(5.1)

$$\text{SR}\begin{cases} TC,\ VC,\ FC \\ AC \\ AVC \\ AFC \\ MC \end{cases} \qquad \text{LR}\begin{cases} TC \\ AC \\ MC \end{cases}$$

<u>Prop.</u> If the production function has constant returns to scale, then

$$c(\underset{\sim}{w}, y) = y\, \underbrace{c(\underset{\sim}{w}, 1)}_{AC}.$$

$c(\underset{\sim}{w}, \lambda y) = \lambda^1 c(\underset{\sim}{w}, y)$ "homog. deg. 1 in y"

<u>Note</u>: $AC = \frac{c(\underset{\sim}{w}, y)}{y} = c(\underset{\sim}{w}, 1)$, not a function of y.

<u>Proof.</u>

Let $\underset{\sim}{x}'$ be opt ... $c(\underset{\sim}{w}', 1) = \underset{\sim}{w}' \cdot \underset{\sim}{x}'$ and $\overbrace{\ldots, \underset{\sim}{x}' \in V(1),}$

$\underset{\sim}{w}$ ... Multiply by y:

$y\, \underset{\sim}{w}'$ ... Rearrange:

$\underset{\sim}{w}' \cdot ($ ...

... $\sim$ $y\, \underset{\sim}{x} \in V(y)$ & vice versa

... $\in V(y)$ and

$\underset{\sim}{w}' \cdot$ ... $) \iff$ (if $\underset{\sim}{x}'' \triangleq y\, \underset{\sim}{x}$)

$\underset{\sim}{w}' \cdot ($ ... $)$. Since $y\, \underset{\sim}{x}' \in V(y)$, $\Rightarrow$

<u>Better proof</u>:

$$c(\underset{\sim}{w}, y) = \min_{\underset{\sim}{x}} \underset{\sim}{w} \cdot \underset{\sim}{x} \ \text{ s.t. } f(\underset{\sim}{x}) = y$$

$$= \min_{\underset{\sim}{x}} \underset{\sim}{w} \cdot \underset{\sim}{x} \ \text{ s.t. } \tfrac{1}{y} f(\underset{\sim}{x}) = 1$$

$$= \min_{\underset{\sim}{x}} \underset{\sim}{w} \cdot \underset{\sim}{x} \ \text{ s.t. } f\left(\tfrac{1}{y} \underset{\sim}{x}\right) = 1$$

$(\because \text{CRS})$ $(f(\lambda \underset{\sim}{x}) = \lambda f(\underset{\sim}{x}))$

$$= y \min_{\underset{\sim}{x}} \underset{\sim}{w} \cdot \tfrac{1}{y} \underset{\sim}{x} \ \text{ s.t. } f\left(\tfrac{1}{y} \underset{\sim}{x}\right) = 1$$

$$= y \min_{\frac{1}{y}\underset{\sim}{x}} \underset{\sim}{w} \cdot \tfrac{1}{y} \underset{\sim}{x} \ \text{ s.t. } f\left(\tfrac{1}{y} \underset{\sim}{x}\right) = 1$$

$\because$ $\underset{\sim}{x}$ & y only appear together $\underset{\sim}{z} \triangleq \frac{\underset{\sim}{x}}{y}$:

$$= y \min_{\underset{\sim}{z}} \underset{\sim}{w} \cdot \underset{\sim}{z} \ \text{ s.t. } f(\underset{\sim}{z}) = 1$$

$$\left(= y \min_{\underset{\sim}{x}} \underset{\sim}{w} \cdot \underset{\sim}{x} \ \text{ s.t. } f(\underset{\sim}{x}) = 1\right)$$

$$= y \cdot \underbrace{c(\underset{\sim}{w}, 1)}_{AC}.$$

<u>Corollary.</u> CRS $\Rightarrow$ MC = AC.

