Section 1:

More than One Topic

2010 Qualifying Exam

Section 1.
Answer all of the following three questions.

1. **[11 points]** Tell me everything you know about:

   (a) quasiconcavity; and

   (b) why it is important in consumer theory (you do not have to say anything about producer theory).

   The essay you write should be understandable to an undergraduate student who has had no mathematics beyond the first year of calculus and one semester of linear algebra; so you will certainly have to define any terms you use which I taught this year.

   You do not have to discuss quasiconvexity, nor dual functions (so there are two reasons why you do not have to discuss the result that "the indirect utility function $v(\mathbf{p}, m)$ is quasiconvex in $\mathbf{p}$").

Answer to Summer 2010

Qualifying Exam, Section 1, Question 1

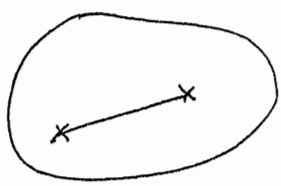
① A quasiconcave function has $\underset{(a)}{\underline{\text{convex}}}$ $\underset{(b)}{\underline{\text{upper level sets}}}$.

(a) A <u>convex set</u> is one for which a line drawn between any two members of the set is itself in the set.

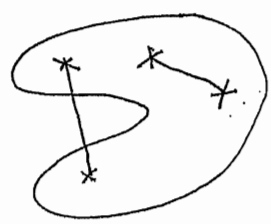
i.e.,

$$\left.\begin{array}{l} \underset{\sim}{x}_1 \in A \\ \underset{\sim}{x}_2 \in A \end{array}\right\} \Rightarrow \lambda \underset{\sim}{x}_1 + (1-\lambda)\underset{\sim}{x}_2 \in A \quad \text{for all } \lambda \in [0,1]$$

and for all $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ pairs in A.



Convex



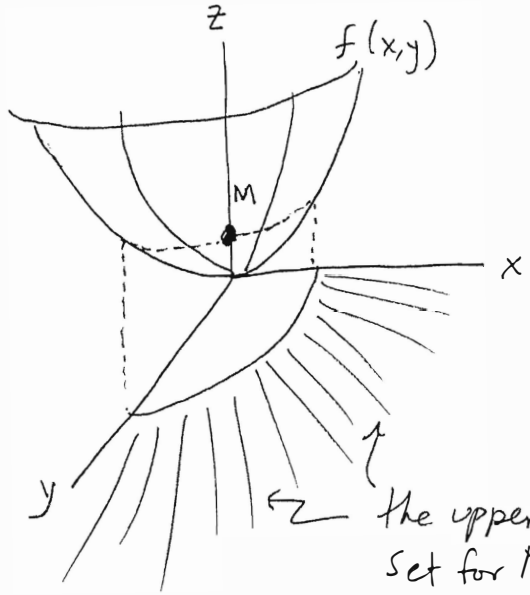
not convex

(I'm assuming that the quasiconcave function maps $\mathbb{R}^n$ into $\mathbb{R}^1$, which is the only case we consider this year.)

(b) An <u>upper level set</u> of a function $f(\underset{\sim}{x})$ is the set of all points in f's domain whose image under f is $\geq$ a particular number, say M. i.e., if $f(\underset{\sim}{x}) : \mathbb{R}^n \to \mathbb{R}^1$ then

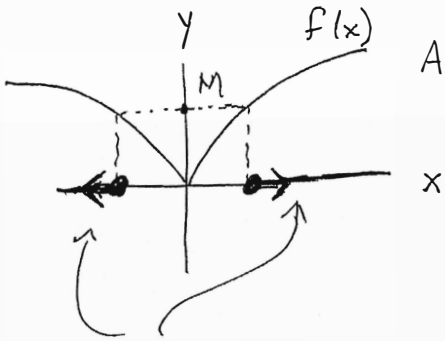
$$\{ \underset{\sim}{x} \in \mathbb{R}^n \mid f(\underset{\sim}{x}) \geq M \}$$

is the upper level set of f for M.



A 3-dimensional example of a function which looks somewhat like a bowl

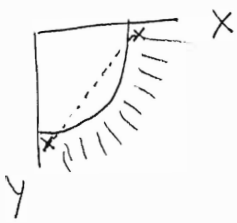
the upper level set for M; all these points in the domain (which is the x-y plane) get mapped by f into a value > M

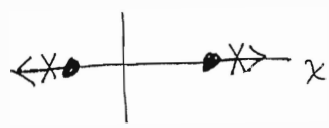


A two-dimensional example (f is a "function of one variable").

The upper level set of f for M; all these points in the domain (which is the x axis) get mapped by f into a value > M.

<u>Neither</u> of the functions I've drawn on this page is quasiconcave because neither of their upper level sets is a convex set.





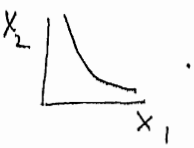
A line between the pair of x's goes outside of the upper level set in both examples.

