(3.1) profit function $\pi(\underset{\sim}{p}) = \max_{\underset{\sim}{y}} \underset{\sim}{p} \cdot \underset{\sim}{y}$ s.t. $\underset{\sim}{y} \in Y$.

1) ↑ in output prices
↓ in input prices

Proof: At prices $\underset{\sim}{p}_1$ & $\underset{\sim}{y}_1$ is optimal: $\pi(\underset{\sim}{p}_1) = \underset{\sim}{p}_1 \cdot \underset{\sim}{y}_1$.

" " $\underset{\sim}{p}_2$ " $\underset{\sim}{y}_2$ " " : $\pi(\underset{\sim}{p}_2) = \underset{\sim}{p}_2 \cdot \underset{\sim}{y}_2$.

$$\underset{\sim}{p}_2 \cdot \underset{\sim}{y}_2 \geq \underset{\sim}{p}_2 \cdot \underset{\sim}{y}_1.$$

$\forall i$ having $y_{1i} \geq 0$, & $p_{2i} \geq p_{1i}$

$\forall i$ having $y_{1i} \leq 0$, & $p_{2i} \leq p_{1i}$.

Alternative proof:
$\frac{\partial \pi(\underset{\sim}{p})}{\partial p_i} = y_i(\underset{\sim}{p}) \begin{cases} <0 \text{ for inputs} \\ >0 \text{ for outputs} \end{cases}$
§3.2 Hotelling's Lemma.
Do this first

Then

$$\underset{\sim}{p}_2 \cdot \underset{\sim}{y}_1 \geq \underset{\sim}{p}_1 \cdot \underset{\sim}{y}_1.$$ change price, hold output/inputs constant

So

$$\underset{\sim}{p}_2 \cdot \underset{\sim}{y}_2 \geq \underset{\sim}{p}_1 \cdot \underset{\sim}{y}_1 \text{ and } \pi(\underset{\sim}{p}_2) \geq \pi(\underset{\sim}{p}_1). \blacksquare$$

2) homogeneous of degree one [in $\underset{\sim}{p}$] : $\pi(t\underset{\sim}{p}) = t\,\pi(\underset{\sim}{p})$ $\forall t > 0$.

At prices $\underset{\sim}{p}_1$, suppose $\underset{\sim}{y}_1$ is optimal. Then $\underset{\sim}{p}_1 \cdot \underset{\sim}{y}_1 \geq \underset{\sim}{p}_1 \cdot \underset{\sim}{y}_2$ for any $\underset{\sim}{y}_2 \in Y$. Hence $t\underset{\sim}{p}_1 \cdot \underset{\sim}{y}_1 \geq t\underset{\sim}{p}_1 \cdot \underset{\sim}{y}_2$ $\forall$ $\underset{\sim}{y}_2 \in Y$. So $\underset{\sim}{y}_1$ maximizes profits at prices $t\underset{\sim}{p}_1$. So $\pi(t\underset{\sim}{p}_1) = t\underset{\sim}{p}_1 \cdot \underset{\sim}{y}_1$ same as $t(\underset{\sim}{p}_1 \cdot \underset{\sim}{y}_1) = t\,\pi(\underset{\sim}{p}_1)$.

Corollary. $y_i(\underset{\sim}{p})$ is homogeneous of degree zero in $\underset{\sim}{p}$.
Proof. Hotelling's Lemma.

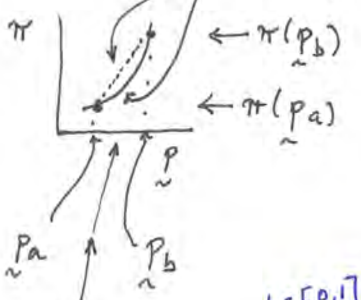
Luenberger's proof: $\pi(t\underset{\sim}{p}) = \max_{\underset{\sim}{y} \in Y} t\underset{\sim}{p} \cdot \underset{\sim}{y} = t \max_{\underset{\sim}{y} \in Y} \underset{\sim}{p} \cdot \underset{\sim}{y} = t\,\pi(\underset{\sim}{p})$. $\blacksquare$

3) $\pi(\underset{\sim}{p})$ is convex.

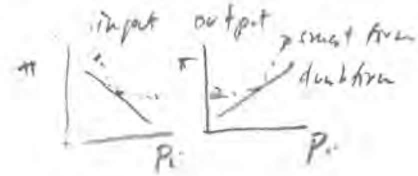
$t\pi(\underset{\sim}{p}_a) + (1-t)\pi(\underset{\sim}{p}_b)$

$\pi(t\underset{\sim}{p}_a + (1-t)\underset{\sim}{p}_b)$

Or



Intuition



$t \in [0,1]$

$$\pi(t\underset{\sim}{p}_a + (1-t)\underset{\sim}{p}_b) \leq t\pi(\underset{\sim}{p}_a) + (1-t)\pi(\underset{\sim}{p}_b)$$

Let $\underset{\sim}{y}_a$ maximize profits at $\underset{\sim}{p}_a$: $\pi(\underset{\sim}{p}_a) = \underset{\sim}{p}_a \cdot \underset{\sim}{y}_a$.

" $\underset{\sim}{y}_b$ " " " $\underset{\sim}{p}_b$: $\pi(\underset{\sim}{p}_b) = \underset{\sim}{p}_b \cdot \underset{\sim}{y}_b$.

