Section 7.1

X: the consumption set

Notation for preferences: $x \succeq y$ or $\mathbf{x} \succeq \mathbf{y}$. Also, $\mathbf{x} \succ \mathbf{y}$, $\mathbf{x} \preceq \mathbf{y}$, $\mathbf{x} \prec \mathbf{y}$, or $\mathbf{x} \sim \mathbf{y}$. (No prices nor income; not a market environment; psychology only.)

A consumer is "rational" if preferences are:

- "complete": assuming $x \neq y$, either $x \succ y$, or $y \succ x$, or $x \sim y$. Implications: no learning. Difficult example: choose which of your children to give up.
- ["reflexive": Varian says this is needed but it's not.]
- "transitive": if $x \succeq y$ and $y \succeq z$ then $x \succeq z$. (Sometimes violated.)

"Continuity" of preferences: suppose an infinite sequence $\mathbf{x}_i$ is convergent and call its limit $\mathbf{x}^*$. If $\mathbf{x}_i \succeq \mathbf{y}$ for all i, then "continuity of preferences" requires $\mathbf{x}^* \succeq \mathbf{y}$.

A theorem (MCWG p. 47): if a consumer's preferences are "rational" and "continuous" then those preferences can be represented by a continuous function mapping X into $\mathbf{R}^1$. In other words, there exists at least one function $u(\mathbf{x}) : X \to \mathbf{R}^1$ which satisfies

$$\mathbf{x} \succ \mathbf{y} \Leftrightarrow u(\mathbf{x}) > u(\mathbf{y}) \quad \forall\, \mathbf{x}, \mathbf{y} \in X.$$

We call this function a "utility function." Cf. Varian p. 97.

First problem with these assumptions: perhaps preferences, instead of being fixed, depend on the question asked. See MCWG p. 7:

> Consider the following example, paraphrased from Kahneman and Tversky (1984):
>
> > Imagine that you are about to purchase a stereo for 125 dollars and a calculator for 15 dollars. The salesman tells you that the calculator is on sale for 5 dollars less at the other branch of the store, located 20 minutes away. The stereo is the same price there. Would you make the trip to the other store?
>
> It turns out that the fraction of respondents saying that they would travel to the other store for the 5 dollar discount is much higher than the fraction who say they would travel when the question is changed so that the 5 dollar saving is on the stereo. This is so even though the ultimate saving obtained by incurring the inconvenience of travel is the same in both cases.

Second problem with these assumptions: people may not know what makes them happy.

1. Daniel Gilbert, Harvard Psychology Dept.: discussed in Sept. 7, 2003 New York Times, "The Futile Pursuit of Happiness." Happiness set points.
2. Baba Shiv `https://whywereason.wordpress.com/tag/baba-shiv/`. Cognitive processing is hard; the brain is not monolithic.

Third problem with these assumptions: lexicographic preferences (MCWG p. 46) violate the continuity assumption. Suppose a consumer always prefers bundles having more chocolate to those having less chocolate regardless of what else is in the bundles. For example, if chocolate is the second element in the consumption vector, this consumer would have:

$$(1, 1 + \tfrac{1}{i}) \succ (2, 1) \quad \text{for all } i > 0.$$

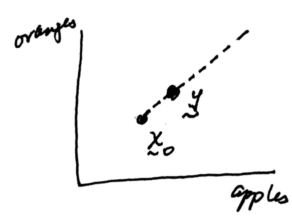
These preferences are lexicographic, and while the above ranking makes sense for all $i < \infty$, it makes no sense in the limit as $i \to \infty$—in other words, the limiting bundle, $\mathbf{x}^* = (1, 1)$, does not satisfy $(1, 1) = \mathbf{x}^* \succ (2, 1)$—so these preferences violate continuity. (Mention Nicholas Georgescu-Roegen.)

