset{\sim}{\hat{p}}, v(\underset{\sim}{\bar{p}}, m))$$

$$= \left(\frac{\hat{p}_x^{\alpha}\,\hat{p}_y^{\beta}}{\alpha^{\alpha}\beta^{\beta}}\right)^{\frac{1}{\alpha+\beta}}(\alpha+\beta)\exp\left(\frac{v(\underset{\sim}{\bar{p}}, m)}{\alpha+\beta}\right) \rightarrow$$

$$= \left(\frac{\hat{p}_x^{\alpha} \; \hat{p}_y^{\beta}}{\alpha^{\alpha} \beta^{\beta}} \right)^{\frac{1}{\alpha+\beta}} (\alpha+\beta) \left[\exp v(\underset{\sim}{\bar{p}}, m) \right]^{\frac{1}{\alpha+\beta}}$$

$$= (\alpha+\beta) \left(\frac{\hat{p}_x^{\alpha} \; \hat{p}_y^{\beta}}{\alpha^{\alpha} \beta^{\beta}} \right)^{\frac{1}{\alpha+\beta}} \left[\frac{\alpha^{\alpha} \beta^{\beta} \; m^{\alpha+\beta}}{\bar{p}_x^{\alpha} \; \bar{p}_y^{\beta} \; (\alpha+\beta)^{\alpha+\beta}} \right]^{\frac{1}{\alpha+\beta}}$$

$$= (\alpha+\beta) \left[\frac{\hat{p}_x^{\alpha} \; \hat{p}_y^{\beta}}{\bar{p}_x^{\alpha} \; \bar{p}_y^{\beta}} \; \frac{\alpha^{\alpha} \beta^{\beta}}{\alpha^{\alpha} \beta^{\beta}} \right]^{\frac{1}{\alpha+\beta}} \frac{m}{\alpha+\beta} = \left[\frac{\hat{p}_x^{\alpha} \; \hat{p}_y^{\beta}}{\bar{p}_x^{\alpha} \; \bar{p}_y^{\beta}} \right]^{\frac{1}{\alpha+\beta}} m .$$

<u>Optional</u>: This can easily be checked by noting that if $\underset{\sim}{\hat{p}} = \underset{\sim}{\bar{p}}$, then μ should equal m, which it does.

d) $\mu(\underset{\sim}{\hat{p}} : \underset{\sim}{\bar{p}}, m) \equiv e(\underset{\sim}{\hat{p}}, v(\underset{\sim}{\bar{p}}, m))$ is the amount of money which the consumer would require at prices $\underset{\sim}{\hat{p}}$ to have the same utility as if the prices were $\underset{\sim}{\bar{p}}$ and his income was m.

1. [**13 points**] Suppose a price-taking consumer consumes two commodities x and y and has an indirect utility function of the form

$$v(\bar{p}_x, \bar{p}_y, m) = \ln\left[\frac{\alpha^\alpha \beta^\beta m^{\alpha+\beta}}{\bar{p}_x^\alpha \bar{p}_y^\beta (\alpha+\beta)^{\alpha+\beta}}\right]$$

where m is the consumer's income, $\bar{p}_x$ is the price of x, and $\bar{p}_y$ is the price of y.

(a) Show that this consumer's expenditure function is

$$e(\bar{p}_x, \bar{p}_y, u) = (\alpha+\beta)\left(\frac{\bar{p}_x^\alpha \bar{p}_y^\beta}{\alpha^\alpha \beta^\beta}\right)^{\frac{1}{\alpha+\beta}} \exp\left(\frac{u}{\alpha+\beta}\right)$$

where exponentiation is denoted by "exp" to avoid confusion with the notation for the expenditure function e.

(b) Form this consumer's money metric indirect utility function,

$$\mu(\hat{p}_x, \hat{p}_y; \bar{p}_x, \bar{p}_y, m) \equiv e(\hat{p}_x, \hat{p}_y, v(\bar{p}_x, \bar{p}_y, m)) .$$

Your final expression should not explicitly involve the indirect utility function v.

(c) Show that this consumer's utility function is either

$$u(x, y) = \alpha \ln x + \beta \ln y$$

or a monotonically increasing function of this (such as $x^\alpha y^\beta$).

Summer 2011 qualifying exam, Sec. 1 Qu. 1

Answers to Microeconomics Qualifying Exam
Questions of Prof. Lozada, Summer 2011

Section 1 Question 1.

[This question is very closely related to Fall 2008 Final Exam Question 1.]

a) We know that $v(\underset{\sim}{p}, e(\underset{\sim}{p}, u)) = u$, or, abbreviating the expenditure function by "e", $v(\underset{\sim}{p}, e) = u$ so

$$u = v(\underset{\sim}{p}, e) = v(\bar{p}_x, \bar{p}_y, \underset{\substack{\uparrow \\ e \text{ in} \\ \text{our case}}}{m}) = \ln \frac{\alpha^\alpha \beta^\beta e^{\alpha+\beta}}{\bar{p}_x^{\alpha} \bar{p}_y^{\beta} (\alpha+\beta)^{\alpha+\beta}} .$$

We now have to solve for e, remembering that e is the expenditure function, not the base of the natural logarithms.

$$\exp u = \frac{\alpha^\alpha \beta^\beta e^{\alpha+\beta}}{\bar{p}_x^{\alpha} \bar{p}_y^{\beta} (\alpha+\beta)^{\alpha+\beta}} \Rightarrow e^{\alpha+\beta} = (\alpha+\beta)^{\alpha+\beta} \frac{\bar{p}_x^{\alpha} \bar{p}_y^{\beta}}{\alpha^\alpha \beta^\beta} (\exp u)$$

$$\Rightarrow e = (\alpha+\beta) \left(\frac{\bar{p}_x^{\alpha} \bar{p}_y^{\beta}}{\alpha^\alpha \beta^\beta} \right)^{\frac{1}{\alpha+\beta}} \exp\left(\frac{u}{\alpha+\beta}\right) .$$

b) Since $e(\hat{p}_x, \hat{p}_y, v) = (\alpha+\beta) \left[\frac{\hat{p}_x^{\alpha} \hat{p}_y^{\beta}}{\alpha^\alpha \beta^\beta} \right]^{\frac{1}{\alpha+\beta}} \exp\left(\frac{v}{\alpha+\beta}\right)$,

$$e(\hat{p}_x, \hat{p}_y, v(\bar{p}_x, \bar{p}_y, m)) = (\alpha+\beta) \left[\frac{\hat{p}_x^{\alpha} \hat{p}_y^{\beta}}{\alpha^\alpha \beta^\beta} \right]^{\frac{1}{\alpha+\beta}} \exp\left[\frac{v(\bar{p}_x, \bar{p}_y, m)}{\alpha+\beta} \right]$$

$$= (\alpha+\beta) \left[\frac{\hat{p}_x^{\alpha} \hat{p}_y^{\beta}}{\alpha^\alpha \beta^\beta} \right]^{\frac{1}{\alpha+\beta}} \left[\exp v(\bar{p}_x, \bar{p}_y, m) \right]^{\frac{1}{\alpha+\beta}} =$$

$$(\alpha+\beta)\left[\frac{\hat{p}_x^{\alpha}\ \hat{p}_y^{\beta}}{\alpha^{\alpha}\beta^{\beta}}\right]^{\frac{1}{\alpha+\beta}}\left[\frac{\alpha^{\alpha}\beta^{\beta}m^{\alpha+\beta}}{\bar{p}_x^{\alpha}\ \bar{p}_y^{\beta}(\alpha+\beta)^{\alpha+\beta}}\right]^{\frac{1}{\alpha+\beta}}$$

. This simplifies to $m\left[\hat{p}_x^{\alpha}\ \hat{p}_y^{\beta}/\bar{p}_x^{\alpha}\ \bar{p}_y^{\beta}\right]^{\frac{1}{\alpha+\beta}}$.

c) $u(\underset{\sim}{x}) = \min\limits_{\underset{\sim}{p}} v(\underset{\sim}{p})$ s.t. $\underset{\sim}{p}\cdot\underset{\sim}{x} = 1$

$$= \min \ln\left[\bar{p}_x^{-\alpha}\ \bar{p}_y^{-\beta}\ \frac{\alpha^{\alpha}\beta^{\beta}\ \overset{\text{setting } m=1}{1}{}^{\alpha+\beta}}{(\alpha+\beta)^{\alpha+\beta}}\right] \text{ s.t. } \underset{\sim}{p}\cdot\underset{\sim}{x} = 1$$

$$= \min\ -\alpha\ln\bar{p}_x - \beta\ln\bar{p}_y + \ln\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}}$$

$$\mathcal{L} = -\alpha\ln\bar{p}_x - \beta\ln\bar{p}_y + \ln\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} + \lambda(\bar{p}_x x + \bar{p}_y y - 1)$$

F.O.C.'s

$$\left.\begin{aligned} 0 &= \frac{\partial\mathcal{L}}{\partial\bar{p}_x} = \frac{-\alpha}{\bar{p}_x} + \lambda x \\ 0 &= \frac{\partial\mathcal{L}}{\partial\bar{p}_y} = \frac{-\beta}{\bar{p}_y} + \lambda y \end{aligned}\right\} \Rightarrow \lambda = \frac{\alpha}{x\bar{p}_x} = \frac{\beta}{y\bar{p}_y}$$

$$\Rightarrow \bar{p}_x = \frac{\alpha}{x}\frac{y}{\beta}\bar{p}_y$$

$$0 = \frac{\partial\mathcal{L}}{\partial\lambda} = \bar{p}_x x + \bar{p}_y y - 1 \rightarrow$$

$$1 = \left(\frac{\alpha}{x}\frac{y}{\beta}\bar{p}_y\right)x + \bar{p}_y y$$

$$= \frac{\alpha y}{\beta}\bar{p}_y + \bar{p}_y y = y\bar{p}_y\left(\frac{\alpha}{\beta}+1\right)$$

$$= y\bar{p}_y\frac{\alpha+\beta}{\beta} \Rightarrow \bar{p}_y = \frac{\beta}{\alpha+\beta}\frac{1}{y} \text{ and}$$

therefore $\bar{p}_x = \frac{\alpha}{x}\frac{y}{\beta}\left[\frac{\beta}{\alpha+\beta}\frac{1}{y}\right] = \frac{\alpha}{\alpha+\beta}\frac{1}{x}$. Substituting these minimizing $\bar{p}_x$ and $\bar{p}_y$ into v results in

$$u = \ln\left[\frac{\alpha}{\alpha+\beta}\frac{1}{x}\right]^{-\alpha}\left[\frac{\beta}{\alpha+\beta}\frac{1}{y}\right]^{-\beta}\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} = \ln\frac{(\alpha+\beta)^{\alpha}}{\alpha^{\alpha}}x^{\alpha}\frac{(\alpha+\beta)^{\beta}}{\beta^{\beta}}y^{\beta}\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}}$$

$= \ln x^{\alpha} y^{\beta} = \alpha \ln x + \beta \ln y.$

Note that since $\ln x$ is increasing in x, one can, instead of minimizing

$$\ln \left[\bar{p}_x^{-\alpha} \, \bar{p}_y^{-\beta} \, \frac{\alpha^{\alpha} \beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} \right] \qquad \text{s.t. } \underset{\sim}{p} \cdot \underset{\sim}{x} = 1,$$

minimize instead

$$\left[\bar{p}_x^{-\alpha} \, \bar{p}_y^{-\beta} \, \frac{\alpha^{\alpha} \beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} \right] \quad \text{s.t. } \underset{\sim}{p} \cdot \underset{\sim}{x} = 1.$$

If you do this, the result is $u = x^{\alpha} y^{\beta}$, which represent the same preferences as $\ln x^{\alpha} y^{\beta} = \alpha \ln x + \beta \ln y$.

Final Exam
2004
Question 1

①

Answer all of the following four questions.

✓ 1. [13 points] Let $h_i(\mathbf{p}, u^0)$ denote a consumer's Hicksian demand curve for good i when the consumer faces prices $\mathbf{p}$ and enjoys utility level u^0. Prove that

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}$$

by using the definition of the expenditure function and by using the Envelope Theorem.

Answers to Final Exam, Econ 7005, Spring 2004

Final Exam
2004
Answer 1

① $e(\underset{\sim}{p}, u) = \min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x}$ s.t. $u(\underset{\sim}{x}) = u$ (fixed)

$$\mathcal{L} = \underset{\sim}{p} \cdot \underset{\sim}{x} + \lambda\left(u - u(\underset{\sim}{x})\right)$$

$\frac{\partial e}{\partial p_i} = \frac{\partial \mathcal{L}^*}{\partial p_i}$ from the Envelope Theorem

$= x_i$ or, in our usual notation, h_i, the Hicksian demand curve

$$\frac{\partial e}{\partial p_i} = h_i \Rightarrow \begin{cases} \frac{\partial h_i}{\partial p_j} = \frac{\partial}{\partial p_j}\frac{\partial e}{\partial p_i} = \frac{\partial^2 e}{\partial p_j \partial p_i} \\ \frac{\partial h_j}{\partial p_i} = \frac{\partial}{\partial p_i}\frac{\partial e}{\partial p_j} = \frac{\partial^2 e}{\partial p_i \partial p_j} \end{cases}$$

→ which are equal (by Young's Theorem).

Qualifying Exam
1997
Question 2

①

Question 2. Let 'i' and 'j' be any two commodities which a consumer buys. If the Hicksian (or "compensated") demand function of a consumer is $\mathbf{h}(\mathbf{p}, u_0)$, prove that $\partial h_i/\partial p_i \leq 0$ and that $\partial h_i/\partial p_j = \partial h_j/\partial p_i$.

You do not have to prove the Envelope Theorem here, but you do have to prove all other results which you use.

Optional Question 2.

Step 1: the analogue of Shephard's Lemma for consumers, namely $\underset{\sim}{h} = \nabla_{\underset{\sim}{p}} e(\underset{\sim}{p}, u_0)$.

Proof: $e(\underset{\sim}{p}, u_0) = \min \underset{\sim}{p} \cdot \underset{\sim}{x}$ s.t. $u(\underset{\sim}{x}) \geq u_0$

The Lagrangian is $\mathcal{L} = \underset{\sim}{p} \cdot \underset{\sim}{x} + \lambda [u(\underset{\sim}{x}) - u_0]$.

Note: Varian has "u" instead of "u_0", which is OK, just a bit confusing

The Envelope Theorem then implies that

$$\frac{\partial e(\underset{\sim}{p}, u_0)}{\partial p_i} = \frac{\partial \mathcal{L}^*}{\partial p_i},$$

which is just x_i^*, the demand for commodity i. The vector of such demands is $\underset{\sim}{h}$, and the vector form of $\partial e(\underset{\sim}{p}, u_0)/\partial p_i$ is $\nabla_{\underset{\sim}{p}} e(\underset{\sim}{p}, u_0)$.

Qualifying Exam 1997

Answer 2

Step 2. Differentiate Step 1's result with respect to $\underset{\sim}{p}$:

$$\nabla_{\underset{\sim}{p}} \underset{\sim}{h} = \nabla^2_{\underset{\sim}{p}} e(\underset{\sim}{p}, u_0).$$

The RHS is a Hessian and so is symmetric; therefore, the LHS is symmetric and $\partial h_i / \partial p_j = \partial h_j / \partial p_i$.

Step 3. $e(\underset{\sim}{p}, u_0)$ is concave in $\underset{\sim}{p}$.

Proof:

$$e(\lambda \underset{\sim}{p_1} + (1-\lambda) \underset{\sim}{p_2}, u_0) = \min_{\underset{\sim}{x}} (\lambda \underset{\sim}{p_1} + (1-\lambda) \underset{\sim}{p_2}) \cdot \underset{\sim}{x} \quad \text{s.t. } u(\underset{\sim}{x}) \geq u_0$$

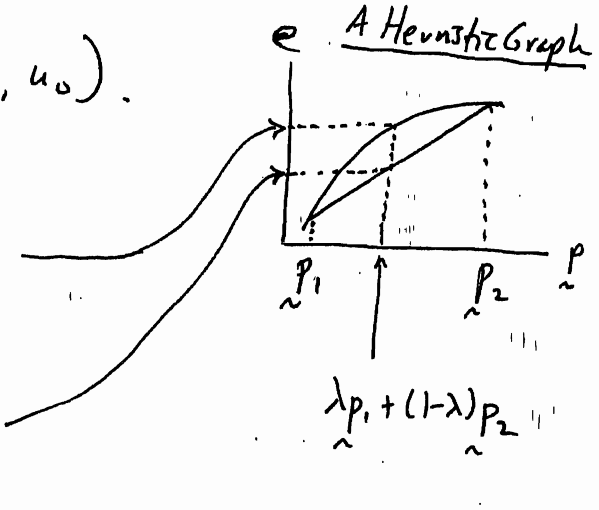
$$\geq \min_{\underset{\sim}{x}} \lambda \underset{\sim}{p_1} \cdot \underset{\sim}{x} \; (\text{s.t. } u(\underset{\sim}{x}) \geq u_0) + \min_{\underset{\sim}{x}} (1-\lambda) \underset{\sim}{p_2} \cdot \underset{\sim}{x} \; (\text{s.t. } u(\underset{\sim}{x}) \geq u_0)$$

because taking two separate minimizations cannot lead to a higher value

$$= \lambda \min_{\underset{\sim}{x}} \underset{\sim}{p_1} \cdot \underset{\sim}{x} \; (\text{s.t. } u(\underset{\sim}{x}) \geq u_0)$$

$$+ (1-\lambda) \min_{\underset{\sim}{x}} \underset{\sim}{p_2} \cdot \underset{\sim}{x} \; (\text{s.t. } u(\underset{\sim}{x}) \geq u_0)$$

$$= \lambda\, e(\underset{\sim}{p_1}, u_0) + (1-\lambda)\, e(\underset{\sim}{p_2}, u_0).$$



Step 4. Since e is concave in $\underset{\sim}{p}$, $\nabla^2_{\underset{\sim}{p}} e$ is negative semidefinite. So its diagonal elements are ≤ 0. From Step 2, this means that the diagonal elements of $\nabla_{\underset{\sim}{p}} \underset{\sim}{h}$ are ≤ 0; therefore $\partial h_i / \partial p_i \leq 0$ for all i.

Qualifying Exam
1997
Answer 2 cont..

2015 Final Exam Qu. 2

2. [18 points] If h_i denotes a competitive consumer's Hicksian demand curve for good i and p_i denotes the price of good i, prove that

$$\partial h_i/\partial p_i \leq 0 \quad \text{and}$$
$$\partial h_i/\partial p_j = \partial h_j/\partial p_i \,.$$

If you use an "Envelope Theorem" result, prove it (by applying the Envelope Theorem). If you contend that a function is concave or convex, prove it.

②

a) $\underset{\sim}{h} = \nabla_{\underset{\sim}{p}}\, e(\underset{\sim}{p}, \bar{u})$. Proof: $e(\underset{\sim}{p}, \bar{u}) = \min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x}$ s.t. $u(\underset{\sim}{x}) = \bar{u}$

$\uparrow$ the expenditure function

so by the Envelope Theorem,

$$\frac{\partial e}{\partial p_i} = \frac{\partial \mathcal{L}^*}{\partial p_i}$$

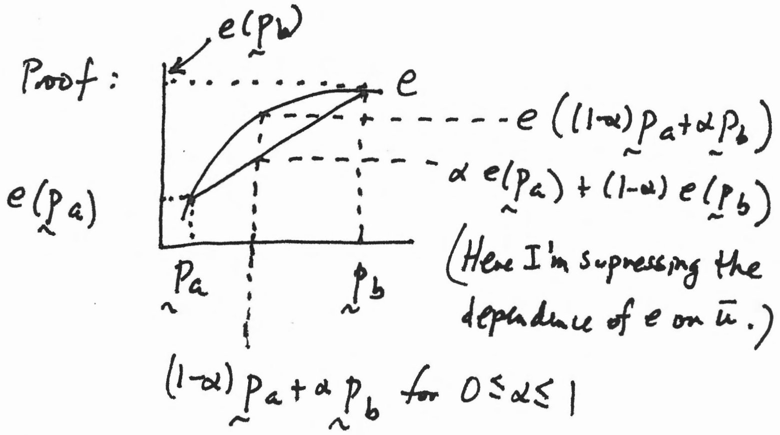
$$= \frac{\partial}{\partial p_i}\left[\underset{\sim}{p} \cdot \underset{\sim}{x} + \lambda(\bar{u} - u(\underset{\sim}{x}))\right]$$

$= x_i$ or, in standard notation, h_i, the Hicksian demand curve for good i.

(a) $\Rightarrow$

$$\nabla_{\underset{\sim}{p}}\, \underset{\sim}{h} = \nabla^2_{\underset{\sim}{p}}\, e(\underset{\sim}{p}, \bar{u}).$$ (b)

(c) $e(\underset{\sim}{p}, \bar{u})$ is concave in $\underset{\sim}{p}$. Proof:



(Here I'm supressing the dependence of e on $\bar{u}$.)

$$e((1-\alpha)\underset{\sim}{p}_a + \alpha\underset{\sim}{p}_b) = \min_{\underset{\sim}{x}} [(1-\alpha)\underset{\sim}{p}_a + \alpha\underset{\sim}{p}_b] \cdot \underset{\sim}{x} \text{ s.t. } u(\underset{\sim}{x}) = \bar{u}$$

I'll supress this from now on

$$= \min_{\underset{\sim}{x}} [(1-\alpha)\underset{\sim}{p}_a \cdot \underset{\sim}{x} + \alpha\underset{\sim}{p}_b \cdot \underset{\sim}{x}]$$

$\geq \min_{\underset{\sim}{x}} (1-\alpha)\underset{\sim}{p}_a \cdot \underset{\sim}{x} + \min_{\underset{\sim}{x}} \alpha\underset{\sim}{p}_b \cdot \underset{\sim}{x}$ (because

$\min_x [f(x)+g(x)] \leq \min_x f(x) + \min_x g(x)$ as shown in class)

$$= (1-\alpha) \min_{\underset{\sim}{x}} \underset{\sim}{p}_a \cdot \underset{\sim}{x} + \alpha \min_{\underset{\sim}{x}} \underset{\sim}{p}_b \cdot \underset{\sim}{x}$$

$$= (1-\alpha)\, e(\underset{\sim}{p}_a) + \alpha\, e(\underset{\sim}{p}_b)$$

which proves concavity.

(c) $\Rightarrow \nabla^2_{\underset{\sim}{p}} e(\underset{\sim}{p}, \bar{u})$ is negative semidefinite, and hence its diagonal terms are ≤ 0. (This was also shown in class: if A is negative semidefinite, $\underset{\sim}{x}^T A \underset{\sim}{x} \leq 0 \ \forall x$; take $\underset{\sim}{x} = (1,0,0,\dots,0)$, then $\underset{\sim}{x} = (0,1,0,0,\dots,0)$, etc.)

Then from (b), the diagonal terms of $\nabla_{\underset{\sim}{p}} \underset{\sim}{h}$ are ≤ 0; but these terms are just $\partial h_i / \partial p_i$. ■

(b) $\Rightarrow \partial h_i / \partial p_j = \partial h_j / \partial p_i$ because $\nabla^2 e$ is symmetric (its (i,j)th element is $\partial^2 e / \partial p_i \partial p_j$ and its (j,i)th element is $\partial^2 e / \partial p_j \partial p_i$, which are equal). ■

Fall 2021 Exam 1 Question 3

3. **[11 points]**

   Suppose a consumer has an $n \times n$ negative semidefinite symmetric Slutsky Substitution Matrix $\mathbf{S}$, but an economist has data only on $n' < n$ of the commodities. Should the $n' \times n'$ matrix $\mathbf{S}'$ (whose i, j entry is $\partial h_i / \partial p_j$) which is formed by using data only from n' of the commodities be negative semidefinite and symmetric? (This question requires a proof, not just a "yes or no" answer.)

Answer to Question 3, Exam 1, Fall 2021, Econ. 7005

Suppose $n = 4$, $n' = 2$, and the commodities one has data on are 1 and 3.