<u>Proof.</u> $MC = \frac{\partial}{\partial y} c(\underset{\sim}{w}, y) = \frac{\partial}{\partial y} y\, c(\underset{\sim}{w}, 1) = \underset{\substack{\uparrow \\ \text{a constant} \\ \text{w.r.t. } y}}{c(\underset{\sim}{w}, 1)} = \frac{c(\underset{\sim}{w}, y)}{y} = AC.$ ∎

(5.2)

<u>Prop.</u> $MC(q = 0) = AVC(q = 0)$. (or "y")

<u>Proof.</u> $AVC(q=0) = \lim_{q \to 0} \frac{VC(q)}{q} = \text{"}\frac{0}{0}\text{"}$

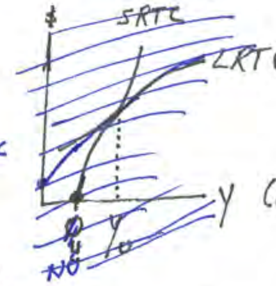
$= \lim_{q \to 0} \frac{MC(q)}{1}$ using L'Hopital's Rule

$= MC(0)$. ∎

(5.3)

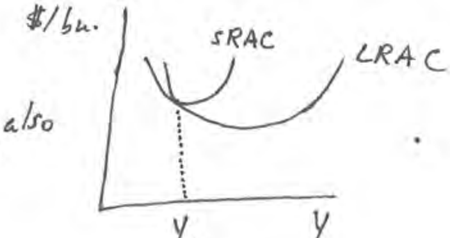
- $LRTC(y) \leq SRTC(y, W_0)$ ← fixed input, "water" (not $\underset{\sim}{w}$)
- $LRTC(y) = \min_{W_0} SRTC(y, W_0)$.
- Suppose that at output level y_0, the cost-minimizing amount of water is W_0^*.

Then:



y (output)

. Clearly also



$$LRTC(y_0) = SRTC(y_0, W_0^*)$$
$$= SRTC(y_0, W_0^*(y_0))$$

(Just divide this by y_0.)

At the y_0 where $LRTC = SRTC$, $LRAC = SRATC$ and $LRMC = SRMC$.

Prop. $LRMC(y_0) = SRMC(y_0)$.

Proof. Envelope Theorem (§7.3, §7.4, but esp. §3.2)

$M(a) = \max_{x_1, x_2} g(x_1, x_2, a)$

$LRTC(y) = \min_{\underset{\sim}{x}\text{ fixed}} SRTC(y, \underset{\sim}{w}, \underset{\sim}{x}\text{ fixed})$

~~$\min_{W_0} SRTC(y, W_0)$~~

$$\frac{d\,M(a)}{da} = \frac{\partial g}{\partial a}^{*} \qquad \frac{d\,LRTC(y)}{dy} = \frac{\partial\,SRTC}{\partial y}^{*}$$

or $LRMC(y) = SRMC(y, \underset{\sim}{W_0}^*)$ ($\underset{\sim}{x}^*$ fixed) ~~when $w = w_0$ (i.e., at the optimum)~~. ∎

(5.4)

$c(\underset{\sim}{w}, y)$ is: $= \min_{\underset{\sim}{x} \in V(y)} \underset{\sim}{w} \cdot \underset{\sim}{x}$

1) Nondecreasing in $\underset{\sim}{w}$: if $\underset{\sim}{w}' \geq \underset{\sim}{w}$ then $c(\underset{\sim}{w}', y) \geq c(\underset{\sim}{w}, y)$.

Proof. Let $\underset{\sim}{x}$ be optimal when prices are $\underset{\sim}{w}$;

" $\underset{\sim}{x}'$ " " " " " $\underset{\sim}{w}'$. & both $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ produce y.

At prices $\underset{\sim}{w}$: $\underset{\sim}{w} \cdot \underset{\sim}{x} \leq \underset{\sim}{w} \cdot \underset{\sim}{x}'$.