Common assumptions on preferences:

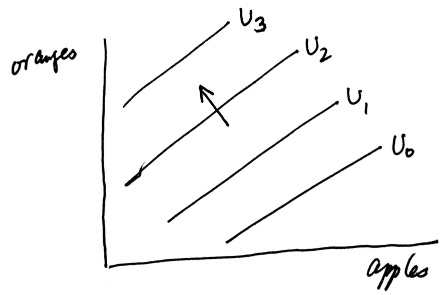
- local nonsatiation
- weak monotonicity ("more is never worse")
- strong monotonicity ("more is strictly better")

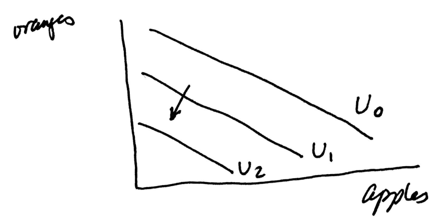
Historical example of violation of monotonicity: the "potlatch" of the Native Americans of the Pacific Northwest.

Claim: strong monotonicity implies local nonsatiation.



Claim: local nonsatiation does not imply strong monotonicity. [In the counterexample, the straight lines are "indifference curves," which are defined to be the contour lines ("level sets") of the utility function, and also, $U_0 < U_1 < U_2 < U_3$.]





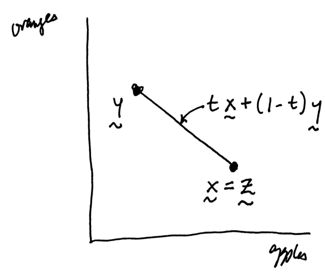
"Convex" or "strictly convex" preferences. Do not confuse this "convexity" with:

- convex combinations (of, especially, vectors)
- convex sets
- convex functions.

"Convex preferences" are a type of convex binary relation: roughly (for the exact definitions see p. 96 of Varian),

$$\mathbf{x} \succeq \mathbf{z} \text{ and } \mathbf{y} \succeq \mathbf{z} \ \Rightarrow\ \forall\, t \in (0,1),\ t\,\mathbf{x}+(1-t)\,\mathbf{y} \begin{cases} \succeq \mathbf{z} & \text{convex preferences} \\ \succ \mathbf{z} & \text{strictly convex preferences.} \end{cases}$$

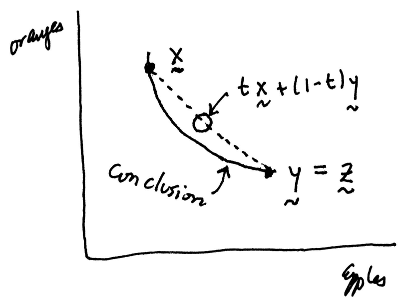
A graph which shows preferences that are convex but not strictly convex (the straight line is an indifference curve):



Claim: if preferences are strictly convex and if $\mathbf{x} \sim \mathbf{y}$, then

$$t\,\mathbf{x} + (1 - t)\,\mathbf{y} \succ \mathbf{x} \sim \mathbf{y}\,.$$

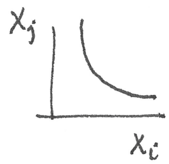
Recall that from Varian's (rigorously correct) definition of strictly convex preferences, in the special case of the diagram below, "$\mathbf{x} \neq \mathbf{y}$, $\mathbf{x} \succeq \mathbf{z}$, $\mathbf{y} \succeq \mathbf{z}$" implies $t\,\mathbf{x} + (1 - t)\,\mathbf{y} \succ \mathbf{z} = \mathbf{y} \sim \mathbf{x}$. (Hence there is a relation between "preferences are a strictly convex binary relation" and "indifference curves are a strictly convex function of, in this graph, the single variable 'apples'.")



Existence of u : MWG p.47 $\left\{ \begin{array}{l} \text{rational: complete \& transitive (Varian)} \\ \text{MWG p.6 too} \end{array} \right.$

Marginal Rate of Substitution : $d\,u(\underset{\sim}{x}) = 0$ or, better: $u(\underset{\sim}{x}) = \text{constant}$

$$\frac{\partial u(\underset{\sim}{x})}{\partial x_i}\, dx_i + \frac{\partial u(\underset{\sim}{x})}{\partial x_j}\, dx_j = 0$$



$$\frac{dx_j}{dx_i} = - \frac{\partial u / \partial x_i}{\partial u / \partial x_j}$$

" strictly increasing "

Invariant to monotonic transformations :

$$d\, v(u(\underset{\sim}{x})) = 0 \quad \text{w/ } v'(u) > 0$$

$$\frac{dv}{du} \frac{\partial u}{\partial x_i} x_i + \frac{dv}{du} \frac{\partial u}{\partial x_j} dx_j = 0$$

$\vdots$

Prop.

