(3.1) profit function $\pi(p) = \max_{y} \; p \cdot y$ s.t. $y \in Y$.

1) $\uparrow$ in output prices
$\downarrow$ in input prices

Proof: At prices p_1 & y_1 is optimal: $\pi(p_1) = p_1 \cdot y_1$.

" " p_2 " y_2 " " : $\pi(p_2) = p_2 \cdot y_2$.

$$p_2 \cdot y_2 \geq p_2 \cdot y_1.$$

$\forall i$ having $y_{1i} \geq 0$, & $p_{2i} \geq p_{1i}$

$\forall i$ having $y_{1i} \leq 0$, & $p_{2i} \leq p_{1i}$.

Alternative proof:
$$\frac{\partial \pi(p)}{\partial p_i} = y_i(p) \begin{cases} <0 \text{ for inputs} \\ >0 \text{ for outputs} \end{cases}$$
§3.2 Hotelling's Lemma.
Do this first

Then

$$p_2 \cdot y_1 \geq p_1 \cdot y_1.$$

change price, hold output/inputs constant

So

$$p_2 \cdot y_2 \geq p_1 \cdot y_1 \text{ and } \pi(p_2) \geq \pi(p_1). \blacksquare$$

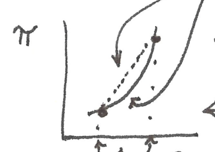
2) homogeneous of degree one [in p] : $\pi(tp) = t\,\pi(p)$ $\forall t > 0$.

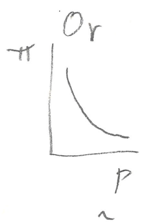
At prices p_1, suppose y_1 is optimal. Then $p_1 \cdot y_1 \geq p_1 \cdot y_2$ for any $y_2 \in Y$. Hence $tp_1 \cdot y_1 \geq tp_1 \cdot y_2$ $\forall$ $y_2 \in Y$. So y_1 maximizes profits at prices tp_1. So $\pi(tp_1) = tp_1 \cdot y_1$ same as $t(p_1 \cdot y_1) = t\,\pi(p_1)$.

Corollary. $y_i(p)$ is homogeneous of degree zero in p.
Proof. Hotelling's Lemma.

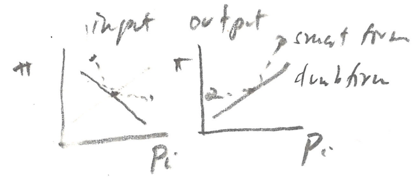
Luenberger's proof: $\pi(tp) = \max_{y \in Y} tp \cdot y = t \max_{y \in Y} p \cdot y = t\,\pi(p)$. $\blacksquare$

3) $\pi(\mathbf{p})$ is convex.





Intuition:



$t\in[0,1]$

$$\pi(t\mathbf{p}_a+(1-t)\mathbf{p}_b) \le t\pi(\mathbf{p}_a)+(1-t)\pi(\mathbf{p}_b)$$

Let $\mathbf{y}_a$ maximize profits at $\mathbf{p}_a$: $\pi(\mathbf{p}_a)=\mathbf{p}_a\cdot\mathbf{y}_a$.

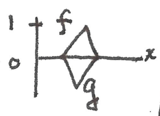
" $\mathbf{y}_b$ " " " $\mathbf{p}_b$: $\pi(\mathbf{p}_b)=\mathbf{p}_b\cdot\mathbf{y}_b$.

" $\mathbf{y}_t$ " " " $t\mathbf{p}_a+(1-t)\mathbf{p}_b$:

$$\pi(t\mathbf{p}_a+(1-t)\mathbf{p}_b)=(t\mathbf{p}_a+(1-t)\mathbf{p}_b)\cdot\mathbf{y}_t.$$
$$= t\,\mathbf{p}_a\cdot\mathbf{y}_t+(1-t)\mathbf{p}_b\cdot\mathbf{y}_t.$$

