

**Problem Set on Normative General Equilibrium**

1. Consider an economy with a single commodity. Let  $X$  be the total amount available and  $x_a$  and  $x_b$  be the allocations between two consumers whose utility functions are given by

$$u^a(x_a, x_b) = x_a^{2/3} x_b^{1/3}$$

$$u^b(x_a, x_b) = x_a^{1/3} x_b^{2/3}.$$

Note that these consumers are altruistic.

- (a) What allocations are Pareto optimal for this economy?
- (b) Suppose the initial endowment is  $(\omega_a, \omega_b) = (5, 1)$ . Assuming that consumer A is allowed to make charitable contributions to consumer B, what contribution will he make?
2. Suppose there are two individuals, A and B, with utility functions

$$u_i(x_{i1}, x_{i2}) = \alpha_i \ln x_{i1} + (1 - \alpha_i) \ln x_{i2} \quad \text{for } i = A, B$$

for two commodities. (So  $x_{ij}$  means person  $i$  and commodity  $j$ .) Suppose the totals for the two commodities available are  $\bar{x}_i$  for  $i = A, B$ .

- (a) Individual  $i$  is given an amount of money  $y_i$  to spend. Prices  $p_j$  for  $j = 1, 2$  are set so that demand equals supply in both markets. What are the prices and allocations of the goods?
- (b) Find the allocation which maximizes  $u_a + u_b$ . What amount of money,  $y_i$ , should be allocated to each consumer so that they will in fact receive the utilitarian optimal allocation if it is obtained by competitive allocation?
- (c) Repeat the entire analysis for the case where the utility functions are  $u_i(x_{i1}, x_{i2}) = \alpha_i(x_{i1})^{1/2} + (1 - \alpha_i)(x_{i2})^{1/2}$ . Show that in this case, as opposed to the logarithmic case, the amounts of  $y_i$  do depend on the commodity totals  $\bar{x}_i$  as well as on the parameter  $\alpha_i$  of the utility function. Thus the distribution of income necessary to induce the utilitarian allocation will depend on the commodity scarcities as well as on the individual utility functions.

3. A two-person economy has one input, labor, and one output,  $x$ , which is produced with labor. Each agent is endowed with one unit of labor. Let the amount of good  $x$  produced by Agent 1 be  $X_1^S$  and the amount of good  $x$  consumed by Agent 1 be  $x_1$ . Similarly, let the amount of good  $x$  produced by Agent 2 be  $X_2^S$  and the amount of good  $x$  consumed by Agent 2 be  $x_2$ .

Agent 1 is less productive than agent 2:

$$\begin{aligned} X_1^S &= b_1 L_1 \\ X_2^S &= b_2 L_2 \quad \text{with } b_1 < b_2. \end{aligned}$$

The utility functions are given by

$$\begin{aligned} u_1 &= \alpha_1 \ln x_1 + (1 - \alpha_1) \ln(1 - L_1) \\ u_2 &= \alpha_2 \ln x_2 + (1 - \alpha_2) \ln(1 - L_2) \end{aligned}$$

with

$$0 < \alpha_1 < 1 \quad \text{and} \quad 0 < \alpha_2 < 1.$$

- (a) A utilitarian social welfare planner wishes to maximize the sum of the two agents' utilities. What allocations of  $x_1$  and  $L_1$  will he choose?
- (b) If the utilitarian social welfare planner were to redistribute income from one agent to the other to achieve the socially optimal allocation computed in part (a) as a competitive equilibrium, what lump sum subsidies or taxes would the planner choose? (Assume each agent is paid the value of his marginal product.)
- (c) Suppose the social planner is restricted to taxes of the form  $t = -a + mz$  where  $z$  is earned income and  $a$  and  $m$  are positive parameters. Further, the sum of the taxes across agents must be zero. Formulate the problem of determining the parameters necessary to achieve the social planners' desired allocations. Compare the efficiency of this scheme with that of the lump sum transfer scheme.

4. Consider an exchange economy with two consumers and two goods.

$$\begin{aligned} U^1(x_{11}, x_{21}) &= x_{11}^{1/2} + x_{21}^{1/2} \quad \text{and} \quad \omega_1 = (1, 0) \\ U^2(x_{12}, x_{22}) &= x_{12}^{1/2} + x_{22}^{1/2} \quad \text{and} \quad \omega_2 = (0, 2). \end{aligned}$$

(So  $x_{ij}$  means commodity  $i$  and person  $j$ , unlike in some earlier questions.) What is the set of core allocations? What is the set of competitive equilibrium allocations?

5. Consider an exchange economy with two consumers and two goods.

$$U^1(x_{11}, x_{21}) = x_{11}^{1/2} x_{21}^{1/2} \quad \text{and} \quad \omega_1 = (2, 1)$$
$$U^2(x_{12}, x_{22}) = x_{12} x_{22} \quad \text{and} \quad \omega_2 = (1, 2).$$

What is the set of core allocations? How is this set modified if we add two consumers, one with preferences and endowments identical to those of individual 1, and one with preferences and endowments identical to those of individual 2?

# Answers to Problem Set on Normative General Equilibrium

① a)

We often find Pareto optimal allocations by solving  $\max U^a$  s.t.  $U^b = \text{constant}$ . That doesn't work here because it's not possible to change  $U^a$  while keeping  $U^b$  constant, due to altruism.