If $u(\underset{\sim}{x})$ represents the preferences of a consumer and if $v(u)$ is a strictly increasing function of u then $v(u(\underset{\sim}{x}))$ also represents the preferences of that consumer.

iff

Not "positive monotonic transformation"

Proof. $v(u(\underset{\sim}{x}_1)) > v(u(\underset{\sim}{x}_2)) \Rightarrow u(\underset{\sim}{x}_1) > u(\underset{\sim}{x}_2) \Rightarrow \underset{\sim}{x}_1 \succ \underset{\sim}{x}_2$.

$\Leftarrow \qquad \Leftarrow$

■

Q-concavity vs. concavity . . . $u = \sqrt{x}$, $v(u) = u^4$ destroys concavity but not quasiconcavity

Altruism, jealousy. Value of giving away stuff ("potlatch") to happiness. (monotonicity)

Methodological Individualism.

'water freezes @ 0°C: derived from H_2O?

7.2

$$B \triangleq \{ \mathbf{x} : \mathbf{x} \in X \text{ and } \mathbf{p}\cdot\mathbf{x} \le m \}$$

$$\max\ \underbrace{u(\mathbf{x})}_{\text{continuous}} \quad \text{s.t.} \quad \mathbf{x} \in \underbrace{B}_{\text{closed \& bdd. (p.505)}}.$$

Under ~~local nonsatiation~~ strong monotonicity, $\mathbf{p}\cdot\mathbf{x} = m$.

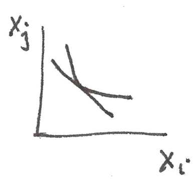
indirect utility function $v(\mathbf{p}, m) = \max u(\mathbf{x})$ s.t. $\mathbf{p}\cdot\mathbf{x} = m$

demand function $\mathbf{x}(\mathbf{p}, m)$ - homogeneous of degree 0 in $(\mathbf{p}, m)$

$$\mathcal{L} = u(\mathbf{x}) - \lambda(\mathbf{p}\cdot\mathbf{x} - m)$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \Rightarrow \frac{\partial u}{\partial x_i} - \lambda p_i = 0$$

$$\underbrace{\frac{\partial u/\partial x_i}{\partial u/\partial x_j}}_{\text{MRS}} = \frac{p_i}{p_j}$$



$$f(\mathbf{x} + \Delta\mathbf{x}) \approx f(\mathbf{x}) + \underbrace{\nabla f(\mathbf{x})}_{\text{gradient}} \cdot \Delta\mathbf{x} + \frac{1}{2} \Delta\mathbf{x}^T \underbrace{\nabla^2 f}_{\text{Hessian}} \Delta\mathbf{x}$$

$$f(\mathbf{x} + \Delta\mathbf{x}) - f(\mathbf{x}) = \underbrace{\nabla f}_{=0} \cdot \Delta\mathbf{x} + \frac{1}{2} \Delta\mathbf{x}^T \nabla^2 f \Delta\mathbf{x} \quad \le 0 ?$$

so $\nabla^2 f$ should be neg. semi definite $\forall$ admissible $\Delta\mathbf{x}$.

Here: $\nabla^2 \mathcal{L}(\mathbf{x})$ should be neg.

$\Delta\mathbf{x}^T \nabla^2 \mathcal{L}(\mathbf{x}) \Delta\mathbf{x} \le 0$

Don't cover; go to handout on S.O.C.

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7.3

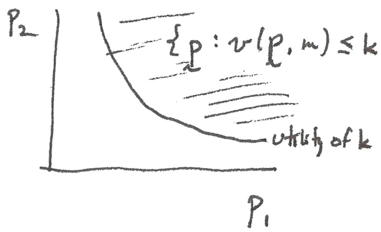
Indirect utility function $v(\underset{\sim}{p}, m)$

1) nonincreasing in $\underset{\sim}{p}$; nondecreasing in m

2) homogeneous of degree 0 in $(\underset{\sim}{p}, m)$

to prove, look at the affordable set

3) quasiconvex in p



(proof omitted)

4) continuous (p.506 Thm.(1).)