We know that $\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} = S$ is negative semidefinite and

symmetric. But the order of commodities is arbitrary. Put them in this order: 1, 3, 2, 4. (That is, put the commodities you have data on first.) Then

$$\hat{S} = \begin{bmatrix} S_{11} & S_{13} & S_{12} & S_{14} \\ S_{31} & S_{33} & S_{32} & S_{34} \\ S_{21} & S_{23} & S_{22} & S_{24} \\ S_{41} & S_{43} & S_{42} & S_{44} \end{bmatrix}$$

is also negative semidefinite and symmetric. Hence $S_{13} = S_{31}$. Also, being negative semidefinite implies that:

all the Δ_1 of $\hat{S}$ are ≤ 0

all the Δ_2 of $\hat{S}$ are ≥ 0. But $S' = \begin{bmatrix} S_{11} & S_{13} \\ S_{31} & S_{33} \end{bmatrix}$ is the upper

left-hand corner of $\hat{S}$, so the above conditions on the Δ's of $\hat{S}$ imply that

all the Δ_1 of S' are ≤ 0

all the Δ_2 of S' are ≥ 0. Hence S' is negative semidefinite. Its symmetry was proven above when we observed that $S_{13} = S_{31}$.

Extending this proof to other values of n and n' is straightforward. //

Note: there may be other ways to prove that the leading principal minors of a negative definite symmetric matrix are negative definite symmetric.

3. [11 points] Starting from the identity

$$h_k(\mathbf{p}, u) \equiv x_k\big(\mathbf{p}, e(\mathbf{p}, u)\big)$$

where x_k is the Marshallian demand for good k and h_k is the Hicksian demand for good k, determine the conditions on x under which

$$\frac{\partial x_j(\mathbf{p}, m)}{\partial p_i} = \frac{\partial x_i(\mathbf{p}, m)}{\partial p_j}.$$

[Fall 2004 Exam 1.]

3)

$$h_k(\underset{\sim}{p}, u) \equiv x_k(\underset{\sim}{p}, e(\underset{\sim}{p}, u))$$

$$\frac{\partial h_k}{\partial p_e} = \frac{\partial x_k}{\partial p_e} + \frac{\partial x_k}{\partial e} \frac{\partial e}{\partial p_e}$$

$$= \frac{\partial x_k}{\partial p_e} + \frac{\partial x_k}{\partial m} (x_e)$$

actually h_e but the quantities are the same though the functions differ

$$\Rightarrow \frac{\partial x_k}{\partial p_e} = \frac{\partial h_k}{\partial p_e} - x_e \frac{\partial x_k}{\partial m}$$ the Slutsky Equation.

So $$\frac{\partial x_j}{\partial p_i} \overset{?}{=} \frac{\partial x_i}{\partial p_j} \iff$$

$$\frac{\partial h_j}{\partial p_i} - x_i \frac{\partial x_j}{\partial m} \overset{?}{=} \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial m}$$

cancel because the Slutsky substitution matrix is symmetric

$$\iff \quad x_i \frac{\partial x_j}{\partial m} = x_j \frac{\partial x_i}{\partial m}$$

or $$\frac{1}{x_j} \frac{\partial x_j}{\partial m} = \frac{1}{x_i} \frac{\partial x_i}{\partial m} .$$

(or $\frac{m}{x_j} \frac{\partial x_j}{\partial m} = \frac{m}{x_i} \frac{\partial x_i}{\partial m}$, in other words, i and j have the same income elasticity.)

2017 Final Exam Qu. 1; resembles 1997 Ex. 1 Qu. 1, and has the same answer as it (see next problem)

1. **[16 points]**
   Prove that a consumer's expenditure function $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$. Fully explain your work.

Exam 1
1997
Question 1

①

Answer all of the following three questions.

1. Prove that a consumer's expenditure function $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$. Fully explain your work.

   Hint: Here is the beginning and middle of a bare, unexplained proof that a firm's cost function $c(\mathbf{w}, y)$ is concave in $\mathbf{w}$:

   To prove: $c\big(t\mathbf{w}_a + (1-t)\mathbf{w}_b,\ y\big) \geq t\,c(\mathbf{w}_a, y) + (1-t)\,c(\mathbf{w}_b, y)$ for $0 \leq t \leq 1$.

   Proof:

$$\begin{aligned} c\big(t\mathbf{w}_a + (1-t)\mathbf{w}_b,\ y\big) &= \min_{\mathbf{x}\in V(y)} \big[t\mathbf{w}_a + (1-t)\mathbf{w}_b\big]\cdot\mathbf{x} \\ &= \min_{\mathbf{x}\in V(y)} \big[t\mathbf{w}_a\cdot\mathbf{x} + (1-t)\mathbf{w}_b\cdot\mathbf{x}\big] \\ &\geq \min_{\mathbf{x}\in V(y)} t\mathbf{w}_a\cdot\mathbf{x} + \min_{\mathbf{x}\in V(y)} (1-t)\mathbf{w}_b\cdot\mathbf{x}\,. \end{aligned}$$

Exam 1
1997
Answer 1

Answers to Exam 1, Econ 621, Winter 1997

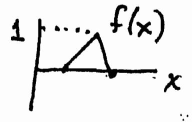
① $e(t\underset{\sim}{p}_a + (1-t)\underset{\sim}{p}_b, u) = \min_{\underset{\sim}{x}} [t\underset{\sim}{p}_a + (1-t)\underset{\sim}{p}_b]\cdot \underset{\sim}{x}$ s.t. $u(\underset{\sim}{x}) = u$

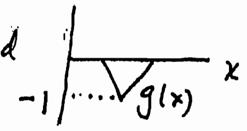
by the definition of the expenditure function, where $\underset{\sim}{x}$ is the commodity bundle purchased and $u(\underset{\sim}{x})$ is the utility function

$= \min_{\underset{\sim}{x}} [t\underset{\sim}{p}_a\cdot\underset{\sim}{x} + (1-t)\underset{\sim}{p}_b\cdot\underset{\sim}{x}]$ s.t. $u(\underset{\sim}{x}) = u$

$\geq \min_{\underset{\sim}{x}} t\underset{\sim}{p}_a\cdot\underset{\sim}{x}$ [s.t. $u(\underset{\sim}{x}) = u$]

$+ \min_{\underset{\sim}{x}} (1-t)\underset{\sim}{p}_b\cdot\underset{\sim}{x}$ [s.t. $u(\underset{\sim}{x}) = u$]

since $\min_x [f(x)+g(x)] \geq \min_x f(x) + \min_x g(x)$, as illustrated by 

and 

, where $\min(f+g) = \min(0) = 0$

but $\min f + \min g = 0 + (-1) = -1$.

$= t \min_{\underset{\sim}{x}} \underset{\sim}{p}_a\cdot\underset{\sim}{x}$ [s.t. $u(\underset{\sim}{x}) = u$] $+ (1-t)\min_{\underset{\sim}{x}} \underset{\sim}{p}_b\cdot\underset{\sim}{x}$ [s.t. $u(\underset{\sim}{x}) = u$]

$= t\, e(\underset{\sim}{p}_a, u) + (1-t)\, e(\underset{\sim}{p}_b, u)$

Exam 1
1997
Answer 1 cont...

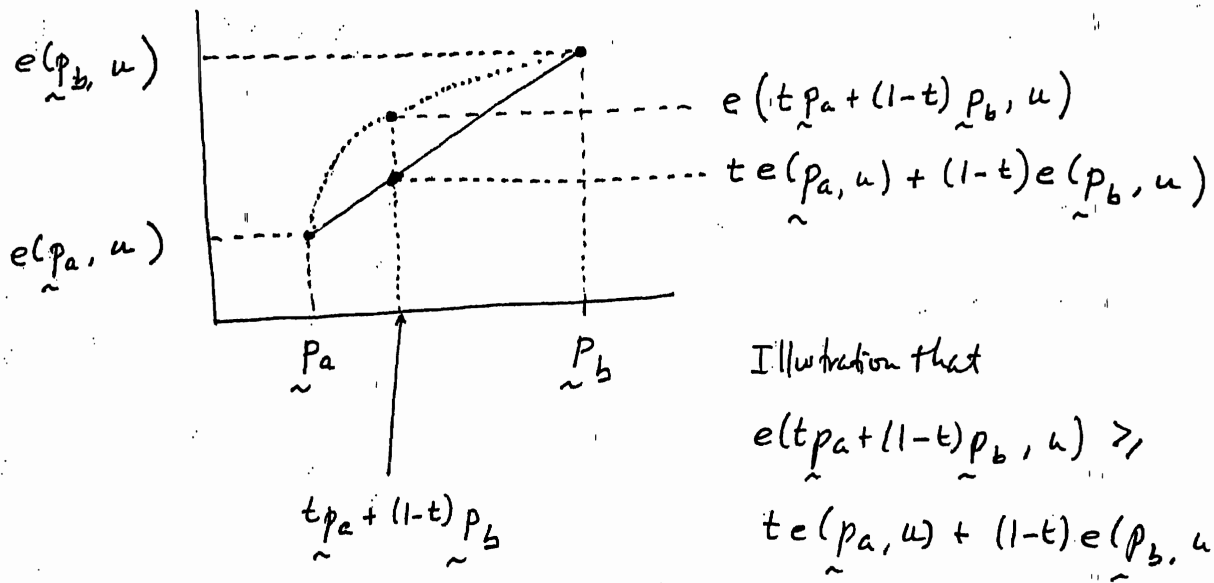


Illustration that

$$e(t\underset{\sim}{p}_a + (1-t)\underset{\sim}{p}_b, u) \geq te(\underset{\sim}{p}_a, u) + (1-t)e(\underset{\sim}{p}_b, u)$$

and therefore that $e(\underset{\sim}{p}, u)$ is concave.

Qualifying. Exam
1997
Question 1 ①

621 Section (must answer one)

Question 1. Suppose a price-taking consumer buys n commodities q which are indexed by i, as in: $q_1, q_2, \ldots, q_i, \ldots, q_n$. Let the price of commodity i be p_i. Let the consumer's income be m. Let the consumer's budget share for item i be

$$\alpha_i = \frac{p_i\, q_i}{m}\,.$$

Let

$$\epsilon_{ij} = \frac{\partial \ln q_i}{\partial \ln p_j}\,.$$

What is this?

Let

$$\eta_i = \frac{\partial \ln q_i}{\partial \ln m}\,.$$

What is this?

a) Prove the so-called Cournot Aggregation Condition:

$$\sum_j \alpha_j\, \epsilon_{ji} = -\alpha_i\,.$$

As a hint: differentiate the budget constraint with respect to p_i.

b) Prove the so-called Engel Aggregation Condition:

$$\sum_j \alpha_j\, \eta_j = 1\,.$$

As a hint: differentiate the budget constraint with respect to m.

Optional Question 1. $\mathcal{E}_{ij}$ is the elasticity of demand for good i with respect to the price of good j. η_i is the income elasticity of good i. Also,

$$\mathcal{E}_{ij} = \frac{\partial \ln q_i}{\partial \ln p_j} = \frac{p_j}{q_i}\frac{\partial q_i}{\partial p_j} \quad \text{and} \quad \eta_i = \frac{\partial \ln q_i}{\partial \ln m} = \frac{m}{q_i}\frac{\partial q_i}{\partial m}.$$

a) Differentiating $\sum_j p_j q_j = m$ with respect to p_i yields

$$\sum_j \left(\underbrace{\frac{dp_j}{dp_i}} q_j + p_j \frac{dq_j}{dp_i} \right) = \frac{dm}{dp_i} .$$

$\hookrightarrow$ 0 except when
$j = i$, in which case
$dp_j/dp_i = dp_i/dp_i = 1$
(and $q_j = q_i$ since $j = i$)

$\hookrightarrow 0$

Qualifying Exam
1997
Answer 1

Therefore

$$1 \cdot q_i + \sum_j p_j \frac{dq_j}{dp_i} = 0 .$$

Multiply by $\frac{p_i}{m}$ and multiply inside the summation by q_j/q_j :

$$\sum_j \frac{p_i}{m} p_j \frac{q_j}{q_j} \frac{dq_j}{dp_i} = -q_i \frac{p_i}{m}$$

$$\sum_j \frac{p_j q_j}{m} \cdot \frac{p_i}{q_j} \frac{dq_j}{dp_i} = -\frac{q_i p_i}{m} \Rightarrow \sum_j \alpha_j \mathcal{E}_{ji} = -\alpha_i .$$

b) Differentiating $\sum_j p_j q_j = m$ with respect to m yields

$\sum_j p_j \frac{dq_j}{dm} = 1$. Multiply the LHS by $\frac{q_j}{m} \frac{m}{q_j}$:

$$\sum_j \left[\frac{p_j q_j}{m} \cdot \frac{m}{j} \frac{dq_j}{dm} \right] = 1$$

$$\Rightarrow \sum_j \alpha_j \eta_j = 1.$$

Qualifying Exam
1997
Answer 1 cont...

new: 2020 Exam 1, Qu. 1.

1. **[11 points]**

Suppose a consumer consumes two goods, "1" and "2," and that if the consumer consumes $\mathbf{x} = (x_1, x_2)$ amounts of those two goods, his utility function is

$$u(\mathbf{x}) = x_1 x_2 .$$

Assume

$$x_1 \geq 0$$
$$x_2 \geq 0 .$$

Consider another family of preferences

$$\hat{u}_\gamma(\mathbf{x}) = (x_1 x_2)^2 + \gamma$$

where $\gamma \in \mathbf{R}^1 = (-\infty, \infty)$.

(a) For what values of γ do $\hat{u}_\gamma(\mathbf{x})$ and $u(\mathbf{x})$ represent the same preferences?

(b) State whether or not u is homogeneous, and if it is, state its degree of homogeneity.

(c) State for what value or values of γ, if any, $\hat{u}_\gamma(\mathbf{x})$ is homogeneous and what the degree of homogeneity is (or the degrees of homogeneity are).

(d) State whether or not u is homothetic. Hint: You could use the definition of homotheticity to prove this, but you need not go through that trouble here: there is a way to answer by using the answer to a previous part of this question and then just appealing to a result that I showed in class and that you do not have to prove here.

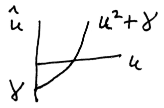
(e) State for what value or values of γ, if any, $\hat{u}_\gamma(\mathbf{x})$ is homothetic. Use the definition of homotheticity given in class as the basis for your answer.

Answers to Exam 1, Econ. 7005, Fall 2020

1. (a) Observe that $\hat{u}$ is the following transformation f of u:

$$\hat{u}_\gamma = f(u) = u^2 + \gamma\,.$$

The domain is $u \geq 0$ because the problem states that $x_1 \geq 0$ and $x_2 \geq 0$, making $u = x_1 x_2 \geq 0$. As the following graph shows, this $f(u)$ is increasing in u over the entire domain $u \geq 0$ for all γ. (The value of γ could be negative or positive or zero.)



This shows that "$\hat{u}_\gamma$ is an increasing transformation of u" (which just means that $\hat{u}_\gamma$ is increasing in u (that is, $d\hat{u}_\gamma/du > 0$), not that $\hat{u}_\gamma > u$; note that the latter is not true in the graph I drew because $\hat{u}_\gamma$ is sometimes negative while u is never negative). Therefore, $\hat{u}_\gamma(\mathbf{x})$ and $u(\mathbf{x})$ represent the same preferences for all values of γ.

(b) A function $f(\mathbf{x})$ is by definition "homogeneous of degree k" if $f(\lambda\mathbf{x}) = \lambda^k f(\mathbf{x})$. Here, we have $u(\lambda\mathbf{x}) = \lambda x_1 \cdot \lambda x_2 = \lambda^2 x_1 x_2$. So u is homogeneous of degree 2.

(c) Here,

$$\begin{aligned}\hat{u}_\gamma(\lambda\mathbf{x}) &= (\lambda x_1\, \lambda x_2)^2 + \gamma \\ &= \lambda^4 x_1^2 x_2^2 \overset{?}{=} \lambda^k\,[x_1^2 x_2^2 + \gamma] = \lambda^k \hat{u}_\gamma(\mathbf{x})\end{aligned}$$

so $\hat{u}_\gamma(\mathbf{x})$ is homogeneous only if $k = 4$ and $\gamma = 0$.

(d) From part (b), u is homogeneous. All homogeneous functions are homothetic, so u is homothetic.

(e) If $\hat{u}_\gamma(\mathbf{x})$ is homothetic then, by definition,

$$\hat{u}_\gamma(\mathbf{x}) = \hat{u}_\gamma(\mathbf{y}) \tag{1}$$

would imply that

$$\hat{u}_\gamma(\lambda\mathbf{x}) = \hat{u}_\gamma(\lambda\mathbf{y})\,. \tag{2}$$

(1) implies that

$$\begin{aligned}x_1^2 x_2^2 + \gamma = y_1^2 y_2^2 + \gamma \quad &\Leftrightarrow \\ x_1^2 x_2^2 = y_1^2 y_2^2\,. \end{aligned} \tag{3}$$

On the other hand, (2) would imply that

$$\begin{aligned} \lambda^2 x_1^2 x_2^2 + \gamma &= \lambda^2 y_1^2 y_2^2 + \gamma \\ \lambda^2 x_1^2 x_2^2 &= \lambda^2 y_1^2 y_2^2 \\ x_1^2 x_2^2 &= y_1^2 y_2^2 \,. \end{aligned} \tag{4}$$

Since (3) is the same as (4), $\hat{u}_\gamma(\mathbf{x})$ is homothetic.

Optional: Thus this problem shows three things:

- u and $\hat{u}_\gamma$ represent the same preferences;
- u and $\hat{u}_\gamma$ are not both homogeneous (in general); and
- u and $\hat{u}_\gamma$ are both homothetic.

This confirms what we said in class about homotheticity rather than homogeneity being the important concept in consumer theory.

2005
Qualifier
Sec. 2

1. (a) Prove that Hicksian demand curves $h(\mathbf{p}, u)$ are homogeneous of degree zero in $\mathbf{p}$.

   (b) As some of you may already know, Euler proved the following: if $f(\mathbf{x})$ is differentiable and is homogeneous of degree k, then

$$\nabla f(\mathbf{x}) \cdot \mathbf{x} = k\, f(\mathbf{x}) .$$

   (Do not forget that the left-hand side has a "$\cdot\, \mathbf{x}$" in it.) What property of Hicksian demand curves can you derive from this result, given what you already know from part (a)?

   (c) Rewrite your answer to part (b) for the special case when the total number of commodities is exactly three.

   (d) For any two commodities j and k, here are two definitions:

$$\partial h_j(\mathbf{p}, u)/\partial p_k \geq 0 \quad \Longleftrightarrow j \text{ and } k \text{ are "substitutes"}$$
$$\partial h_j(\mathbf{p}, u)/\partial p_k < 0 \quad \Longleftrightarrow j \text{ and } k \text{ are "complements."}$$

   [By the way, if instead of using the Hicksian demand curve $h_j(\mathbf{p}, u)$ on the left-hand side, we used Marshallian demand curves $x_j(\mathbf{p}, m)$ where m is income, then we would use the terms "gross substitutes" and "gross complements" on the right-hand side (but this is not important for this exam).]

   Use the previous parts of this question, and other information, to prove that if the total number of commodities is three, then every good has at least one substitute.

   (e) Prove that every good has at least one substitute (regardless of what the total number of commodities may be).

Section 2

① a) Hicksian demand curves solve

$$\min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x} \quad s.t.\ u(\underset{\sim}{x}) = \bar{u}. \qquad (P1)$$

If $\underset{\sim}{p}$ changes to $\lambda \underset{\sim}{p}$, the problem becomes

$$\min_{\underset{\sim}{x}} \lambda \underset{\sim}{p} \cdot \underset{\sim}{x} \quad s.t.\ u(\underset{\sim}{x}) = \bar{u} \qquad (P2)$$

$$\Longleftrightarrow \quad \lambda \min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x} \quad s.t.\ u(\underset{\sim}{x}) = \bar{u}. \qquad (P3)$$

(P3) has the same optimal point $\underset{\sim}{x}^*$ (or "$\underset{\sim}{h}^*$") as (P1). So $\underset{\sim}{h}(\underset{\sim}{p}, \bar{u})$ doesn't change when $\underset{\sim}{p}$ changes.

b) $\nabla_{\underset{\sim}{p}} h_i(\underset{\sim}{p}, u) \cdot \underset{\sim}{p} = k\, h_i(\underset{\sim}{p}, u)$ where k is the degree of homogeneity of $h_i(\underset{\sim}{p}, u)$ in $\underset{\sim}{p}$, which is zero:

$$\nabla_{\underset{\sim}{p}} h_i(\underset{\sim}{p}, u) \cdot \underset{\sim}{p} = 0.$$

c) $$\frac{\partial h_i}{\partial p_1} p_1 + \frac{\partial h_i}{\partial p_2} p_2 + \frac{\partial h_i}{\partial p_3} p_3 = 0 \quad \text{for } i \in \{1, 2, 3\}.$$

d) If $i = 1$ then part (c) $\Rightarrow$ $$\underbrace{\frac{\partial h_1}{\partial p_1} p_1} + \frac{\partial h_1}{\partial p_2} p_2 + \frac{\partial h_1}{\partial p_3} p_3 = 0.$$

Similar reasoning applies for $i = 2$ and $i = 3$

But this is ⊖ (Hicksian own-demand curves always slope downwards). So one of the other terms has to be positive (for all three to add up to zero)

e) One of the terms in $\nabla_{\underset{\sim}{p}} h_i$ is $\partial h_i / \partial p_i$, which is < 0, so for $\nabla_{\underset{\sim}{p}} h \cdot \underset{\sim}{p} = 0$, one of the other $\partial h_i / \partial p_j$ has to be positive.

Qualifying Exam
2000 ①
Question 1

Short Section

6710 Section (must answer one)

Question 1. On p. 147 of Varian's *Microeconomic Analysis*, he writes:

> We saw in our discussion of production theory that if a production function was homogeneous of degree 1, then the cost function could be written as $c(\mathbf{w}, y) = c(\mathbf{w})\, y$. It follows from this observation that if the utility function is homogeneous of degree 1, then the expenditure function can be written as $e(\mathbf{p}, u) = e(\mathbf{p})\, u$.

You may use this information (without proving that it is true) in the questions below.

a) Prove that if the utility function is homogeneous of degree 1, then the indirect utility function can be written as

$$v(\mathbf{p}, m) = v(\mathbf{p})\, m\,.$$

Hint: You may use $e(\mathbf{p},\, v(\mathbf{p}, m)) = m$ without proving it.

b) Prove that if the indirect utility function can be written as $v(\mathbf{p}, m) = v(\mathbf{p})\, m$, then the demand functions can be written as

$$x_i(\mathbf{p}, m) = x_i(\mathbf{p})\, m$$

—i.e., they are linear functions of income. Hint: You need not prove Roy's Identity.

c) Prove that if the demand functions can be written as $x_i(\mathbf{p}, m) = x_i(\mathbf{p})\, m$ —i.e., they are linear functions of income—then

$$\frac{\partial x_i(\mathbf{p}, m)}{\partial p_j} = \frac{\partial x_j(\mathbf{p}, m)}{\partial p_i}\,.$$

Hint: If you use the result that $\mathbf{h} = \nabla_{\mathbf{p}}\, e$ here, then you should prove that $\mathbf{h} = \nabla_{\mathbf{p}}\, e$. If you use the Slutsky equation here, then you should prove the Slutsky equation. If you use $h_i(\mathbf{p}, u) = x_i(\mathbf{p},\, e(\mathbf{p}, u))$ here, you do not need to prove it.