Let  $\alpha =$  social weight on individual A's utility  
 $1-\alpha =$  " " " " B's utility.

Another way to generate the Pareto optimal frontier is to solve  $\max \alpha U^a + (1-\alpha) U^b$  s.t. feasibility. This leads to

$$\mathcal{L} = \alpha x_a^{2/3} x_b^{1/3} + (1-\alpha) x_a^{1/3} x_b^{2/3} + \lambda (x - x_a - x_b)$$

$$0 = \partial \mathcal{L} / \partial \lambda = x - x_a - x_b \quad (1)$$

$$0 = \partial \mathcal{L} / \partial x_a = \frac{2}{3} \alpha \frac{1}{x_a} x_a^{2/3} x_b^{1/3} + \frac{1}{3} (1-\alpha) \frac{1}{x_a} x_a^{1/3} x_b^{2/3} - \lambda \quad (2)$$

$$0 = \partial \mathcal{L} / \partial x_b = \frac{1}{3} \alpha \frac{1}{x_b} x_a^{2/3} x_b^{1/3} + \frac{2}{3} (1-\alpha) \frac{1}{x_b} x_a^{1/3} x_b^{2/3} - \lambda \quad (3)$$

Solving for  $\lambda$  yields

$$\frac{2}{3} \alpha \frac{1}{x_a} x_a^{2/3} x_b^{1/3} + \frac{1}{3} (1-\alpha) \frac{1}{x_a} x_a^{1/3} x_b^{2/3} = \frac{1}{3} \alpha \frac{1}{x_b} x_a^{2/3} x_b^{1/3} + \frac{2}{3} (1-\alpha) \frac{1}{x_b} x_a^{1/3} x_b^{2/3} \quad \text{Multiply by } 3x_a x_b \text{ and rearrange:}$$

$$(2\alpha x_b - \alpha x_a) x_a^{2/3} x_b^{1/3} = [2(1-\alpha)x_a - (1-\alpha)x_b] x_a^{1/3} x_b^{2/3}$$

$$\alpha (2x_b - x_a) x_a^{2/3} x_b^{1/3} = (1-\alpha) (2x_a - x_b) x_a^{1/3} x_b^{2/3}$$

$$\alpha (2x_b - x_a) x_a^{1/3} = (1-\alpha) (2x_a - x_b) x_b^{2/3}$$

Since  $x_a + x_b = x$ ,  $x_b = x - x_a$ ; then  $2x_b - x_a = 2x - 2x_a - x_a = 2x - 3x_a$   
 and  $2x_a - x_b = 2x_a - x + x_a = 3x_a - x$ . Substituting in yields

$$\alpha (2x - 3x_a) x_a^{1/3} = (1-\alpha) (3x_a - x) (x - x_a)^{1/3}. \text{ Cube both sides:}$$

$$\alpha^3 (2x - 3x_a)^3 x_a = (1-\alpha)^3 (3x_a - x)^3 (x - x_a) \quad (4)$$

This is a fourth degree polynomial in  $x_a$ .

It is reasonable to guess that  $\partial x_a / \partial \alpha > 0$ : as  $\alpha \uparrow$ , the weight

on person A goes up, and he values  $x_a$  more than B does. Proving

that  $\partial x_a / \partial \alpha > 0$  would involve differentiating both sides of (4) w.r.t.

$\alpha$ , considering  $x_a$  an implicit function of  $\alpha$ .\*

If  $\partial x_a / \partial \alpha > 0$  then the Pareto optimal range for  $x_a$  is obtained

by letting  $\alpha$  go from 0 to 1.  $\rightarrow$  or  $x_a = x$  from (4), but giving everything to person a when his social weight is zero cannot maximize social welfare.

$$\alpha = 0 \Rightarrow 3x_a - x = 0 \text{ from (4)} \Rightarrow x_a = \frac{x}{3}$$

$$\alpha = 1 \Rightarrow 2x - 3x_a = 0 \text{ from (4)} \Rightarrow x_a = \frac{2}{3}x.$$

So under this assumption the set of Pareto optimal allocations would be

$$\{(x_a, x_b) : x_a + x_b = x, \frac{x}{3} \leq x_a \leq \frac{2x}{3}\}.$$

b) Let  $g =$  gift from A to B. A maximizes  $x_a^{2/3} x_b^{1/3}$  s.t.  $x_a = 5 - g$  and  $x_b = 1 + g$ .

$$\max_g (5-g)^{2/3} (1+g)^{1/3} \Leftrightarrow \max_g (5-g)^2 (1+g) \text{ since cubing is a positive transformation}$$

$$\Rightarrow 0 = -2(5-g)(1+g) + (5-g)^2; g \neq 5 \Rightarrow$$

$$0 = -2(1+g) + (5-g) = -2 - 2g + 5 - g = 3 - 3g,$$

$$\Rightarrow g = 1.$$

\* Or one could consider  $\partial x_a / \partial \alpha$  a comparative statics problem and find it by taking the differentials of (1), (2), and (3).