Consider the "bordered Hessian" of $f(\underset{\sim}{x})$:

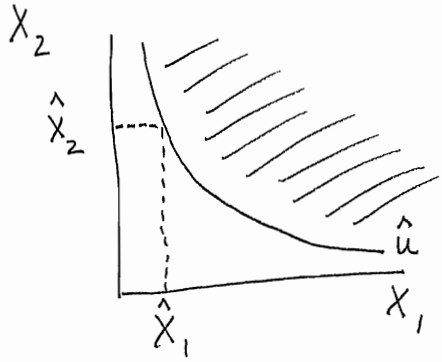
$$\begin{bmatrix} 0 & \nabla f \\ \nabla f^T & \nabla^2 f \end{bmatrix} = \begin{bmatrix} 0 & f_1' & f_2' & f_3' & \cdots & f_n' \\ f_1' & f_{11}'' & f_{12}'' & f_{13}'' & \cdots & f_{1n}'' \\ f_2' & f_{21}'' & f_{22}'' & f_{23}'' & \cdots & f_{2n}'' \\ f_3' & f_{31}'' & f_{32}'' & f_{33}'' & \cdots & f_{3n}'' \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ f_n' & f_{n1}'' & f_{n2}'' & f_{n3}'' & \cdots & f_{nn}'' \end{bmatrix}.$$

Let δ_r be the $(r+1)^{ST}$ leading principal minor of this bordered Hessian. A sufficient condition for f to be quasiconcave is that δ_r alternate in sign beginning with >0 for $r = 2, 3, \ldots, n$.

(Note that $\delta_1 = \begin{vmatrix} 0 & f_1' \\ f_1' & f_{11}'' \end{vmatrix} = -(f_1')^2 < 0$ for any function f so it never needs to be tested.)

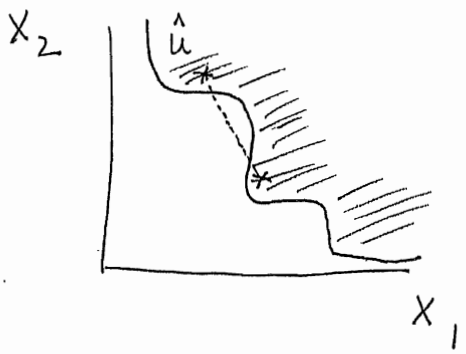
In consumer theory, we would like to characterize utility functions $u(\underset{\sim}{x})$ which give rise to the typical indifference curve shape studied in undergraduate texts: 

. Consider the set of all (x_1, x_2) which the consumer prefers to, say, $(\hat{x}_1, \hat{x}_2)$, and suppose the utility level at $(\hat{x}_1, \hat{x}_2)$ is $\hat{u}$. This set of bundles looks like



In order for the $\hat{u}$ indifference curve to have its "usual" shape, this "better than" set must be convex. This "better than" set is simply an upper level set for u. So, saying that the indifference curves have their usual shape is equivalent to saying that u is quasiconcave. (This ensures a unique utility-maximizing bundle.)

If the indifference curves have a strange shape,



the upper level set is not convex, so u is not quasiconcave.

Optional: Consider $\max_{\tilde{x}} f(\underset{\sim}{x})$ s.t. $\underset{\sim}{m} \cdot \underset{\sim}{x} = c$ (i.e., a linear constraint).

It can be shown that if $f(\underset{\sim}{x})$ is quasiconcave, then any $\underset{\sim}{x}^*$ satisfying this problem's first-order conditions will also satisfy its second-order conditions. An important such problem is maximizing utility of a price-taking consumer.

2017 Final Exam Qu. 4 (Same answer as Fall 2011 Ex. 1 Qu. 2.)

4. **[17 points]**

(a) Suppose a competitive profit-maximizing firm produces an output from two inputs $\mathbf{x} = (x_1, x_2)$ according to the production function $f(\mathbf{x})$. By how much would the firm's purchases of x_1 change if the price it paid for x_1 changed?

Would the firm's purchases of x_1 go up or go down if the price it paid for x_1 rose? Why?

(b) Suppose an expenditure-minimizing (*not* utility-maximizing) consumer obtains utility from two commodities $\mathbf{x} = (x_1, x_2)$ according to the utility function $u(\mathbf{x})$. By how much would the consumer's purchases of x_1 change if the price he paid for x_1 changed?

Would the consumer's purchases of x_1 go up or go down if the price he paid for x_1 rose? Why?

(c) Suppose a utility-maximizing consumer obtains utility from two commodities $\mathbf{x} = (x_1, x_2)$ according to the utility function $u(\mathbf{x})$. By how much would the consumer's purchases of x_1 change if the price he paid for x_1 changed?

Would the consumer's purchases of x_1 go up or go down if the price he paid for x_1 rose? Why?

(d) Explain in one sentence, with little or no mathematics, why some results you learned about in lectures make you unsurprised by the answers to parts (a), (b) and (c).

(2) a) $$\max_{x_1, x_2} \underbrace{pf(x_1, x_2) - w_1 x_1 - w_2 x_2}_{\text{profit "}\pi\text{"}}$$

F.O.C.: $$0 = \frac{\partial \pi}{\partial x_1} = p f_1' - w_1$$

$$0 = \frac{\partial \pi}{\partial x_2} = p f_2' - w_2$$

Take differentials of the First Order Conditions:

$$0 = p f_{11}'' dx_1 + p f_{12}'' dx_2 - dw_1$$

$$0 = p f_{21}'' dx_1 + p f_{22}'' dx_2 \qquad - dw_2$$

zero by assumption

$$\underset{\sim}{0} = \begin{bmatrix} p f_{11}'' & p f_{12}'' \\ p f_{21}'' & p f_{22}'' \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} - \begin{bmatrix} dw_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} dw_1 \\ 0 \end{bmatrix} = \begin{bmatrix} p f_{11}'' & p f_{12}'' \\ p f_{21}'' & p f_{22}'' \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p f_{11}'' & p f_{12}'' \\ p f_{21}'' & p f_{22}'' \end{bmatrix} \begin{bmatrix} \partial x_1 / \partial w_1 \\ \partial x_2 / \partial w_1 \end{bmatrix}$$

so by Cramer's Rule,

$$\frac{\partial x_1}{\partial w_1} = \frac{\begin{vmatrix} 1 & p f_{12}'' \\ 0 & p f_{22}'' \end{vmatrix}}{\begin{vmatrix} p f_{11}'' & p f_{12}'' \\ p f_{21}'' & p f_{22}'' \end{vmatrix}} = \frac{p f_{22}''}{p^2 f_{11}'' f_{22}'' - p^2 f_{12}'' f_{21}''} = \frac{1}{p} \frac{f_{22}''}{f_{11}'' f_{22}'' - (f_{12}'')^2}$$

The second-order sufficient conditions for a maximum:

D_{2m+1} of $\nabla^2\mathcal{L}$ has the sign of $(-1)^{m+1}$

then the larger D's alternate in sign

until D_{m+n} is reached.