Expenditure function $e(\underset{\sim}{p}, \bar{u}) = \min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x}$ s.t. $u(\underset{\sim}{x}) \geq \bar{u}$

1) nondecreasing in $\underset{\sim}{p}$:

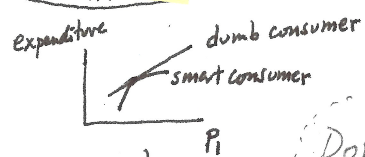
$\frac{\partial e}{\partial p_i} = h_i \geq 0$

(Shephard's Lemma) DO next page first! →

2) homog. degree 1 in $\underset{\sim}{p}$

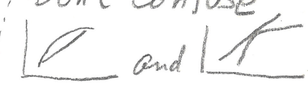
$\min_{\underset{\sim}{x}} \lambda \underset{\sim}{p} \cdot \underset{\sim}{x} = \ldots$ "λ" notation clash (Use α)

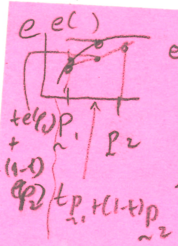
3) concave in $\underset{\sim}{p}$ (~~proof omitted~~)



4) continuous in $\underset{\sim}{p}$ for $\underset{\sim}{p} \gg 0$ (proof omitted)

Don't confuse





$e(t\underset{\sim}{p}_1 + (1-t)\underset{\sim}{p}_2)$

$= \min_{\underset{\sim}{x}} (t\underset{\sim}{p}_1 + (1-t)\underset{\sim}{p}_2)\hat{x}$

$\geq \min t\underset{\sim}{p}_1 \cdot \hat{x} + \min (1-t)\underset{\sim}{p}_2 \cdot \hat{x}$

$= t\, e(\underset{\sim}{p}_1) + (1-t)\, e(\underset{\sim}{p}_2)$

max(f+g) vs. max f + max g

5)

Hicksian demand function $\underset{\sim}{h}(\underset{\sim}{p}, u)$ is the answer to the exp. min. problem.

term? ↗

Shephard's Lemma: $h_i(\underset{\sim}{p}, u) = \dfrac{\partial e(\underset{\sim}{p}, u)}{\partial p_i}$. Proof ⟶

Marshallian is $\underset{\sim}{x}(\underset{\sim}{p}, m)$.

↓

MC,W,G: "Walrasian"

501-
502

Envelope Theorem. (p.501)

$M(a) =$

$$\max_{x_1, x_2} g(x_1, x_2, a)$$

s.t.

$$h(x_1, x_2, a) = 0.$$

$$\mathcal{L} = g - \lambda h.$$

$$\frac{d\,M(a)}{d\,a} = \frac{\partial \mathcal{L}^*}{\partial a}.$$

Expenditure Function.

$e(\underset{\sim}{p}, \hat{u}) =$

$$\min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x}$$

s.t.

$$u(\underset{\sim}{x}) - \hat{u} = 0.$$

$$\mathcal{L} = \underset{\sim}{p} \cdot \underset{\sim}{x} - \lambda \left[u(\underset{\sim}{x}) - \hat{u} \right].$$

$$\frac{\partial e}{\partial p_i} = \frac{\partial \mathcal{L}^*}{\partial p_i}$$

$$= x_i \quad . \quad \blacksquare$$

7.4

Do p.4 (Roy's Identity) first.

3 pages from now, i.e.:

$v(\underset{\sim}{p}, m) = \max_{\underset{\sim}{x}} u(\underset{\sim}{x})$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} = m$

$\partial v / \partial p_i = \ldots$ Envelope Thm.

$\partial v / \partial m = \ldots$

Solve for x_i^*.

1) exp. fnctn.

$$e(\underset{\sim}{p}, u) = m$$

$u \leftarrow v(\underset{\sim}{p}, m)$

shows how to go from e to v!

2) ind. u. fnctn

$$v(\underset{\sim}{p}, m) = u$$

$m \leftarrow e(\underset{\sim}{p}, u)$

shows how to go from v to e!