Qualifying Exam
2000
Answer 1

Short Section

Question 1.

utility function is homogeneous of degree 1 $\Rightarrow$ $e(\underset{\sim}{p}, u) = e(\underset{\sim}{p})\, u.$ (1)

a) $e(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) = m$. From (1),

$$e(\underset{\sim}{p})\, v(\underset{\sim}{p}, m) = m$$

$$v(\underset{\sim}{p}, m) = \frac{1}{e(\underset{\sim}{p})} \cdot m .$$

So if we let $v(\underset{\sim}{p})$ equal $\frac{1}{e(\underset{\sim}{p})}$, the claim is proven.

b). $v(\underset{\sim}{p}, m) = v(\underset{\sim}{p})\, m.$

Roy's Identity is $x_i(\underset{\sim}{p}, m) = - \dfrac{\partial v(\underset{\sim}{p}, m)/\partial p_i}{\partial v(\underset{\sim}{p}, m)/\partial m}$; in this case,

$$= - \frac{\frac{\partial}{\partial p_i}\left[v(\underset{\sim}{p})\, m\right]}{\frac{\partial}{\partial m}\left[v(\underset{\sim}{p})\, m\right]} = - \frac{\partial v(\underset{\sim}{p})/\partial p_i \cdot m}{v(\underset{\sim}{p})}$$

$$= \left[\frac{-\partial v(\underset{\sim}{p})/\partial p_i}{v(\underset{\sim}{p})}\right] \cdot m .$$

So if we let this ↑ (in brackets) be $x_i(\underset{\sim}{p})$, the claim is proven.

c) $$h_i(\underset{\sim}{p}, u) = x_i(\underset{\sim}{p}, e(\underset{\sim}{p}, u))$$

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial m}\frac{\partial e}{\partial p_j} \Rightarrow$$

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - \underbrace{\frac{\partial e}{\partial p_j}}\ \underbrace{\frac{\partial x_i}{\partial m}}. \qquad (1)$$

$= h_j$ (see proof below), or, as Varian writes, x_j, understood not as a function but as a single quantity.

$$= \frac{\partial}{\partial m} x_i(\underset{\sim}{p}, m); \text{ here,}$$
$$= \frac{\partial}{\partial m}\left[x_i(\underset{\sim}{p})\cdot m\right]$$
$$= x_i(\underset{\sim}{p}).$$

So (1) $\Rightarrow$

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - x_j x_i. \qquad (2)$$

By writing "j" for "i" and "i" for "j" in the above proof, one derives

$$\frac{\partial x_j}{\partial p_i} = \frac{\partial h_j}{\partial p_i} - x_i x_j. \qquad (3)$$

We wish to prove that the LHS of (2) equals the LHS of (3). Looking at their RHS's, all we need to show is that $\partial h_i/\partial p_j = \partial h_j/\partial p_i$. This is straightforward: since $\underset{\wedge}{h} = \nabla_{\underset{\sim}{p}} e$ (see proof below),

$\nabla_{\underset{\sim}{p}} \underset{\wedge}{h} = \nabla^2_{\underset{\wedge}{p}} e$. The latter is symmetric because it is a Hessian, so $\nabla_{\underset{\sim}{p}} \underset{\wedge}{h}$ is symmetric, which is what needed to be shown.

Above, we used $\partial e/\partial p_j = h_j$ and $\nabla_{\underset{\sim}{p}} e = \underset{\sim}{h}$ which are the same result: Shepherd's Lemma applied to consumer theory. It is proven by posing the consumer's problem

$$e(\underset{\sim}{p},\bar{u}) = \min_{\underset{\sim}{x}} \underset{\sim}{p}\cdot\underset{\sim}{x} \quad s.t. \quad u(\underset{\sim}{x}) \geq \bar{u} \quad \Rightarrow$$

$$\mathcal{L} = \underset{\sim}{p}\cdot\underset{\sim}{x} + \lambda(u(\underset{\sim}{x})-\bar{u}),$$

then applying the envelope theorem:

$$\frac{\partial e(\underset{\sim}{p},\bar{u})}{\partial p_i} = \frac{\partial \mathcal{L}^*}{\partial p_i}$$

$= x_i$, which is the Hicksian demand curve here (usually denoted h_i), not the Marshallian demand curve, because this is an expenditure-minimization problem, and this "x_i" depends on $\underset{\sim}{p}$ and $\bar{u}$, not $\underset{\sim}{p}$ and m.

Exam 1
1996
Question 1 ①

Answer all of the following three questions.

1. If $\mathbf{h}(\mathbf{p}, u)$ are the Hicksian demand curves for a consumer, prove that $d\mathbf{p} \cdot d\mathbf{h} \leq 0$.

   Also: very briefly discuss why this result is or is not surprising.

   Hint: use the analogue for consumers of Shephard's Lemma; find a total differential. You do not have to *prove* Shephard's Lemma, nor do you have to *prove* the convexity or concavity of a function, although if this were the qualifying exam then I would want you to prove these things.

Answers to Exam 1, Econ. 621, Winter 1996

Exam 1
1996
Answer 1

① $\underset{\sim}{h}(\underset{\sim}{p}, u) = \nabla_{\underset{\sim}{p}}\, e(\underset{\sim}{p}, u)$ is the analogue of Shephard's Lemma, for consumers.

Taking the differential of both sides,

$$d\underset{\sim}{h}(\underset{\sim}{p}, u) = \nabla^2_{\underset{\sim}{p}}\, e(\underset{\sim}{p}, u) \cdot d\underset{\sim}{p} + \left(\frac{\partial}{\partial u} \nabla_{\underset{\sim}{p}}\, e(\underset{\sim}{p}, u)\right) du.$$

But $du = 0$ here, so

$$d\underset{\sim}{h} = \nabla^2_{\underset{\sim}{p}}\, e(\underset{\sim}{p}, u) \cdot d\underset{\sim}{p}$$

and left-multiplying by $d\underset{\sim}{p}$ (actually, by the transpose of $d\underset{\sim}{p}$) gives

$$d\underset{\sim}{p} \cdot d\underset{\sim}{h} = d\underset{\sim}{p} \cdot \nabla^2_{\underset{\sim}{p}}\, e(\underset{\sim}{p}, u) \cdot d\underset{\sim}{p} \leq 0$$

because $d\underset{\sim}{p} \cdot \nabla^2_{\underset{\sim}{p}} e \cdot d\underset{\sim}{p}$ is a quadratic form and $\nabla^2 e$ is negative definite since $e(\underset{\sim}{p}, u)$ is concave in $\underset{\sim}{p}$. The result isn't surprising because we already knew that $\partial h_i / \partial p_i \leq 0$ (Hicksian demand curves are downward sloping): this result extends the simpler one to cases when more than one price or quantity demanded changes.

Summer 2011 Qualifying Exam, Sec. 1 Qu. 2.

2. [**13 points**] On August 22, 2010, the *Los Angeles Times* published an opinion piece entitled “Disincentivizing Greed” written by Neal Gabler (who was then a public policy scholar at the Woodrow Wilson Center in Washington, DC). Here is an excerpt of the piece; it essentially argues that decreasing tax rates increases the amount of dishonest labor, which is an assertion about a comparative statics derivative.

> To a surprising degree, economic misfortune has correlated with low top marginal tax rates. The top marginal tax rate at the time of the 1929 crash was 24%. After his election, Roosevelt promptly raised it to 63% and then to 94%, and one could easily make the case that it was this rise, rather than financial regulation, that played the primary. . . role in curbing abuses by attacking greed at its source, without, by the way, damaging the economy. Roosevelt essentially taxed away big money.
>
> During the long postwar economic boom, the top marginal rates hovered at 91%, removing a lot of the incentive to game the financial system. There was no point in scheming if you couldn’t profit from it. Still, the country prospered. So did Wall Street.
>
> Then came the greed deluge. . . .[W]hen President Reagan cut the top marginal tax rate drastically from 70% to 50% in 1981 and then to 28% in 1988 (putting aside for the moment the cut in the capital gains tax and other investment incentives), that’s when the troubles began—from the S&L crisis right through to the fall of Lehman Bros. It wasn’t enough for the rich to be rich. Human nature being what it is, they had to be super-rich. Or put another way, tax cuts, including the Bush tax cuts, fed some of the worst aspects of human nature and led to some of the worst excesses. It was just a matter of time before Wall Street went wild.
>
> When the fire of greed is stoked this way, financial reforms cannot possibly bank it. . . .We now live in a country that seems to worship wealth, and we may just have to live with the consequences—a Bernie Madoff, an Enron, a Lehman Bros., and a steep recession when the super-rich overplay

> their hand. The alternative is regulation that goes to the source by raising those marginal tax rates (and capital gains taxes) and forcing the super-rich to merely be rich again. . . .

(a) Argue that a reasonable way—certainly not the only way, but a reasonable way—to model the (indirect) utility that the “rich” or “super-rich” people described in this article get from their pretax income is

$$\text{“honest income”} + \sqrt{\text{“dishonest income”}}\,.$$

(This is not a standard way of modeling indirect utility, of course.)

(b) If the tax rate is t, interpret

$$\text{“honest income”} + \sqrt{\text{“dishonest income”}} \\ - t \cdot (\text{“honest income”} + \text{“dishonest income”})\,.$$

(c) Modelling “income” as a wage rate (consider an “honest wage” and a “dishonest wage”) times a number of hours worked (consider “honest labor time” and “dishonest labor time”) and imposing some constraint on the number of hours humans work, discuss whether or not the expression in part (b) supports Gabler’s hypothesis by calculating an appropriate comparative statics derivative. Does the appropriate second-order condition hold?

Hint: If you substitute the constraint on working hours into the objective function, the new problem has only one endogenous variable, which is much easier to work with. You may ignore leisure (and hence any leisure-labor tradeoff) in your answer.

Section 1 Question 2 .

a) Both kinds of income increase (indirect) utility. This captures the reason some super-rich people did unethical things. However, the square root function means that $1 of unethically-earned income contributes less to (indirect) utility than $1 of ethically-earned income : so the super-rich do have some moral misgivings about unethically-earned income.

b) The utility gained from pre-tax income, minus taxes. An objective function for the super-rich.

c) W_h wage rate of honest work
W_d " " " dishonest "
l_h hours worked doing honest labor
l_d " " " dishonest "

Working time constraint : $l_h + l_d = 1$ (the "1" stands for "one working day"; instead of "1" you could use "18 hours" or "8 hours" or "24 hours) .

honest income $= W_h l_h$

dishonest income $= W_d l_d = W_d (1 - l_h)$.

Objective: max $w_h \ell_h + \sqrt{w_d(1-\ell_h)} - t\left[w_h \ell_h + w_d(1-\ell_h)\right]$

over ℓ_h :

$$0 = \frac{d\,(\text{objective})}{d\ell_h} = w_h + \frac{1}{2}\,\frac{-w_d}{\sqrt{w_d(1-\ell_h)}} - t\left[w_h - w_d\right]. \quad (1)$$

Solution Method 1 : No need to solve for ℓ_h.

$$0 = dt\left[-w_h + w_d\right] + d\ell_h\left[\frac{-1}{4}\,\frac{-w_d}{(w_d(1-\ell_h))^{+3/2}}\,(-w_d)\right]$$

$$(w_h - w_d)\,dt = \left[\frac{-1}{4}\,w_d^{2}\,w_d^{-3/2}\,\frac{1}{(1-\ell_h)^{3/2}}\right] d\ell_h$$

$$= \frac{-\sqrt{w_d}}{4(1-\ell_h)^{3/2}}\,d\ell_h$$

$$\Rightarrow \quad \frac{d\ell_h}{dt} = \frac{w_h - w_d}{-\sqrt{w_d}}\,4(1-\ell_h)^{3/2} = \frac{w_h - w_d}{-\sqrt{w_d}}\,4\ell_d^{3/2}$$

$$= \frac{4\ell_d^{3/2}}{\sqrt{w_d}}\,(w_d - w_h).$$

So if dishonest labor pays more than honest labor (if it didn't, then in this model no one would do dishonest labor, which contradicts the article's opinion), one has $w_d - w_h > 0$, so $d\ell_h / dt > 0$, therefore (due to the working hour constraint) we obtain $d\ell_d / dt < 0$, supporting the author's hypothesis.

Solution Method 2: Solving for ℓ_h.

From (1),

$$t(w_h - w_d) = w_h - \frac{1}{2} \frac{w_d}{\sqrt{w_d}\sqrt{1-\ell_h}} = w_h - \frac{1}{2} \frac{\sqrt{w_d}}{\sqrt{1-\ell_h}} \Rightarrow$$

$$\frac{1}{2} \frac{\sqrt{w_d}}{\sqrt{1-\ell_h}} = w_h - t(w_h - w_d)$$

$$\frac{\sqrt{w_d}}{2[w_h - t(w_h - w_d)]} = \sqrt{1-\ell_h} \Rightarrow$$

$$1 - \ell_h = \frac{w_d}{4[w_h - t(w_h - w_d)]^2}.$$

It would be trivial to solve this for ℓ_h, but it's even easier to solve it for ↙

$$\ell_d = \frac{w_d}{4[w_h - t(w_h - w_d)]^2}$$

and then calculate

$$\frac{d\ell_d}{dt} = \frac{w_d}{4} \frac{-2}{[w_h - t(w_h - w_d)]^3} [-(w_h - w_d)]$$

$$= \frac{w_d}{4} \frac{1}{[w_h - t(w_h - w_d)]^2} \frac{-2}{[w_h - t(w_h - w_d)]} [-(w_h - w_d)]$$

$$= \ell_d \frac{-2}{w_h - t(w_h - w_d)} [-(w_h - w_d)] = \ell_d \frac{2(w_h - w_d)}{w_h - t(w_h - w_d)}$$

$$= \frac{2\,l_d\,(W_h - W_d)}{W_h + t(W_d - W_h)} = \frac{-2\,l_d\,(W_d - W_h)}{W_h + t(W_d - W_h)} < 0$$

since $W_d - W_h > 0$. This is the same sign obtained using Method 1.

2008 Qualifier

Section 1.
Answer all of the following three questions.

1. **[12 points]**
   There is an old saying,

   > "Idle hands are the devil's workshop."

   In other words, excessive idleness ("leisure") is a bad thing for people.

   Suppose a price-taking consumer's utility depends on his purchases of a good $x \geq 0$ and his consumption of leisure z (the notation "z" recalls the sound at the beginning of the second syllable of "leisure").

   (a) Argue that postulating a utility function of

   $$u = \ln x + \left[-(z-1)^2 + 1\right] \quad \text{with } 0 \leq z \leq 2$$

   is a reasonable way of modeling the idea that excessive idleness is bad.

   (b) Suppose the wage rate (that is, the payment for the opposite of leisure) is w. Assume that all of the consumer's income comes from selling his non-leisure time. Suppose the price of x is p. What is the consumer's budget constraint?

   (c) From the first-order conditions, argue that in order for the optimal x to be positive, the optimal z must be in $[0, 1)$.

   (d) Find the optimal x and z explicitly in terms of exogenous variables.

   (e) What does this consumer's labor supply curve look like?

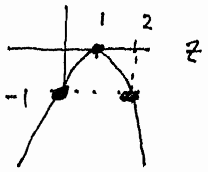
   (f) Check the second-order conditions for the optimization problem.

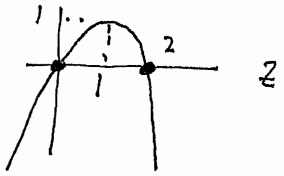
   (g) What is this consumer's indirect utility function?

Answers to 7005 section of 2008 Microeconomics Qualifying Exam

① a) The function $-z^2$ looks like  .

The function $-(z-1)^2$ thus looks like



and $-(z-1)^2+1$ looks like  .

$z=0$ is no leisure; $z=2$ is 100% leisure. For $z \in [0,1]$, increasing leisure increases utility. So if you have rather little leisure, more leisure is better. But once you have the "best possible" amount of leisure, $z=1$, getting any more leisure decreases utility; hence the negative slope for $z>1$.

The last, "+1" term of "$-(z-1)^2+1$" does nothing mathematically and could be left out. I just put it in because positive values for utility are slightly easier to understand.

Optional: We can predict that z^* will not be greater than 1, because increasing z beyond 1 decreases utility <u>and</u> decreases income. See also part (c).

b) income = expenditure

$$w\,\underbrace{(2-z)} = p\,x$$

price of x
(any variable name is
acceptable)

wage rate
(any variable
name is
acceptable)

Amount of work; $z=0 \Rightarrow$ no leisure $\Rightarrow$ all time is spent on work,
$z=2 \Rightarrow$ no work $\Rightarrow$ no income

c) $u = \ln x + [-(z^2-2z+1)+1]$

$= \ln x + [-z^2+2z-1+1]$

$= \ln x - z^2 + 2z.$

$$\mathcal{L} = \ln x - z^2 + 2z + \lambda[w(2-z) - px]$$

F.O.C.

$0 = \mathcal{L}'_\lambda = w(2-z) - px$

$0 = \mathcal{L}'_x = \frac{1}{x} - \lambda p \Rightarrow \lambda = \frac{1}{xp}$

$0 = \mathcal{L}'_z = -2z+2-\lambda w \Rightarrow \lambda = \frac{-2z+2}{w}$

$\frac{1}{xp} = \frac{2-2z}{w}$

$xp = \frac{w}{2-2z}$

$x = \frac{1}{2-2z}\frac{w}{p}$

$= \frac{1}{2}\frac{1}{1-z}\frac{w}{p}$

over →

Recall that the range of z is $[0,2]$. If $z>1$, then $x = \frac{1}{2}\frac{1}{1-z}\frac{w}{p}$ would be negative. If $z=1$, x would be ∞. So to make sense, z needs to be in $[0,1)$.

d) $w(2-z) = px$ from (b) or the first F.O.C.

$$= p \cdot \frac{1}{2}\frac{1}{1-z}\frac{w}{p}$$

$$= \frac{1}{2}\frac{1}{1-z}w$$

$$2w(2-z)(1-z) = w$$

$$2(2-z)(1-z) = 1$$

$$(z-2)(z-1) = \frac{1}{2}$$

$$z^2-3z+2 = \frac{1}{2}$$

$$z^2-3z+\frac{3}{2} = 0$$

$$z = \frac{3 \pm \sqrt{9-4\cdot\frac{3}{2}}}{2} = \frac{3\pm\sqrt{9-6}}{2} = \frac{3\pm\sqrt{3}}{2}.$$

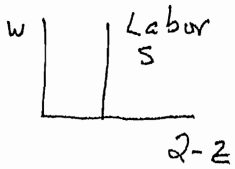
Choosing the "+" sign would yield $z = \frac{3+\sqrt{3}}{2} > \frac{3+0}{2} = 1.5$ violating part (c)'s conclusion that z has to be between 0 and 1.

So $\boxed{z^* = \frac{3-\sqrt{3}}{2}}$. (This is about 0.6.)

$$x^* = \frac{1}{2}\frac{1}{1-z^*}\frac{w}{p} = \frac{w}{2p}\frac{1}{1-\frac{3-\sqrt{3}}{2}} = \frac{w}{2p}\frac{2}{2-(3-\sqrt{3})} = \frac{w}{2p}\frac{2}{2-3+\sqrt{3}}$$

$$= \boxed{\frac{w}{p}\frac{1}{\sqrt{3}-1}}.$$ (This is >0 since $\sqrt{3} > \sqrt{1} = 1$.)

e) Since $z^* = \frac{3-\sqrt{3}}{2}$ is a constant, z^* does not depend on w. Work hours $(2-z^*)$ hence do not depend on w either. So the labor supply curve is vertical:



.

f)

$$D^2\mathcal{L} = \begin{bmatrix} \mathcal{L}''_{\lambda\lambda} & \mathcal{L}''_{\lambda x} & \mathcal{L}''_{\lambda z} \\ \mathcal{L}''_{x\lambda} & \mathcal{L}''_{xx} & \mathcal{L}''_{xz} \\ \mathcal{L}''_{z\lambda} & \mathcal{L}''_{zx} & \mathcal{L}''_{zz} \end{bmatrix} = \begin{bmatrix} 0 & -p & -w \\ -p & -1/x^2 & 0 \\ -w & 0 & -2 \end{bmatrix}$$

S.O.C. for a maximum: $D_{2m+1}, \dots, D_{m+n}$ alternate in sign beginning with $(-1)^{m+1}$.

$n = 2$ # variables
$m = 1$ # constraints

$2m+1 = 3$
$m+n = 3$

So D_3 should have the sign of $(-1)^2 > 0$.

D_3 is $(-1)^{2+1}(-p)\left[2p - 0\right] + (-1)^{3+1}(-w)\left[0 - \frac{w}{x^2}\right]$

$= p(2p) - w\left(\frac{-w}{x^2}\right) = 2p^2 + \frac{w^2}{x^2}$ which, because all the variables are squared, is clearly positive even without substituting the optimal x of part (d) into this expression.

g)

$$
\begin{aligned}
v &= \ln x^{*} + \left[-(z^{*}-1)^{2}+1\right] \\
&= \ln\left[\frac{w}{P}\,\frac{1}{\sqrt{3}-1}\right] + \left[-\left(\frac{3-\sqrt{3}}{2}-1\right)^{2}+1\right] \\
&= \quad \text{''} \quad + \left[-\left(\frac{3-\sqrt{3}}{2}-\frac{2}{2}\right)^{2}+1\right] \\
&= \quad \text{''} \quad + \left[-\left(\frac{1-\sqrt{3}}{2}\right)^{2}+1\right] \\
&= \quad \text{''} \quad + \left[-\left(\frac{1-2\sqrt{3}+3}{4}\right)+1\right] \\
&= \quad \text{''} \quad + \left[-\left(\frac{4-2\sqrt{3}}{4}\right)+1\right] \\
&= \quad \text{''} \quad + \left[-\left(1-\frac{\sqrt{3}}{2}\right)+1\right] \\
&= \quad \text{''} \quad + \left[-1+\frac{\sqrt{3}}{2}+1\right] \\
&= \quad \text{''} \quad + \frac{\sqrt{3}}{2}\,.
\end{aligned}
$$

This is the end of the Consumer Theory positive questions, and the beginning of the Consumer Theory normative questions.

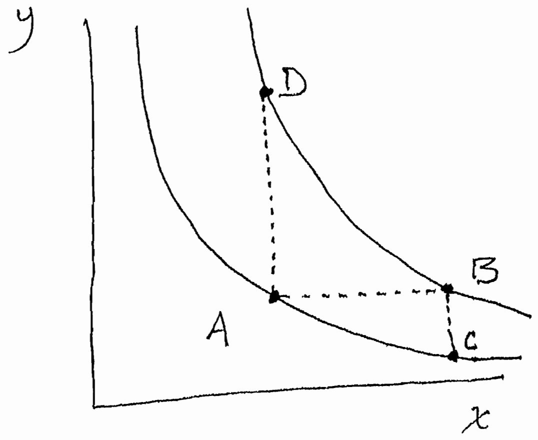
The normative questions are not covered on the midterm exam. So if you are studying for the midterm exam, you need not go beyond this point.

2007
Final

5. [11 points] Suppose a consumer buys only two commodities, called x and y, and suppose this consumer's preferences for x and y are strongly monotonic. Suppose the government is contemplating giving (for free) the consumer a certain amount more of commodity x. By drawing *one* graph with x on the horizontal axis and y on the vertical axis, illustrate the typical result that for this contemplated change, "willingness to pay" is less than "willingness to accept." Also, describe how your graph would have to change in order to obtain the atypical result that WTP is equal to WTA.

2007
Final

⑤



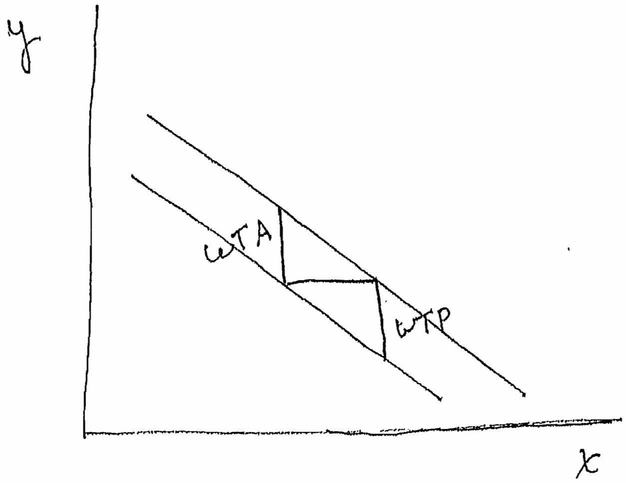
A: original point

B: new point if given x for free

BC: WTP to get the new amount of x (i.e., how much y you'd be willing to give up)

AD: WTA compensation (i.e., more y) in return for not being given the extra x

If the indifference curves were parallel straight lines, then WTP = WTA:



1. [17 points] Using $\mathbf{p}$ to denote prices, m to denote income, e to denote the expenditure function, and v to denote the indirect utility function, consider the expression

   $$e\big(\,\cdot\,, v(\mathbf{p}', m')\big) - e\big(\,\cdot\,, v(\mathbf{p}_0, m_0)\big) \quad ;$$

   if the dot ("$\cdot$") is replaced by $\mathbf{p}_0$, this expression measures "Equivalent Variation," and if it is replaced by $\mathbf{p}'$, it measures "Compensating Variation."

   If a consumer buys two goods, a and b, at prices p_a and p_b, and has utility function $u = a \times b$, then find the Compensating Variation (not the Equivalent Variation) for this consumer.