Here:

$m = 0$

$D_{2m+1} = D_1$

$(-1)^{m+1} = -1^1 < 0$

$D_{m+n} = D_2 > 0$

In this problem, D_2 of $\nabla^2\mathcal{L}$ is $p^2 f_{11}'' f_{22}'' - p^2 (f_{12}'')^2$. This ought to be > 0, so the denominator of $\partial x_1/\partial w_1$ should be > 0.

D_1 of $\nabla^2\mathcal{L}$ is $p f_{11}''$, which ought to be < 0. Since $p > 0$, this implies $f_{11}'' < 0$. Since the numbering of the "first" or "second" input is arbitrary, this also implies $f_{22}'' < 0$.

So
$$\frac{\partial x_1}{\partial w_1} = \underbrace{\frac{1}{p}}_{+} \frac{\overbrace{f_{22}''}^{-}}{\underbrace{f_{11}'' f_{22}'' - (f_{12}'')^2}_{+}} < 0.$$

b)
$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \quad \text{s.t. } u(x_1, x_2) \geq \bar{u} \quad . \Rightarrow \mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda\left[\bar{u} - u(x_1, x_2)\right]$$

First-Order Conditions:
$$0 = \partial\mathcal{L}/\partial\lambda = \bar{u} - u(x_1, x_2)$$
$$0 = \partial\mathcal{L}/\partial x_1 = p_1 - \lambda u_1'$$
$$0 = \partial\mathcal{L}/\partial x_2 = p_2 - \lambda u_2'$$

Differentials:
$$0 = 0\, d\lambda - u_1' dx_1 - u_2' dx_2 + \cancel{d\bar{u}} \rightarrow \text{set} = 0$$
$$0 = -u_1' d\lambda - \lambda u_{11}'' dx_1 - \lambda u_{12}'' dx_2 + dp_1 - u_1' i$$
$$0 = -u_2' d\lambda - \lambda u_{21}'' dx_1 - \lambda u_{22}'' dx_2 + \cancel{dp_2} \rightarrow \text{set} = 0$$

$$\begin{bmatrix} 0 \\ -dp_1 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -u_1' & -u_2' \\ -u_1' & -\lambda u_{11}'' & -\lambda u_{12}'' \\ -u_2' & -\lambda u_{21}'' & -\lambda u_{22}'' \end{bmatrix}}_{\nabla^2 \mathcal{L}} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -u_1' & -u_2' \\ -u_1' & -\lambda u_{11}'' & -\lambda u_{12}'' \\ -u_2' & -\lambda u_{21}'' & -\lambda u_{22}'' \end{bmatrix} \begin{bmatrix} \partial\lambda/\partial p_1 \\ \partial x_1/\partial p_1 \\ \partial x_2/\partial p_1 \end{bmatrix}.$$ By Cramer's Rule,

$$\frac{\partial x_1}{\partial p_1} = \frac{\begin{vmatrix} 0 & 0 & -u_2' \\ -u_1' & -1 & -\lambda u_{12}'' \\ -u_2' & 0 & -\lambda u_{22}'' \end{vmatrix}}{|\nabla^2 \mathcal{L}|} = \frac{-1\,(-1)^{2+2} \begin{vmatrix} 0 & -u_2' \\ -u_2' & -\lambda u_{22}'' \end{vmatrix}}{|\nabla^2 \mathcal{L}|}$$

$$= \frac{(-1)(+1)\left[0 - (u_2')^2\right]}{|\nabla^2 \mathcal{L}|} = \frac{(u_2')^2}{|\nabla^2 \mathcal{L}|}.$$

The second-order sufficient conditions for a minimum:

D_{2m+1} of $\nabla^2 \mathcal{L}$ has the sign of $(-1)^m$, as do all the other D's, up to D_{m+n}.

Here: $m = 1$, $(-1)^m < 0$, so $D_{2m+1} = D_3 < 0$; $D_3 = \nabla^2 \mathcal{L}$; so

$$\frac{\partial x_1}{\partial p_1} = \frac{(u_2')^2 \ \oplus}{|\nabla^2 \mathcal{L}| \ \ominus} < 0.$$

c) $\max\ u(x_1, x_2)$ s.t. $p_1 x_1 + p_2 x_2 = m$

$\mathcal{L} = u(x_1, x_2) + \lambda(m - p_1 x_1 - p_2 x_2)$

F.O.C. $0 = \partial\mathcal{L}/\partial\lambda = m - p_1 x_1 - p_2 x_2$

$0 = \partial\mathcal{L}/\partial x_1 = u_1' - \lambda p_1$

$0 = \partial\mathcal{L}/\partial x_2 = u_2' - \lambda p_2$

Differentials:

$$
\begin{aligned}
0 &= -p_1\, dx_1 - p_2\, dx_2 + dm - x_1\, dp_1 - x_2\, dp_2 \\
0 &= -p_1\, d\lambda + u''_{11}\, dx_1 + u''_{12}\, dx_2 - \lambda\, dp_1 \\
0 &= -p_2\, d\lambda + u''_{21}\, dx_1 + u''_{22}\, dx_2 - \lambda\, dp_2
\end{aligned}
$$