If you work it out the other way you get

$$0 = -\frac{2}{3} (5-g)^{-1/3} (1+g)^{1/3} + \frac{1}{3} (5-g)^{2/3} (1+g)^{-2/3}$$

$$2(5-g)^{-1/3} (1+g)^{1/3} = (5-g)^{2/3} (1+g)^{-2/3}$$

$$2(1+g) = (5-g)$$

$$2+2g = 5-g \Rightarrow 3g = 3 \Rightarrow g = 1 \text{ as before.}$$

②

Consumers here are given exogenously determined levels of income,  $y_1$  and  $y_2$ , so prices are not normalizable as they usually are.

$$a) \max \alpha_i \ln x_{i1} + (1-\alpha_i) \ln x_{i2} \quad \text{s.t.} \quad y - p_1 x_{i1} - p_2 x_{i2} = 0.$$

$$\mathcal{L} = \alpha_i \ln x_{i1} + (1-\alpha_i) \ln x_{i2} + \lambda (y - p_1 x_{i1} - p_2 x_{i2})$$

$$0 = \partial \mathcal{L} / \partial \lambda = y - p_1 x_{i1} - p_2 x_{i2}$$

$$0 = \partial \mathcal{L} / \partial x_{i1} = \frac{\alpha_i}{x_{i1}} - \lambda p_1$$

$$0 = \partial \mathcal{L} / \partial x_{i2} = \frac{1-\alpha_i}{x_{i2}} - \lambda p_2$$

$$\left. \begin{array}{l} \frac{p_1}{p_2} = \frac{\alpha_i}{x_{i1}} \frac{x_{i2}}{1-\alpha_i} \end{array} \right\}$$

$$\Rightarrow p_1 x_{i1} = \frac{\alpha_i}{1-\alpha_i} p_2 x_{i2}$$

$$\Rightarrow y = \frac{\alpha_i}{1-\alpha_i} p_2 x_{i2} + p_2 x_{i2} = \left( \frac{\alpha_i}{1-\alpha_i} + 1 \right) p_2 x_{i2}$$

$$= \frac{1}{1-\alpha_i} p_2 x_{i2} \Rightarrow x_{i2} = (1-\alpha_i) \frac{y}{p_2} \quad \text{and}$$

$$x_{i1} = \frac{\alpha_i}{1-\alpha_i} \frac{p_2}{p_1} \cdot \frac{y}{p_2} (1-\alpha_i) = \alpha_i \frac{y}{p_1}.$$

$$\text{Clear good 1: } \alpha_1 \frac{y_1}{p_1} + \alpha_2 \frac{y_2}{p_1} = \bar{x}_1 \Rightarrow p_1 = \frac{\alpha_1 y_1 + \alpha_2 y_2}{\bar{x}_1}$$

$$\text{Clear good 2: } (1-\alpha_1) \frac{y_1}{p_2} + (1-\alpha_2) \frac{y_2}{p_2} = \bar{x}_2 \Rightarrow p_2 = \frac{(1-\alpha_1) y_1 + (1-\alpha_2) y_2}{\bar{x}_2}$$

For allocations see part (b), near top of next page.

$$b) \max (u_a + u_b)$$

$$\max \alpha_1 \ln x_{11} + (1-\alpha_1) \ln x_{12} + \alpha_2 \ln x_{21} + (1-\alpha_2) \ln x_{22} \quad \text{s.t.}$$

$$x_{11} + x_{21} = \bar{x}_1$$

$$x_{12} + x_{22} = \bar{x}_2. \quad \text{Substitute the constraints into the objective function:}$$

$$\max \alpha_1 \ln x_{11} + (1-\alpha_1) \ln x_{12} + \alpha_2 \ln (\bar{x}_1 - x_{11}) + (1-\alpha_2) \ln (\bar{x}_2 - x_{12})$$

$$\text{FOC wrt } x_{11}: 0 = \frac{\alpha_1}{x_{11}} - \frac{\alpha_2}{\bar{x}_1 - x_{11}} \Rightarrow \alpha_1 (\bar{x}_1 - x_{11}) = \alpha_2 x_{11}$$

$$\text{FOC wrt } x_{12}: 0 = \frac{1-\alpha_1}{x_{12}} - \frac{1-\alpha_2}{\bar{x}_2 - x_{12}} \Rightarrow (1-\alpha_1) (\bar{x}_2 - x_{12}) = (1-\alpha_2) x_{12}$$

$$\left\{ \begin{aligned} \Rightarrow \alpha_1 \bar{x}_1 &= \alpha_1 x_{11} + \alpha_2 x_{11} \Rightarrow x_{11} = \frac{\alpha_1 \bar{x}_1}{\alpha_1 + \alpha_2} \\ \Rightarrow (1 - \alpha_1) \bar{x}_2 &= (1 - \alpha_1) x_{12} + (1 - \alpha_2) x_{12} \end{aligned} \right.$$

$$\Downarrow \\ x_{12} = \frac{(1 - \alpha_1) \bar{x}_2}{1 - \alpha_1 + 1 - \alpha_2}$$

From part (a), under competition,

$$x_{1i} = \alpha_i \frac{y_i}{p_i} = \alpha_i y_i \frac{\bar{x}_i}{\alpha_1 y_1 + \alpha_2 y_2}$$