Fall 2010, Final Exam, Qu. 1

Answer to Question 1 of Econ 7005 Final Exam, Fall 2010

① $u = ab$

To find v :

$\max\ u$ s.t. $p_a a + p_b b = m$

$$\mathcal{L} = ab + \lambda\left[m - p_a a - p_b b\right]$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - p_a a - p_b b$$

$$0 = \frac{\partial \mathcal{L}}{\partial a} = b + \lambda\left[-p_a\right] \quad \Rightarrow \quad \lambda p_a = b \Rightarrow \lambda = \frac{b}{p_a}$$

$$0 = \frac{\partial \mathcal{L}}{\partial b} = a + \lambda\left[-p_b\right] \quad \Rightarrow \quad \lambda p_b = a$$

$\frac{b}{p_a} p_b = a$; substitute into the budget constraint :

$$m = p_a a + p_b b = p_a\left(\frac{p_b}{p_a} b\right) + p_b b = 2 p_b b \Rightarrow$$

$$b^* = \frac{m}{2p_b} \text{ and } a^* = \frac{p_b}{p_a} b^* = \frac{p_b}{p_a} \frac{m}{2p_b} = \frac{m}{2p_a}.$$

$$\text{So } u^* = a^* b^* = \frac{m}{2p_a} \frac{m}{2p_b} = \boxed{\frac{m^2}{4p_a p_b} = v(\underset{\sim}{p}, m).}$$

To find e : Either use $v(\underset{\sim}{p}, e(\underset{\sim}{p}, \hat{u})) = \hat{u} \Rightarrow \frac{e^2}{4p_a p_b} = \hat{u} \Rightarrow e = \boxed{2\sqrt{p_a p_b \hat{u}}}$ or:

$\min\ p_a a + p_b b$ s.t. $u = \hat{u}$

$\mathcal{L} = p_a a + p_b b + \mu\left[\hat{u} - ab\right]$ where I use μ instead of λ for the Lagrange multiplier just to avoid confusion with the "λ" of the previous part.

$$0 = \frac{\partial \mathcal{L}}{\partial \mu} = \hat{u} - ab$$

$$\left.\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial a} = p_a - \mu b \\ 0 &= \frac{\partial \mathcal{L}}{\partial b} = p_b - \mu a \end{aligned}\right\} \Rightarrow \quad \frac{p_a}{p_b} = \frac{b}{a} \Rightarrow b = \frac{p_a}{p_b} a \Rightarrow$$

$$\hat{u} = ab = a\left(\frac{p_a}{p_b} a\right) = \frac{p_a}{p_b} a^2 \Rightarrow$$

$$a^* = \sqrt{p_b \hat{u} / p_a} \text{ and}$$

$$b^* = \frac{p_a}{p_b} a^* = \sqrt{\frac{p_a^2}{p_b^2} \frac{p_b \hat{u}}{p_a}} = \sqrt{p_a \hat{u} / p_b} .$$

So $e(\underset{\sim}{p}, \hat{u}) = p_a a^* + p_b b^* = p_a \sqrt{\frac{p_b \hat{u}}{p_a}} + p_b \sqrt{\frac{p_a \hat{u}}{p_b}}$

$$= \sqrt{p_a p_b \hat{u}} + \sqrt{p_a p_b \hat{u}} = \boxed{2 \sqrt{p_a p_b \hat{u}} .} \quad *$$

<u>To find CV:</u>

$$CV = e(\underset{\sim}{p}', v(\underset{\sim}{p}', m')) - e(\underset{\sim}{p}', v(\underset{\sim}{p}_0, m_0))$$

$$= 2\sqrt{p_a' p_b' \, v(\underset{\sim}{p}', m')} - 2\sqrt{p_a' p_b' \, v(\underset{\sim}{p}_0, m_0)}$$

$$= 2\sqrt{p_a' p_b' \frac{(m')^2}{4 p_a' p_b'}} - 2\sqrt{p_a' p_b' \frac{m_0^2}{4 p_{a0} p_{b0}}}$$

$$= m' - m_0 \sqrt{\frac{p_a' p_b'}{p_{a0} p_{b0}}} .$$

* Remark: If you found $e(\underset{\sim}{p}, \hat{u})$ first, you can obtain $v(\underset{\sim}{p}, m)$ via: $m = e(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) = 2\sqrt{p_a p_b v} \Rightarrow m^2 = 4 p_a p_b v \Rightarrow v = m^2/(4 p_a p_b)$.

1/3

Qualifying Exam
2004
Question 1

6 ?

Section 2.
Answer two of the following three questions.

1. Attached to this exam is an excerpt from pages 167 and 168 of Varian's textbook. This excerpt ends with Varian stating that

$$p^0 > p^1 \Longrightarrow EV > CV\,.$$

   (a) If $p^0 < p^1$, is EV greater than or less than CV?

   (b) Related to compensating and equivalent variation are:
      - "willingness to pay" an amount of money in order to avoid suffering the price increase from p^0 to p^1; and
      - "willingness to accept" an amount of money in order to accept the price increase from p^0 to p^1.

   Is "willingness to pay" equal to CV or to EV? Is "willingness to accept" equal to CV or to EV? Why?

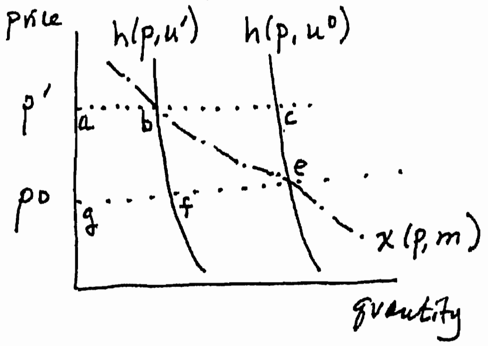
Section 2 #1.

Qualifying Exam
2004
Answer 1

$$EV = \int_{p'}^{p^0} h(p,u')\,dp$$

$$CV = \int_{p'}^{p^0} h(p,u^0)\,dp$$

a) $p^0 < p' \Rightarrow u^0 > u' \Rightarrow h(p,u^0) > h(p,u')$



$|EV| = abfg$ $|CV| = aceg > abfg = |EV|$

$EV = \ominus$ because $h > 0$ $CV = \ominus$ for similar reasons

and the integral's lower limit (p') is larger than its upper limit (p^0)

So $|EV| < |CV|$, $EV < 0$, $CV < 0$. This implies that $\underline{CV < EV} < 0$.

Hence $CV < EV$ regardless of whether p^0 is > or < p'.

b) From the second paragraph of the excerpt from p. 167 of Varian,

EV uses base year prices and

CV uses final year prices.

The move from p^0 to p' is a loss to the consumer. In this case of a loss, WTP to avoid the loss must be measured using base year prices (since the loss is avoided, the base prices are relevant). So EV = WTP to avoid the loss. Similarly, CV, since it uses final year prices, goes with the loss having already occurred. So CV = WTA compensation for the loss.

Optional: For a gain the situation differs.

WTA to forego a gain involves base year prices, so it's EV.

WTP for the gain if you get the gain involves final year prices, so it's CV.

Summary:

	Loss	Gain
EV	WTP	WTA
CV	WTA	WTP

Qualifying Exam
2004
Answer 1 cont...

new: 2019 Final Exam, Qu. 2.

2. **[17 points]**

   Suppose a consumer consumes apples "a" and bananas "b" and has utility function $u = \ln a + 3 \ln b$. Suppose this consumer takes the price of apples p_a and the price of bananas p_b as given.

   (a) Find this consumer's expenditure function.

   (b) Find this consumer's indirect utility function "v" by using the result of part (a), not by solving an optimization problem. (If you were not able to solve part (a), then make up a hypothetical solution for it in order to work this part of the problem.)

   (c) One way of expressing the welfare change experienced by a consumer who faced an initial price of apples p_{0a} and an initial price of bananas p_{0b} and had an initial income m_0 and then is placed in a new economic environment in which his utility becomes u_1 is the "equivalent variation," denoted EV, which can be expressed in several ways, one of which is

   $$v(p_{0a}, p_{0b}, m_0 + EV) = u_1 \, .$$

   Find an expression for the EV of this consumer.

Answer to Econ. 7005 Fall 2019 Final Exam, Qu. 2

$u = \ln a + 3 \ln b$

a) $\min\ p_a a + p_b b$ s.t. $\ln a + 3 \ln b = \bar{u}$

$$\mathcal{L} = p_a a + p_b b + \lambda \left[\bar{u} - \ln a - 3 \ln b \right]$$

$$\left.\begin{array}{l} 0 = \mathcal{L}'_a = p_a - \dfrac{\lambda}{a} \\ 0 = \mathcal{L}'_b = p_b - \dfrac{3\lambda}{b} \end{array}\right\} \Rightarrow \lambda = a p_a = \frac{b p_b}{3} \Rightarrow b = \frac{3 a p_a}{p_b} \text{ and}$$

$$0 = \mathcal{L}'_\lambda = \bar{u} - \ln a - 3 \ln b \quad \Rightarrow \bar{u} = \ln a + 3 \ln \frac{3 a p_a}{p_b}$$

$$= \ln a + \ln \frac{3^3 a^3 p_a^3}{p_b^3} = \ln \frac{3^3 a^4 p_a^3}{p_b^3}$$

$$E^{\bar{u}} = 3^3 a^4 p_a^3 / p_b^3$$

I'll use "E" rather than "e" to represent the number which is the base of natural logarithms, because this problem asks for the expenditure function, whose traditional notation is "e."

$$\Rightarrow a = \left(\frac{p_b^3 E^{\bar{u}}}{3^3 p_a^3} \right)^{1/4} \text{ and } b = \frac{3 p_a}{p_b} \left(\frac{p_b^3 E^{\bar{u}}}{3^3 p_a^3} \right)^{1/4} = \left[\frac{3^4 p_a^4 p_b^3 E^{\bar{u}}}{p_b^4 3^3 p_a^3} \right]^{1/4}$$

$$= \left(\frac{3 p_a E^{\bar{u}}}{p_b} \right)^{1/4}.$$

Then expenditure is

$$e = p_a a + p_b b = \left(p_a^4 \frac{p_b^3 E^{\bar{u}}}{3^3 p_a^3} \right)^{1/4} + \left(p_b^4 \frac{3 p_a E^{\bar{u}}}{p_b} \right)^{1/4}$$

$$= \left(\frac{p_a p_b^3 E^{\bar{u}}}{3^3} \right)^{1/4} + \left(3 p_a p_b^3 E^{\bar{u}} \right)^{1/4} = \boxed{\left(\frac{1}{3^{3/4}} + 3^{1/4} \right) \left(p_a p_b^3 E^{\bar{u}} \right)^{1/4}.}$$

Note that $\frac{1}{3^{3/4}} + 3^{1/4} = \frac{1+3^{\frac{1}{4}}3^{\frac{3}{4}}}{3^{3/4}} = \frac{1+3}{3^{3/4}} = \frac{4}{3^{3/4}}$ so alternatively

$$e(\underset{\sim}{p}, \bar{u}) = 4 \cdot 3^{-3/4} (p_a p_b^3 E^{\bar{u}})^{1/4}.$$

b) $e(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) \equiv m$ so from part (a),

$4 \cdot 3^{-3/4} (p_a p_b^3 E^{v})^{1/4} = m$. Solving for v:

$$4^4 3^{-3} p_a p_b^3 E^v = m^4$$

$$E^v = \frac{3^3 m^4}{4^4 p_a p_b^3}$$

$$\underline{v = \ln \frac{3^3 m^4}{4^4 p_a p_b^3}.}$$

c) Set u_1 equal to $v(p_{0a}, p_{0b}, m_0 + EV) = \ln \frac{3^3 (m_0 + EV)^4}{4^4 p_{0a} p_{0b}^3}$

$$E^{u_1} = \frac{3^3 (m_0 + EV)^4}{4^4 p_{0a} p_{0b}^3}$$

$$(m_0 + EV)^4 = 4^4 p_{0a} p_{0b}^3 E^{u_1} / 3^3$$

$$m_0 + EV = 4 \left(p_{0a} p_{0b}^3 E^{u_1} / 3^3\right)^{1/4}$$

$$\underline{EV = 4 \left(p_{0a} p_{0b}^3 E^{u_1} / 3^3\right)^{1/4} - m_0.}$$

Fall 2021 Final Exam Question 1
Ch. 10 Partial Equilibrium

1. **[17 points]**

   Suppose a consumer takes prices as given and consumes two goods, x and y, which generate utility $u(x, y) = xy$. Suppose this consumer faces an "original" situation with prices $\mathbf{p}^0 = (p_x^0, p_y^0)$ and income m^0, or a "new" situation with prices $\mathbf{p}' = (p_x', p_y')$ and income m'. One definition of compensating variation is

   $$CV = e(\mathbf{p}', v(\mathbf{p}', m')) - e(\mathbf{p}', v(\mathbf{p}^0, m^0))$$

   and one definition of equivalent compensating variation is

   $$EV = e(\mathbf{p}^0, v(\mathbf{p}', m')) - e(\mathbf{p}^0, v(\mathbf{p}^0, m^0)) .$$

   (a) Find CV and EV if $p_x' = p_x^0 + \epsilon$, $p_y' = p_y^0$, and $m' = m^0$.

   (b) If $\epsilon > 0$, what is the sign of CV and what is the sign of EV?

   (c) In the case when $\epsilon > 0$, argue that EV is "willingness and ability to pay," and show mathematically that the absolute value of its formula in part (a) is always less than income. In the case when $\epsilon > 0$, argue that CV is "willingness to accept," and show mathematically that the absolute value of its formula in part (a) is not always less than income.

Answer to Fall 2021 Econ. 7005 Final Exam, Question 1

a) $CV = e(\underset{\sim}{p}', v(\underset{\sim}{p}', m')) - e(\underset{\sim}{p}', v(\underset{\sim}{p}^0, m^0))$

$= m' - e(\underset{\sim}{p}', v(\underset{\sim}{p}^0, m^0))$. Similarly,

$EV = e(\underset{\sim}{p}^0, v(\underset{\sim}{p}', m')) - e(\underset{\sim}{p}^0, v(\underset{\sim}{p}^0, m^0))$

$= e(\underset{\sim}{p}^0, v(\underset{\sim}{p}', m')) - m^0$. I use these simplifications below.

Method 1:

Find e.

$\min p_x x + p_y y$ s.t. $xy = u(x,y) = \bar{u}$

(or $xy \geq \bar{u}$)

$\mathcal{L} = p_x x + p_y y + \lambda(\bar{u} - xy)$

F.O.C.

$0 = \mathcal{L}'_x = p_x - \lambda y$

$0 = \mathcal{L}'_y = p_y - \lambda x$

$0 = \bar{u} - xy = \mathcal{L}'_\lambda$

$\lambda = \frac{p_x}{y} = \frac{p_y}{x}$

$\Rightarrow \quad x = \frac{y p_y}{p_x}$ and

$\bar{u} = xy = \frac{y p_y}{p_x} y = \frac{p_y}{p_x} y^2$

Method 2:

Find v.

$\max \underset{\overset{\uparrow}{xy}}{u(x,y)}$ s.t. $p_x x + p_y y = m$

$\mathcal{L} = xy + \lambda(m - p_x x - p_y y)$

F.O.C.

$0 = \mathcal{L}'_x = y - \lambda p_x$

$0 = \mathcal{L}'_y = x - \lambda p_y$

$0 = \mathcal{L}'_\lambda = m - p_x x - p_y y$

$\lambda = \frac{y}{p_x} = \frac{x}{p_y}$

$\Rightarrow \quad x = \frac{p_y}{p_x} y$ and

$m = p_x x + p_y y = p_x \frac{p_y}{p_x} y + p_y y$

$= 2 p_y y$

Method 1, continued

So $y = \sqrt{\frac{p_x}{p_y}\bar{u}}$

and $x = \frac{y\,p_y}{p_x} = \sqrt{\frac{p_x}{p_y}\bar{u}\,\frac{p_y^2}{p_x^2}}$

$= \sqrt{\frac{p_y}{p_x}\bar{u}}$,

making

$$e = p_x x^* + p_y y^*$$

$$= p_x\sqrt{\frac{p_y}{p_x}\bar{u}} + p_y\sqrt{\frac{p_x}{p_y}\bar{u}}$$

$$= \sqrt{p_x p_y \bar{u}} + \sqrt{p_x p_y \bar{u}}$$

$$= 2\sqrt{p_x p_y \bar{u}}$$

Knowing e, we can find v using

$$e(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) = m.$$

$$2\sqrt{p_x p_y v} = m$$

$$p_x p_y v = (m/2)^2$$

$$v = \frac{m^2}{4 p_x p_y}.$$

Method 2, continued

So $y = \frac{m}{2p_y}$

and $x = \frac{p_y}{p_x} y = \frac{p_y}{p_x}\,\frac{m}{2p_y}$

$= \frac{m}{2p_x}$,

making

$$v = x^* y^* = \frac{m}{2p_x}\,\frac{m}{2p_y} = \frac{m^2}{4p_x p_y}$$

Knowing v, we can find e using

$$v(\underset{\sim}{p}, e(\underset{\sim}{p}, u)) = u.$$

$$\frac{e^2}{4p_x p_y} = u$$

$$e^2 = 4p_x p_y u$$

$$e(p_x, p_y, \bar{u}) = 2\sqrt{p_x p_y \bar{u}}.$$

Clearly the two methods result in the same e and v functions.

The problem says $p_x' = p_x^0 + \varepsilon$

$$p_y' = p_y^0$$

$$m' = m^0$$

so $CV = m' - e(\underset{\sim}{p}', v(\underset{\sim}{p}^0, m^0)) = m' - 2\sqrt{p_x' p_y' \, v(\underset{\sim}{p}^0, m^0)}$

$$= m' - 2\sqrt{p_x' p_y' \frac{(m^0)^2}{4 p_x^0 p_y^0}}$$

$$= m^0 - 2\sqrt{(p_x^0 + \varepsilon) p_y^0 \frac{(m^0)^2}{4 p_x^0 p_y^0}} = m^0 - m^0\sqrt{\frac{p_x^0 + \varepsilon}{p_x^0}}$$

$$= m^0 - m^0\sqrt{1 + \frac{\varepsilon}{p_x^0}}.$$

Similarly,

$$EV = e(\underset{\sim}{p}^0, v(\underset{\sim}{p}', m')) - m^0 = 2\sqrt{p_x^0 p_y^0 \, v(\underset{\sim}{p}', m')} - m^0$$

$$= 2\sqrt{p_x^0 p_y^0 \frac{(m')^2}{4 p_x' p_y'}} - m^0 = m'\sqrt{p_x^0 p_y^0 \frac{1}{(p_x^0 + \varepsilon) p_y^0}} - m^0$$

$$= m^0\sqrt{\frac{p_x^0}{p_x^0 + \varepsilon}} - m^0 = m^0\sqrt{\frac{1}{1 + \varepsilon/p_x^0}} - m^0.$$

b) (If $\varepsilon > 0$, $p_x' > p_x^0$, and since p_y and m stay the same, the consumer must be worse off.) $CV = m^0 - m^0\sqrt{1 + \frac{\varepsilon}{p_x^0}}$ and $\sqrt{1 + \frac{\varepsilon}{p_x^0}} > 1$ so $CV < 0$.

$$EV = m^0 \sqrt{\frac{1}{1+\varepsilon/p_x^0}} - m^0 .$$ With $\varepsilon > 0$, $1 + \frac{\varepsilon}{P_0} > 1$

$$\frac{1}{1+\frac{\varepsilon}{P_0}} < 1$$

$$\sqrt{\frac{1}{1+\varepsilon/p_0}} < 1$$

so $EV < 0$.

hence, original prices are used ↓

c) With $\varepsilon > 0$, EV is, if the policy is not carried out (the price is not increased), the WATP in gratitude for not increasing the price of x.

From part (a) $EV = m^0 \left[\sqrt{\frac{1}{1+\varepsilon/p_0}} - 1 \right]$. From part (b), $EV < 0$;

so $|EV| = -EV = m^0 \left[1 - \sqrt{\frac{1}{1+\varepsilon/p_0}} \right]$.

At $\varepsilon = 0$, this is 0.
As $\varepsilon \to \infty$, this is m^0.
For ε between 0 and ∞, this is between 0 and m^0.

From part (a), $CV = m^0 - m^0 \sqrt{1 + \frac{\varepsilon}{p_x^0}}$. From part (b), $CV < 0$;

so $|CV| = -CV = m^0 \sqrt{1 + \frac{\varepsilon}{p_x^0}} - m^0$. At $\varepsilon = 0$, this is 0.
As $\varepsilon \to \infty$, this is ∞. So it is not bounded by income.

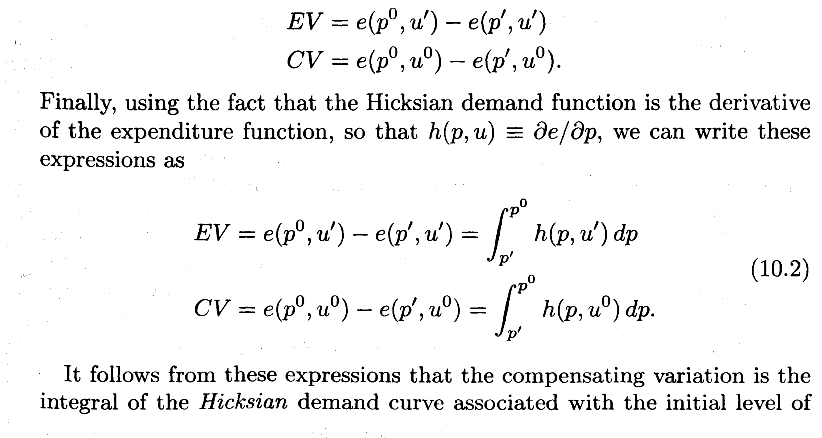
With $\varepsilon > 0$, CV is, if the policy is carried out (the price is increased – so, using new prices) the WTA compensation for the $\uparrow p_x$.

2021 Qualifying Exam Sec. 3 Qu. 1

1. **[10 points]** Let p^0 be an "initial" price vector, p' be a "final" price vector, m^0 be "initial" income, and m' be "final" income. One definition of equivalent variation, EV, and compensating variation, CV, is

$$EV = e(p^0, v(p', m')) - m^0$$
$$CV = m' - e(p', v(p^0, m^0)$$

where $e(\cdot)$ denotes the expenditure function of a consumer and $v(\cdot)$ denotes the indirect utility function of that consumer. Let $u^0 = v(p^0, m^0)$ and $u' = v(p', m')$. Under the assumption that $m^0 = m'$, Varian writes on his page 167 that:



$$EV = e(p^0, u') - e(p', u')$$
$$CV = e(p^0, u^0) - e(p', u^0).$$

Finally, using the fact that the Hicksian demand function is the derivative of the expenditure function, so that $h(p, u) \equiv \partial e/\partial p$, we can write these expressions as

$$\begin{aligned} EV &= e(p^0, u') - e(p', u') = \int_{p'}^{p^0} h(p, u')\, dp \\ CV &= e(p^0, u^0) - e(p', u^0) = \int_{p'}^{p^0} h(p, u^0)\, dp. \end{aligned} \tag{10.2}$$

It follows from these expressions that the compensating variation is the integral of the *Hicksian* demand curve associated with the initial level of

Show that if $m^0 \neq m'$, Varian's (10.2) have to be modified to

$$EV = \int_{p'}^{p^0} h(p, u')\, dp + \text{something more}$$
$$CV = \int_{p'}^{p^0} h(p, u^0)\, dp + \text{something more}\,.$$

by finding what the "something more" parts are.

Answer to Summer 2021 Qualifying Exam, Section 3 Question 1

Let $u' = v(p', m')$. Then

$$EV = e(p^0, v(p', m')) - m^0 = e(p^0, u') - m^0 \,. \tag{1}$$

Next, notice that the second equality in the first equation of Varian's (10.2) is still true when $m^0 \neq m'$:

$$e(p^0, u') - e(p', u') = \int_{p'}^{p^0} \frac{\partial e(p, u')}{\partial p}\, dp = \int_{p'}^{p^0} h(p, u')\, dp \tag{2}$$

(the first equality here is the Fundamental Theorem of Calculus, that $e = \int (de/dp)\, dp$, and the second is, as Varian points out, $h(p, u) \equiv \partial e(p, u)/\partial p$). So, if we can get the right-hand side of (1) to look more like the left-hand side of (2), we should be able to express EV in terms of the right-hand side of (2). To do this, add and subtract $e(p', u')$ from (1):

$$\begin{aligned} EV &= e(p^0, u') - m^0 + [e(p', u') - e(p', u')] \\ &= e(p^0, u') - e(p', u') - m^0 + e(p', u') \end{aligned}$$

and from (2),

$$= \int_{p'}^{p^0} h(p, u')\, dp - m^0 + e(p', u') \,. \tag{3}$$

This is good enough for an answer. However, it's easier to interpret if you use the fact that $m' = e(p', u')$, which is true because one of the four basic identities on Varian's p. 106 in Ch. 7 says $m = e(p, v(p, u))$ and because at the beginning of this answer I set $u' = v(p', m')$. This results in

$$EV = \int_{p'}^{p^0} h(p, u')\, dp - m^0 + m' \,, \tag{4}$$

showing that the "something more" asked for in the question is just the change in income, $m' - m^0$.