Luenberger's proof:

$$\pi(t\mathbf{p}_a+(1-t)\mathbf{p}_b)=\max_{\mathbf{y}\in Y}[t\mathbf{p}_a+(1-t)\mathbf{p}_b]\cdot\mathbf{y}$$
$$=\max_{\mathbf{y}\in Y}\left[(t\mathbf{p}_a\cdot\mathbf{y})+(1-t)\mathbf{p}_b\cdot\mathbf{y}\right]$$
$$\le \max_{\mathbf{y}\in Y} t\mathbf{p}_a\cdot\mathbf{y}+\max_{\mathbf{y}\in Y}(1-t)\mathbf{p}_b\cdot\mathbf{y}$$
$$=t\max_{\mathbf{y}\in Y}\mathbf{p}_a\cdot\mathbf{y}+(1-t)\max_{\mathbf{y}\in Y}\mathbf{p}_b\cdot\mathbf{y}=t\pi(\mathbf{p}_a)+(1-t)\pi(\mathbf{p}_b).\ \blacksquare$$



$\max(f+g) = \max 0 = 0$

$\max f + \max g = 1+0=1$

$$\pi(\mathbf{p}_a)=\mathbf{p}_a\cdot\mathbf{y}_a \ge \mathbf{p}_a\cdot\mathbf{y}_t$$
$$t\pi(\mathbf{p}_a) \ge t\mathbf{p}_a\cdot\mathbf{y}_t.$$

$$\pi(\mathbf{p}_b)=\mathbf{p}_b\cdot\mathbf{y}_b \ge \mathbf{p}_b\cdot\mathbf{y}_t$$
$$(1-t)\pi(\mathbf{p}_b) \ge (1-t)\mathbf{p}_b\cdot\mathbf{y}_t$$

$$t\mathbf{p}_a\cdot\mathbf{y}_t+(1-t)\mathbf{p}_b\cdot\mathbf{y}_t \le t\pi(\mathbf{p}_a)+(1-t)\pi(\mathbf{p}_b).$$

$\blacksquare$

4) $\pi(\mathbf{p})$ is continuous.

Discuss Fig. 3.1.

Do example at the top of p. 43. It assumes p is realized <u>before</u> the output decision is made.

(3.2) net supply function $y_i(\underset{\sim}{p}) = \dfrac{\partial \pi(\underset{\sim}{p})}{\partial p_i}$ $\forall i = 1, 2, \ldots, n$.

Hotelling's Lemma.

Proof.

Envelope theorem. (§7.3)

$$M(a) = \max_{x_1, x_2} g(x_1, x_2, a).$$

$$\frac{d M(a)}{d a} = \frac{\partial g^*}{\partial a}.$$

Hotelling's Lemma.

$$\pi(\underset{\sim}{p}) = \max_{\underset{\sim}{y}} \underset{\sim}{p} \cdot \underset{\sim}{y}.$$

$$\frac{d \pi(\underset{\sim}{p})}{d \underset{\sim}{p}} = \frac{\partial\, \underset{\sim}{p} \cdot \underset{\sim}{y}^*}{\partial \underset{\sim}{p}}$$

$$= \underset{\sim}{y}^*. \blacksquare$$

Hotelling: this
exhaustible resources
spatial imperfect competition
travel cost method
accounting
taxation
integral equations

3.3 already done

3.4

$$y_i(\underset{\sim}{p}) = \frac{\partial \pi(\underset{\sim}{p})}{\partial p_i}$$

↑ $\pi(\underset{\sim}{p})$ is homog. deg. 1, ∴

→ homog. deg. 0. (See my note to §3.1.)

Better:

$\underset{\sim}{y} = \nabla_{\underset{\sim}{p}} \pi$

$\nabla_p \underset{\sim}{y} = \nabla_p^2 \pi$
↑ convex
p.s.d.

π convex $\Rightarrow \nabla_{\underset{\sim}{p}}^2 \pi$ positive semidefinite symmetric

$$= \nabla_{\underset{\sim}{p}} \nabla_{\underset{\sim}{p}} \pi$$

$$= \nabla_{\underset{\sim}{p}} \underset{\sim}{y} = \begin{bmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_2}{\partial p_1} & \cdots \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$ positive semidefinite symmetric

← actually T of this

$$\Rightarrow \boxed{\frac{\partial y_i}{\partial p_i} \geq 0}$$

i an output: ↑ sloping S curve

i an input: ↓ sloping input D curve

$$\frac{\partial y_i}{\partial p_j} = \frac{\partial y_j}{\partial p_i}$$

already done in §2.4