(9.1)

endowment $\underset{\sim}{\omega}$

income $= \underset{\sim}{p} \cdot \underset{\sim}{\omega}$

utility-max. problem: $\max_{\underset{\sim}{x}} u(\underset{\sim}{x})$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = \underset{\sim}{p} \cdot \underset{\sim}{\omega}$.

Marshallian demand $x_i(\underset{\sim}{p}, \underset{\sim}{p} \cdot \underset{\sim}{\omega})$.

Differentiate w.r.t. p_j:

$$\frac{\partial x_i(\underset{\sim}{p}, \underset{\sim}{p} \cdot \underset{\sim}{\omega})}{\partial p_j} = \underbrace{\left.\frac{\partial x_i(\underset{\sim}{p}, \underset{\sim}{p} \cdot \underset{\sim}{\omega})}{\partial p_j}\right|_{\underset{\sim}{p} \cdot \underset{\sim}{\omega} \text{ const.}}}_{\frac{\partial h_i(\underset{\sim}{p}, u)}{\partial p_j} - \frac{\partial x_i(\underset{\sim}{p}, \underset{\sim}{p} \cdot \underset{\sim}{\omega})}{\partial m} x_j \text{ Slutsky}} + \underbrace{\frac{\partial x_i(\underset{\sim}{p}, \underset{\sim}{p} \cdot \underset{\sim}{\omega})}{\partial \underset{\sim}{p} \cdot \underset{\sim}{\omega}} \frac{\partial \underset{\sim}{p} \cdot \underset{\sim}{\omega}}{\partial p_j}}_{\frac{\partial x_i(\underset{\sim}{p}, \underset{\sim}{p} \cdot \underset{\sim}{\omega})}{\partial m} \omega_j}.$$

9.2

$k \in$ integers only??

Homogeneous of degree k: $f(\lambda \underset{\sim}{x}) = \lambda^k f(\underset{\sim}{x})$ for all $\lambda > 0$.

Prop 0 (p482) If $f(\underset{\sim}{x})$ is homogeneous of degree k then

$$\frac{\partial f(\underset{\sim}{x})}{\partial x_i} \text{ is homogeneous of degree } k-1.$$

Proof. Differentiate the first equation on this page w.r.t. x_i:

$f(x) = g(x) \not\Rightarrow$ $f'(x) = g'(x)$ but $f(x) \equiv g(x) \Rightarrow$ $f'(x) = g'(x)$.

$$\text{RHS}: \; \lambda^k \left.\frac{\partial f(\underset{\sim}{x})}{\partial x_i}\right|_{\underset{\sim}{x}}$$

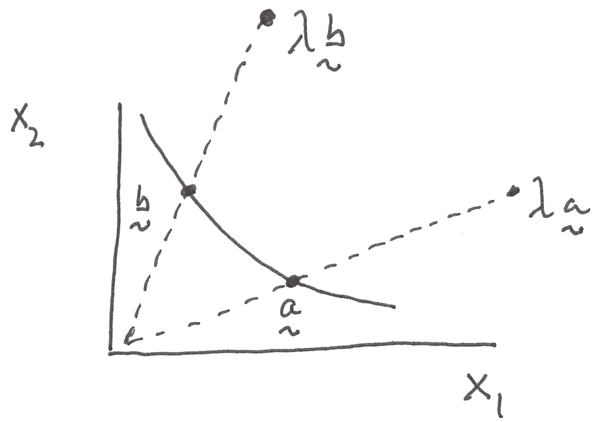
$$\text{LHS}: \; \frac{\partial}{\partial x_i} f(\lambda \underset{\sim}{x}) = \frac{\partial f(\lambda \underset{\sim}{x})}{\partial (\lambda x_i)} \; \frac{d \lambda x_i}{d x_i} \quad \text{Chain Rule}$$

$$= \left.\frac{\partial f(\underset{\sim}{x})}{\partial x_i}\right|_{\lambda \underset{\sim}{x}} \lambda .$$