Take $dm = 0$ and $dp_2 = 0$

$$
\begin{bmatrix} x_1 \\ \lambda \\ 0 \end{bmatrix} dp_1 = \begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{11} & u''_{12} \\ -p_2 & u''_{21} & u''_{22} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix}
$$

$$
\begin{bmatrix} x_1 \\ \lambda \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{11} & u''_{12} \\ -p_2 & u''_{21} & u''_{22} \end{bmatrix} \begin{bmatrix} \partial\lambda/\partial p_1 \\ \partial x_1/\partial p_1 \\ \partial x_2/\partial p_1 \end{bmatrix}
$$

Cramer's Rule:

$$
\frac{\partial x_1}{\partial p_1} = \frac{\begin{vmatrix} 0 & x_1 & -p_2 \\ -p_1 & \lambda & u''_{12} \\ -p_2 & 0 & u''_{22} \end{vmatrix}}{\begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{11} & u''_{12} \\ -p_2 & u''_{21} & u''_{22} \end{vmatrix}} = \frac{x_1 (-1)^{1+2} \begin{vmatrix} -p_1 & u''_{12} \\ -p_2 & u''_{22} \end{vmatrix} + \lambda (-1)^{2+2} \begin{vmatrix} 0 & -p_2 \\ -p_2 & u''_{22} \end{vmatrix}}{|\nabla^2 \mathcal{L}|}
$$

$$
= \frac{x_1 (-1)\left[-p_1 u''_{22} + p_2 u''_{12}\right] + \lambda (+1)\left[0 + p_2^2\right]}{(D_3 \text{ of } \nabla^2 \mathcal{L})}
$$

$$
= \frac{x_1 (p_1 u''_{22} - p_2 u''_{12}) + \lambda p_2^2}{(D_3 \text{ of } \nabla^2 \mathcal{L})}.
$$

The S.O.C. for a maximum are given in part a. Here $m = 1$, $D_{2m+1} = D_3$, $(-1)^{m+1} = (-1)^2 > 0$, and $D_{m+n} = D_3$. So the denominator of $\partial x_1/\partial p_1$ is

positive. The numerator, however, cannot be signed. (u''_{12} is especially hard to say anything about.) (Note that by the Envelope Theorem, $\partial u^*/\partial m = \partial \mathcal{L}/\partial m = \lambda$, and it's clear that $\partial u^*/\partial m \geq 0$, so $\lambda \geq 0$.)

optional

d) Firms have downward-sloping input demand curves (part a), and consumers have downward-sloping Hicksian (compensated) demand curves (part b), but because Giffen goods could exist, consumer's Marshallian demand curves might not be downward sloping (part c).

6710 Section

Answer one of the following two questions.

1999 Qualifier

1. (a) A consumer consumes n commodities labeled 1, 2, 3, ..., n. This consumer takes the prices of commodities 2, 3, 4, ..., n as given, but this consumer can affect the price he pays for commodity 1 because the more of commodity 1 he buys, the lower the price he has to pay for commodity 1.

   For what commodities i, if any, is it true that this consumer's commodity demands obey

$$x_i = -\frac{\partial v/\partial p_i}{\partial v/\partial m}\,?$$

   The letter v denotes the indirect utility function.

   (b) A firm has market power over commodities 1, 2, 3, ..., m, but this firm has no market power over commodities $m+1$, $m+2$, ..., n. Suppose that $m < n$.

   For what commodities i and j, if any, is it true that this firm's supply and demands obey

$$\frac{dy_i}{dp_j} = \frac{dy_j}{dp_i}\,?$$

   For what commodities i, if any, is it true that this firm's supply and demands obey

$$\frac{dy_i}{dp_i} \geq 0\,?$$

Optional Question 1.

a) $v(p_2, p_3, \ldots, p_n, m) = \max_{\underset{\sim}{x}} u(\underset{\sim}{x})$ s.t. $p_1(x_1)x_1 + p_2 x_2 + \cdots + p_n x_n = m$.

$$\mathcal{L} = u(\underset{\sim}{x}) + \lambda\left[m - p_1(x_1)x_1 - p_2 x_2 - p_3 x_3 - \cdots - p_n x_n\right]$$

Envelope Theorem: $\partial v/\partial p_i = \partial \mathcal{L}^*/\partial p_i \quad \forall\, i = 2, 3, \ldots, n$

$\partial v/\partial m = \partial \mathcal{L}^*/\partial m$.

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Therefore $\dfrac{\partial v}{\partial p_i} = -\lambda x_i \quad \forall\, i = 2, 3, \ldots, n$

$$\frac{\partial v}{\partial m} = \lambda$$

and dividing one by the other, then multiplying both sides by negative one;

$$x_i = -\frac{\partial v/\partial p_i}{\partial v/\partial m} \quad \forall\, i = 2, 3, \ldots, n.$$

There is no similar result for x_1. Note that p_1 is not an exogenous variable in this problem, so $\partial v/\partial p_1$ cannot be expected to be equal to $\partial \mathcal{L}^*/\partial p_1$ in general.

b) $\pi = \max_{\underset{\sim}{y}} \underset{\sim}{p} \cdot \underset{\sim}{y}$ s.t. $\underset{\sim}{y} \in Y$, or more explicitly,

$$\pi(p_{m+1}, p_{m+2}, \ldots, p_n) = \max_{y_i,\, 1 \le i \le m} P_i(y_i) \cdot y_i + \max_{y_i,\, m+1 \le i \le n} P_i y_i$$

$$\text{s.t. } \underset{\sim}{y} \in Y$$

Envelope Theorem: $\partial \pi/\partial p_i = \partial \mathcal{L}^*/\partial p_i, \quad m+1 \le i \le n$.

$$\mathcal{L} = \sum_{i=1}^{m} p_i(y_i)\cdot y_i + \sum_{i=m+1}^{n} p_i y_i$$

so $\dfrac{\partial \pi}{\partial p_i} = y_i$ for $m+1 \le i \le n$ (since the profit function is only a function of the i's between $m+1$ and n).