Also  $x_{i2} = (1 - \alpha_i) y_i / p_2$   
 $= (1 - \alpha_i) y_i \frac{\bar{x}_2}{(1 - \alpha_1) y_1 + (1 - \alpha_2) y_2}$

If  $x_{1i}$  for competition =  $x_{1i}$  for welfare maximization then

$$\alpha_1 y_1 \frac{\bar{x}_1}{\alpha_1 y_1 + \alpha_2 y_2} = \frac{\alpha_1 \bar{x}_1}{\alpha_1 + \alpha_2}$$

$$\frac{y_1}{\alpha_1 y_1 + \alpha_2 y_2} = \frac{1}{\alpha_1 + \alpha_2} \Rightarrow \alpha_1 y_1 + \alpha_2 y_1 = \alpha_1 y_1 + \alpha_2 y_2$$

$$\alpha_2 y_1 = \alpha_2 y_2$$

$$\boxed{y_1 = y_2}$$

(c) Competitive:

$$\max \alpha_i x_{i1}^{1/2} + (1 - \alpha_i) x_{i2}^{1/2} \quad \text{s.t.} \quad y - p_1 x_{i1} - p_2 x_{i2} = 0$$

$$\mathcal{L} = \alpha_i \sqrt{x_{i1}} + (1 - \alpha_i) \sqrt{x_{i2}} + \lambda (y - p_1 x_{i1} - p_2 x_{i2})$$

$$0 = \partial \mathcal{L} / \partial \lambda = y - p_1 x_{i1} - p_2 x_{i2}$$

$$0 = \partial \mathcal{L} / \partial x_{i1} = \frac{1}{2} \alpha_i x_{i1}^{-1/2} - \lambda p_1 \quad \left\{ \begin{aligned} \frac{p_1}{p_2} &= \frac{\alpha_i}{1 - \alpha_i} \frac{x_{i2}^{1/2}}{x_{i1}^{1/2}} \end{aligned} \right.$$

$$0 = \partial \mathcal{L} / \partial x_{i2} = \frac{1}{2} (1 - \alpha_i) x_{i2}^{-1/2} - \lambda p_2$$



$$x_{i1}^{1/2} = \frac{p_2}{p_1} \frac{\alpha_i}{1-\alpha_i} x_{i2}^{1/2}$$

$$x_{i1} = \left( \frac{p_2}{p_1} \frac{\alpha_i}{1-\alpha_i} \right)^2 x_{i2} \text{ into budget constraint } \Rightarrow$$

$$y = p_1 \left( \frac{p_2}{p_1} \frac{\alpha_i}{1-\alpha_i} \right)^2 x_{i2} + p_2 x_{i2}$$

$$= \left( \frac{p_2^2}{p_1} \frac{\alpha_i^2}{(1-\alpha_i)^2} + p_2 \right) x_{i2} = \frac{p_2^2 \alpha_i^2 + p_1 p_2 (1-\alpha_i)^2}{p_1 (1-\alpha_i)^2} x_{i2}$$

$$x_{i2} = \frac{p_1 (1-\alpha_i)^2}{p_2^2 \alpha_i^2 + p_1 p_2 (1-\alpha_i)^2} y_i$$

$$x_{i1} = \frac{p_2^2}{p_1^2} \frac{(\alpha_i)^2}{(1-\alpha_i)^2} \frac{p_1 (1-\alpha_i)^2}{p_2^2 \alpha_i^2 + p_1 p_2 (1-\alpha_i)^2} y_i$$

$$= \frac{p_2 \alpha_i^2}{p_1 p_2 \alpha_i^2 + p_1^2 (1-\alpha_i)^2} y_i$$

Clear good 1:

$$\bar{x}_1 = x_{11} + x_{21} = \frac{p_2 \alpha_1^2 y_1}{p_1 p_2 \alpha_1^2 + p_1^2 (1-\alpha_1)^2} + \frac{p_2 \alpha_2^2 y_2}{p_1 p_2 \alpha_2^2 + p_1^2 (1-\alpha_2)^2}$$

Clear good 2:

$$\bar{x}_2 = x_{12} + x_{22} = \frac{p_1 (1-\alpha_1)^2 y_1}{p_2^2 \alpha_1^2 + p_1 p_2 (1-\alpha_1)^2} + \frac{p_1 (1-\alpha_2)^2 y_2}{p_2^2 \alpha_2^2 + p_1 p_2 (1-\alpha_2)^2}$$

this is a system of two equations in the two unknowns  $p_1$  and  $p_2$ .

Unfortunately, I have not been able to solve this system (and neither has "Mathematica" despite several hours of computer run time).

If I were able to solve it, I'd get

$$p_1 = p_1(\alpha_1, \alpha_2, \bar{x}_1, \bar{x}_2, y_1, y_2)$$

$$p_2 = p_2(\alpha_1, \alpha_2, \bar{x}_1, \bar{x}_2, y_1, y_2).$$

Substituting these values of  $p_1$  and  $p_2$  into the demand functions would yield

$$x_{11} = x_{11}(\alpha_1, \alpha_2, \bar{x}_1, \bar{x}_2, y_1, y_2)$$

$$x_{12} = x_{12}(\quad \text{---} \quad \text{---} \quad \text{---} \quad)$$

$$x_{21} = x_{21}(\quad \text{---} \quad \text{---} \quad \text{---} \quad)$$

$$x_{22} = x_{22}(\quad \text{---} \quad \text{---} \quad \text{---} \quad).$$

On the next page, the utilitarian  $x_{11}$ ,  $x_{12}$ ,  $x_{21}$ , and  $x_{22}$  are found.