" $\underset{\sim}{y}_t$ " " " $t\underset{\sim}{p}_a + (1-t)\underset{\sim}{p}_b$:

$$\pi(t\underset{\sim}{p}_a + (1-t)\underset{\sim}{p}_b) = (t\underset{\sim}{p}_a + (1-t)\underset{\sim}{p}_b)\cdot \underset{\sim}{y}_t.$$

$$= t\,\underset{\sim}{p}_a \cdot \underset{\sim}{y}_t + (1-t)\underset{\sim}{p}_b \cdot \underset{\sim}{y}_t.$$

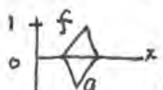
Luenberger's proof:

$$\pi(t\underset{\sim}{p}_a + (1-t)\underset{\sim}{p}_b) = \max_{\underset{\sim}{y}\in Y} [t\underset{\sim}{p}_a + (1-t)\underset{\sim}{p}_b]\cdot \underset{\sim}{y}$$

$$= \max_{\underset{\sim}{y}\in Y} [(t\underset{\sim}{p}_a \cdot \underset{\sim}{y}) + (1-t)\underset{\sim}{p}_b \cdot \underset{\sim}{y}]$$

$$\leq \max_{\underset{\sim}{y}\in Y} t\underset{\sim}{p}_a \cdot \underset{\sim}{y} + \max_{\underset{\sim}{y}\in Y} (1-t)\underset{\sim}{p}_b \cdot \underset{\sim}{y}$$

$$= t\max_{\underset{\sim}{y}\in Y} \underset{\sim}{p}_a \cdot \underset{\sim}{y} + (1-t)\max_{\underset{\sim}{y}\in Y} \underset{\sim}{p}_b \cdot \underset{\sim}{y} = t\pi(\underset{\sim}{p}_a) + (1-t)\pi(\underset{\sim}{p}_b). \blacksquare$$



$\max(f+g) = \max 0 = 0$

$\max f + \max g = 1 + 0 = 1$

$$\pi(\underset{\sim}{p}_a) = \underset{\sim}{p}_a \cdot \underset{\sim}{y}_a \geq \underset{\sim}{p}_a \cdot \underset{\sim}{y}_t$$

$$t\pi(\underset{\sim}{p}_a) \geq t\underset{\sim}{p}_a \cdot \underset{\sim}{y}_t.$$

$$\pi(\underset{\sim}{p}_b) = \underset{\sim}{p}_b \cdot \underset{\sim}{y}_b \geq \underset{\sim}{p}_b \cdot \underset{\sim}{y}_t$$

$$(1-t)\pi(\underset{\sim}{p}_b) \geq (1-t)\underset{\sim}{p}_b \cdot \underset{\sim}{y}_t$$

$$t\underset{\sim}{p}_a \cdot \underset{\sim}{y}_t + (1-t)\underset{\sim}{p}_b \cdot \underset{\sim}{y}_t \leq t\pi(\underset{\sim}{p}_a) + (1-t)\pi(\underset{\sim}{p}_b). \blacksquare$$

4) $\pi(\underset{\sim}{p})$ is continuous.

Discuss Fig. 3.1.

Do example at the top of p. 43. It assumes p is realized before the output decision is made.

(3.2) net supply function $y_i(\underset{\sim}{p}) = \dfrac{\partial \pi(\underset{\sim}{p})}{\partial p_i} \quad \forall i = 1, 2, \ldots, n.$

Hotelling's Lemma.*

Proof.

Envelope Theorem. (§7.3)

$M(a) =$

$$\max_{x_1, x_2} g(x_1, x_2, a).$$

$$\frac{d M(a)}{d a} = \frac{\partial g^*}{\partial a}.$$

Hotelling's Lemma.

$\pi(\underset{\sim}{p}) =$

$$\max_{\underset{\sim}{y}} \underset{\sim}{p} \cdot \underset{\sim}{y}.$$

$$\frac{d \pi(\underset{\sim}{p})}{d \underset{\sim}{p}} = \frac{\partial\, \underset{\sim}{p} \cdot \underset{\sim}{y}^*}{\partial \underset{\sim}{p}}$$

$$= \underset{\sim}{y}^*. \quad \blacksquare$$

Hotelling: this
exhaustible resources
spatial imperfect competition
travel cost method
accounting
taxation
integral equations

(3.3) already done

(3.4)

$$y_i(\underset{\sim}{p}) = \frac{\partial \pi(\underset{\sim}{p})}{\partial p_i}$$

↑ $\pi(\underset{\sim}{p})$ is homog. deg. 1, ∴

→ homog. deg. 0. (See my note to §3.1.)

Better:
$\underset{\sim}{y} = \nabla_{\underset{\sim}{p}} \pi$

$\nabla_p \underset{\sim}{y} = \nabla_p^2 \pi$ ← π convex
p.s.d.

π convex $\Rightarrow \nabla_{\underset{\sim}{p}}^2 \pi$ positive semidefinite symmetric

$= \nabla_{\underset{\sim}{p}} \nabla_{\underset{\sim}{p}} \pi$

$$= \nabla_{\underset{\sim}{p}} \underset{\sim}{y} = \begin{bmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_2}{\partial p_1} & \cdots \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

positive semidefinite symmetric

← actually T of this

$$\Rightarrow \boxed{\frac{\partial y_i}{\partial p_i} \geq 0}$$

i an output : ↑ sloping S curve
i an input : ↓ sloping input D curve

$$\frac{\partial y_i}{\partial p_j} = \frac{\partial y_j}{\partial p_i}$$

already done in §2.4