Let the vector $(p_{m+1}, p_{m+2}, \dots, p_n)$ be denoted $\underset{\sim}{\bar{p}}$ and let the vector $(y_{m+1}, y_{m+2}, \dots, y_n)$ be denoted by $\underset{\sim}{\bar{y}}$. Then

$\nabla_{\underset{\sim}{\bar{p}}}\, \pi(\underset{\sim}{\bar{p}}) = \underset{\sim}{\bar{y}}$, and differentiating both sides,

Qualifying Exam 1999

Answer 1 cont...

$$\nabla^2_{\underset{\sim}{\bar{p}}}\, \pi = \nabla_{\underset{\sim}{\bar{p}}}\, \underset{\sim}{\bar{y}}.$$

this is
$$\begin{bmatrix} \frac{\partial y_{m+1}}{\partial p_{m+1}} & \frac{\partial y_{m+2}}{\partial p_{m+1}} & \cdots & \frac{\partial y_n}{\partial p_{m+1}} \\ \frac{\partial y_{m+1}}{\partial p_{m+2}} & \frac{\partial y_{m+2}}{\partial p_{m+2}} & \cdots & \frac{\partial y_n}{\partial p_{m+2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_{m+1}}{\partial p_n} & \frac{\partial y_{m+2}}{\partial p_n} & \cdots & \frac{\partial y_n}{\partial p_n} \end{bmatrix}$$

The left-hand side is a Hessian, which by Young's theorem is symmetric, so

$\dfrac{\partial y_i}{\partial p_j} = \dfrac{\partial y_j}{\partial p_i}$ for all i and j between $m+1$ and n.

I claim that the profit function $\pi(\underset{\sim}{\bar{p}})$ is convex. The proof is that

$$\pi(\lambda \underset{\sim}{\bar{p}}_a + (1-\lambda)\underset{\sim}{\bar{p}}_b) = \max_{\underset{\sim}{y}} \Big[(p_1, p_2, \dots, p_m)\cdot(y_1, y_2, \dots, y_m) + (\lambda \underset{\sim}{\bar{p}}_a + (1-\lambda)\underset{\sim}{\bar{p}}_b)\cdot \underset{\sim}{\bar{y}} \Big]$$

$$= \max_{\underset{\sim}{y}} \left[(p_1, \dots, p_m) \cdot (y_1, \dots, y_m) + \lambda \underset{\sim}{\bar{p}_a} \cdot \underset{\sim}{\bar{y}} + (1-\lambda) \underset{\sim}{\bar{p}_b} \cdot \underset{\sim}{\bar{y}} \right]$$

$$= \max_{\underset{\sim}{y}} \Big[\lambda (p_1, \dots, p_m) \cdot (y_1, \dots, y_m)$$
$$+ (1-\lambda) (p_1, \dots, p_m) \cdot (y_1, \dots, y_m) + \lambda \underset{\sim}{\bar{p}_a} \cdot \underset{\sim}{\bar{y}} + (1-\lambda) \underset{\sim}{\bar{p}_b} \cdot \underset{\sim}{\bar{y}} \Big]$$

$$\leq \max_{\underset{\sim}{y}} \left[\lambda (p_1, \dots, p_m) \cdot (y_1, \dots, y_m) + \lambda \underset{\sim}{\bar{p}_a} \cdot \underset{\sim}{\bar{y}} \right]$$
$$+ \max_{\underset{\sim}{y}} \left[(1-\lambda) (p_1, \dots, p_m) \cdot (y_1, \dots, y_m) + (1-\lambda) \underset{\sim}{\bar{p}_b} \cdot \underset{\sim}{\bar{y}} \right]$$

$$= \lambda \max_{\underset{\sim}{y}} \left[(p_1, \dots, p_m) \cdot (y_1, \dots, y_m) + \underset{\sim}{\bar{p}_a} \cdot \underset{\sim}{\bar{y}} \right]$$
$$+ (1-\lambda) \max_{\underset{\sim}{y}} \left[(p_1, \dots, p_m) \cdot (y_1, \dots, y_m) + \underset{\sim}{\bar{p}_b} \cdot \underset{\sim}{\bar{y}} \right]$$

$$= \lambda \pi(\underset{\sim}{\bar{p}_a}) + (1-\lambda) \pi(\underset{\sim}{\bar{p}_b}) .$$

This completes the proof that $\pi(\underset{\sim}{\bar{p}})$ is convex. It follows that $\nabla^2 \pi$ is positive semidefinite, and hence that its diagonal terms, which are equal to $\partial y_i / \partial p_i$ for all i between $m+1$ and n, are ≥ 0.

1/3

Qualifying Exam
1994
Question 3
① ②

3. One form of the Envelope Theorem is: if

$$M(a) = \max_{x_1, x_2} g(x_1, x_2, a) \quad \text{such that } h(x_1, x_2, a) = 0$$

and if $\mathscr{L} = g - \lambda h$ then

$$\frac{dM(a)}{da} = \frac{\partial \mathscr{L}^*}{\partial a}.$$

Another form of the Envelope Theorem is: if

$$M(a) = \max_{x_1, x_2} g(x_1, x_2, a)$$

then

$$\frac{dM(a)}{da} = \frac{\partial g^*}{\partial a}.$$

Use the Envelope Theorem to prove the following four results. Explain what each symbol means.

