

5.1

$$SR \begin{cases} TC, VC, FC \\ AC \\ AVC \\ AFC \\ MC \end{cases}$$

$$LR \begin{cases} TC \\ AC \\ MC \end{cases}$$

Prop. If the production function has constant returns to scale, then

$$c(\underline{w}, y) = y \underbrace{c(\underline{w}, 1)}_{AC \dots \dots \dots}$$

$c(\underline{w}, \lambda y) = \lambda^1 c(\underline{w}, y)$  "homog. deg. 1 in y"  
Note:  $AC = \frac{c(\underline{w}, y)}{y} = c(\underline{w}, 1)$ , w/o function of y.

Proof.

Let  $\underline{x}'$  be opt

Better proof:

$$\begin{aligned} c(\underline{w}, y) &= \min_{\underline{x}} \underline{w} \cdot \underline{x} \text{ s.t. } f(\underline{x}) = y \\ &= \min_{\underline{x}} \underline{w} \cdot \underline{x} \text{ s.t. } \frac{1}{y} f(\underline{x}) = 1 \\ &= \min_{\underline{x}} \underline{w} \cdot \underline{x} \text{ s.t. } f\left(\frac{1}{y} \underline{x}\right) = 1 \\ &\quad (\because \text{CRS}) (f(\lambda \underline{x}) = \lambda f(\underline{x})) \\ &= y \min_{\underline{x}} \underline{w} \cdot \frac{1}{y} \underline{x} \text{ s.t. } f\left(\frac{1}{y} \underline{x}\right) = 1 \\ &= y \min_{\frac{1}{y} \underline{x}} \underline{w} \cdot \frac{1}{y} \underline{x} \text{ s.t. } f\left(\frac{1}{y} \underline{x}\right) = 1 \\ &\quad \because \underline{x} \text{ \& only appear together } \underline{z} \triangleq \frac{\underline{x}}{y} \\ &= y \min_{\underline{z}} \underline{w} \cdot \underline{z} \text{ s.t. } f(\underline{z}) = 1 \\ &= y \min_{\underline{x}} \underline{w} \cdot \underline{x} \text{ s.t. } f(\underline{x}) = 1 \\ &= y \cdot \underbrace{c(\underline{w}, 1)}_{AC} \end{aligned}$$

Multiply by y:  
 Rearrange:  
 $c(\underline{w}', 1) = \underline{w}' \cdot \underline{x}'$  and  
 $\underline{x}' \in V(1)$ ,  
 $\underline{x} \in V(y)$  & vice versa  
 $\Leftrightarrow$  (if  $\underline{x}'' \triangleq y \underline{x}$ )  
 Since  $y \underline{x}' \in V(y)$ ,  $\Rightarrow$

Corollary. CRS  $\Rightarrow$  MC = AC.

Proof.  $MC = \frac{\partial}{\partial y} c(\underline{w}, y) = \frac{\partial}{\partial y} y c(\underline{w}, 1) = c(\underline{w}, 1) = \frac{c(\underline{w}, y)}{y} = AC.$

↑  
a constant w.r.t. y

5.2

Prop.  $MC(\overset{\text{or "y" } \curvearrowright}{q}=0) = AVC(\overset{\curvearrowright}{q}=0).$