Setting LHS = RHS yields

$$\left.\frac{\partial f(\underset{\sim}{x})}{\partial x_i}\right|_{\lambda \underset{\sim}{x}} \lambda = \lambda^k \left.\frac{\partial f(\underset{\sim}{x})}{\partial x_i}\right|_{\underset{\sim}{x}} \Rightarrow$$

$$\left.\frac{\partial f(\underset{\sim}{x})}{\partial x_i}\right|_{\lambda \underset{\sim}{x}} = \lambda^{k-1} \left.\frac{\partial f(\underset{\sim}{x})}{\partial x_i}\right|_{\underset{\sim}{x}} . \quad \blacksquare$$



Prop. 1. $\left.\begin{array}{l} f(\underset{\sim}{x}) \text{ homogeneous} \\ f(\underset{\sim}{a}) = f(\underset{\sim}{b}) \end{array}\right\} \Rightarrow f(\lambda \underset{\sim}{a}) = f(\lambda \underset{\sim}{b}).$

Proof. $f(\lambda \underset{\sim}{a}) = \lambda^k f(\underset{\sim}{a})$

$f(\lambda \underset{\sim}{b}) = \lambda^k f(\underset{\sim}{b})$. If $f(\underset{\sim}{a}) = f(\underset{\sim}{b})$ then these two expressions are equal. ∎

Prop. 2. $f(\underset{\sim}{x})$ homogeneous (of degree k) $\Rightarrow \dfrac{f_1'(\underset{\sim}{a})}{f_2'(\underset{\sim}{a})} = \dfrac{f_1'(\lambda \underset{\sim}{a})}{f_2'(\lambda \underset{\sim}{a})}$.

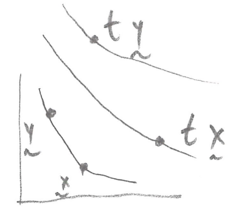
Proof. $$\frac{f_1'(\lambda \underset{\sim}{a})}{f_2'(\lambda \underset{\sim}{a})} = \frac{\lambda^{k-1} f_1'(\underset{\sim}{a})}{\lambda^{k-1} f_2'(\underset{\sim}{a})} = \frac{f_1'(\underset{\sim}{a})}{f_2'(\underset{\sim}{a})}.$$ ∎

See Fig. 1.7, p. 18, Panel A. The fact that the second contour line represents utility (or f) which is 2^k times more than the first contour line is a corollary of Proposition 1's proof.

Homotheticity

S&H (16.29): A function f is homothetic if

$$\left.\begin{array}{l} f(\underset{\sim}{x}) = f(\underset{\sim}{y}) \\ t > 0 \end{array}\right\} \Rightarrow f(t\underset{\sim}{x}) = f(t\underset{\sim}{y}).$$



Not homothetic.

Prop. 2a. f homogeneous $\Rightarrow f$ homothetic. Proof. $f(\underset{\sim}{x}) = f(\underset{\sim}{y})$, $t > 0$; $f(t\underset{\sim}{x}) = t^k f(\underset{\sim}{x})$ $\Rightarrow$ $f(t\underset{\sim}{y}) = t^k f(\underset{\sim}{y})$ ‖

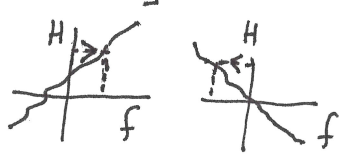
Prop. 3. If $f(\underset{\sim}{x})$ is homogeneous and

$H(f): \mathbb{R}^1 \to \mathbb{R}^1$ is strictly increasing everywhere or
" " decreasing "

then $H(f(\underset{\sim}{x}))$ is homothetic.

Proof. [Minor extension of S&H p. 573.] Pick $\underset{\sim}{a}$ and $\underset{\sim}{b}$ such that

$H(f(\underset{\sim}{a})) = H(f(\underset{\sim}{b}))$.