3) Hicksian demand $h_i(\underset{\sim}{p}, u) = x(\underset{\sim}{p}, m)$

$u \leftarrow v(\underset{\sim}{p}, m)$

4) Marshallian demand $x(\underset{\sim}{p}, m) = h(\underset{\sim}{p}, u)$

$m \leftarrow e(\underset{\sim}{p}, u)$

Hicksian d. fnctn. is
Marshallian " " "compensated"
to keep utility constant.

Appendix on p.113 (over)

Example. If $v(\underset{\sim}{p}, m) = \dfrac{m}{2\sqrt{p_1 p_2}}$ and $h_2(\underset{\sim}{p}, \hat{u}) = \hat{u}\sqrt{p_1/p_2}$ then find x_2.

Ans: from (3) $x_2(\underset{\sim}{p}, m) = h_2(\underset{\sim}{p}, v) = \dfrac{m}{2\sqrt{p_1 p_2}} \cdot \sqrt{\dfrac{p_1}{p_2}} = \dfrac{m}{2 p_2}$.

<u>Prop</u>. Utility maximization $\Rightarrow$ expenditure minimization.

Assume local nonsatiation

Let $\underset{\sim}{x}^{*u}$ solve $\max_{\underset{\sim}{x}} u(\underset{\sim}{x})$ s.t. $\underset{\sim}{p}\cdot\underset{\sim}{x} \le m$.

Let $\hat{u} = u(\underset{\sim}{x}^{*u})$.

Then $\underset{\sim}{x}^{*u}$ solves $\min_{\underset{\sim}{x}} \underset{\sim}{p}\cdot\underset{\sim}{x}$ s.t. $u(\underset{\sim}{x}) \geq \hat{u}$.

<u>Proof</u>. Suppose not. Let $\underset{\sim}{x}^{*e}$ solve $\min_{\underset{\sim}{x}} \underset{\sim}{p}\cdot\underset{\sim}{x}$ s.t. $u(\underset{\sim}{x}) \geq \hat{u}$. Note that $\underset{\sim}{x}^{*u}$ is admissible for this problem since $u(\underset{\sim}{x}^{*u}) = \hat{u}$.

Then $\underset{\sim}{p}\cdot\underset{\sim}{x}^{*e} < \underset{\sim}{p}\cdot\underset{\sim}{x}^{*u}$ and $u(\underset{\sim}{x}^{*e}) \geq \hat{u} = u(\underset{\sim}{x}^{*u})$.

Merely pedagogically helpful; else $\geq$

If $$u(\underset{\sim}{x}^{*e}) > u(\underset{\sim}{x}^{*u})$$
we'd be done because then $\underset{\sim}{x}^{*e}$ would solve the utility-maximization problem, which contradicts our assumption that $\underset{\sim}{x}^{*u}$ solves that problem.

How about if $$u(\underset{\sim}{x}^{*e}) = u(\underset{\sim}{x}^{*u})?$$

Then by local nonsatiation $\exists\, \underset{\sim}{x}''$ close enough to $\underset{\sim}{x}^{*e}$ that it still satisfies $\underset{\sim}{p}\cdot\underset{\sim}{x}'' < \underset{\sim}{p}\cdot\underset{\sim}{x}^{*u} \le m$, but $\underset{\sim}{x}''$ improves on $\underset{\sim}{x}^{*e}$:
$$u(\underset{\sim}{x}'') > u(\underset{\sim}{x}^{*e}) = u(\underset{\sim}{x}^{*u}).$$

Then $\underset{\sim}{x}''$, not $\underset{\sim}{x}^{*u}$, solves the utility-maximization problem, a contradiction. ∎

<u>Prop</u>. Expenditure minimization ⇒ utility maximization.

Assume continuity of u.

Let $\underset{\sim}{x}^{*e}$ solve $\min_{\underset{\sim}{x}} \underset{\sim}{p} \cdot \underset{\sim}{x}$ s.t. $u(\underset{\sim}{x}) \geq \hat{u}$.

Let $m = \underset{\sim}{p} \cdot \underset{\sim}{x}^{*e}$.