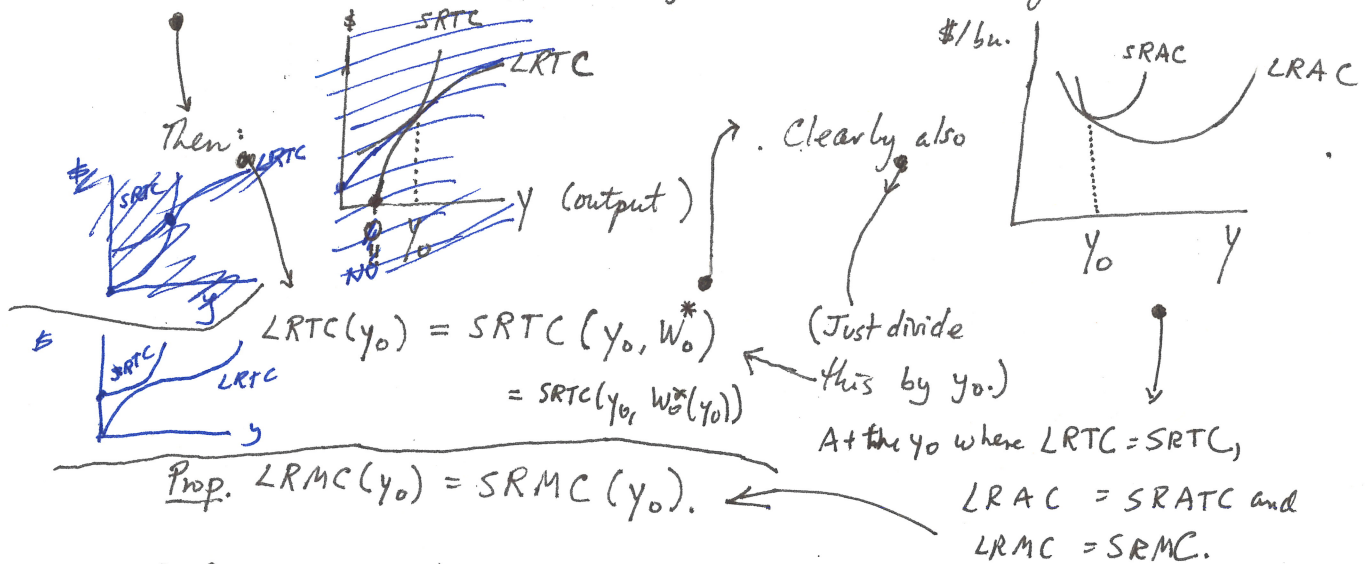
Proof.  $AVC(q=0) = \lim_{q \rightarrow 0} \frac{VC(q)}{q} = \frac{0}{0}$

$= \lim_{q \rightarrow 0} \frac{MC(q)}{1} \text{ using L'Hopital's Rule}$

$= MC(0). \quad \blacksquare$

5.3

- $LRTC(y) \leq SRTC(y, w_0)$  fixed input, "water" (not  $w$ )
- $LRTC(y) = \min_{w_0} SRTC(y, w_0)$ .
- Suppose that at output level  $y_0$ , the cost-minimizing amount of water is  $w_0^*$ .



Proof. Envelope Theorem  
(§7.3, §7.4, but esp. §3.2)

$$M(a) = \max_{x_1, x_2} q(x_1, x_2, a)$$

$$\frac{dM(a)}{da} = \frac{\partial q}{\partial a}^*$$

$$LRTC(y) = \min_{w_0} SRTC(y, w_0, x \text{ fixed})$$

$$\frac{dLRTC(y)}{dy} = \frac{\partial SRTC}{\partial y}^*$$

$$\text{or } LRMC(y) = SRMC(y, w_0^*) \text{ when } w_0 = w_0^*(y)$$

$w_0 = w_0^*$  (i.e., at the optimum) . ■

(5.4)

$c(\underline{w}, y)$  is:

$$= \min_{\underline{x} \in V(y)} \underline{w} \cdot \underline{x}$$

1) Nondecreasing in  $\underline{w}$ : if  $\underline{w}' \geq \underline{w}$  then  $c(\underline{w}', y) \geq c(\underline{w}, y)$ .

Proof. Let  $\underline{x}$  be optimal when prices are  $\underline{w}$ ;

"  $\underline{x}'$  " " " "  $\underline{w}'$ .  $\nexists$  both  $\underline{x}$  and  $\underline{x}'$  produce  $y$ .

At prices  $\underline{w}$ :  $\underline{w} \cdot \underline{x} \leq \underline{w} \cdot \underline{x}'$ .

By assumption:

$$\underline{w} \cdot \underline{x}' \leq \underline{w}' \cdot \underline{x}'$$

$$\therefore \underline{w} \cdot \underline{x} \leq \underline{w}' \cdot \underline{x}' \quad \blacksquare$$

Alternative proof:

$$\frac{\partial c(\underline{w}, y)}{\partial w_i} = x_i \geq 0 \text{ by assumption.}$$

Shephard's Lemma, §5.4.

2) Homogeneous of degree one in  $\underline{w}$ :  $c(t\underline{w}, y) = t c(\underline{w}, y)$  for  $t > 0$ .

Proof. Let  $\underline{x}$  be cost-minimizing at prices  $\underline{w}$  for output  $y$ .