The CV proof is completely analogous. Let $u^0 = v(p^0, m^0)$. Then

$$CV = m' - e(p', v(p^0, m^0)) = m' - e(p', u^0) \,. \tag{5}$$

The second equality in the second equation of Varian's (10.2) is still true when $m^0 \neq m'$, for the same reason as given for the second equality in the first equation of Varian's (10.2) being still true when $m^0 \neq m'$, so

$$e(p^0, u^0) - e(p', u^0) = \int_{p'}^{p^0} \frac{\partial e(p, u^0)}{\partial p}\, dp = \int_{p'}^{p^0} h(p, u^0)\, dp \,. \tag{6}$$

Add and subtract $e(p^0, u^0)$ from (5):

$$\begin{aligned} CV &= m' - e(p', u^0) + [e(p^0, u^0) - e(p^0, u^0)] \\ &= e(p^0, u^0) - e(p', u^0) + m' - e(p^0, u^0) \end{aligned}$$

and from (6),

$$= \int_{p'}^{p^0} h(p, u')\, dp + m' - e(p^0, u^0)\,. \tag{7}$$

This is good enough for an answer. However, it's easier to interpret if you use $m^0 = e(p^0, u^0)$ (setting $u^0 = v(p^0, m^0)$), resulting in

$$CV = \int_{p'}^{p^0} h(p, u')\, dp + m' - m^0\,, \tag{8}$$

showing that the "something more" asked for in the question is again just the change in income, $m' - m^0$.

1. **[12 points]** Suppose a consumer gets utility from consumption of apples "a" and bananas "b" according to a strictly quasiconcave utility function $u(a, b)$. Suppose the consumer's initial consumption bundle is (a_0, b_0), and let U_0 be $u(a_0, b_0)$ where u is increasing in a and b. The purpose of this question is to investigate one way to define the "value" to this consumer of giving this consumer more apples, moving his consumption bundle to (a_1, b_0), where $a_1 > a_0$.

   (a) (2 points) Let compensating variation "CV" for this environment which lacks prices and incomes be implicitly defined by

   $$u(a_0, b_0) = u(a_1, b_0 - CV).$$

   (In microeconomic theory textbooks, CV is only defined in environments with prices and incomes.) How could CV, defined in this way, be interpreted as a measure of the value of moving from a_0 to a_1?

   (b) (2 points) Sketch a graph with a on the horizontal axis and b on the vertical axis, illustrating CV. Hint: begin by drawing the indifference curve which $u(a_0, b_0)$ lies on.

   (c) (3 points) By calculating the appropriate (total) differential, show that

   $$\frac{\partial\, CV}{\partial\, a_1} = \frac{\partial u(a_1, b_0 - CV)/\partial a_1}{\partial u(a_1, b_0 - CV)/\partial (b_0 - CV)}$$

   which is an abbreviated notation for

   $$\frac{\partial\, CV}{\partial\, a_1} = \frac{\left.\frac{\partial u(a,b)}{\partial a}\right|_{(a_1, b_0 - CV)}}{\left.\frac{\partial u(a,b)}{\partial b}\right|_{(a_1, b_0 - CV)}}. \tag{1}$$

   (d) (3 points) As a_1 increases, does the strict quasiconcavity of u imply that the right-hand side of (1) increases, decreases, or remains constant? Why? (If you make an assertion about how the strict quasiconcavity of u affects the indifference curves, you do not have to prove that assertion.) Hint: Rather than calculate any derivatives, think about what happens to the Marginal Rate of Substitution of a for b. It is completely acceptable for you to simply assert, rather than rigorously prove, the connection between the Marginal Rate of Substitution and this problem, because that proof is an undergraduate-level exercise.

(e) (2 points) Sketch a rough graph of CV versus a_1. Make sure the first derivative of your sketch of $CV(a_1)$ is consistent with (1) and the second derivative of your sketch of $CV(a_1)$ is consistent with your answer to part (d).

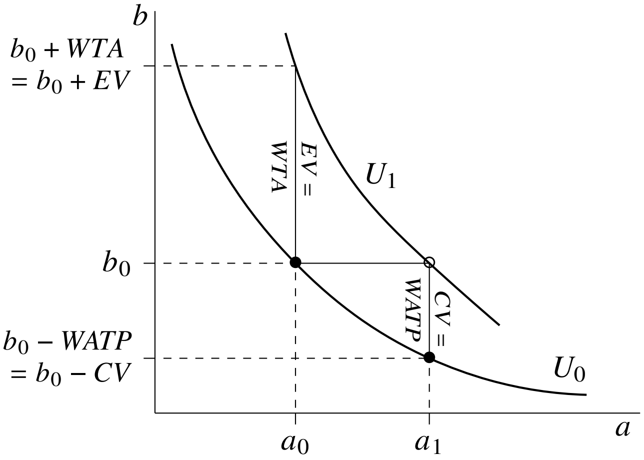


Figure 1. Two non-price- nor income-based measures of the value of a policy of moving from a_0 to $a_1 > a_0$. Only the compensating variation (CV) version of this measure of value was asked for in this question.

Answer to Summer 2023 Qualifying Exam, Section 3 Question 1

(a) When apples increase from a_0 to a_1, utility goes up. What is the maximum number of bananas the consumer is willing to give up for this increase in apples? The answer is CV as defined in this equation, because if the consumer had to give up more than this number of bananas, the consumer would be worse off than if he had not gotten the increased number of apples, whereas if the consumer only had to give up less than this amount of bananas, he would be strictly better off than he was originally, so he would not have given up a maximum number of bananas.

(b) The graph is Figure 1, where the initial position is (a_0, b_0), receiving the extra apples moves the consumer to the open circle at (a_1, b_0), and if the consumer then gives up CV bananas, the consumer ends up at the solid circle at $(a_1, b_0{-}CV)$, which is on the same U_0 indifference curve as when the process began.

The graph has extra information that was not asked for in this question, and it is not expected that you included this extra information in your graph. The extra, not needed information is the "willingness and ability to pay“ (*WATP*), the “willingness to accept” (*WTA*), the “equivalent variation” (*EV*), and the “new” indifference curve U_1.

(c) To take the differential of the equation defining CV, it is convenient to rewrite it as being equal to zero:

$$u(a_1, b_0{-}CV) - u(a_0, b_0) = 0.$$

In taking the differential, a_0 and b_0 are fixed, while a_1 and CV vary:

$$\frac{\partial u(a_1, b_0{-}CV)}{\partial a_1}\, da_1 + \frac{\partial u(a_1, b_0{-}CV)}{\partial (b_0 - CV)}\, \frac{\partial (b_0 - CV)}{\partial CV}\, dCV = 0.$$

Since $\partial(b_0 - CV)/\partial CV = -1$, this leads to

$$\frac{\partial u(a_1, b_0{-}CV)}{\partial a_1}\, da_1 = \frac{\partial u(a_1, b_0{-}CV)}{\partial (b_0 - CV)}\, dCV$$

and the expression given in the exam follows.

(d) The key insight here is that the right-hand side of (1) *is* the marginal rate of substitution of a for b, as is shown in undergraduate Intermediate Microeconomics textbooks (and in Varian’s §7.1). (Note that the right-hand side of (1) is positive.)

Next, the marginal rate of substitution is -1 times the slope of the indifference curve, as is also shown in those texts.

Next, a quasiconcave function is defined as a function having convex upper level sets. Since the utility function in this problem was specified as being quasiconcave, its upper levels sets—and in particular, its upper level set for U_0—is a convex set. This means that the indifference curve U_0, thought of as a function of a, is a convex function. Since the utility function in this problem was specified as being *strictly* quasiconcave, the indifference curve U_0, thought of as a function of a, is a *strictly* convex function, as drawn in Figure 1. Hence its slope gets closer to zero as a_1 increases. But this slope is just -1 times (the right-hand side of) (1); hence (1) also gets closer to zero as a_1 increases.

It follows that $\partial CV/\partial a_1$ gets closer to zero as a_1 increases, so $\partial^2 CV/\partial a_1^2 < 0$.

(e) From the previous part, CV is a positive, increasing, concave function of a_1 for $a_1 > a_0$. ("Positive" is implied by parts (a) and (b); "increasing" is implied by (1) because its right-hand side is positive; and "concave" is implied by part (d).) The rough graph should have these properties. (Optional: at $a_1 = a_0$, CV is zero.)

Completely Optional Remarks

The problem has shown that the value of apples (in terms of bananas, measured using CV) (I would call this the "use value" of apples, as opposed to their "exchange value," which is their price; see `https://en.wikipedia.org/wiki/Use_value`) is an increasing, concave, cardinal function of apple consumption. In the late nineteenth and early twentieth centuries, an individual's utility was seen by the Utilitarians (or at least by Marshall and Pigou) as being an increasing, concave, cardinal function of wealth. If in addition one assumes either that everyone has the same utility function, or that the social planner wishes to act as if everyone has the same utility function when it comes to decisions on distribution, then maximizing social welfare implies giving everyone the same amount of each commodity. The rise of ordinalism destroyed this egalitarian argument. This problem's definition of use value resurrects it.

Historical comments: In his 1973 book "On Economic Inequality," Nobel laureate Amartya Sen calls cardinal "utilitarianism, the dominant faith of 'old' welfare economics" (p. 23). He explains how cardinal Utilitarianism came to have an egalitarian reputation (pp. 15–17):

> Once the information content of individual preferences has been broadened to include interpersonally comparable cardinal welfare functions, many methods of social judgement become available. The most widely used approach is that of utilitarianism in which the sum of the individual utilities is taken as the measure of social welfare, and alternative social states are ordered in terms of the value of the sum of individual utilities. Pioneered by Bentham (1789), this approach has been widely used in economics for social judgements, notably by Marshall (1890), Pigou (1920), and Robertson (1952). In the context of the measurement of inequality of income distribution, and in

> that of judging alternative distributions of income, it has been used by Dalton (1920), Lange (1938), Lerner (1944), Aigner and Heins (1967), and Tinbergen (1970), among others.
>
> The trouble with this approach is that maximizing the sum of individual utilities is supremely unconcerned with the interpersonal distribution of that sum. This should make it a particularly unsuitable approach to use for measuring or judging inequality. Interestingly enough, however, not only has utilitarianism been fairly widely used for distributional judgements, it has—somewhat amazingly—even developed the reputation of being an egalitarian criterion. This seems to have come about through a peculiar dialectical process whereby such adherents of utilitarianism as Marshall and Pigou were attacked by Robbins and others for their supposedly egalitarian use of the utilitarian framework. This gave utilitarianism a ready-made reputation for being equality-conscious.
>
> The whole thing arises from a very special coincidence under some extremely simple assumptions. The maximization of the sum of individual utilities through the distribution of a given total of income between different persons requires equating the marginal utilities from income of different persons, and if the special assumption is made that everyone has the same utility function, then equating marginal utilities amounts to equating total utilities as well. Marshall and others noted this particular aspect of utilitarianism, though they were in no particular hurry to draw any radical distributive policy prescription out of this. But when the attack on utilitarianism came, this particular aspect of it was singled out for an especially stern rebuke.
>
> While this dialectical process gave utilitarianism its ill-deserved egalitarian reputation, the true character of that approach can be seen quite easily by considering a case where one person A derives exactly twice as much utility as person B. . .

Sen *really* did not think utilitarianism deserved its egalitarian reputation, continuing (p. 18):

> It seems fairly clear that fundamentally utilitarianism is very far from an egalitarian approach. It is, therefore, odd that virtually all attempts at measuring inequality from a welfare point of view, or exercises in deriving optimal distributional rules, have concentrated on the utilitarian approach. It might be thought

> that this criticism would not apply at all if utilitarianism were combined with the assumption that everyone has the same utility function. But this is not quite the case. The distribution of welfare between persons is a relevant aspect of any problem of income distribution, and our evaluation of inequality will obviously depend on whether we are concerned only with the loss of the sum of individual utilities through a bad distribution of income, or also with the inequality of welfare levels of different individuals. Its lack of concern with the latter tends to make utilitarianism a blunt approach to measuring and judging different extents of inequality even if the assumption is made that everyone has the same utility function. As a framework of judging inequality, utilitarianism is indeed a non-starter, despite the spell that this approach seems to have cast on this branch of normative economics.

However, this criticism by Sen of the consequences of "the assumption that everyone has the same utility function" seems unconvincing.

2022 Qualifying Exam Sec. 3 Qu. 1

1. **[12 points]** Suppose a consumer gets utility from two commodities, apples a and bananas b, according to the utility function

$$u(a, b) = ab^2 .$$

Suppose the consumer takes as given the price of apples, p_a, and the price of bananas, p_b, and that the consumer has income denoted by m.

(a) Find this consumer's demand for apples and demand for bananas. Do not bother checking the second-order conditions.

(b) For the rest of this problem, suppose that the price of apples is $p_a = 1/3$ and the price of bananas is $p_b = 2/3$.

If this consumer's initial income is $m_0 = 3$, determine his initial consumption of apples and of bananas.

(c) If this consumer's income changes to $m' = 4$, determine his new consumption of apples and of bananas.

(d) [Optional motivation for the rest of the problem: You might think that all "valuation" methods for the \$1 in extra income the consumer gets when going from $m_0 = 3$ to $m' = 4$ would assign this change a value of exactly \$1, but do they?]

Make a sketch of this consumer's indifference curve through his initial consumption point (the one for $m_0 = 3$) and this consumer's indifference curve through his new consumption point (the one for $m' = 4$), graphing apples on the horizontal axis and bananas on the vertical axis. You do not draw this graph very precisely. In particular, you do not need to use the mathematical form of this consumer's actual utility function, $u = ab^2$, in making your graph; it is fine to draw your graph in a generic way, using the sort of indifference curves discussed in undergraduate textbooks.

(e) On the graph you just drew, indicate this consumer's "Willingness (and Ability) to Pay," measured in terms of *apples*, to move from his initial consumption point to his new consumption point. This is equal to this consumer's compensating variation, measured in terms of apples.

(f) Numerically calculate this consumer's Willingness (and Ability) to Pay, measured in terms of apples, to move from his initial consumption point to his new consumption point. To do this, you need to use the fact that the consumer's utility function is $u = ab^2$, and you need to use the coordinates in the (apples, bananas) plane of the initial and final consumption points.

Since you do not have calculators, you do not need to simplify expressions that are purely numerical. For example, you would not need to simplify $4 - 27/16$.

(g) In the previous part of this problem, you calculated this consumer's Willingness (and Ability) to Pay, measured in terms of apples, to move from his initial consumption point to his new consumption point. What is the dollar value of this WATP number of apples, using the prevailing price of apples? (Again, you do not need to simplify expressions that are purely numerical.)

(h) On the graph you drew above, indicate this consumer's "Willingness (and Ability) to Pay," measured in terms of *bananas*, to move from his initial consumption point to his new consumption point. This is equal to this consumer's compensating variation, measured in terms of bananas.

(i) Numerically calculate this consumer's Willingness (and Ability) to Pay, measured in terms of bananas, to move from his initial consumption point to his new consumption point. To do this, you need to use the fact that the consumer's utility function is $u = ab^2$, and you need to use the coordinates in the (apples, bananas) plane of the initial and final consumption points.

You do not need to simplify expressions that are purely numerical.

(j) In the previous part of this problem, you calculated this consumer's Willingness (and Ability) to Pay, measured in terms of bananas, to move from his initial consumption point to his new consumption point. What is the dollar value of this WATP number of bananas, using the prevailing price of bananas? (Again, you do not need to simplify expressions that are purely numerical.)

Answer to Summer 2021 Qualifying Exam, Section 3 Question 1

(a) The Lagrangian is

$$\mathscr{L} = ab^2 + \lambda\,(m - p_a a - p_b b)\,.$$

So

$$\begin{aligned} 0 &= \mathscr{L}'_a = b^2 - \lambda p_a \quad \text{and} \\ 0 &= \mathscr{L}'_b = 2ab - \lambda p_b\,. \end{aligned}$$

From the first and second equations, respectively, it follows that

$$\lambda = \frac{b^2}{p_a} = \frac{2ab}{p_b}$$

so $b = 2ap_a/p_b$ and $m = p_a a + p_b b = p_a a + p_b(2ap_a/p_b) = 3ap_a$ and the demand for apples is

$$a^D = \frac{m}{3p_a}\,.$$

Then the demand for bananas is

$$b^D = \frac{2p_a a}{p_b} = \frac{2p_a}{p_b}\frac{m}{3p_a} = \frac{2m}{3p_b}\,.$$

(b) With $p_a = 1/3$ and $p_b = 2/3$, and $m_0 = 3$, the demand curves from part (a) result in $a = 3$ and $b = 3$.

(c) With income changing to $m' = 4$, the demand curves from part (a) result in $a = 4$ and $b = 4$.

(d) See Figure 1. The relevant characteristics of that figure for this question are that the original indifference curve U_0 passes through $(3, 3)$ and that the new indifference curve U' passes through $(4, 4)$.

(e) See the $WATP_a$ of Figure 1. This answers the question "if the consumer were able to move to the new point $(4, 4)$, how many apples would he then be willing to give up," because if he had to give up any more apples than this, his utility would fall below U_0, which he would not willingly do.

(Optional: The question says this is compensating variation because it assumes the consumer first moves to the new point.)

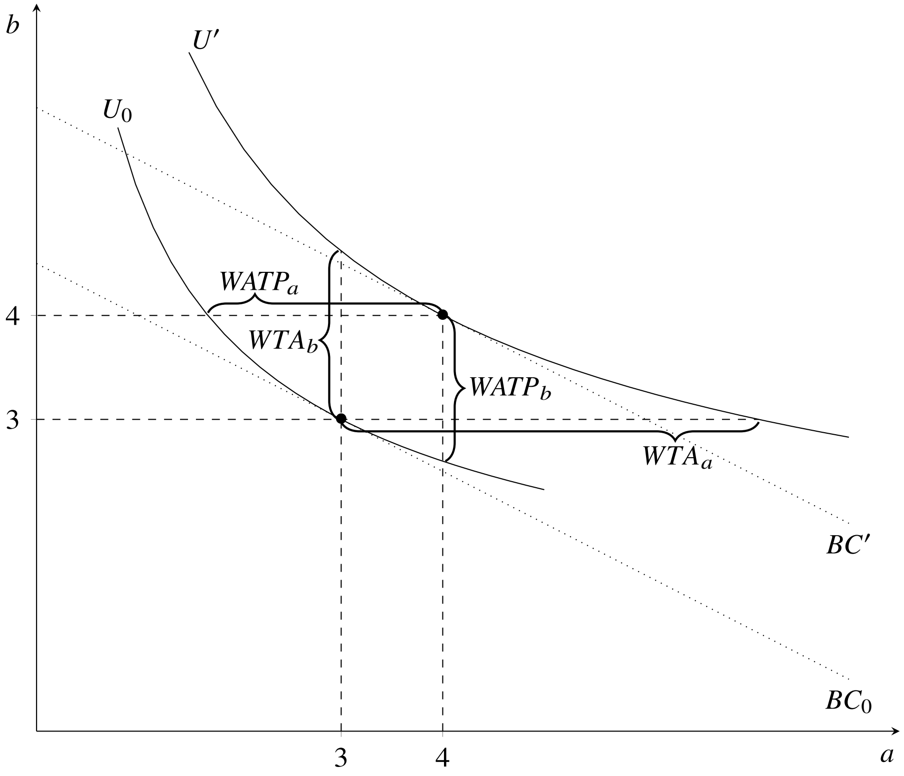


Figure 1. With an initial income of \$3, a consumer with utility function ab^2 (where a is apples and b is bananas) facing prices $p_a = 1/3$ and $p_b = 2/3$ consumes at $(3, 3)$, where budget constraint BC_0 is tangent to indifference curve U_0. With a new income of \$4, that consumer consumes at $(4, 4)$, where budget constraint BC' is tangent to indifference curve U'. Four measures of the consumer's value of this \$1 increase in income are shown. The *WATP* measures are for the exam's Section 3 Question 1, and the *WTA* measures are for the exam's Section 3 Question 2. **Optional:** The horizontal distance between BC_0 and BC' is one dollar's worth of apples. Since this graph has been drawn precisely, this means that $WATP_a$ is less than \$1 (to see this, compare $WATP_a$ to the gap between the dotted lines along the dashed line at $b = 4$), and WTA_a is more than \$1 (to see this, compare WTA_a to the gap between the dotted lines along the dashed line at $b = 3$). Similarly, the vertical distance between BC_0 and BC' is one dollar's worth of bananas. Hence $WATP_b$ is less than \$1 (to see this, compare $WATP_b$ to the gap between the dotted lines along the dashed line at $a = 4$), and WTA_b is more than \$1 (to see this, compare WTA_b to the gap between the dotted lines along the dashed line at $a = 3$).

(f) Starting from the new point $(4, 4)$, he would be willing to give up apples up to, but not beyond, the point where his utility after giving up those apples, $u(4 - WATP_a, 4)$, was equal to his original utility, $u(3, 3)$. So we have

$$\begin{aligned} u(3,3) &= u(4 - WATP_a, 4) \\ 3 \cdot 3^2 &= (4 - WATP_a) \cdot 4^2 \\ 27/16 &= 4 - WATP_a \\ WATP_a &= 4 - 27/16 = 2\tfrac{5}{16} \approx 2.3125\,. \end{aligned}$$

Throughout this problem, you were told that you did not have to simplify purely numerical expressions.

(g) It is p_a times $WATP_a$, so $1/3$ times $2\frac{5}{16}$, which is $\$37/48 \approx \$0.77 < \$1$. The "less than \$1" part is optional; it is interesting that "willingness and ability to pay," when measured in apples, is less than one dollar's worth of apples.

(h) See the $WATP_b$ of Figure 1. This answers the question "if the consumer were able to move to the new point $(4, 4)$, how many bananas would he be willing to give up," because if he had to give up any more bananas than this, his utility would fall below U_0, which he would not willingly do.

(Optional: The question says this is compensating variation because it assumes the consumer moves first to the new point.)

(i) Starting from the new point $(4, 4)$, he would be willing to give up bananas up to, but not beyond, the point where his utility after giving up those bananas, $u(4, 4 - WATP_b)$, was equal to his original utility, $u(3, 3)$. So we have

$$\begin{aligned} u(3,3) &= u(4, 4 - WATP_b) \\ 3 \cdot 3^2 &= 4 \cdot (4 - WATP_a)^2 \\ \frac{\sqrt{3} \cdot 3}{2} &= 4 - WATP_b \\ WATP_b &= 4 - \frac{\sqrt{3} \cdot 3}{2} = \frac{8 - 3\sqrt{3}}{2} \approx 1.40\,. \end{aligned}$$

(j) It is p_b times $WATP_b$, so $2/3$ times $\frac{8-3\sqrt{3}}{2}$, which is $\$\frac{8-3\sqrt{3}}{3} \approx \$0.93 < \$1$. The "less than \$1" part is optional; it is interesting that "willingness and ability to pay," when measured in bananas, is less than one dollar's worth of bananas.

Also optional: the dollar value of “$WATP_a$ for \$1 in extra income” is not the same as the dollar value of “$WATP_b$ for \$1 in extra income.”

The first four parts of this question are identical to the first four parts of the previous question.

2022 Qualifying Exam Sec. 3 Qu. 2

2. **[12 points]** Suppose a consumer gets utility from two commodities, apples a and bananas b, according to the utility function

$$u(a, b) = ab^2 .$$

Suppose the consumer takes as given the price of apples, p_a, and the price of bananas, p_b, and that the consumer has income denoted by m.

(a) Find this consumer's demand for apples and demand for bananas. Do not bother checking the second-order conditions.

(b) For the rest of this problem, suppose that the price of apples is $p_a = 1/3$ and the price of bananas is $p_b = 2/3$.