$(H \circ f)(\underset{\sim}{a}) = (H \circ f)(\underset{\sim}{b})$

Since H is strictly increasing or strictly decreasing, $f(\underset{\sim}{a}) = f(\underset{\sim}{b})$. Also,

$$H(f(t\underset{\sim}{a})) = \underbrace{H(t^k f(\underset{\sim}{a}))}_{\text{homogeneity of } f} = \underbrace{H(t^k f(\underset{\sim}{b}))}_{f(\underset{\sim}{a}) = f(\underset{\sim}{b})} = \underbrace{H(f(t\underset{\sim}{b}))}_{\text{homogeneity of } f}$$

so $(H \circ f)(t\underset{\sim}{a}) = (H \circ f)(t\underset{\sim}{b})$, that is, $H \circ f$ is homothetic. ∎

Corollary. If $f(x)$ is homogeneous then it is homothetic. } But Prop. 2a is better.
Proof. Above, take $H(f) = f$. ∎

Remark. Varian (p. 146) defines $H \circ f$ to be homothetic if

f is homogeneous of degree 1 and
H is strictly increasing.

This seems to be restrictive but actually it isn't.

Exercise. Show that $u(\underset{\sim}{x}) = x_1^\alpha x_2^\beta + 1$ is not homogeneous but is homothetic. (Assume $\alpha \neq -\beta$.)

Answer. Not homogeneous:

$$u(\lambda \underset{\sim}{x}) = (\lambda x_1)^\alpha (\lambda x_2)^\beta + 1 = \lambda^\alpha x_1^\alpha \lambda^\beta x_2^\beta + 1$$
$$= \lambda^{\alpha+\beta} x_1^\alpha x_2^\beta + 1 \qquad [\text{if } \alpha = -\beta, \text{ this would be } \lambda^0 x_1^\alpha x_2^\beta + 1 = u(\underset{\sim}{x})]$$
$$\neq \lambda^k u(\underset{\sim}{x}) \text{ for any } k.$$

Is homothetic, using the definition of homotheticity but not Prop. 3:

$$\left.\begin{array}{l} u(\underset{\sim}{a}) = u(\underset{\sim}{b}) \\ t > 0 \end{array}\right\} \overset{?}{\Rightarrow} u(t\underset{\sim}{a}) = u(t\underset{\sim}{b})$$

$$\Leftrightarrow \quad \left.\begin{array}{l} a_1^\alpha a_2^\beta + 1 = b_1^\alpha b_2^\beta + 1 \\ t > 0 \end{array}\right\} \overset{?}{\Rightarrow} (t a_1)^\alpha (t a_2)^\beta + 1 = (t b_1)^\alpha (t b_2)^\beta + 1$$

$$\Leftrightarrow$$

$$t^{\alpha+\beta} a_1^\alpha a_2^\beta + 1 \overset{?}{=} t^{\alpha+\beta} b_1^\alpha b_2^\beta + 1 \quad \Leftrightarrow$$

$$t^{\alpha+\beta} a_1^\alpha a_2^\beta \overset{?}{=} t^{\alpha+\beta} b_1^\alpha b_2^\beta$$

$$a_1^\alpha a_2^\beta \overset{?}{=} b_1^\alpha b_2^\beta$$

$$a_1^\alpha a_2^\beta + 1 \overset{?}{=} b_1^\alpha b_2^\beta + 1 \quad \text{OK.}$$

Is homothetic, using Prop. 3: Let

$$f(\underset{\sim}{x}) = x_1^{\alpha} x_2^{\beta} \text{ and}$$

$$H(f) = f + 1.$$

$f(\underset{\sim}{x})$ is homogeneous of degree $\alpha + \beta$; $H(f)$ is a strictly increasing function of f; and $H(f) = x_1^{\alpha} x_2^{\beta} + 1 = u(\underset{\sim}{x})$. ∎

Remark. We saw in an earlier section that if H is a strictly increasing function from $\mathbb{R}^1$ to $\mathbb{R}^1$ then

$H(u(\underset{\sim}{x}))$ represents the same preferences as $u(\underset{\sim}{x})$.