Then $\underset{\sim}{x}^{*e}$ solves $\max_{\underset{\sim}{x}} u(\underset{\sim}{x})$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} \leq m$.

<u>Proof</u>. Suppose not, and instead let $\underset{\sim}{x}^{*u}$ solve $\max_{\underset{\sim}{x}} u(\underset{\sim}{x})$ s.t. $\underset{\sim}{p} \cdot \underset{\sim}{x} \leq m$. Note that $\underset{\sim}{x}^{*e}$ is admissible for this problem since $\underset{\sim}{p} \cdot \underset{\sim}{x}^{*e} = m$.

Then $u(\underset{\sim}{x}^{*u}) > u(\underset{\sim}{x}^{*e})$ and $\underset{\sim}{p} \cdot \underset{\sim}{x}^{*u} \leq m = \underset{\sim}{p} \cdot \underset{\sim}{x}^{*e}$.

By the definition of $\underset{\sim}{x}^{*e}$, $u(\underset{\sim}{x}^{*e}) \geq \hat{u}$, so

$$u(\underset{\sim}{x}^{*u}) > u(\underset{\sim}{x}^{*e}) \geq \hat{u}.$$

[If $\underset{\sim}{p} \cdot \underset{\sim}{x}^{*u} < \underset{\sim}{p} \cdot \underset{\sim}{x}^{*e}$ we'd be done because then $\underset{\sim}{x}^{*u}$ would solve the expenditure minimization problem, contradicting our assumption that $\underset{\sim}{x}^{*e}$ does so.]

Let $\lambda \in (0,1)$ (think of it being very close to 1). Then

$$\underbrace{\underset{\sim}{p} \cdot (\lambda \underset{\sim}{x}^{*u})}_{\text{strictly cheaper than}} < \underset{\sim}{p} \cdot \underset{\sim}{x}^{*u} \leq \underset{\sim}{p} \cdot \underset{\sim}{x}^{*e}$$

and by continuity of $u(\cdot)$, since $u(\underset{\sim}{x}^{*u}) > u(\underset{\sim}{x}^{*e})$,

$$u(\lambda \underset{\sim}{x}^{*u}) > u(\underset{\sim}{x}^{*e}) \text{ for } \lambda \text{ close to } 1.$$

Then $\lambda \underset{\sim}{x}^{*u}$ solves the expenditure-minimization problem, not $\underset{\sim}{x}^{*e}$, a contradiction. ∎

Roy's Identity. $\quad x_i(\underset{\sim}{p}, m) = -\dfrac{\partial v(\underset{\sim}{p}, m)/\partial p_i}{\partial v(\underset{\sim}{p}, m)/\partial m}$ for $i = 1, 2, \ldots, k$.

Proof.

Envelope thm. (from §7.3)

$M(\underset{\sim}{a}) =$

$$\max_{x_1, x_2} g(x_1, x_2, \underset{\sim}{a})$$

s.t.

$$h(x_1, x_2, \underset{\sim}{a}) = 0.$$

$$\mathcal{L} = g - \lambda h.$$

$$\frac{\partial M(a)}{\partial a_i} = \frac{\partial \mathcal{L}^*}{\partial a_i}.$$

Indirect Utility Function.

$v(\underset{\sim}{p}, m) =$

$$\max_{\underset{\sim}{x}} u(\underset{\sim}{x})$$

s.t.

$$\underset{\sim}{p} \cdot \underset{\sim}{x} - m = 0.$$

$$\mathcal{L} = u(\underset{\sim}{x}) - \lambda[\underset{\sim}{p} \cdot \underset{\sim}{x} - m].$$

$$\begin{cases} \dfrac{\partial v}{\partial m} = \dfrac{\partial}{\partial m}(\mathcal{L}^*) = \lambda & \text{(sensitivity analysis)} \\ \dfrac{\partial v}{\partial p_i} = \dfrac{\partial}{\partial p_i}(\mathcal{L}^*) = -\lambda x_i. \end{cases}$$

Substituting for λ, one has

$$\frac{\partial v}{\partial p_i} = -\frac{\partial v}{\partial m} x_i. \quad \blacksquare$$