If this consumer's initial income is $m_0 = 3$, determine his initial consumption of apples and of bananas.

(c) If this consumer's income changes to $m' = 4$, determine his new consumption of apples and bananas.

(d) [Optional motivation for the rest of the problem: You might think that all "valuation" methods for the $1 in extra income the consumer gets when going from $m_0 = 3$ to $m' = 4$ would assign this change a value of exactly $1, but do they?]

Make a sketch of this consumer's indifference curve through his initial consumption point (the one for $m_0 = 3$) and this consumer's indifference curve through his new consumption point (the one for $m' = 4$), graphing apples on the horizontal axis and bananas on the vertical axis. You do not draw this graph very precisely. In particular, you do not need to use the mathematical form of this consumer's actual utility function, $u = ab^2$, in making your graph; it is fine to draw your graph in a generic way, using the sort of indifference curves discussed in undergraduate textbooks.

(e) On the graph you just drew, indicate this consumer's "Willingness to Accept" compensation, measured in terms of *apples*, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. This is equal to this consumer's equivalent variation, measured in terms of apples.

(f) Numerically calculate this consumer's Willingness to Accept compensation, measured in terms of apples, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. To do this, you need to use the fact that the consumer's utility function is $u = ab^2$, and you need to use the coordinates in the (apples, bananas) plane of the initial and final consumption points.

Since you do not have calculators, you do not need to simplify expressions that are purely numerical. For example, you would not need to simplify $4 - 27/16$.

(g) In the previous part of this problem, you calculated this consumer's Willingness to Accept compensation, measured in terms of apples, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. What is the dollar value of this WTA number of apples, using the prevailing price of apples? (Again, you do not need to simplify expressions that are purely numerical.)

(h) On the graph you drew above, indicate this consumer's "Willingness to Accept" compensation, measured in terms of *bananas*, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. This is equal to this consumer's equivalent variation, measured in terms of bananas.

(i) Numerically calculate this consumer's Willingness to Accept compensation, measured in terms of bananas, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. To do this, you need to use the fact that the consumer's utility function is $u = ab^2$, and you need to use the coordinates in the (apples, bananas) plane of the initial and final consumption points.

You do not need to simplify expressions that are purely numerical.

(j) In the previous part of this problem, you calculated this consumer's Willingness to Accept compensation, measured in terms of bananas, in return for being denied the opportunity to move from his initial consumption point to his new consumption point. What is the dollar value of this WTA number of bananas, using the prevailing price of bananas? (Again, you do not need to simplify expressions that are purely numerical.)

Answer to Summer 2022 Qualifying Exam, Section 3 Question 2

The answers of parts (a)–(d) are the same as for Summer 2022 Qualifying Exam Section 3 Question 1.

For Figure 1 see the previous question.

(e) See the WTA_a of Figure 1. This answers the question "if the consumer were not able to move to the new point $(4, 4)$, how many apples would he require in compensation," because if he is given this many apples, his utility would be the same, U', as it would have been if he had been able to move to the new point.

(Optional: The question says this is equivalent variation because it assumes the consumer is not allowed to move to the new point.)

(f) Starting from the old point $(3, 3)$, he would need to be given apples up to the point where his utility after being given those apples, $u(3 + WTA_a, 3)$, was equal to the utility he would have if he had been allowed to move to the new point, $u(4, 4)$. So we have

$$\begin{aligned} u(3 + WTA_a, 3) &= u(4, 4) \\ (3 + WTA_a) \cdot 3^2 &= 4 \cdot 4^2 \\ 64/9 &= 3 + WTA_a \\ WTA_a &= 64/9 - 3 = 37/9 = 4\tfrac{1}{9} \approx 4.11\,. \end{aligned}$$

Throughout this problem, you were told that you did not have to simplify purely numerical expressions.

(g) It is p_a times WTA_a, so $1/3$ times $37/9$, which is $\$37/27 \approx \$1.37 > \$1$. The "greater than \$1" part is optional; it is interesting that "willingness to accept," when measured in apples, is greater than one dollar's worth of apples.

(h) See the WTA_b of Figure 1. This answers the question "if the consumer were not able to move to the new point $(4, 4)$, how many bananas would he require in compensation," because if he is given this many bananas, his utility would be the same, U', as it would have been if he had been able to move to the new point.

(Optional: The question says this is equivalent variation because it assumes the consumer is not allowed to move to the new point.)

(i) Starting from the old point $(3, 3)$, he would need to be given bananas up to the point where his utility after being given those bananas,

$u(3, 3 + WTA_b)$, was equal to the utility he would have if he had been allowed to move to the new point, $u(4, 4)$. So we have

$$\begin{aligned}
u(3, 3 + WTA_b) &= u(4, 4) \\
3 \cdot (3 + WTA_b)^2 &= 4 \cdot 4^2 \\
(3 + WTA_b)^2 &= 4^3/3 \\
3 + WTA_b &= 4\sqrt{4/3} = 8/\sqrt{3} \\
WTA_b &= 8/\sqrt{3} - 3 = (8\sqrt{3} - 9)/3 \approx 1.62\,.
\end{aligned}$$

(j) It is p_b times WTA_b, so $2/3$ times $(8\sqrt{3} - 9)/3$, which is $2\,(8\sqrt{3} - 9)/9 \approx \$1.08 > \$1$. The “greater than \$1” part is optional; it is interesting that “willingness to accept,” when measured in apples, is greater than one dollar’s worth of apples.

Also optional: the dollar value of “WTA_a for \$1 in extra income” is not the same as the dollar value of “WTA_b for \$1 in extra income.”

Optional: Note that in Figure 1, the Compensating Variation measures, $WATP_a$ and $WATP_b$, are measured from the new point $(4, 4)$, whereas the Equivalent Variation measures, WTA_a and WTA_b, are measured from the old point, $(3, 3)$. We can summarize the answers to this question and the previous question as:

- The change in income is \$1, which could buy, among other possibilities, “3 more apples and no more bananas,” or “no more apples and 1.5 more bananas,” or “1 more apple and 1 more banana.” The consumer in this problem does the last of these.
- $WATP_a \approx 2.31$ apples, worth approximately \$0.77
- $WATP_b \approx 1.40$ bananas, worth approximately \$0.93
- $WTA_a \approx 4.11$ apples, worth approximately \$1.37
- $WTA_b \approx 1.62$ bananas, worth approximately \$1.08.

These results are related to the “lump sum” principle taught in Intermediate Microeconomics, which is the superiority of lump sum taxes or subsidies over taxes or subsidies on just one good. In this problem, receiving \$1 worth of apples is worth less than receiving \$1 in cash, so if you received the cash (compensating variation), you would be *WATP* less than \$1 worth of apples in return. The same is true for bananas. On the other hand, if you did not receive the \$1 in cash (equivalent variation), you would need more than \$1 worth of apples or bananas (*WTA*) to compensate you for not receiving the \$1 cash.

1. **[16 points]** Suppose a consumer consumes two goods, x and y, has income m, and has the "Constant Elasticity of Substitution" utility function $u(x, y) = (x^{1/2} + y^{1/2})^2$. Suppose

   - the price of y, denoted p_y, is always equal to one.

   You may use without proof the facts that Varian's book states on its p. 112, namely that:

   - if a consumer has the CES utility function $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$
   - and if we define $r = \frac{\rho}{\rho - 1}$
   - then the consumer's expenditure function is $e(\mathbf{p}, u) = (p_1^r + p_2^r)^{1/r}\, u$,
   - their indirect utility function is $v(\mathbf{p}, m) = (p_1^r + p_2^r)^{-1/r}\, m$,
   - their demand for x_1 is $x_1(\mathbf{p}, m) = \dfrac{m p_1^{r-1}}{p_1^r + p_2^r}$,
   - and their money metric utility function is $\mu(\mathbf{p}; \mathbf{q}, m) = (p_1^r + p_2^r)^{1/r} (q_1^r + q_2^r)^{-1/r} m$.

   On p. 161 of Varian's book, upon defining

   $$\mu(\mathbf{q}; \mathbf{p}, m) = e(\mathbf{q}, v(\mathbf{p}, m)),$$

   Varian gives the equivalent and compensating variation as, respectively,

   $$EV = \mu(\mathbf{p}^0; \mathbf{p}', m') - m^0 \quad \text{and}$$
   $$CV = m' - \mu(\mathbf{p}'; \mathbf{p}^0, m^0).$$

   (a) The consumer's "expenditure on x" is the price of x times the quantity of x which he buys, in other words, $x p_x$. Assuming that $p_x = 3$, find an expression for this consumer's expenditures on x. (This expression will depend on m.)

   (b) Assuming that $p_x = 3$, find an expression for this person's consumer surplus generated from x. (This expression will depend on m.) You may use without proof the following result:

   $$\int \frac{p^{-2}}{p^{-1} + 1}\, dp = \int \frac{dp}{p + p^2} = \int \frac{1 + p - p}{p\,(1 + p)}\, dp$$

$$= \int \Big[\frac{1+p}{p\,(1+p)} - \frac{p}{p\,(1+p)}\Big]\, dp = \int \Big(\frac{1}{p} - \frac{1}{1+p}\Big)\, dp$$

$$= \ln p - \ln(1{+}p)\,.$$

(c) Express EV and CV in terms of the expenditure function [better: in terms of income and the old and new prices] and describe the relationship between EV and CV, on the one hand, and "willingness to pay" ("WTP") and "willingness to accept" ("WTA"), on the other hand.

(d) Find this consumer's WTP for a decrease in the price of x from infinity to 3. Assume as before that $p_y = 1$. (Your expression for WTP will depend on m.)

(e) Find this consumer's WTA for an increase in the price of x from 3 to infinity. Assume as before that $p_y = 1$. (Your expression for WTA will depend on m.)

(f) Sketch a graph showing how this person's expenditures on x, consumer surplus generated from x, WTP, and WTA all depend on m. Use your results from (a), (b), (d), and (e) to do this.

Answer to Summer 2020 Qualifying Exam, Section 3 Question 1

$u(x,y) = \left(x^{1/2}+y^{1/2}\right)^2 \qquad P_y = 1$

a) The translation between Varian's notation and the notation of this problem is

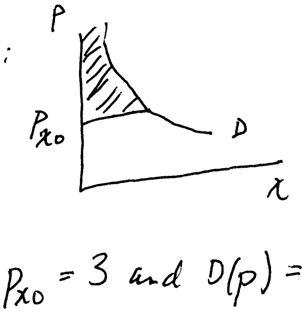
this problem	Varian
x	x_1
y	x_2
$\frac{1}{2}$	ρ
-1	$r = \frac{\rho}{\rho-1} = \frac{1/2}{1/2-1} = \frac{1/2}{-1/2} = -1$
P_x	P_1
P_y	P_2
m	m

So Varian's $x_1 = \frac{m\,p_1^{r-1}}{p_1^r + p_2^r}$ becomes $x = \frac{m\,P_x^{-1-1}}{P_x^{-1}+P_y^{-1}} = \frac{m\,P_x^{-2}}{P_x^{-1}+1} = \frac{m}{P_x + P_x^2}$.

Expenditure on x is $x P_x = \frac{m P_x}{P_x + P_x^2} = \frac{m}{1+P_x}$.

Expenditure on x with $P_x = 3$ is $\frac{m}{1+3} = \boxed{\frac{m}{4}\,.}$

b) Consumer surplus is this area:



This is $\int_{P_{x0}}^{\infty} D(p)\,dp$. Here $P_{x0} = 3$ and $D(p) = \frac{m}{P_x + P_x^2}$ so consumer surplus is

$$\int_3^{\infty} \frac{m\,dp}{P_x + P_x^2} = m\left[\ln p - \ln(p+1)\right]\Big|_3^{\infty} = m \ln \frac{p}{p+1}\Big|_3^{\infty} = m\left(\ln 1 - \ln \tfrac{3}{4}\right)$$

$\uparrow$ from the formula given in the question

$$= \boxed{m \ln \tfrac{4}{3}} \approx 0.29m.$$

c) $EV = \mu(\underset{\sim}{p}^{0} ; \underset{\sim}{p}', m') - m^{0}$

$CV = m' - \mu(\underset{\sim}{p}' ; \underset{\sim}{p}^{0}, m^{0})$

with Varian's $\mu(\underset{\sim}{p} : \underset{\sim}{q}, m) = (p_1^{r} + p_2^{r})^{1/r} (q_1^{r} + q_2^{r})^{-1/r} m$ imply (moving the "0" superscripts down to subscripts to avoid the exponents):

$$EV = (p_{01}^{r} + p_{02}^{r})^{1/r} (p_1'^{r} + p_2'^{r})^{-1/r} m' - m_0$$

$$CV = m' - (p_1'^{r} + p_2'^{r})^{1/r} (p_{01}^{r} + p_{02}^{r})^{-1/r} m_0$$

So in our problem

$$EV = (p_{x0}^{-1} + p_{y0}^{-1})^{-1} (p_x'^{(-1)} + p_y'^{(-1)})^{\frac{-1}{-1}} m - m$$

$$= (p_{x0}^{-1} + 1)^{-1} (p_x'^{(-1)} + 1) m - m$$ ← OK to leave in this unsimplified form

$$= m \frac{1 + 1/p_x'}{1 + 1/p_{x0}} - m = m \frac{1 + \frac{1}{p_x'} - 1 - \frac{1}{p_{x0}}}{1 + 1/p_{x0}}$$

$$= \left(\frac{1}{p_x'} - \frac{1}{p_{x0}}\right) \frac{m}{1 + 1/p_{x0}}$$ and

$$CV = m - (p_x'^{(-1)} + p_y'^{(-1)})^{\frac{1}{-1}} (p_{x0}^{-1} + p_{y0}^{-1})^{\frac{-1}{-1}} m$$

$$= m - (p_x'^{(-1)} + 1)^{-1} (p_{x0}^{-1} + 1) m$$ ← OK to leave in this unsimplified form

$$= m \left[1 - \frac{1 + 1/p_{x0}}{1 + 1/p_x'}\right] = \frac{m}{1 + 1/p_x'} \left[1 + \frac{1}{p_x'} - 1 - \frac{1}{p_{x0}}\right]$$

$$= \left(\frac{1}{p_x'} - \frac{1}{p_{x0}}\right) \frac{m}{1 + 1/p_x'} .$$

On this page, this level of detail is not needed; the main point is being able to get from WTP/WTA to EV/CV or vice versa.

Also,

$$EV = \mu(\underset{\sim}{p}^0; \underset{\sim}{p}', m') - m^0 = e(\underset{\sim}{p}^0, v(\underset{\sim}{p}', m')) - m^0$$

$$CV = m' - \mu(\underset{\sim}{p}'; \underset{\sim}{p}^0, m^0) = m' - e(\underset{\sim}{p}', v(\underset{\sim}{p}^0, m^0)).$$

Thus EV uses base year prices, asking:

if we did not do this, what would you have to $\left\{\begin{array}{l}\text{be paid (if "this" is a gain)}\\ \text{pay (if "this" is a loss)}\end{array}\right\}$

to make you as well off as if we had done it?

CV uses new year prices, asking:

if we did this, what would you have to $\left\{\begin{array}{l}\text{pay (if "this" is a gain)}\\ \text{be paid (if "this" is a loss)}\end{array}\right\}$

to make you as well off as if we had not done it?

Therefore:	Contemplated Gain	Contemplated Loss
if we didn't do this: EV	"be paid": WTA	"pay": WTP
if we did this: CV	"pay": WTP	"be paid": WTA

WTP: willingness to pay

WTA: willingness to accept

(A better term than WTP would be "willingness and ability to pay," but that would be nonstandard.)

↑ ↑

(optional)

Conclusion:

In both parts (b) and (c), we are asking what happens if a policy is undertaken, so we will use CV not EV.

d) Old p_x : ∞

New p_x : 3

This is a gain, hence the question asks WTP and CV.

↑ which actually happens, not which we refrain from carrying out

From above, $CV = \left(\frac{1}{p_x'} - \frac{1}{p_{x0}}\right)\frac{m}{1+1/p_x'} = \left(\frac{1}{3} - \frac{1}{\infty}\right)\frac{m}{1+\frac{1}{3}}$

$$= m\,\frac{1/3}{1+1/3} = m\,\frac{1/3}{4/3} = \frac{m}{4}.$$

e) Old p_x : 3

New p_x : ∞

This is an actual loss (not a loss we refrain from carrying out),

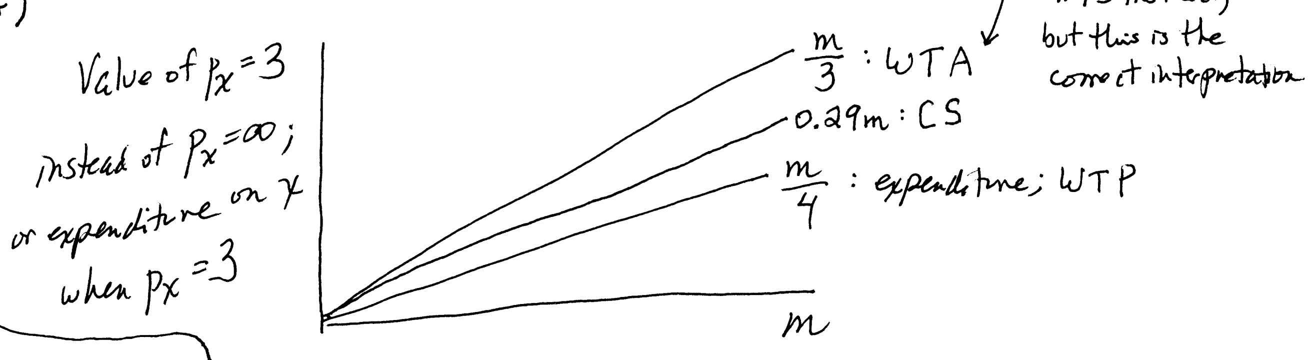
hence the question asks WTA and CV.

From above,

$$CV = \left(\frac{1}{p_x'} - \frac{1}{p_{x0}}\right)\frac{m}{1+1/p_x'} = \left(\frac{1}{\infty} - \frac{1}{3}\right)\frac{m}{1+\frac{1}{\infty}}$$

$$= \frac{-1}{3}\,\frac{m}{1} = \frac{-m}{3}.$$

f)



WTA, WTP, CS

optional { These are three ways of measuring the consumer's value of a "$p_x = 3$" policy. If society adopts this policy valuation, richer people's value has greater weight than poorer people's value.

THIS PAGE COMPLETELY OPTIONAL!

To obtain the figure on the next page, which illustrates these results when $m = \$100$ and $p_x = 3$, obtain the Marshallian demand curve from part (a) above:

$$x = \frac{m}{p_x + p_x^2} = \frac{100}{3+9} = \frac{100}{12} = \frac{25}{3}.$$

From part (b), $CS \approx m \ln \frac{4}{3} \approx 29$.

To obtain the Hicksian demand curves,

$$h_x = \frac{\partial e}{\partial p_x} = \frac{\partial}{\partial p_x} (p_x^{-1} + 1)^{-1} u = -(p_x^{-1}+1)^{-2}(-1 p_x^{-2}) u$$

$$= \frac{+u}{[(p_x^{-1}+1)p_x]^2} = \frac{u}{(1+p_x)^2}.$$

- u when $p_x = 3$: From the Marshallian demand curve, $x = \frac{25}{3}$, and expenditures on x are $p_x \cdot x = 3 \cdot \frac{25}{3} = 25$. Since $m = 100$, expenditures on y must be $100 - 25 = 75$. Since $p_y = 1$, this means that $y = 75$. Then $u = \left(\sqrt{\frac{25}{3}} + \sqrt{75}\right)^2 = \left(\frac{5}{\sqrt{3}} + \sqrt{25 \cdot 3}\right)^2 = \left(\frac{5}{\sqrt{3}} + 5\sqrt{3}\right)^2 = \frac{25}{3} + 2 \cdot \frac{5}{\sqrt{3}} \cdot 5\sqrt{3} + 25 \cdot 3 = \frac{25}{3} + 50 + 75$ $= 8\frac{1}{3} + 125 = 133\frac{1}{3}$. Hence $h_x = \frac{133\frac{1}{3}}{(1+p_x)^2}$.
- u when $p_x = \infty$. From the Marshallian demand curve, $x = 0$, so expenditures on x are zero, so the entire income of $m = 100$ is spent on y, meaning, since $p_y = 1$, that $y = 100$. Then $u = (\sqrt{0} + \sqrt{100})^2 = 100$. Hence $h_x = \frac{100}{(1+p_x)^2}$.

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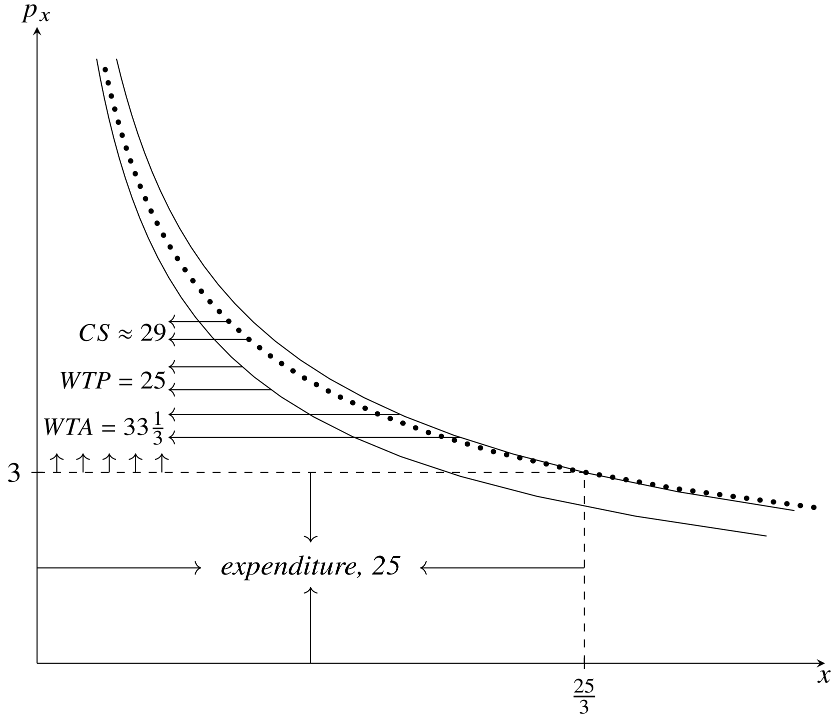

Figure 6. Demand curves for an income of $m = 100$ and a utility function of $u(x, y) = (x^{1/2} + y^{1/2})^2$ when $p_y = 1$. **Dotted curve**: the Marshallian demand curve, $x = p_x^{-2}/(p_x^{-1} + 1) = 1/(p_x + p_x^2)$. Consumer surplus when $p_x = 3$ is the area left of this curve and above the line $p_x = 3$; it is $m \ln(4/3) \approx \$29$. **Solid curves**: Hicksian demand curves, $u/(1 + p_x)^2$. **Right-most solid curve**: Hicksian demand curve with utility fixed at its level when $p_x = 3$, therefore $u(25/3, 75) = ((25/3)^{1/2} + 75^{1/2})^2 = 133\frac{1}{3}$. WTA when $p_x = 3$ is the area left of this curve and above the line $p_x = 3$; it is $m/3 = \$33\frac{1}{3}$. **Left-most solid curve**: Hicksian demand curve with utility fixed at its level when $p_x = \infty$, therefore $u(0, 100) = (0^{1/2} + 100^{1/2})^2 = 100$. WTP when $p_x = 3$ is the area left of this curve and above the line $p_x = 3$; it is $m/4 = \$25$. When $p_x = 3$, **expenditure** is $3 \cdot 25/3 = \$25$. The **value** of "$p_x = 3$" or "$x = 25/3$" is the \$25 expenditure plus the measure of the surplus (either WTA or WTP or CS).

2020 Qualifying Exam Sec. 3 Qu. 2 [Open-book exam due to the pandemic]

2. **[16 points]** Suppose a consumer consumes two goods, x and y, has income m, and has the "Constant Elasticity of Substitution" utility function $u(x, y) = (x^{1/2} + y^{1/2})^2$. Suppose

   - the price of y, denoted p_y, is always equal to one;
   - and the consumer's income $m = 100$.