Then  $\underline{w} \cdot \underline{x} \leq \underline{w} \cdot \underline{x}' \quad \forall \underline{x}' \text{ s.t. } f(\underline{x}') = y$ .

$$(t\underline{w}) \cdot \underline{x} \leq (t\underline{w}) \cdot \underline{x}' \quad \text{--- " ---}$$

So at prices  $t\underline{w}$ ,  $\underline{x}$  is the cost minimizing bundle, and

$$\begin{aligned} c(t\underline{w}, y) &= t\underline{w} \cdot \underline{x} \\ &= t c(\underline{w}, y). \quad \blacksquare \end{aligned}$$

Luenberger's proof:

$$c(t\underline{w}, y) = \min_{\underline{x} \in V(y)} t\underline{w} \cdot \underline{x} = t \min_{\underline{x} \in V(y)} \underline{w} \cdot \underline{x} = t c(\underline{w}, y). \quad \blacksquare$$

CRS:  
 $c(\underline{w}, ty) = t c(\underline{w}, y)$   
 $= t c(\underline{w}, y)$

Corollary:  $x_c(\underline{w}, y)$  is homogeneous of degree zero in  $\underline{w}$ .

Proof: Shephard's Lemma.



3) Concave in  $\underline{w}$ :

$\Rightarrow$  Intuition ...

$$c(t\underline{w} + (1-t)\underline{w}', y) \geq t c(\underline{w}, y) + (1-t) c(\underline{w}', y)$$

Luenberger's proof:

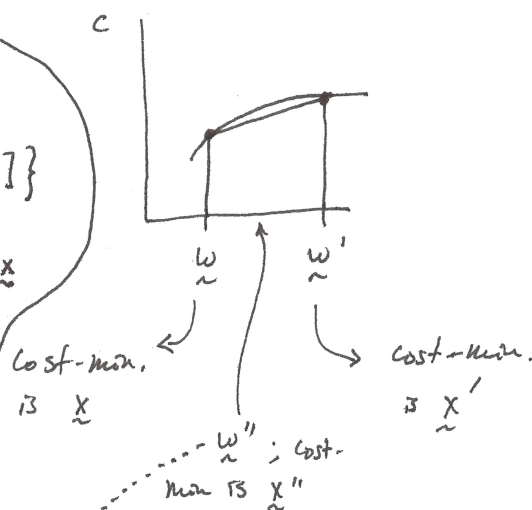
$$\text{LHS} = \min_{\underline{x} \in V(y)} [t\underline{w} + (1-t)\underline{w}'] \cdot \underline{x}$$

$$= \min_{\underline{x} \in V(y)} \{ [t\underline{w} \cdot \underline{x}] + [(1-t)\underline{w}' \cdot \underline{x}] \}$$

$$\geq \min_{\underline{x} \in V(y)} t\underline{w} \cdot \underline{x} + \min_{\underline{x} \in V(y)} (1-t)\underline{w}' \cdot \underline{x}$$

$$= t \min_{\underline{x} \in V(y)} \underline{w} \cdot \underline{x} + (1-t) \min_{\underline{x} \in V(y)} \underline{w}' \cdot \underline{x}$$

$$= t c(\underline{w}, y) + (1-t) c(\underline{w}', y) \quad \blacksquare$$



$$y = \overset{f}{\mathcal{N}}(\underline{x}) = \overset{f}{\mathcal{N}}(\underline{x}') \\ = \overset{f}{\mathcal{N}}(\underline{x}'')$$

$$c(t\underline{w} + (1-t)\underline{w}') \triangleq [t\underline{w} + (1-t)\underline{w}'] \cdot \underline{x}'' \quad \swarrow \text{optimal}$$

$$\underline{w}'' \triangleq t\underline{w} + (1-t)\underline{w}'$$

$$c(\underline{w}'') = \underline{w}'' \cdot \underline{x}'' = t\underline{w} \cdot \underline{x}'' + (1-t)\underline{w}' \cdot \underline{x}''$$

$$\text{At prices } \underline{w}, \quad \underline{w} \cdot \underline{x}'' \geq \underline{w} \cdot \underline{x} = c(\underline{w}, y).$$

$$t\underline{w} \cdot \underline{x}'' \geq t\underline{w} \cdot \underline{x} = t c(\underline{w}, y)$$

At prices  $\underline{w}'$ ,

$$\underline{w}' \cdot \underline{x}'' \geq \underline{w}' \cdot \underline{x}' = c(\underline{w}', y)$$

$$(1-t)\underline{w}' \cdot \underline{x}'' \geq (1-t)\underline{w}' \cdot \underline{x}' = (1-t) c(\underline{w}', y).$$

$$\text{Adding:} \quad t\underline{w} \cdot \underline{x}'' + (1-t)\underline{w}' \cdot \underline{x}'' \geq t c(\underline{w}, y) + (1-t) c(\underline{w}', y)$$

$$\hookrightarrow [t\underline{w} + (1-t)\underline{w}'] \cdot \underline{x}'' \triangleq c(t\underline{w} + (1-t)\underline{w}'). \quad \blacksquare$$

4)  $c(\underline{w}, y)$  is continuous in  $\underline{w}$  [and in  $y$ ].