For example,  $x_{11}^U = x_{11}^U(\alpha_1, \alpha_2, \bar{x}_1)$

$$x_{12}^U = x_{12}^U(\alpha_1, \alpha_2, \bar{x}_2).$$

Setting  $x_{11}^U$  equal to  $x_{11}(\alpha_1, \alpha_2, \bar{x}_1, \bar{x}_2, y_1, y_2)$  and

setting  $x_{12}^U$  " "  $x_{12}(\alpha_1, \alpha_2, \bar{x}_1, \bar{x}_2, y_1, y_2)$  yields a system of two equations in the two unknowns  $y_1$  and  $y_2$ . Solving for  $y_1$  and  $y_2$  would complete the problem.

utilitarian:

$$\begin{aligned} & \max (u_a + u_b) \\ & = \max \alpha_1 x_{11}^{1/2} + (1-\alpha_1) x_{12}^{1/2} + \alpha_2 x_{21}^{1/2} + (1-\alpha_2) x_{22}^{1/2} \\ & \text{s.t. } x_{11} + x_{21} = \bar{x}_1 \\ & \quad x_{12} + x_{22} = \bar{x}_2. \end{aligned}$$

Substitute the constraints into the objective function (this is easier than writing a Lagrangian with two constraints):

$$\max \alpha_1 x_{11}^{1/2} + (1-\alpha_1) x_{12}^{1/2} + \alpha_2 (\bar{x}_1 - x_{11})^{1/2} + (1-\alpha_2) (\bar{x}_2 - x_{12})^{1/2}$$

$$\text{FOC wrt } x_{11}: 0 = \frac{1}{2} \alpha_1 x_{11}^{-1/2} - \frac{1}{2} \alpha_2 (\bar{x}_1 - x_{11})^{-1/2}$$

$$\text{FOC wrt } x_{12}: 0 = \frac{1}{2} (1-\alpha_1) x_{12}^{-1/2} - \frac{1}{2} (1-\alpha_2) (\bar{x}_2 - x_{12})^{-1/2}$$

$$\rightarrow \frac{1}{2} \alpha_1 x_{11}^{-1/2} = \frac{1}{2} \alpha_2 (\bar{x}_1 - x_{11})^{-1/2}$$

$$\alpha_1^2 x_{11}^{-1} = \alpha_2^2 (\bar{x}_1 - x_{11})^{-1}$$

$$\alpha_1^2 (\bar{x}_1 - x_{11}) = \alpha_2^2 x_{11}$$

$$\alpha_1^2 \bar{x}_1 = (\alpha_2^2 + \alpha_1^2) x_{11}$$

$$x_{11} = \frac{\alpha_1^2 \bar{x}_1}{\alpha_1^2 + \alpha_2^2}$$

$$\frac{1}{2} (1-\alpha_1) x_{12}^{-1/2} = \frac{1}{2} (1-\alpha_2) (\bar{x}_2 - x_{12})^{-1/2}$$

$$(1-\alpha_1)^2 x_{12}^{-1} = (1-\alpha_2)^2 (\bar{x}_2 - x_{12})^{-1}$$

$$(1-\alpha_1)^2 (\bar{x}_2 - x_{12}) = (1-\alpha_2)^2 x_{12}$$

$$(1-\alpha_1)^2 \bar{x}_2 = [(1-\alpha_1)^2 + (1-\alpha_2)^2] x_{12}$$

$$x_{12} = \frac{(1-\alpha_1)^2 \bar{x}_2}{(1-\alpha_1)^2 + (1-\alpha_2)^2}$$

$$x_{21} = \bar{x}_1 - x_{11} = \frac{\alpha_2^2 \bar{x}_1}{\alpha_1^2 + \alpha_2^2}$$

$$x_{22} = \bar{x}_2 - x_{12} = \frac{(1-\alpha_2)^2 \bar{x}_2}{(1-\alpha_1)^2 + (1-\alpha_2)^2}$$

$$\textcircled{3} \quad a) \max u_1 + u_2 \quad \text{s.t.} \quad x^D = x^S \\ x_1 + x_2 = b_1 L_1 + b_2 L_2$$

$$\mathcal{L} = \alpha_1 \ln x_1 + (1-\alpha_1) \ln(1-L_1) + \alpha_2 \ln x_2 + (1-\alpha_2) \ln(1-L_2) \\ + \lambda (b_1 L_1 + b_2 L_2 - x_1 - x_2)$$

$$0 = \partial \mathcal{L} / \partial \lambda = b_1 L_1 + b_2 L_2 - x_1 - x_2$$

$$0 = \partial \mathcal{L} / \partial x_1 = \frac{\alpha_1}{x_1} - \lambda \\ 0 = \partial \mathcal{L} / \partial x_2 = \frac{\alpha_2}{x_2} - \lambda \quad \left. \begin{array}{l} \\ \end{array} \right\} \frac{\alpha_1}{x_1} = \frac{\alpha_2}{x_2} \Rightarrow x_1 = \frac{\alpha_1}{\alpha_2} x_2. \text{ Put other variables in terms of } x_2:$$