By assumption: $\underset{\sim}{w} \cdot \underset{\sim}{x}' \leq \underset{\sim}{w}' \cdot \underset{\sim}{x}'$.

$\therefore \quad \underset{\sim}{w} \cdot \underset{\sim}{x} \leq \underset{\sim}{w}' \cdot \underset{\sim}{x}'$. ∎

Alternative proof:
$\frac{\partial c(\underset{\sim}{w}, y)}{\partial w_i} = x_i \geq 0$ by assumption.
Shephard's Lemma, §5.4.

2) Homogeneous of degree one in $\underset{\sim}{w}$: $c(t\underset{\sim}{w}, y) = t\, c(\underset{\sim}{w}, y)$ for $t > 0$.

Proof. Let $\underset{\sim}{x}$ be cost-minimizing at prices $\underset{\sim}{w}$ for output y.

Then $\underset{\sim}{w} \cdot \underset{\sim}{x} \leq \underset{\sim}{w} \cdot \underset{\sim}{x}' \quad \forall\, \underset{\sim}{x}'$ s.t. $f(\underset{\sim}{x}') = y$.

$(t\underset{\sim}{w}) \cdot \underset{\sim}{x} \leq (t\underset{\sim}{w}) \cdot \underset{\sim}{x}'$ —— " ——.

CRS:
$c(w, ty) = t g\, c(w,1)$
$= t\, c(w,y)$

So at prices $t\underset{\sim}{w}$, $\underset{\sim}{x}$ is the cost minimizing bundle, and

$$c(t\underset{\sim}{w}, y) = t\, \underset{\sim}{w} \cdot \underset{\sim}{x}$$
$$= t\, c(\underset{\sim}{w}, x). \blacksquare$$

Corollary: $x_i(\underset{\sim}{w}, y)$ is homogeneous of degree zero in $\underset{\sim}{w}$.
Proof: Shephard's Lemma.

Luenberger's proof:

$$c(t\underset{\sim}{w}, y) = \min_{\underset{\sim}{x} \in V(y)} t\underset{\sim}{w} \cdot \underset{\sim}{x} = t \min_{\underset{\sim}{x} \in V(y)} \underset{\sim}{w} \cdot \underset{\sim}{x} = t\, c(\underset{\sim}{w}, y). \blacksquare$$

$\Rightarrow$ Intuition ...

3) Concave in $\underset{\sim}{w}$:

$$c(t\underset{\sim}{w} + (1-t)\underset{\sim}{w}', y) \geq t\, c(\underset{\sim}{w}, y) + (1-t)\, c(\underset{\sim}{w}', y)$$

Luenberger's proof:

$$LHS = \min_{\underset{\sim}{x} \in V(y)} [t\underset{\sim}{w} + (1-t)\underset{\sim}{w}'] \cdot \underset{\sim}{x}$$

$$= \min_{\underset{\sim}{x} \in V(y)} \{ [t\underset{\sim}{w} \cdot \underset{\sim}{x}] + [(1-t)\underset{\sim}{w}' \cdot \underset{\sim}{x}] \}$$

$$\geq \min_{\underset{\sim}{x} \in V(y)} t\underset{\sim}{w} \cdot \underset{\sim}{x} + \min_{\underset{\sim}{x} \in V(y)} (1-t)\underset{\sim}{w}' \cdot \underset{\sim}{x}$$

$$= t \min_{\underset{\sim}{x} \in V(y)} \underset{\sim}{w} \cdot \underset{\sim}{x} + (1-t) \min_{\underset{\sim}{x} \in V(y)} \underset{\sim}{w}' \cdot \underset{\sim}{x}$$

$$= t\, c(\underset{\sim}{w}, y) + (1-t)\, c(\underset{\sim}{w}', y). \blacksquare$$