(a) $h_i(\mathbf{p}, u) = \partial e(\mathbf{p}, u)/\partial p_i$.

(b) $x_i(\mathbf{p}, m) = -[\partial v(\mathbf{p}, m)/\partial p_i] \div [\partial v(\mathbf{p}, m)/\partial m]$ [Roy's Identity].

(c) $y_i(\mathbf{p}) = \partial \pi(\mathbf{p})/\partial p_i$ [Hotelling's Lemma].

(d) $x_i(\mathbf{w}, y) = \partial c(\mathbf{w}, y)/\partial w_i$ [Shephard's Lemma].

Qualifying Exam
1994
Answer 3

(3) a) $\min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x}$ s.t. $u(\underset{\sim}{x}) \geq u$ $= e(\underset{\sim}{p}, u)$

$$\mathcal{L} = \underset{\sim}{p} \cdot \underset{\sim}{x} + \lambda \left(u(\underset{\sim}{x}) - u \right)$$

$$\frac{dM(a)}{da} = \frac{\partial \mathcal{L}^*}{\partial a} \Rightarrow \frac{\partial e(\underset{\sim}{p}, u)}{\partial p_i} = \frac{\partial \mathcal{L}^*}{\partial p_i}$$

$\underset{\sim}{p}$: price $= x_i$

$\underset{\sim}{x}$: consumer demand

$u(\cdot)$: utility function $= h_i(\underset{\sim}{p}, u)$ by definition.

e : expenditure function

h_i : Hicksian demand curve

b) $\max_{\underset{\sim}{x}} u(\underset{\sim}{x})$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} \leq m$ ← income $= v(\underset{\sim}{p}, m)$ ← indirect utility function

Other variables as in part (a).

$$\mathcal{L} = u(\underset{\sim}{x}) + \lambda \left[m - \underset{\sim}{p} \cdot \underset{\sim}{x} \right]$$

Marshallian demand curve

$$\frac{dM(a)}{da} = \frac{\partial \mathcal{L}^*}{\partial a} \Rightarrow \begin{cases} \dfrac{\partial v}{\partial p_i} = \dfrac{\partial \mathcal{L}^*}{\partial p_i} = -\lambda x_i = -\dfrac{\partial v}{\partial m} x_i \\ \dfrac{\partial v}{\partial m} = \dfrac{\partial \mathcal{L}^*}{\partial m} = \lambda \end{cases}$$

; solve for x_i

Qualifying Exam
1994

Answer 3 cont...

c) $\pi(\underset{\sim}{p}) = \max_{\underset{\sim}{y}} \underset{\sim}{p} \cdot \underset{\sim}{y}$

profit function

prices faced by firms

firm's inputs and outputs
(<0) (>0)

$$\frac{dM(a)}{da} = \frac{\partial g^*}{\partial a} \Rightarrow \frac{\partial \pi(\underset{\sim}{p})}{\partial p_i} = \frac{\partial \underset{\sim}{p} \cdot \underset{\sim}{y}}{\partial p_i} = y_i(\underset{\sim}{p})$$

firm's input supply or output demand function

d) $\min_{\underset{\sim}{x}} \underset{\sim}{w} \cdot \underset{\sim}{x}$ s.t. $f(\underset{\sim}{x}) \geq y = c(\underset{\sim}{w}, y)$; $\mathcal{L} = \underset{\sim}{w} \cdot \underset{\sim}{x} + \lambda[f(\underset{\sim}{x}) - y]$

input prices

inputs

production function

output

cost function

$$\frac{dM(a)}{da} = \frac{\partial \mathcal{L}^*}{\partial a} \Rightarrow \frac{\partial c(\underset{\sim}{w}, y)}{\partial w_i} = \frac{\partial \mathcal{L}^*}{\partial w_i}$$

$$= x_i(\underset{\sim}{w}, y)$$

input demand function

Qualifying Exam
1998
Question 1
(0)

621 Section (must answer one)

Question 1. What is the Envelope Theorem? (You do not have to prove the Envelope Theorem.)

What four important results is the Envelope Theorem used to prove? What are the proofs?

Qualifying Exam
1998
Answer 1

Envelope Theorem with Constraint

$$M(a) = \max_{x_1, x_2} g(x_1, x_2, a)$$

s.t.

$$h(x_1, x_2, a) = 0.$$

$$\Rightarrow \mathcal{L} = g - \lambda h \text{ and}$$

$$\frac{dM(a)}{da} = \frac{\partial \mathcal{L}^*}{\partial a}.$$

Expenditure Function

$$e(\underset{\sim}{p}, u) = \min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x}$$

prices; utility level; commodities

s.t. $u(\underset{\sim}{x}) - u = 0.$ (utility function)

$$\Rightarrow \mathcal{L} = \underset{\sim}{p} \cdot \underset{\sim}{x} - \lambda[u(\underset{\sim}{x}) - u]$$

and

$$\frac{\partial e}{\partial p_i} = \frac{\partial \mathcal{L}^*}{\partial p_i} = x_i.$$

Hicksian demand

The Analogue of Shephard's Lemma for Consumers

u: utility function; utility level
$\underset{\sim}{x}$: commodity bundle
$\underset{\sim}{p}$: price of $\underset{\sim}{x}$