$$\rightarrow \text{Also, } \lambda = \alpha_2 / x_2.$$

$$0 = \partial \mathcal{L} / \partial L_1 = -\frac{1-\alpha_1}{1-L_1} + \lambda b_1 \rightarrow \frac{1-\alpha_1}{1-L_1} = \lambda b_1 \Rightarrow 1-L_1 = \frac{1-\alpha_1}{\lambda b_1},$$

$$0 = \partial \mathcal{L} / \partial L_2 = -\frac{1-\alpha_2}{1-L_2} + \lambda b_2 \quad L_1 = 1 - \frac{1-\alpha_1}{\lambda b_1} = \frac{\lambda b_1 - 1 + \alpha_1}{\lambda b_1}$$

$$\downarrow \\ \frac{1-\alpha_2}{1-L_2} = \lambda b_2$$

$$1-L_2 = \frac{1-\alpha_2}{\lambda b_2}$$

$$L_2 = 1 - \frac{1-\alpha_2}{\lambda b_2} = \frac{\lambda b_2 - 1 + \alpha_2}{\lambda b_2} = \frac{\frac{\alpha_2}{x_2} b_2 - 1 + \alpha_2}{\frac{\alpha_2}{x_2} b_2}$$

$$= \frac{\frac{\alpha_2}{x_2} b_1 - 1 + \alpha_1}{\frac{\alpha_2}{x_2} b_1} = \frac{\alpha_2 b_1 - x_2 + \alpha_1 x_2}{\alpha_2 b_1}$$

$$= \frac{\alpha_2 b_2 - x_2 + \alpha_2 x_2}{\alpha_2 b_2}$$

$$x^D = x^S \Leftrightarrow \frac{\alpha_1}{\alpha_2} x_2 + x_2 = \frac{\alpha_2 b_1 - x_2 + \alpha_1 x_2}{\alpha_2} + \frac{\alpha_2 b_2 - x_2 + \alpha_2 x_2}{\alpha_2}$$

$$\left( \frac{\alpha_1}{\alpha_2} + 1 \right) x_2 = b_1 + b_2 + x_2 \left( \frac{-1}{\alpha_2} + \frac{\alpha_1}{\alpha_2} - \frac{1}{\alpha_2} + 1 \right)$$

$$\frac{2}{\alpha_2} x_2 = b_1 + b_2$$

$$x_2^* = \frac{\alpha_2}{2} (b_1 + b_2).$$

$$\Rightarrow x_1^* = \frac{\alpha_1}{\alpha_2} x_2 = \frac{\alpha_1}{2} (b_1 + b_2)$$

$$L_1^* = \frac{1}{\alpha_2 b_1} (\alpha_2 b_1 - x_2 + \alpha_1 x_2)$$

$$= \frac{1}{\alpha_2 b_1} \left( \alpha_2 b_1 - \frac{\alpha_2}{2} (b_1 + b_2) + \frac{\alpha_1 \alpha_2}{2} (b_1 + b_2) \right)$$

$$= \frac{2\alpha_2 b_1 - \alpha_2 (b_1 + b_2) + (\alpha_1 \alpha_2) (b_1 + b_2)}{2\alpha_2 b_1}$$

$$= \frac{2b_1 + (b_1 + b_2)(-1 + \alpha_1)}{2b_1} = 1 + \frac{(\alpha_1 - 1)(b_1 + b_2)}{2b_1}$$

$$= \frac{2b_1 - b_1 + \alpha_1 b_1 - b_2 + \alpha_1 b_2}{2b_1} = \frac{b_1 - b_2 + \alpha_1 (b_1 + b_2)}{2b_1}$$

$$L_2^* = \frac{1}{\alpha_2 b_2} (\alpha_2 b_2 - x_2 + \alpha_2 x_2)$$

$$= \frac{1}{\alpha_2 b_2} \left[ \alpha_2 b_2 + (\alpha_2 - 1) x_2 \right]$$

$$= \frac{1}{\alpha_2 b_2} \left[ \alpha_2 b_2 + (\alpha_2 - 1) \frac{\alpha_2}{2} (b_1 + b_2) \right]$$

$$= \frac{1}{2b_2} \left[ 2b_2 + (\alpha_2 - 1) (b_1 + b_2) \right]$$

$$= \frac{1}{2b_2} \left[ 2b_2 + \alpha_2 b_1 + \alpha_2 b_2 - b_1 - b_2 \right] = \frac{b_2 - b_1 + \alpha_2 (b_1 + b_2)}{2b_2}$$

$$\lambda^* = \frac{\alpha_2}{x_2} = \frac{2}{b_1 + b_2}$$

(b) Notice that  $\lambda^* = \partial (u_1^* + u_2^*) / \partial x_i$  by the Envelope Theorem. So  $\lambda^*$  is the shadow price of  $x$  at the maximum of social welfare. To achieve the social-welfare-maximizing allocation with a decentralized market mechanism,

See p. 13  
for an  
alternative  
method

the planner announces  $\lambda^*$  as the price of good  $x$  and allows individuals to maximize their utility after giving lump-sum transfers  $S_1$  to the first individual and  $S_2$  to the second individual.