$\underset{\sim}{w}$ → cost-min. is $\underset{\sim}{x}$

$\underset{\sim}{w}'$ → cost-min. is $\underset{\sim}{x}'$

$\underset{\sim}{w}''$: cost-min is $\underset{\sim}{x}''$

$$y = f(\underset{\sim}{x}) = f(\underset{\sim}{x}') = f(\underset{\sim}{x}'').$$

$$c(t\underset{\sim}{w} + (1-t)\underset{\sim}{w}') \triangleq [t\underset{\sim}{w} + (1-t)\underset{\sim}{w}'] \cdot \underset{\sim}{x}'' \quad \leftarrow \text{optimal}$$

$$\underset{\sim}{w}'' \triangleq t\underset{\sim}{w} + (1-t)\underset{\sim}{w}'$$

$$c(\underset{\sim}{w}'') = \underset{\sim}{w}'' \cdot \underset{\sim}{x}'' = t\underset{\sim}{w} \cdot \underset{\sim}{x}'' + (1-t)\underset{\sim}{w}' \cdot \underset{\sim}{x}''$$

At prices $\underset{\sim}{w}$, $\quad \underset{\sim}{w} \cdot \underset{\sim}{x}'' \geq \underset{\sim}{w} \cdot \underset{\sim}{x} = c(\underset{\sim}{w}, y)$.

$$t\underset{\sim}{w} \cdot \underset{\sim}{x}'' \geq t\underset{\sim}{w} \cdot \underset{\sim}{x} = t\, c(\underset{\sim}{w}, y)$$

At prices $\underset{\sim}{w}'$,

$$\underset{\sim}{w}' \cdot \underset{\sim}{x}'' \geq \underset{\sim}{w}' \cdot \underset{\sim}{x}' = c(\underset{\sim}{w}', y)$$

$$(1-t)\underset{\sim}{w}' \cdot \underset{\sim}{x}'' \geq (1-t)\underset{\sim}{w}' \cdot \underset{\sim}{x}' = (1-t)\, c(\underset{\sim}{w}', y).$$

Adding: $\quad t\underset{\sim}{w} \cdot \underset{\sim}{x}'' + (1-t)\underset{\sim}{w}' \cdot \underset{\sim}{x}'' \geq t\, c(\underset{\sim}{w}, y) + (1-t)\, c(\underset{\sim}{w}', y)$

$$\hookrightarrow = [t\underset{\sim}{w} + (1-t)\underset{\sim}{w}'] \cdot \underset{\sim}{x}'' \triangleq c(t\underset{\sim}{w} + (1-t)\underset{\sim}{w}'). \blacksquare$$

4) $c(\underset{\sim}{w}, y)$ is continuous in $\underset{\sim}{w}$ [and in y].

$\Rightarrow$ Fig. 5.4

<u>Shephard's Lemma</u>: $x_i(\underset{\sim}{w}, y) = \dfrac{\partial c(\underset{\sim}{w}, y)}{\partial w_i}$.

<u>Proof</u>.

§§7.3, 7.4, 3.2, 5.3

<u>Envelope theorem</u>

$M(a) =$

$$\max_{x_1, x_2} g(x_1, x_2, a)$$

s.t.

$$h(x_1, x_2, a) = 0.$$

$$\mathcal{L} = g - \lambda h$$

$$\frac{dM(a)}{da} = \frac{\partial \mathcal{L}^*}{\partial a}.$$

<u>Cost Function.</u>

$c(\underset{\sim}{w}, y) =$

$$\min_{\underset{\sim}{x}} \underset{\sim}{w} \cdot \underset{\sim}{x}$$

s.t.

$$f(x_1, x_2) - y = 0.$$

← $\underset{\sim}{x}$ list x_1, x_2

$$\mathcal{L} = \underset{\sim}{w} \cdot \underset{\sim}{x} - \lambda \left[f(x_1, x_2) - y \right]$$

$$\frac{\partial c(\underset{\sim}{w}, y)}{\partial w_i} = \frac{\partial \mathcal{L}^*}{\partial w_i}$$

$$= x_i. \blacksquare$$

Input Demands $y_i^*(\underset{\sim}{p})$ Ch. 3

Input Demands $x_i^*(\underset{\sim}{w}, y)$ Ch. 5

<u>Also</u>: $\dfrac{\partial c(\underset{\sim}{w}, y)}{\partial y} = \dfrac{\partial \mathcal{L}^*}{\partial y} \Rightarrow$

$$MC = \frac{\partial c(\underset{\sim}{w}, y)}{\partial y} = \lambda^*.$$

(5.5) done already

(5.6)

1)

1& 2 just repeat §5.4.