Cost Function

$$c(\underset{\sim}{w}, y) = \min_{\underset{\sim}{x}} \underset{\sim}{w} \cdot \underset{\sim}{x}$$

input prices; output; inputs

s.t. $f(\underset{\sim}{x}) - y = 0.$ (production function)

$$\Rightarrow \mathcal{L} = \underset{\sim}{w} \cdot \underset{\sim}{x} - \lambda[f(\underset{\sim}{x}) - y]$$

and

$$\frac{\partial c}{\partial w_i} = \frac{\partial \mathcal{L}^*}{\partial w_i} = x_i$$

input demand

Shephard's Lemma

$\underset{\sim}{x}$: inputs
$\underset{\sim}{w}$: price of $\underset{\sim}{x}$
f: production function
y: output

Indirect Utility Function

$$v(\underset{\sim}{p}, m) = \max_{\underset{\sim}{x}} u(\underset{\sim}{x})$$

prices; income; utility function; commodities

s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} - m = 0.$

$$\Rightarrow \mathcal{L} = u(\underset{\sim}{x}) - \lambda[\underset{\sim}{p} \cdot \underset{\sim}{x} - m]$$

and

$$\begin{cases} \dfrac{\partial v}{\partial m} = \dfrac{\partial \mathcal{L}^*}{\partial m} = \lambda \\[2ex] \dfrac{\partial v}{\partial p_i} = \dfrac{\partial \mathcal{L}^*}{\partial p_i} = -\lambda x_i. \end{cases}$$

Substituting for λ yields

$$x_i = \frac{-\partial v / \partial p_i}{\partial v / \partial m}.$$

demand

Roy's Identity

m: income
rest of symbols like "expenditure function" column

Envelope Theorem with no Constraints

$$M(a) = \max_{x_1, x_2} g(x_1, x_2, a).$$

$$\Rightarrow \frac{dM(a)}{da} = \frac{\partial g^*}{\partial a}.$$

Qualifying Exam
1998
Answer 1 cont...

Profit Function

input & output prices

$$\pi(\underset{\sim}{p}) = \max_{\underset{\sim}{y}} \underset{\sim}{p} \cdot \underset{\sim}{y}$$

input/output vector

$$\Rightarrow \frac{\partial \pi(\underset{\sim}{p})}{\partial p_i} = y_i^* \text{ . input & outputs}$$

Hotelling's Lemma

$\underset{\sim}{y}$: input-output vector

$\underset{\sim}{p}$: price of $\underset{\sim}{y}$

Analogues

M	e	c	v	π
a	$\underset{\sim}{p}, u$	$\underset{\sim}{w}, y$	$\underset{\sim}{p}, m$	$\underset{\sim}{p}$
x_1, x_2	$\underset{\sim}{x}$	$\underset{\sim}{x}$	$\underset{\sim}{x}$	$\underset{\sim}{y}$
g	$\underset{\sim}{p} \cdot \underset{\sim}{x}$	$\underset{\sim}{w} \cdot \underset{\sim}{x}$	$u(\underset{\sim}{x})$	$\underset{\sim}{p} \cdot \underset{\sim}{y}$
h	$u(\underset{\sim}{x}) - u$	$f(\underset{\sim}{x}) - y$	$\underset{\sim}{p} \cdot \underset{\sim}{x} - m$	

Fall 2005 Final

2. [8 points] Illustrate the use of the Envelope Theorem in a problem of economic interest. Explain how the Envelope Theorem is being used.

(2) The Envelope Theorem states that if

$$M(a) = \max_{\underset{\sim}{x}} g(\underset{\sim}{x}, a) \quad \text{s.t.} \quad h(\underset{\sim}{x}, a) = 0$$

then

$$\frac{dM(a)}{da} = \frac{\partial \mathcal{L}^*}{\partial a}$$ where $\mathcal{L}^*$ is the maximized Lagrangian of the optimization problem.

There are two classic Envelope Theorem results for the firm, namely

a) $X_i = \dfrac{\partial c}{\partial w_i}$ Shephard's Lemma

b) $Y_i = \dfrac{\partial \pi}{\partial p_i}$ Hotelling's Lemma,

and two for the consumer:

c) $h_i = \dfrac{\partial e}{\partial p_i}$ resembles Shephard's Lemma

d) $x_i = -\dfrac{\partial v/\partial p_i}{\partial v/\partial m}$ Roy's Identity

Their proofs are as follows.

a) $c = \min_{\underset{\sim}{x}} \underset{\sim}{w} \cdot \underset{\sim}{x}$ s.t. $f(\underset{\sim}{x}) \geq y$ (cost-minimization)

cost; input price; inputs; production function; fixed level of output

$$\mathcal{L} = \underset{\sim}{w} \cdot \underset{\sim}{x} + \lambda [y - f(\underset{\sim}{x})]$$

$$\frac{\partial c}{\partial w_i} = \frac{\partial \mathcal{L}^*}{\partial w_i} = x_i^*$$

b) $\pi = \max_{\underset{\sim}{y}} \underset{\sim}{p} \cdot \underset{\sim}{y}$ (profit maximization)

profit; input & output price vector; input/output vector

$$\frac{\partial \pi}{\partial p_i} = \frac{\partial \mathcal{L}^*}{\partial p_i} = \frac{\partial (\underset{\sim}{p} \cdot \underset{\sim}{y})^*}{\partial p_i} = y_i^*$$

c) $e = \min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x}$ s.t. $u(\underset{\sim}{x}) \geq u_0$ (expenditure minimization)

expenditure; prices; consumption; utility; fixed utility level

$$\mathcal{L} = \underset{\sim}{p} \cdot \underset{\sim}{x} + \lambda [u_0 - u(\underset{\sim}{x})]$$