$\Rightarrow$  Fig. 5.4

Shephard's Lemma:  $x_i(\underline{w}, y) = \frac{\partial c(\underline{w}, y)}{\partial w_i}$ .

Proof.

Envelope Theorem <sup>§§ 7.3, 7.4,  
3.2, 5.3</sup>

$$M(a) =$$

$$\max_{x_1, x_2} g(x_1, x_2, a)$$

s.t.

$$h(x_1, x_2, a) = 0.$$

$$\mathcal{L} = g - \lambda h$$

$$\frac{dM(a)}{da} = \frac{\partial \mathcal{L}^*}{\partial a}$$

Cost Function

$$c(\underline{w}, y) =$$

$$\min_{\underline{x}} \underline{w} \cdot \underline{x}$$

s.t.

$$f(x_1, x_2) - y = 0.$$

$\leftarrow \underline{x}$  not  $x_1, x_2$

$$\mathcal{L} = \underline{w} \cdot \underline{x} - \lambda [f(x_1, x_2) - y]$$

$$\frac{\partial c(\underline{w}, y)}{\partial w_i} = \frac{\partial \mathcal{L}^*}{\partial w_i}$$

$$= x_i. \blacksquare$$

Input Derives  $y_i^*(p)$  Ch. 3

Input Derives  $x_i^*(\underline{w}, y)$  Ch. 5

$$\text{Also: } \frac{\partial c(\underline{w}, y)}{\partial y} = \frac{\partial \mathcal{L}^*}{\partial y} \Rightarrow$$

$$MC = \frac{\partial c(\underline{w}, y)}{\partial y} = \lambda^*$$

(5.5) done already

5.6

1)

1 & 2 just repeat § 5.4.

$x_i$  Shephard's Lemma

$\geq 0$  by assumption. ■

Note:

2) 3 is weaker than § 3.4 'y' is fixed.

$w$ .

is homogeneous of degree one

$$3a) \frac{\partial x_i(w, y)}{\partial w_j} = \frac{\partial x_j(w, y)}{\partial w_i}.$$

Proof:  $\underline{x} = \nabla_w c(w, y)$  Shephard's Lemma

$\nabla_w \underline{x} = \nabla_w^2 c(w, y)$  which is symmetric. ■

$$3b) \frac{\partial x_i(w, y)}{\partial w_i} \leq 0 \quad \forall i.$$

Proof: This is negative semidefinite  $\because$  from § 5.4 # 3,  $c(w, y)$  is concave in  $w$ . It follows that the diagonal terms are nonpositive. ■

$$3c) d_w \cdot d_x \leq 0.$$

Proof:  $d_x = \nabla_w \underline{x} \cdot dw$  (Chain Rule). Left-multiply by  $d_w$ :

$$d_w \cdot d_x = d_w \cdot \nabla_w \underline{x} \cdot dw$$

$$= d_w \cdot \nabla_w^2 c(w, y) \cdot dw \leq 0 \text{ since } \nabla_w^2 c$$

is negative semidefinite

( $\because c$  is concave in  $w$ ). ■

Example 1.  $w_1 \uparrow, \bar{w}_2, \bar{w}_3$ . Then we know  $x_1 \downarrow$ . What about  $x_2$  and  $x_3$ ?

$$0 \geq dw \cdot dx = dw_1 dx_1 + dw_2 dx_2 + dw_3 dx_3 = dw_1 dx_1$$

No new information here.

Example 2.  $w_1 \uparrow, w_2 \downarrow$ . Can't use partial derivative results like  $\frac{\partial x_i}{\partial w_i} \leq 0 \because$  more than

Most intuitive:  $x_1 \downarrow, x_2 \uparrow$

$$dw_1 dx_1 + dw_2 dx_2 \leq 0$$

(-)      (-)

could happen one input price is changing.

Most counter-intuitive:

$x_1 \uparrow, x_2 \downarrow$

$$(+) \quad (+) \quad \leq 0 \text{ impossible}$$

Third possibility:

$x_1 \downarrow, x_2 \downarrow$

$$(-) \quad (+) \quad \leq 0 \text{ could happen}$$

Last possibility:

$x_1 \uparrow, x_2 \uparrow$

$$(+) \quad (-) \quad \leq 0 \text{ could happen}$$

Firms have no Budget Constraint.

Mirowski - Cons prod same-why?