Note:
3 is weaker than §3.4 ∵ y is fixed.

$= x_i$ Shephard's Lemma

≥ 0 by assumption. ∎

2) ... $\underset{\sim}{w}$.

is homogeneous of degree one

∎

3a) $$\frac{\partial x_i(\underset{\sim}{w}, y)}{\partial w_j} = \frac{\partial x_j(\underset{\sim}{w}, y)}{\partial w_i}.$$

Proof: $\underset{\sim}{x} = \nabla_{\underset{\sim}{w}}\, c(\underset{\sim}{w}, y)$ Shephard's Lemma

$\nabla_{\underset{\sim}{w}}\, \underset{\sim}{x} = \nabla^2_{\underset{\sim}{w}}\, c(\underset{\sim}{w}, y)$ which is symmetric. ∎

3b) $$\boxed{\frac{\partial x_i(\underset{\sim}{w}, y)}{\partial w_i} \leq 0 \;\; \forall i.}$$

Proof: This is negative semidefinite ∵ from §5.4 #3, $c(\underset{\sim}{w}, y)$ is concave in $\underset{\sim}{w}$. It follows that the diagonal terms are nonpositive. ∎

3c) $\boxed{d\underset{\sim}{w} \cdot d\underset{\sim}{x} \leq 0.}$

Proof: $d\underset{\sim}{x} = \nabla_{\underset{\sim}{w}}\, \underset{\sim}{x} \cdot d\underset{\sim}{w}$ (← Jacobian) (Chain Rule). Left-multiply by $d\underset{\sim}{w}$:

$$d\underset{\sim}{w} \cdot d\underset{\sim}{x} = d\underset{\sim}{w} \cdot \nabla_{\underset{\sim}{w}}\, \underset{\sim}{x} \cdot d\underset{\sim}{w}$$

$$= d\underset{\sim}{w} \cdot \nabla^2_{\underset{\sim}{w}}\, c(\underset{\sim}{w}, y) \cdot d\underset{\sim}{w} \leq 0$$ since $\nabla^2_{\underset{\sim}{w}} c$ is negative semidefinite (∵ c is concave in $\underset{\sim}{w}$). ∎

Example 1. $w_1 \uparrow, \overline{w_2}, \overline{w_3}$. Then we know $x_1 \downarrow$. What about x_2 and x_3?

$$0 \geq d\underset{\sim}{w} \cdot d\underset{\sim}{x} = dw_1\, dx_1 + dw_2\, dx_2 + dw_3\, dx_3 = dw_1\, dx_1 .$$

No new information here.

Example 2. $w_1 \uparrow, w_2 \downarrow$. Can't use partial derivative results like $\frac{\partial x_i}{\partial w_i} \leq 0$ ∵ more than one input price is changing.

$$dw_1\, dx_1 + dw_2\, dx_2 \leq 0$$

Most intuitive: $x_1 \downarrow, x_2 \uparrow$ (−) (−) could happen

Most counter-intuitive:

$x_1 \uparrow, x_2 \downarrow$ (+) (+) ≤ 0 impossible.

Third possibility:

$x_1 \downarrow, x_2 \downarrow$ (−) (+) ≤ 0 could happen

Last possibility:

$x_1 \uparrow, x_2 \uparrow$ (+) (−) ≤ 0 could happen

Firms have no Budget Constraint.

Mirowski − cons / prod same − why?

TAMU rat lab