   You may use without proof the facts that Varian's book states on its p. 112, namely that:

   - if a consumer has the CES utility function $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$
   - and if we define $r = \frac{\rho}{\rho - 1}$
   - then the consumer's expenditure function is $e(\mathbf{p}, u) = (p_1^r + p_2^r)^{1/r}\, u$,
   - their indirect utility function is $v(\mathbf{p}, m) = (p_1^r + p_2^r)^{-1/r}\, m$,
   - their demand curve for x_1 is $x_1(\mathbf{p}, m) = \dfrac{m p_1^{r-1}}{p_1^r + p_2^r}$,
   - and their money metric utility function is $\mu(\mathbf{p}; \mathbf{q}, m) = (p_1^r + p_2^r)^{1/r} (q_1^r + q_2^r)^{-1/r} m$.

   On p. 161 of Varian's book, upon defining

   $$\mu(\mathbf{q}; \mathbf{p}, m) = e(\mathbf{q}, v(\mathbf{p}, m)),$$

   Varian gives the equivalent and compensating variation as, respectively,

   $$\begin{aligned} EV &= \mu(\mathbf{p}^0; \mathbf{p}', m') - m^0 \quad \text{and} \\ CV &= m' - \mu(\mathbf{p}'; \mathbf{p}^0, m^0). \end{aligned}$$

   If there are other results which Varian proves which you want to use, simply cite the result and its page number; since this is an open-book exam, I think it's pointless to ask you to copy a proof from Varian's book straight onto your exam paper.

   Please look at Figure 3.

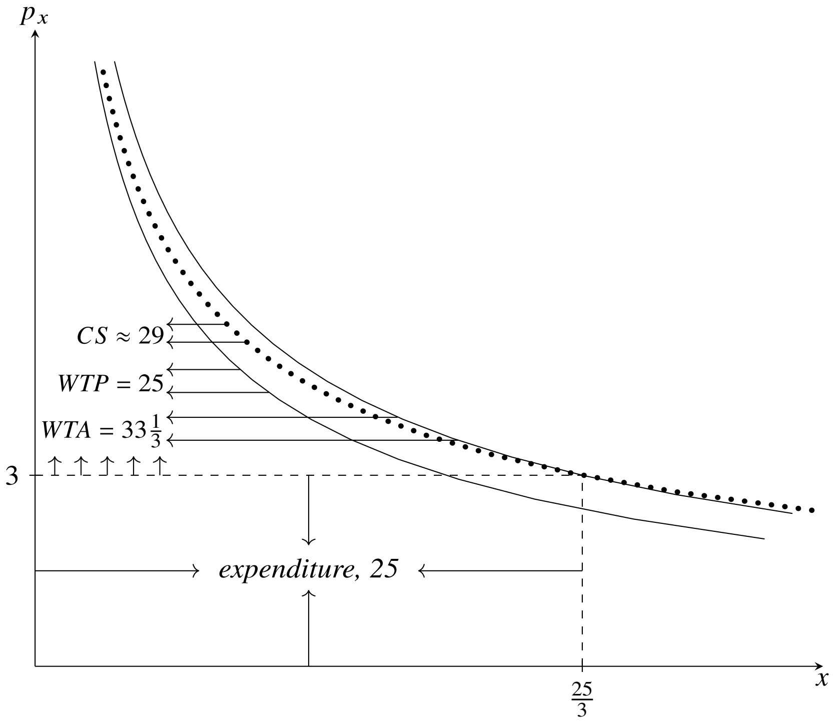

Figure 3. Demand curves for an income of $m = 100$ and a utility function of $u(x, y) = (x^{1/2} + y^{1/2})^2$ when $p_y = 1$.

(a) The "$WTA = 33\frac{1}{3}$" label of Figure 3 denotes willingness to accept the price of x increasing from three to infinity, and $33\frac{1}{3}$ is the area to the left of the right-most curve and above the line $p_x = 3$. Show that the equation of this right-most curve is

$$\frac{u}{(1+p_x)^2}$$

where

$$u = \left(\sqrt{\frac{25}{3}} + \sqrt{75}\right)^2 = 133\tfrac{1}{3}\,.$$

(b) The "$WTP = 25$" label of Figure 3 denotes willingness to pay in return for the price of x decreasing from infinity to three, and 25 is the area to the left of the left-most curve and above the line $p_x = 3$. Show that the equation of this left-most curve is

$$\frac{u}{(1+p_x)^2}$$

where

$$u = \left(\sqrt{0} + \sqrt{100}\right)^2 = 100\,.$$

(c) The "$CS \approx 29$" label of Figure 3 denotes consumer surplus, and $100 * \ln(4/3) \approx 29$ is the area to the left of the dotted curve and above the line $p_x = 3$. What is the equation of this dotted curve? (You may express this either as a function of p_x or as a function of x.)

(d) Consider the following two situations:

   i. The situation depicted in Figure 3, with $p_x = 3$ and $p_y = 1$ and $m = 100$ and consumer surplus from consumption of x approximately equal to 29 and consumer surplus from consumption of y (not illustrated); or

   ii. A situation in which the consumer faces a price-discriminating seller who charges the consumer \$25 + \$29 for consuming $x = 25/3$ (which the consumer does consume and does pay) and the consumer faces a uniform price of $p_y = 1$ for y and the consumer is given an income of \$100 plus \$34.

In which situation does this consumer have the larger consumer surplus (considering both goods x and y)? There is no need to provide a numerically-calculated answer; one arrived at just by logical reasoning is what is being asked for. (This question may be too easy but it becomes interesting in light of the next question.)

(e) In which of the situations described in part (d) does this consumer have the larger utility? There is no need to provide a numerically-calculated answer; one arrived at just by logical reasoning is what is being asked for.

Answer to Summer 2020 Qualifying Exam, Section 3 Question 2

First we need to connect EV and CV with WTP and WTA. From the equations given in the problem,

$$EV = \mu(\underset{\sim}{p}^0 ; \underset{\sim}{p}', m') - m^0 = e(\underset{\sim}{p}^0, v(\underset{\sim}{p}', m')) - m^0$$

$$CV = m' - \mu(\underset{\sim}{p}' ; \underset{\sim}{p}^0, m^0) = m' - e(\underset{\sim}{p}', v(\underset{\sim}{p}^0, m^0)).$$

Thus EV uses base year prices, asking:

if we did not do this, what would you have to $\left\{ \begin{array}{l} \text{be paid (if "this") is a gain} \\ \text{pay (if "this") is a loss} \end{array} \right\}$ to make you as well off as if we had done it?

CV uses new year prices, asking:

if we did this, what would you have to $\left\{ \begin{array}{l} \text{pay (if "this" is a gain)} \\ \text{be paid (if "this" is a loss)} \end{array} \right\}$ to make you as well off as if we had not done it?

Therefore:

	Contemplated Gain	Contemplated Loss
if we didn't do this : EV	"be paid" : WTA	"pay" : WTP
if we did this : CV	"pay" : WTP	"be paid" : WTA

Furthermore, equation (10.2) on p. 167 of Varian gives $EV = \int_{p'}^{p^0} h(p, u')\,dp$ and $CV = \int_{p'}^{p^0} h(p, u^0)\,dp$. It follows that WTP and WTA are integrals of Hicksian demand curves. WTP and WTA would be areas under Hickian demand curves if those curves were graphed with p on the horizontal axis, but with p graphed on the vertical axis as is traditional, WTP and WTA would be areas to the left of a Hicksian demand curve.

a) The problem gives the expenditure function as

$$e(\underset{\sim}{p}, u) = (p_1^r + p_2^r)^{1/r} u$$

where $r = \frac{\ell}{\ell - 1}$ and $u = (x_1^\ell + x_2^\ell)^{1/\ell}$. In our problem, $u(x, y) = (x^{1/2} + y^{1/2})^2$, so our x and y correspond to Varian's x_1 and x_2, our $\frac{1}{2}$ to Varian's ℓ, and $r = \frac{\ell}{\ell - 1} = \frac{1/2}{1/2 - 1} = \frac{1/2}{-1/2} = -1$. So

$$e(\underset{\sim}{p}, u) = (p_x^{-1} + p_y^{-1})^{-1} u$$

and the Hicksian demand curve

$$h_i(\underset{\sim}{p}, u) = \frac{\partial e(\underset{\sim}{p}, u)}{\partial p_i}$$

$$h_x(\underset{\sim}{p}, u) = \frac{\partial e}{\partial p_x} = -(p_x^{-1} + p_y^{-1})^{-2} \cdot (-p_x^{-2}) u$$

$$= \frac{u}{(p_x^{-1} + p_y^{-1})^2 \, p_x^2} = \frac{u}{[(p_x^{-1} + p_y^{-1}) p_x]^2}$$

$$= \frac{u}{(1 + p_x/p_y)^2} = \frac{u}{(1 + p_x)^2} \quad \text{since } p_y = 1.$$

To find u, note that the curve related to WTA goes through the point ($p_x = 3$, $x = 25/3$). So at this point the consumer spent $3 \cdot 25/3 = \$25$ on x; since $m = \$100$, the consumer must spend $\$100 - \$25 = \$75$ on y, which, at a price of \$1, means 75 units of y. So

$$u = (x^{1/2} + y^{1/2})^2 = ((25/3)^{1/2} + (75)^{1/2})^2 = \left(\frac{5}{\sqrt{3}} + 5\sqrt{3}\right)^2$$

$$= \frac{25}{3} + 2 \cdot \frac{5}{\sqrt{3}} \cdot 5\sqrt{3} + 25 \cdot 3 = \frac{25}{3} + 50 + 75$$

$$= 8\tfrac{1}{3} + 125 = 133\tfrac{1}{3}.$$

Hence the relevant Hicksian demand curve through $(p_x = 3,\ x = 25/3)$ is

$$h_x = \frac{133\frac{1}{3}}{(1+p_x)^2}. \qquad (\text{Note. } p_x = 3 \Rightarrow h_x = 25/3.)$$

b) As in (a), $h_x = \frac{u}{(1+p_x)^2}$, so we only need to find u. This curve corresponds with an initial situation of $p_x = \infty$, so $x = 0$, so the consumer spends all his income (of \$100) on y, whose price is $p_y = 1$; thus the consumer buys $y = 100$. Utility then is $(x^{1/2} + y^{1/2})^2 = (0^{1/2} + 100^{1/2})^2 = 100$ and the relevant Hicksian demand curve is

$$h_x = \frac{100}{(1+p_x)^2}.$$

c) Consumer suplus is the area to the left of the Marshallian demand curve, which is given by the question as $x_1 = \frac{m p_1^{r-1}}{p_1^r + p_2^r}$, so in our problem it is

$$x = \frac{m\, p_x^{-1-1}}{p_x^{-1} + p_y^{-1}} = \frac{100 \cdot p_x^{-2}}{p_x^{-1} + 1} = \frac{100}{(p_x^{-1}+1)\, p_x^2} = \frac{100}{p_x + p_x^2}.$$

(Note: $p_x = 3 \Rightarrow x = 25/3$.)

d) In (i) the consumer has \$29 of consumer surplus from food x. In (ii) the consumer has no consumer surplus from x because it all goes to the price-discriminating seller. However the consumer's income is increased enough (\$34 > WTP or WTA or CS) to make the consumer a little bit better off in (ii); this must mean his consumption of y goes up a bit, since his consumption of x is still 25/3. However, y does not increase much (\$34 is not much more than WTP or WTA or CS), so any small increase in consumer surplus from y will be overshadowed by the \$29 loss in consumer surplus from x. So the consumer surplus is larger in (i).

e) In (ii), the consumer is over-compensated for having to face price discrimination, so if x is still 25/3, purchases of y will go up; so utility is higher in (ii).

Completely Optional Remarks

The question never asks you to confirm that CS actually is approximately 29, or that WTP actually is 25, or that WTA actually is $33\frac{1}{3}$. There are two ways to do that.

Method 1. Use the results of the previous problem (not this problem). In that problem, with the same basic setup we found that with $p_x = 3$:

	so if $m = 100 \leftarrow$ as in this problem
expenditure $= m/4$	25
$CS = m \ln \frac{4}{3}$	$100 \ln \frac{4}{3} \approx 29$
$WTP = m/4$	25
$\lvert WTA \rvert = m/3$	$33\frac{1}{3}$

Method 2.

CS : the Marshallian demand curve is, from part (c), $\frac{100}{p_x + p_x^2}$. So $CS = \int_3^\infty \frac{100}{p_x + p_x^2} dp_x$. This is solved in part (b) of the previous question (not this question).

WTP, WTA : The area "under" a Hicksian demand curve (to its left if price is on the vertical axis) is, using parts (a) and (b),

$$\int_3^\infty \frac{\bar{u}}{(1+p_x)^2} dp_x = \frac{-\bar{u}}{1+p_x}\Big|_3^\infty = 0 + \frac{\bar{u}}{1+3} = \frac{\bar{u}}{4}.$$

In part (a), $\bar{u} = 133\frac{1}{3}$, so this is $133\frac{1}{3}/4 = 33\frac{1}{3}$, the WTA.

In part (b), $\bar{u} = 100$, so this is $100/4 = 25$, the WTP.

Expenditure: As with C.S., the Marshallian demand curve is $\frac{100}{p_x + p_x^2}$. At $p_x = 3$ this is $\frac{100}{3+9} = \frac{100}{12} = \frac{25}{3} = x$; then expenditure $x p_x = \frac{25}{3} \cdot 3 = 25$.

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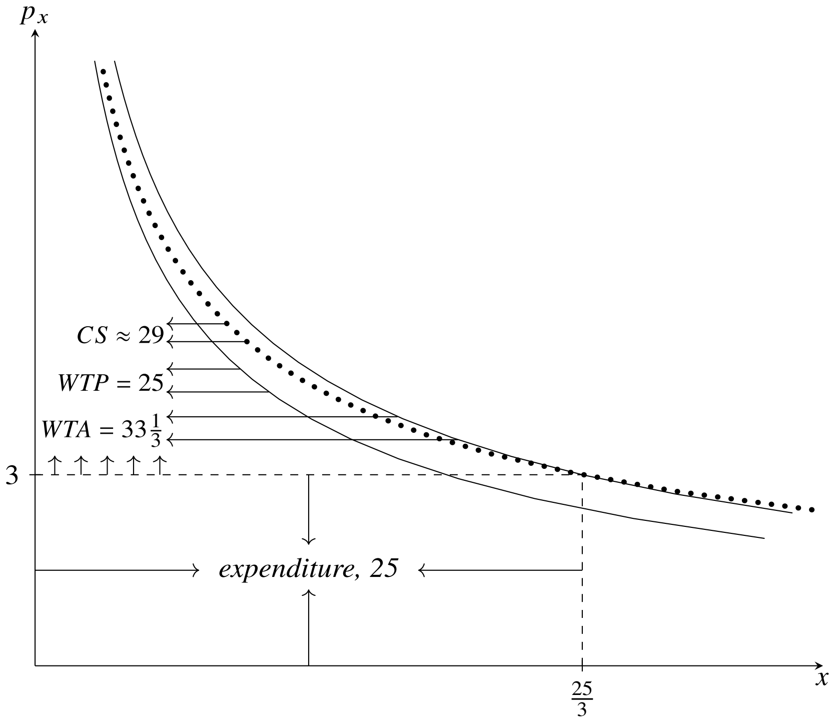


Figure 6. Demand curves for an income of $m = 100$ and a utility function of $u(x, y) = (x^{1/2} + y^{1/2})^2$ when $p_y = 1$. **Dotted curve**: the Marshallian demand curve, $x = p_x^{-2}/(p_x^{-1} + 1) = 1/(p_x + p_x^2)$. Consumer surplus when $p_x = 3$ is the area left of this curve and above the line $p_x = 3$; it is $m \ln(4/3) \approx \$29$. **Solid curves**: Hicksian demand curves, $u/(1 + p_x)^2$. **Right-most solid curve**: Hicksian demand curve with utility fixed at its level when $p_x = 3$, therefore $u(25/3, 75) = ((25/3)^{1/2} + 75^{1/2})^2 = 133\frac{1}{3}$. WTA when $p_x = 3$ is the area left of this curve and above the line $p_x = 3$; it is $m/3 = \$33\frac{1}{3}$. **Left-most solid curve**: Hicksian demand curve with utility fixed at its level when $p_x = \infty$, therefore $u(0, 100) = (0^{1/2} + 100^{1/2})^2 = 100$. WTP when $p_x = 3$ is the area left of this curve and above the line $p_x = 3$; it is $m/4 = \$25$. When $p_x = 3$, **expenditure** is $3 \cdot 25/3 = \$25$. The **value** of "$p_x = 3$" or "$x = 25/3$" is the \$25 expenditure plus the measure of the surplus (either WTA or WTP or CS).

2019 Qualifying Exam Sec. 3 Qu. 1

1. **[16 points]**

   [Completely optional introduction: This is the beginning and the middle but not the end of a demonstration that George Stigler's "Coase Theorem" is false in the general case of goods having arbitrary income effects. (This would please Nobel Laureate Ronald Coase but it profoundly challenges followers of Stigler.)]

   (a) Suppose a consumer's welfare depends on the number of apples a which he consumes and on the amount of clean air in his environment. There is a polluting firm in the consumer's environment and the amount of air pollution it emits is proportional to the level of its output Q. For some fixed level of output $\overline{Q} > 0$, argue that

   $$u(a, Q) = a \cdot (\overline{Q} - Q)$$

   is a reasonable specification for this consumer's utility function.

   (b) Suppose this consumer sets out one day with m dollars to visit the marketplace and buy some apples. Before he gets to the marketplace, he encounters the owner of the polluting firm. He may strike up a conversation with this owner in the hopes of affecting how much the firm pollutes. Perhaps he and the firm owner exchange money for a change in Q. Let m_a denote the amount of money the consumer has when he takes leave of the firm owner and proceeds to the marketplace, at which time the amount of Q, and therefore air pollution, is irrevocably fixed (it will never change again).

   Show that his utility at this point is destined to be

   $$\frac{m_a}{p_a} (\overline{Q} - Q)$$

   where p_a is the price of apples.

   (c) Suppose that in this country, firms have the right to emit pollution at will. (One could say that the firm has the "property right" to pollute.) Suppose that in the absence of any interaction or bargaining between the firm and the consumer,

   the firm sees fit to produce $\overline{Q}/2$ units of output.

   Show that with that level of output, the (indirect) utility of the customer in this initial situation would be

   $$v_0 = \frac{m\overline{Q}}{2p_a} .$$

(d) Upon meeting the firm owner, the consumer contemplates offering the firm owner money in return for a reduction of Q. If the consumer offered the firm owner T dollars and in return the firm owner reduced output to Q, show that the consumer would, after making the bargain and then buying apples, have a utility level of

$$v' = \frac{m - T}{p_a} (\overline{Q} - Q) .$$

(e) Suppose that, for a given Q, the consumer is indifferent between paying $T(Q)$ in return for the firm producing only Q, on the one hand, and paying nothing and having the firm produce $\overline{Q}/2$, on the other hand. Find T as a function of Q.
Hint: I get

$$T = m \frac{\overline{Q} - 2Q}{2\overline{Q} - 2Q} > 0 \quad \text{for } Q < \overline{Q}/2. .$$

(f) Show that

$$\frac{dT}{dQ} = \frac{-m\overline{Q}}{2\,(Q - \overline{Q})^2} < 0 \quad \text{for } Q < \overline{Q}/2\text{, and that}$$
$$\frac{d^2T}{dQ^2} = \frac{m\overline{Q}}{(Q - \overline{Q})^3} < 0 \quad \text{for } Q < \overline{Q}/2.$$

(g) Make a rough sketch of $T(Q)$, indicating the values of $T(0)$ and of $T(\overline{Q}/2)$.

(h) If *EC* denotes the “external cost” which pollution imposes on this consumer, argue that

$$EC(Q) = T(0) - T(Q) .$$

(i) Show that the “marginal external cost”

$$MEC = \frac{d\,EC}{dQ} = \frac{m\overline{Q}}{2\,(Q - \overline{Q})^2} > 0 .$$

(Prove the second equality.) Also show that

$$\frac{d\,MEC}{dQ} = \frac{-m\overline{Q}}{(Q - \overline{Q})^3} = \frac{m\overline{Q}}{(\overline{Q} - Q)^3} > 0 .$$

(Prove at least one of the equalities and prove the inequality.)

(j) Now we contrast this situation to that under a different constitution in which consumers have the right to clean air and firms cannot pollute the air without obtaining permission from the consumer. (One could say that consumers have the "property right" to clean air.) Show that in the absence of any interaction or bargaining between the firm and the consumer, (indirect) utility of the customer in this initial situation would be

$$v_0 = \frac{m\overline{Q}}{p_a}\,.$$

(k) Upon meeting the firm owner, the consumer contemplates offering to allow the firm to increase output to Q in return for the firm paying the consumer $\widehat{T}$ dollars. Show that the consumer would, after making the bargain and then buying apples, have a utility level of

$$v' = \frac{m + \widehat{T}}{p_a}\,(\overline{Q} - Q)\,.$$

(l) Suppose that, for a given Q, the consumer is indifferent between receiving $\widehat{T}(Q)$ in return for allowing the firm to increase its production to Q, on the one hand, and receiving nothing and making no bargain with the firm, on the other hand. Show that

$$\widehat{T} = \frac{mQ}{\overline{Q} - Q} > 0$$

and show that

$$\begin{aligned} \frac{d\widehat{T}}{dQ} &= \frac{m\overline{Q}}{(\overline{Q} - Q)^2} > 0 \quad \text{and that} \\ \frac{d^2\widehat{T}}{dQ^2} &= \frac{2m\overline{Q}}{(\overline{Q} - Q)^3} > 0\,. \end{aligned}$$

(m) In this situation argue that external cost $\widehat{EC}(Q) = \widehat{T}(Q)$.

(n) Show that

$$\widehat{MEC} = \frac{d\,\widehat{MEC}}{dQ} = 2\,MEC \quad \text{and} \quad \frac{d\,\widehat{MEC}}{dQ} = 2\,\frac{d\,MEC}{dQ}\,.$$

Answer to 2019 Micro Qualifying Exam, Section 3 Qu. 1

a) If $u = a(\bar{Q} - Q)$ then as $Q\uparrow$, the air becomes dirtier and utility falls. Formally, $\frac{\partial u}{\partial Q} = \frac{\partial}{\partial Q}\left[a\bar{Q} - aQ\right] = -a < 0$. Optional: $\partial u / \partial a$ is also positive; the formulation is just Cobb-Douglas utility in apples and clean air.

b) By the time has come to purchase apples, Q is exogenously fixed. Let m_a be the amount of money the consumer has left at that time. He will spend all of m_a buying apples, so $a^* = \frac{m_a}{p_a}$ and his utility will be $\frac{m_a}{p_a}(\bar{Q} - Q)$.

c) In this situation, $Q_0 = \frac{1}{2}\bar{Q}$ is given. If the consumer does not bargain with the polluter, his utility will be

$$v_0 = \frac{m}{p_a}\left(\bar{Q} - \tfrac{1}{2}\bar{Q}\right) \quad \text{using the notation for indirect utility;}$$

$$= \frac{m\bar{Q}}{2p_a}.$$

d) If the consumer does bargain with the polluter, suppose he pays the polluter T dollars ("T" for "transfer"), and in return the polluter reduces output to Q. Then

$$v' = \frac{m - T}{p_a}(\bar{Q} - Q) \quad \text{since } m - T = m_a.$$

e) The maximum T which the consumer would be willing to pay for a given Q would satisfy the property that $v' = v_0$, because if v' were any lower than v_0, the consumer would prefer to stay at v_0. [In game theory, this is called the "participation constraint" or the "individual rationality constraint."]