Individual budget constraint: expenditures  $\lambda^* x_i$

income:  $S_i + \text{wage income}$

wage income: "value of marginal product" (or "marginal revenue product") times labor supplied =  
"price of  $x$ " \* marginal product of labor \* labor =  
 $\lambda^* b_i L_i$ .

Individual consumer:

$$\mathcal{L} = \alpha_i \ln x_i + (1 - \alpha_i) \ln(1 - L_i) + \mu [S_i + \lambda^* b_i L_i - \lambda^* x_i]$$

$$= \alpha_i \ln x_i + (1 - \alpha_i) \ln(1 - L_i) + \mu [S_i + \frac{2}{b_1 + b_2} b_i L_i - \frac{2}{b_1 + b_2} x_i]$$

$$0 = \partial \mathcal{L} / \partial \mu = S_i + \frac{2}{b_1 + b_2} b_i L_i - \frac{2}{b_1 + b_2} x_i$$

$$0 = \partial \mathcal{L} / \partial x_i = \frac{\alpha_i}{x_i} - \mu \frac{2}{b_1 + b_2}$$

$$0 = \partial \mathcal{L} / \partial L_i = -\frac{1 - \alpha_i}{1 - L_i} + \mu \frac{2}{b_1 + b_2} b_i$$

Lagrange multiplier for  
this problem

$$\mu \frac{2}{b_1 + b_2} = \frac{\alpha_i}{x_i} = \frac{1}{b_i} \frac{1 - \alpha_i}{1 - L_i}$$

$$x_i = \alpha_i b_i \frac{1 - L_i}{1 - \alpha_i}$$

$$\frac{1 - \alpha_i}{\alpha_i} x_i = b_i - b_i L_i$$

$$b_i L_i = b_i - \frac{1 - \alpha_i}{\alpha_i} x_i$$

Substitute this into the budget constraint:

$$0 = S_i + \frac{2}{b_1 + b_2} (b_i - \frac{1 - \alpha_i}{\alpha_i} x_i) - \frac{2}{b_1 + b_2} x_i$$

$$x_i \left( \frac{2}{b_1 + b_2} \frac{1 - \alpha_i}{\alpha_i} + \frac{2}{b_1 + b_2} \right) = S_i + \frac{2 b_i}{b_1 + b_2}$$

$$x_i \frac{2(1-\alpha_i) + 2\alpha_i}{\alpha_i(b_1+b_2)} = S_i + \frac{2b_i}{b_1+b_2}$$

$$x_i \frac{2}{\alpha_i(b_1+b_2)} = S_i + \frac{2b_i}{b_1+b_2}$$

$$x_i = \frac{1}{2} \alpha_i (b_1+b_2) \left( S_i + \frac{2b_i}{b_1+b_2} \right)$$

$$= \frac{1}{2} \alpha_i (b_1+b_2) S_i + \alpha_i b_i$$

$$= \alpha_i \left[ b_i + \frac{b_1+b_2}{2} S_i \right]$$

Utilitarian  $x_i = \frac{\alpha_i}{2} (b_1+b_2)$ . Equate the utilitarian and competitive

$$x_i \text{'s: } \frac{\alpha_i}{2} (b_1+b_2) = \alpha_i \left( b_i + \frac{b_1+b_2}{2} S_i \right)$$

$$\frac{1}{2} (b_1+b_2) = b_i + \frac{b_1+b_2}{2} S_i$$

$$b_1+b_2 = 2b_i + (b_1+b_2) S_i \text{ ; divide by } b_1+b_2 \Rightarrow$$

$$1 = \frac{2b_i}{b_1+b_2} + S_i \Rightarrow S_i = 1 - \frac{2b_i}{b_1+b_2}$$

$$= \frac{b_1+b_2 - 2b_i}{b_1+b_2}$$

$$\Rightarrow S_1 = \frac{b_1+b_2 - 2b_1}{b_1+b_2} = \frac{b_2 - b_1}{b_1+b_2} > 0$$

$$S_2 = \frac{b_1+b_2 - 2b_2}{b_1+b_2} = \frac{b_1 - b_2}{b_1+b_2} < 0.$$

(Recall that  $b_1 < b_2$ .)

# Homework on Normative General Equilibrium

## Problem 3b, Alternative Answer.

Suppose one doesn't use  $\lambda^*$  from the social planner's problem as a price.