$$\frac{\partial e}{\partial p_i} = \frac{\partial \mathcal{L}^*}{\partial p_i} = x_i^*$$ which is usually called h_i^* (the Hicksian demand)

d) $v = \max_{\underset{\sim}{x}} u(\underset{\sim}{x}) \text{ s.t. } \underset{\sim}{p} \cdot \underset{\sim}{x} \leq m$ (utility maximization)

indirect utility — utility — consumption — prices — income

$$\mathcal{L} = u(\underset{\sim}{x}) + \lambda [m - \underset{\sim}{p} \cdot \underset{\sim}{x}]$$

$$\left.\begin{aligned} \frac{\partial v}{\partial p_i} &= \frac{\partial \mathcal{L}^*}{\partial p_i} = -\lambda^* x_i^* \\ \frac{\partial v}{\partial m} &= \frac{\partial \mathcal{L}^*}{\partial m} = \lambda^* \end{aligned}\right\} \quad \frac{\partial v}{\partial p_i} = -\frac{\partial v}{\partial m} x_i^* \text{ and solve for } x_i^*.$$

There are other possible answers.

Final Exam
1997
Question 4

① ③

4. (a) Suppose a consumer's utility function is given by $u(x_1, x_2) = x_1^\alpha x_2^\beta$. The consumer's income is fixed at m and the prices of goods 1 and 2 are fixed at p_1 and p_2, respectively.
Find the consumer's indirect utility function.

(b) Suppose instead that the income of the consumer in part (a) is a random variable; with probability 1/4 his income is m_1, and with probability 3/4 his income is m_2. Suppose this consumer's preferences obey the Expected Utility Hypothesis.
Find the consumer's indirect utility function.

Final Exam
1997
Answer 4

(4) $u = x_1^{\alpha} x_2^{\beta}$

a) $\mathcal{L} = x_1^{\alpha} x_2^{\beta} + \lambda(m - p_1 x_1 - p_2 x_2)$

F.O.C. $0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2$

$$\left.\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial x_1} = \alpha \frac{u}{x_1} - \lambda p_1 \\ 0 &= \frac{\partial \mathcal{L}}{\partial x_2} = \beta \frac{u}{x_2} - \lambda p_2 \end{aligned}\right\} \quad \frac{\alpha u}{x_1} \div \frac{\beta u}{x_2} = \lambda p_1 \div \lambda p_2$$

$$\frac{\alpha u}{x_1} \cdot \frac{x_2}{\beta u} = \frac{\lambda p_1}{\lambda p_2} \Rightarrow \frac{\alpha x_2}{\beta x_1} = \frac{p_1}{p_2}$$

therefore $x_2 = \frac{\beta}{\alpha} \frac{p_1}{p_2} x_1$. Substituting into the budget constraint,

$$m = p_1 x_1 + p_2 \left(\frac{\beta}{\alpha} \frac{p_1}{p_2} x_1\right) = \left(p_1 + \frac{\beta}{\alpha} p_1\right) x_1 = \left(1 + \frac{\beta}{\alpha}\right) p_1 x_1$$

$$= \frac{\alpha+\beta}{\alpha} p_1 x_1 \Rightarrow x_1^* = \frac{\alpha}{\alpha+\beta} \frac{m}{p_1}.$$

Then $x_2^* = \frac{\beta}{\alpha} \frac{p_1}{p_2} x_1^* = \frac{\beta}{\alpha} \frac{p_1}{p_2} \frac{\alpha}{\alpha+\beta} \frac{m}{p_1} = \frac{\beta}{\alpha+\beta} \frac{m}{p_2}$.

Finally, $v(\underset{\sim}{p}, m) = (x_1^*)^{\alpha} (x_2^*)^{\beta} = \left[\frac{\alpha}{\alpha+\beta} \frac{m}{p_1}\right]^{\alpha} \left[\frac{\beta}{\alpha+\beta} \frac{m}{p_2}\right]^{\beta}$

$$= \frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} \frac{m^{\alpha+\beta}}{p_1^{\alpha} p_2^{\beta}} = \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{m}{\alpha+\beta}\right)^{\alpha+\beta}.$$

b) For the case where utility is assumed to be just a function of wealth, $u(w)$, then if $w = \pi \circ w_1 \oplus (1-\pi) w_2$ the Expected Utility Hypothesis gives utility equal to $\pi\, u(w_1) + (1-\pi)\, u(w_2)$. In this problem, the indirect utility function $v(\underset{\sim}{p}, m)$ plays the role of $u(w)$, where m (income) corresponds to w (wealth) and the prices $\underset{\sim}{p}$ are constant.

Therefore,

$$v\left(\underset{\sim}{p}, \tfrac{1}{4} \circ m_1 \oplus \tfrac{3}{4} \circ m_2\right) = \tfrac{1}{4} v(\underset{\sim}{p}, m_1) + \tfrac{3}{4} v(\underset{\sim}{p}, m_2)$$

$$= \frac{1}{4}\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{m_1}{\alpha+\beta}\right)^{\alpha+\beta} + \frac{3}{4}\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{m_2}{\alpha+\beta}\right)^{\alpha+\beta}$$

$$= \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{1}{\alpha+\beta}\right)^{\alpha+\beta}\left[\frac{m_1^{\alpha+\beta}}{4} + \frac{3\, m_2^{\alpha+\beta}}{4}\right]$$

$$= \frac{1}{4}\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{1}{\alpha+\beta}\right)^{\alpha+\beta}\left(m_1^{\alpha+\beta} + 3 m_2^{\alpha+\beta}\right).$$