So $v_0 = v' \Rightarrow \frac{m\bar{Q}}{2p_a} = \frac{m-T}{p_a}(\bar{Q}-Q)$ and solving for T as a function of Q,

$$\frac{m}{2} = (m-T)\left(1-\frac{Q}{\bar{Q}}\right)$$

$$\frac{m/2}{1-\frac{Q}{\bar{Q}}} = m-T$$

$$T = m - \frac{m/2}{1-\frac{Q}{\bar{Q}}} = m - \frac{\frac{m}{2}}{1-\frac{Q}{\bar{Q}}}\cdot\frac{2\bar{Q}}{2\bar{Q}}$$

$$= m - \frac{m\bar{Q}}{2\bar{Q}-2Q} = m\left(1-\frac{\bar{Q}}{2\bar{Q}-2Q}\right) = m\,\frac{2\bar{Q}-2Q-\bar{Q}}{2\bar{Q}-2Q}$$

$$= m\,\frac{\bar{Q}-2Q}{2\bar{Q}-2Q} = m\,\frac{2Q-\bar{Q}}{2Q-2\bar{Q}}.$$

This is valid for reductions in output below its initial level of $\frac{1}{2}\bar{Q}$, that is, for $Q \leq \frac{1}{2}\bar{Q}$. Note that this makes both T's numerator and its denominator have identical signs, so $T > 0$.

f) For ever-larger decreases in Q, we expect the consumer to be willing to spend more T, so we expect $dT/dQ < 0$. ← optional

Not optional

$$\frac{dT}{dQ} = m\left[\frac{2}{2Q-2\bar{Q}} - \frac{2}{(2Q-2\bar{Q})^2}(2Q-\bar{Q})\right]$$

$$= m\left[\frac{1}{Q-\bar{Q}} - \frac{2(2Q-\bar{Q})}{2(Q-\bar{Q})\cdot 2(Q-\bar{Q})}\right] = m\left[\frac{1}{Q-\bar{Q}} - \frac{2Q-\bar{Q}}{2(Q-\bar{Q})^2}\right]$$

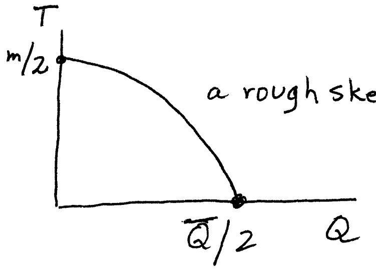
$$= m\,\frac{2(Q-\bar{Q})-2Q+\bar{Q}}{2(Q-\bar{Q})^2} = m\,\frac{2Q-2\bar{Q}-2Q+\bar{Q}}{2(Q-\bar{Q})^2} = \frac{-m\bar{Q}}{2(Q-\bar{Q})^2} < 0.$$

Also

$$\frac{d^2T}{dQ^2} = \frac{-m\bar{Q}}{2}\frac{d}{dQ}(Q-\bar{Q})^{-2} = \frac{-m\bar{Q}}{2}(-2)(Q-\bar{Q})^{-3}(1)$$

$$= \frac{m\bar{Q}}{(Q-\bar{Q})^3}$$ which is negative because $Q \leq \frac{1}{2}\bar{Q} < \bar{Q}$.

g)



h) It is worthwhile for the consumer to spend T to reduce output to Q because output imposes "external costs" "EC" on the consumer. We have

$$EC(Q) = T(0) - T(Q) \qquad (1)$$

zero pollution imposes no EC

because when $Q = 0$ we should have $EC(0) = 0$, which (1) satisfies; and when $Q = \bar{Q}/2$ we should have $EC(\bar{Q}/2) = T(0)$, the consumer's entire willingness-to-pay to reduce output from $\bar{Q}/2$ to zero, and (1) gives this also because it gives $EC(\bar{Q}/2) = T(0) - T(\bar{Q}/2) = T(0) - 0 = T(0)$.

i) Then marginal external cost

$$MEC = \frac{dEC}{dQ} \overset{\text{from (1)}}{=} -\frac{dT}{dQ} = \frac{m\bar{Q}}{2(Q-\bar{Q})^2} > 0 \text{ and}$$

$$\frac{d\,MEC}{dQ} = -\frac{d^2T}{dQ^2} = \frac{-m\bar{Q}}{(Q-\bar{Q})^3} > 0 \text{ because } Q \leq \frac{1}{2}\bar{Q} < \bar{Q}.$$

It may be easier to express this as

$$\frac{d\ MEC}{dQ} = \frac{-m\bar{Q}}{[(-1)(\bar{Q}-Q)]^3} = \frac{-m\bar{Q}}{(-1)^3(\bar{Q}-Q)^3} = \frac{m\bar{Q}}{(\bar{Q}-Q)^3} > 0.$$

$\uparrow$

$m>0$, $\bar{Q}>0$, and

$\bar{Q}-Q>0$ since $Q<\bar{Q}/2$.

j) In this situation, $Q_0 = 0$ is given. If the consumer does not bargain with the polluter, the consumer's utility will be

$$v_0 = \frac{m}{P_a}(\bar{Q}-0) = \frac{m}{P_a}\bar{Q}.$$

k) If the consumer does bargain with the polluter, suppose that in return for allowing the polluter to increase output to Q, the consumer accepts a transfer of $\hat{T}$. Then

$$v' = \frac{m+\hat{T}}{P_a}(\bar{Q}-Q).$$

l) The minimum $\hat{T}$ which the consumer would be willing to accept for a given Q would satisfy the property that $v' = v_0$, because if v' were any lower than v_0, the consumer would prefer to stay at v_0. So

$$v_0 = v' \Rightarrow \frac{m\bar{Q}}{P_a} = \frac{m+\hat{T}}{P_a}(\bar{Q}-Q)$$ and solving for $\hat{T}$ as a function of Q,

$$m\bar{Q} = (m+\hat{T})(\bar{Q}-Q)$$

$$\frac{m\bar{Q}}{\bar{Q}-Q} = m+\hat{T}$$

$$\hat{T} = \frac{m\bar{Q}}{\bar{Q}-Q} - m = m\left[\frac{\bar{Q}}{\bar{Q}-Q} - \frac{\bar{Q}-Q}{\bar{Q}-Q}\right] = \frac{mQ}{\bar{Q}-Q}$$

which is positive in the relevant range of $0 \le Q \le \bar{Q}/2$. If the polluter wants Q to be larger, the consumer will demand that $\hat{T}$ become larger, so we expect that $d\hat{T}/dQ > 0$.

$$\frac{d\hat{T}}{dQ} = \frac{m}{\bar{Q}-Q} - \frac{(-1)}{(\bar{Q}-Q)^2}\, mQ = m\left[\frac{1}{\bar{Q}-Q} + \frac{Q}{(\bar{Q}-Q)^2}\right]$$

$$= m\,\frac{\bar{Q}-Q+Q}{(\bar{Q}-Q)^2} = \frac{m\bar{Q}}{(\bar{Q}-Q)^2} > 0.$$

Also $$\frac{d^2\hat{T}}{dQ^2} = \frac{-2m\bar{Q}}{(\bar{Q}-Q)^3}(-1) = \frac{2m\bar{Q}}{(\bar{Q}-Q)^3} > 0.$$

m) In this situation, $\widehat{EC}(Q) = \hat{T}(Q)$ because the reason the consumer is willing to accept $\underbrace{\hat{T}}_{\text{and not less}}$ is because Q causes $\widehat{EC}(Q)$ in damage.

n) Because of part (m), $\widehat{MEC} = d\hat{T}/dQ$.

In summary,

from part (i)

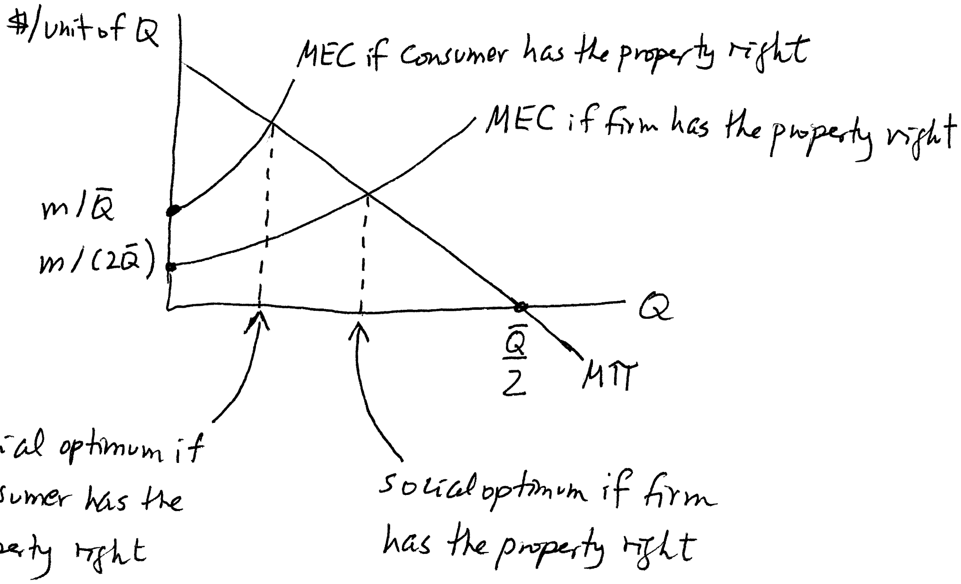
and from part (l)

$$MEC = \frac{m\bar{Q}}{2(Q-\bar{Q})^2} \qquad \widehat{MEC} = \frac{m\bar{Q}}{(\bar{Q}-Q)^2} = 2\,MEC$$

$$\frac{d\,MEC}{dQ} = \frac{m\bar{Q}}{(\bar{Q}-Q)^3} \qquad \frac{d\,\widehat{MEC}}{dQ} = \frac{2m\bar{Q}}{(\bar{Q}-Q)^3} = 2\,\frac{d\,MEC}{dQ}.$$

This Page Is Completely Optional!

A rough sketch would be the following, noting that the polluter's marginal profit $M\Pi$ hits zero at $Q = \bar{Q}/2$ because when the polluter has the property right, $Q = \bar{Q}/2$ is in absence of bargaining.



So the assignment of the property right affects the location of the social optimum, which contradicts the so-called "Coase Theorem," but is after all not suprising if one thinks of, in an Edgeworth Box, the effect of the endowment point on the allocations in the core.

The only case in which the "Coase Theorem" would be true is when, using the language of this example, the utility function is quasilinear, so that the demand for clean air is completely unaffected by one's income. There are very few commodities consumed in equal amounts by the rich and the poor!

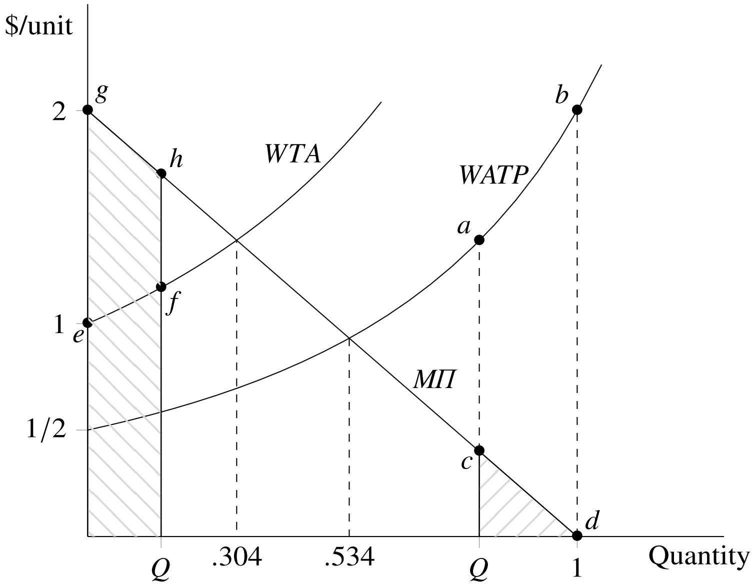


Figure 1. Willingness (and ability) to Pay, "*WATP*," is $m\overline{Q}/[2\cdot(Q-\overline{Q})^2]$ from (i) of the 2019 exam, and $2/(Q-2)^2$ substituting in the parameters here. Willingness to Accept, "*WTA*," is $m\overline{Q}/(\overline{Q}-Q)^2)$ from (l) (that's the letter 'l' not the number '1') of the 2019 exam, and $4/(2-Q)^2$ substituting in the parameters here. Marginal Profit, "*MΠ*," is assumed here to be $2-2Q$.

2021 Qualifying Exam Sec. 3 Qu. 2

2. **[10 points]**

   [Completely optional introduction: This is the end of a demonstration that George Stigler's "Coase Theorem" is false in the general case of goods having arbitrary income effects. (This would please Nobel Laureate Ronald Coase but it profoundly challenges followers of Stigler.)]

   You have been given the question and answer to the 2019 Qualifying Exam's Section 3 Question 1. Adopt all the notation and situations described in that problem. In addition, in that problem, set $\overline{Q}=2$, $m=2$, and suppose the marginal profit of the firm is given by $M\Pi = 2-2Q$. The figure on the last, "optional" page of the answer to the 2019 question then becomes Figure 1.

   (see the previous question)

   (a) Suppose, as in part (c) of the 2019 question, firms have the right to emit pollution at will. The maximum amount of money the consumer is willing and able to pay for the output level "Q" lying directly below points c and a is the area under the WATP curve, that is, under ab. However, in bargaining with the firm, the consumer might not have to pay this maximum amount of money. Explain briefly why the *minimum* amount of money the consumer would have to pay for the output to be reduced to that level of Q is the area under the "cd" segment of the $M\Pi$ curve.

(b) Show that the area under the cd segment of the $M\Pi$ curve is $1 - 2Q + Q^2$.

(c) From (c) of the 2019 exam, $v_0 = m\overline{Q}/(2p_a)$, which under our assumptions is equal to $2/p_a$. From (d) of the 2019 exam, and with our assumptions,

$$v' = \frac{m - T}{p_a}\left(\overline{Q} - Q\right) = \frac{2 - T}{p_a}(2 - Q)\,.$$

If T corresponds not to WATP but to the minimum consumer payment, what value of Q makes v_0 equal to v'? It is sufficient to find an equation that defines Q implicitly; you do not have to find Q explicitly.

(d) Suppose, as in part (j) of the 2019 question, consumers have the right to clean air and firms cannot pollute the air without obtaining permission from the consumer. The minimum amount of money the consumer is willing to accept for the output level "Q" lying directly below points h and f is the area under the WTA curve, that is, under ef. However, in bargaining with the firm, the consumer might not have to accept this minimum amount of money. The *maximum* amount of money firm would pay to the consumer for the output to be increased to that level of Q is the area under the "gh" segment of the $M\Pi$ curve. Show that that area under the gh segment of the $M\Pi$ curve is $2Q - Q^2$.

(e) From (j) of the 2019 exam, $v_0 = m\overline{Q}/p_a$, which under our assumptions is equal to $4/p_a$. From (k) of the 2019 exam, and with our assumptions,

$$v' = \frac{m + \widehat{T}}{p_a}\left(\overline{Q} - Q\right) = \frac{2 + \widehat{T}}{p_a}(2 - Q)\,.$$

If $\widehat{T}$ corresponds not to WTA but to the maximum firm payment, what value of Q makes v_0 equal to v'? It is sufficient to find an equation that defines Q implicitly; you do not have to find Q explicitly.

Answer to Summer 2021 Qualifying Exam, Section 3 Question 2

(a) The area under *cd* is the profit earned when the firm increases output from Q to 1. This is simply because total profit is the area under the marginal profit curve: $\int_Q^1 M\Pi \, dQ = \int_0^1 (d\pi/dQ)\, dQ = \pi(1) - \pi(Q)$ by the Fundamental Theorem of Calculus. The firm will refuse to reduce output to this Q unless it receives payment equal to this lost profit.

(b)

$$\begin{aligned}\int_Q^1 (2 - 2\hat{Q})\, d\hat{Q} &= [2\hat{Q} - \hat{Q}^2]\Big|_Q^1 \\ &= (2-1) - (2Q - Q^2) = 1 - 2Q + Q^2\,.\end{aligned}$$

(This happens to be a perfect square, $(1-Q)^2$.)

(c) Substitute the answer to part (b) (which you know, even if you were not able to solve part (b), because its answer was given in the exam) for T in the equation given in this part of the question for v', then set $v' = v_0$ and recall that $\overline{Q} = 2$ and that $m = 2$:

$$\begin{aligned}\frac{2 - (1 - 2Q + Q^2)}{p_a}(2 - Q) &= \frac{2\overline{Q}}{2p_a} \\ (1 + 2Q - Q^2)(2 - Q) &= \overline{Q} \qquad (9) \\ 2 + 4Q - 2Q^2 - Q - 2Q^2 + Q^3 &= 2 \\ 3Q - 4Q^2 + Q^3 &= 0\,.\end{aligned}$$

Therefore one solution is $Q = 0$. Other solutions are:

$$\begin{aligned}Q^2 - 4Q + 3 &= 0 \\ (Q-3)(Q-1) &= 0\,.\end{aligned}$$

The $Q = 3$ solution makes no sense in this context. The $Q = 1$ solution would mean that *less* pollution reduction would be achieved when pollution victims had to pay little money to polluters than when they had to pay more money ($Q = 0.534$), which also makes no sense. Therefore, the correct answer is $Q = 0$. (The question did not ask you to find such an explicit value for Q, only an implicit definition of it, so any of the displayed equations would be an adequate answer.)

Optional: If you have a computer, you can graph v_0 or, more straightforwardly from (9), $p_a v_0 = \overline{Q} = 2$, and also from (9), $p_a v' = (1 + 2Q - Q^2)(2 - Q)$. This looks like Figure 2: starting from $Q = 1$ (which is

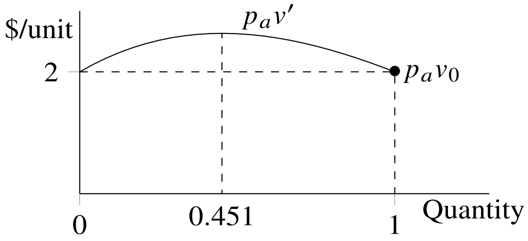


Figure 2. Initial (v_0) and final (v') utility of the pollution victim (times the price of apples) if the firm has the property right to pollute and Coasian bargains are made at the minimum amount of money needed to induce the firm to lower production. If a quantity level has $p_a v' > p_a v_0$, the consumer would strictly prefer being at that quantity level rather than remaining at $Q = 1$. By construction, the firm is indifferent between $Q \in [0, 1]$.

where $M\Pi = 0$), the pollution victim gains from decreasing Q and making the minimal payment to the polluter whenever $p_a v' > p_a v_0$.

Optional: Figure 2 implies that any $Q \in [0, 1)$ would be accepted by both parties as an alternative to $Q = 1$. Bargaining starting from $Q = 1$ and incrementally going left would stop at the maximum of v', which is at $(4 - \sqrt{7})/3 \approx 0.451$.

(d) $\int_0^Q (2 - 2\hat{Q})\, d\hat{Q} = [2\hat{Q} - \hat{Q}^2] \big|_0^Q = (2Q - Q^2) - (0 - 0) = 2Q - Q^2$.

(e) Substitute the answer to part (d) (which you know, even if you were not able to solve part (d), because its answer was given in the exam) for $\widehat{T}$ in the equation given in this part of the question for v', then set $v' = v_0$ and recall that $\overline{Q} = 2$ and that $m = 2$:

$$\begin{aligned}\frac{2 + \widehat{T}}{p_a}(2 - Q) &= \frac{4}{p_a}\\ [2 + (2Q - Q^2)](2 - Q) &= 4\\ (4 + 4Q - 2Q^2) - (2Q + 2Q^2 - Q^3) &= 4\\ Q^3 - 4Q^2 + 2Q + 4 &= 4 \qquad (10)\\ Q^3 - 4Q^2 + 2Q &= 0\,.\end{aligned}$$

Therefore one solution is $Q = 0$. Other solutions solve $Q^2 - 4Q + 2 = 0$ so

$$Q = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

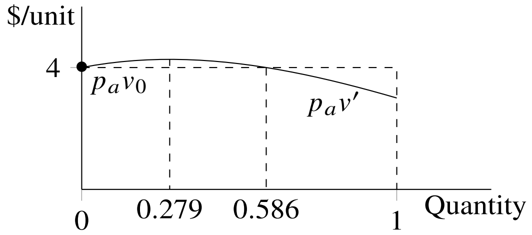


Figure 3. Initial (v_0) and final (v') utility of the pollution victim (times the price of apples) if the pollution victim has the property right to clean air and Coasian bargains are made at the maximum amount of money the firms are willing to pay to increase production. If a quantity level has $p_a v' > p_a v_0$, the consumer would strictly prefer being at that quantity level rather than remaining at $Q = 0$. By construction, the firm is indifferent between $Q \in [0, 1]$.

of which only $2 - \sqrt{2} \approx 0.586$ makes sense in this context. (The question did not ask you to find such an explicit value for Q, only an implicit definition of it, so any of the displayed equations would be an adequate answer.)

Optional: If you have a computer, you can graph v_0 or, more straightforwardly from (10), $p_a v_0 = 4$, and also from (10), $p_a v' = Q^3 - 4Q^2 + 2Q + 4$. This looks like Figure 3: starting from $Q = 0$, the pollution victim gains from increasing Q and receiving the maximal payment from the polluter whenever $p_a v' > p_a v_0$.

Optional: Figure 3 implies that any $Q \in (0, 0.586]$ would be accepted by both parties as an alternative to $Q = 0$. Bargaining starting from $Q = 0$ and incrementally going right would stop at the maximum of v', which is at $(4 - \sqrt{10})/3 \approx 0.279$.

Optional: If the polluter has the property rights, we predict the outcome of Coasian bargaining will be between 0.534 (when pollution victims have to make maximal transfers to the firms, the area under *WATP*) and 0.451 (when pollution victims only have to make minimal transfers to the firms, the area under *MΠ*). If the pollution victim has the property rights, we predict the outcome of Coasian bargaining will be between 0.279 (when firms have to make maximal transfers to the pollution victims, the area under *MΠ*) and 0.304 (when firms only have to make minimal transfers to the firms, the area under *WTA*).

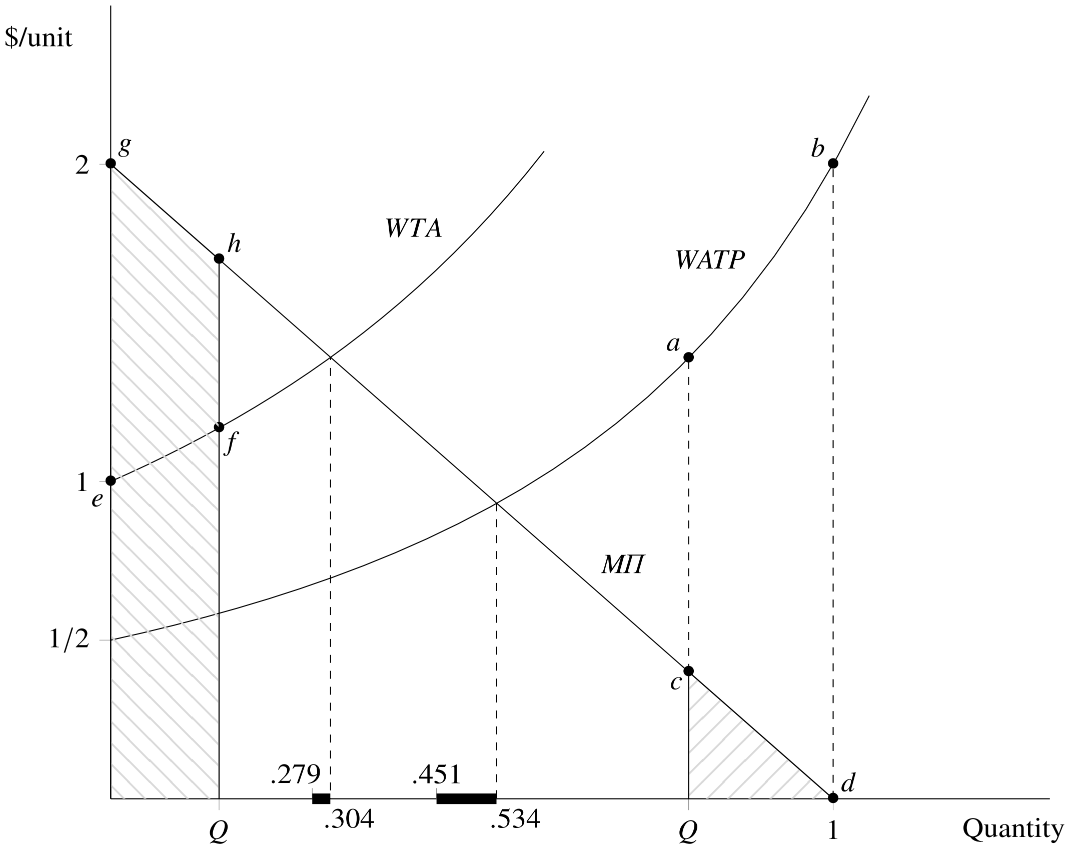


Figure 4. The dark intervals on the Q axis represent, respectively, the possible outcomes of Coasian bargaining when pollution victims have the property right to clean air (left) or when polluters have the property right to pollute (right).