With such an H function, if $u(\underset{\sim}{x})$ is homogeneous, then

$$H(u(\underset{\sim}{x}))$$

is homothetic and represents the same preferences as $u(\underset{\sim}{x})$. In other words, the homogeneous u and the homothetic $H(u)$ represent the same preferences. This is interesting: homogeneity is cardinal but homotheticity not so much. If u is homothetic, an ↑ transformation of it is also homothetic. This is false for homogeneity.

Remark. ~~Prop. 4~~. If the utility function is homothetic, then* ~~given the figure~~ for Prop. 1, ~~2~~$\lambda\underset{\sim}{a}$ and ~~2~~$\lambda\underset{\sim}{b}$ lie ~~on the same~~ indifference curve as each other.

~~[Also true for "$\lambda\underset{\sim}{a}$" and "$\lambda\underset{\sim}{b}$."]~~

$$* \left.\begin{array}{l} u(\underset{\sim}{a}) = u(\underset{\sim}{b}) \\ \lambda > 0 \end{array}\right\} \Rightarrow u(\lambda\underset{\sim}{a}) = u(\lambda\underset{\sim}{b}).$$

Proof. ~~Write the utility function as~~ Trivially follows from the definition of homotheticity.

Prop. 5. If $g(\underset{\sim}{x})$ is homothetic then $\dfrac{g_1'(\underset{\sim}{a})}{g_2'(\underset{\sim}{a})} = \dfrac{g_1'(\lambda \underset{\sim}{a})}{g_2'(\lambda \underset{\sim}{a})}$.

Remark. Prop. 2 showed this for homogeneous functions.

Proof. Without loss of generality, $g(\underset{\sim}{x}) = H(u(\underset{\sim}{x}))$ for H increasing and u homogeneous of degree k.

LHS is
$$\frac{g_1'(\underset{\sim}{a})}{g_2'(\underset{\sim}{a})} = \frac{H'(u(\underset{\sim}{a}))\, u_1'(\underset{\sim}{a})}{H'(u(\underset{\sim}{a}))\, u_2'(\underset{\sim}{a})} \quad \text{(Chain Rule)}$$
$$= \frac{u_1'(\underset{\sim}{a})}{u_2'(\underset{\sim}{a})}.$$

RHS is
$$\frac{g_1'(\lambda\underset{\sim}{a})}{g_2'(\lambda\underset{\sim}{a})} = \frac{H'(u(\lambda\underset{\sim}{a}))\, u_1'(\lambda\underset{\sim}{a})}{H'(u(\lambda\underset{\sim}{a}))\, u_2'(\lambda\underset{\sim}{a})} \quad \text{(Chain Rule)}$$
$$= \frac{u_1'(\lambda\underset{\sim}{a})}{u_2'(\lambda\underset{\sim}{a})} = \frac{\lambda^{k-1} u_1'(\underset{\sim}{a})}{\lambda^{k-1} u_2'(\underset{\sim}{a})} = \frac{u_1'(\underset{\sim}{a})}{u_2'(\underset{\sim}{a})}.$$

So the LHS = the RHS. ∎

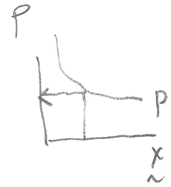
See Fig. 1.7, p. 18, Panel B.

9.3 omit

9.4 omit

9.5

Can we go from $\underset{\sim}{x}$ to $\underset{\sim}{p}$?



w/ vectors!

Inverse D function.

WLOG, Set $m=1$. then

$$\frac{\partial u(\underset{\sim}{x})}{\partial x_i} - \lambda p_i = 0 \qquad (1)$$

$$\sum_{i=1}^{k} p_i x_i = 1. \qquad (2)$$

Multiply (1) by x_i and sum:

$$\sum_{i=1}^{k} \frac{\partial u(\underset{\sim}{x})}{\partial x_i} x_i - \lambda \underbrace{\sum_{i=1}^{k} p_i x_i}_{=1 \text{ from } (2)} = 0$$

$$\Rightarrow \lambda = \sum \frac{\partial u}{\partial x_i} x_i .$$

(1) $\Rightarrow$

$$p_i^{(x)} = \frac{\partial u/\partial x_i}{\lambda} = \frac{\partial u(\underset{\sim}{x})/\partial x_i}{\sum_{j=1}^{k} \frac{\partial u(\underset{\sim}{x})}{\partial x_j} x_j} . \qquad (9.5)$$

This p_i is a % of income "1".

$\underset{\sim}{x}, u \rightarrow p_i$

From § 8.6, $u(\underset{\sim}{x}) = \min_{\underset{\sim}{p}} v(\underset{\sim}{p})$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = 1$.

$$\mathcal{L} = v(\underset{\sim}{p}) - \mu(\underset{\sim}{p} \cdot \underset{\sim}{x} - 1)$$

$0 = \frac{\partial \mathcal{L}}{\partial p_i} = \frac{\partial v}{\partial p_i} - \mu x_i$. Multiply by p_i and sum: $0 = \sum \frac{\partial v}{\partial p_i} p_i - \sum \mu \underbrace{x_i p_i}_{=1}$.

$$x_i = \frac{\partial v/\partial p_i}{\mu} = \frac{\partial v(\underset{\sim}{p})/\partial p_i}{\sum \frac{\partial v}{\partial p_j} p_j} . \qquad (9.6)$$

Cf. Roy's Identity, § 7.4:

$$x_i = -\frac{\partial v/\partial p_i}{\partial v/\partial m} .$$

(9.6) $\underset{\sim}{x}(\underset{\sim}{p}, m)$ is continuous as long as it is "well-defined" (where "well defined" means unique). Proof: Thm. of Max., p. 506.

if the affordable set B is convex,

Prop. If preferences are strictly convex, and if $\underset{\sim}{p} \gg \underset{\sim}{0}$ then $\exists$ a unique $\underset{\sim}{x}$ that maximizes u on the budget set B.

Proof.

Suppose not; then $\underset{\sim}{x}'$ and $\underset{\sim}{x}''$ both maximize u on B.

Defn. of convexity of preferences: (p. 96)

$$\underset{\uparrow}{\underset{\sim}{x}} \succsim \underset{\sim}{z}, \quad \underset{\uparrow}{\underset{\sim}{y}} \succsim \underset{\uparrow}{\underset{\sim}{z}} \Rightarrow t\underset{\sim}{x} + (1-t)\underset{\sim}{y} \succ \underset{\sim}{z}. \quad t \in (0,1)$$

$$\underset{\sim}{x}' \qquad \underset{\sim}{x}'' \quad \underset{\sim}{x}'$$

So that

$$\underset{\text{OK}}{\underset{\sim}{x}' \succsim \underset{\sim}{x}'}, \quad \underset{\text{OK}}{\underset{\sim}{x}'' \succsim \underset{\sim}{x}'} \Rightarrow \underbrace{t\underset{\sim}{x}' + (1-t)\underset{\sim}{x}''}_{\text{affordable}} \succ \underset{\sim}{x}'$$ which violates the assumption that $\underset{\sim}{x}'$ maximizes u on B. ∎

(first line of proof)

affordable { since $t\underset{\sim}{x}' + (1-t)\underset{\sim}{x}'' \in B$ since B is convex.

To get discontinuous (multi-valued) demand curves, preferences must not be strictly convex. → Fig. 9.2.