Using the notation of Problem 3b's first answer, each consumer will

$$\max U_i \text{ s.t. } \underbrace{S_i}_{\text{income}} + p b_i L_i - \underbrace{p X_i}_{\text{expenditures}} = 0 \quad \text{letting "p" be the price of } x_i.$$

$$\mathcal{L} = \alpha_i \ln x_i + (1 - \alpha_i) \ln(1 - L_i) + \lambda (S_i + p b_i L_i - p X_i)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = S_i + p b_i L_i - p X_i$$

$$0 = \frac{\partial \mathcal{L}}{\partial X_i} = \frac{\alpha_i}{X_i} - \lambda p$$

$$0 = \frac{\partial \mathcal{L}}{\partial L_i} = -\frac{1 - \alpha_i}{1 - L_i} + \lambda p b_i$$

$$\left. \begin{array}{l} 0 = \frac{\partial \mathcal{L}}{\partial X_i} = \frac{\alpha_i}{X_i} - \lambda p \\ 0 = \frac{\partial \mathcal{L}}{\partial L_i} = -\frac{1 - \alpha_i}{1 - L_i} + \lambda p b_i \end{array} \right\} \lambda p = \frac{\alpha_i}{X_i} = \frac{1}{b_i} \frac{1 - \alpha_i}{1 - L_i}$$

$$\Rightarrow X_i = \alpha_i b_i \frac{1 - L_i}{1 - \alpha_i}$$

Substitute this into the budget constraint:

$$S_i + p b_i L_i - p \frac{b_i (1 - L_i) \alpha_i}{(1 - \alpha_i)} = 0. \quad \text{Solve for } L_i:$$

$$S_i + p b_i L_i - p \frac{b_i \alpha_i}{1 - \alpha_i} + p \frac{b_i L_i \alpha_i}{1 - \alpha_i} = 0$$

$$p b_i L_i + p \frac{b_i \alpha_i}{1 - \alpha_i} L_i = -S_i + p \frac{b_i \alpha_i}{1 - \alpha_i}. \quad \text{Divide by } p b_i:$$

$$L_i + \frac{\alpha_i}{1 - \alpha_i} L_i = \frac{-S_i}{p b_i} + \frac{\alpha_i}{1 - \alpha_i}$$

$$\left(1 + \frac{\alpha_i}{1 - \alpha_i}\right) L_i = \quad "$$

$$\frac{1}{1 - \alpha_i} L_i = \quad "$$

$$\boxed{L_i = \frac{-S_i}{p b_i} (1 - \alpha_i) + \alpha_i.}$$



$$\begin{aligned}
 \text{Then } X_1 &= \frac{b_1 (1-L_1) \alpha_1}{1-\alpha_1} = \frac{b_1 \alpha_1}{1-\alpha_1} \left[ 1 + \frac{S_1}{p b_1} (1-\alpha_1) - \alpha_1 \right] \\
 &= \frac{b_1 \alpha_1}{1-\alpha_1} \left[ (1-\alpha_1) + \frac{S_1}{p b_1} (1-\alpha_1) \right] \\
 &= b_1 \alpha_1 \left[ 1 + \frac{S_1}{p b_1} \right]
 \end{aligned}$$

$$X_1 = b_1 \alpha_1 + \alpha_1 \frac{S_1}{p} .$$

$L_2$  and  $X_2$  are similar because  $S_2 = -S_1$ ; so

$$\begin{aligned}
 L_2 &= \frac{S_2}{p b_2} (1-\alpha_2) + \alpha_2 \\
 X_2 &= b_2 \alpha_2 + \alpha_2 \frac{S_2}{p} .
 \end{aligned}$$

Equate demand and supply for  $X$ :

$$\begin{aligned}
 (b_1 \alpha_1 + \alpha_1 \frac{S_1}{p}) + (b_2 \alpha_2 + \alpha_2 \frac{S_2}{p}) &= b_1 L_1 + b_2 L_2 \\
 &= b_1 \left[ \alpha_1 - \frac{S_1}{p b_1} (1-\alpha_1) \right] + b_2 \left[ \alpha_2 - \frac{S_2}{p b_2} (1-\alpha_2) \right] .
 \end{aligned}$$

$$S_2 = -S_1 \Rightarrow$$

$$\alpha_1 b_1 + \alpha_1 \frac{S_1}{p} + \alpha_2 b_2 - \alpha_2 \frac{S_1}{p} = \alpha_1 b_1 - \frac{S_1}{p} (1-\alpha_1) + \alpha_2 b_2 + \frac{S_1}{p} (1-\alpha_2)$$

← terms cancel

$$\alpha_1 \frac{S_1}{p} - \alpha_2 \frac{S_1}{p} = -\frac{S_1}{p} + \frac{S_1}{p} \alpha_1 + \frac{S_1}{p} - \frac{S_1}{p} \alpha_2$$

$$0 = -\frac{S_1}{p} + \frac{S_1}{p} \text{ so this is already assured.}$$

The Social Planner wanted  $X_i = \frac{\alpha_i}{2} (b_1 + b_2)$

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$$L_1 = \frac{b_1 - b_2 + \alpha_1 (b_1 + b_2)}{2b_1}$$

$$L_2 = \frac{b_2 - b_1 + \alpha_2 (b_1 + b_2)}{2b_2}$$

Equate  $X_1$ 's:  $\frac{\alpha_1}{2} (b_1 + b_2) = b_1 \alpha_1 + \alpha_1 \frac{s_1}{p}$

$$\frac{1}{2} \alpha_1 b_1 + \frac{1}{2} \alpha_1 b_2 = \alpha_1 b_1 + \alpha_1 \frac{s_1}{p}$$

$$\frac{1}{2} \alpha_1 b_2 = \frac{1}{2} \alpha_1 b_1 + \alpha_1 \frac{s_1}{p}$$

$$\frac{1}{2} b_2 = \frac{1}{2} b_1 + \frac{s_1}{p} \Rightarrow \frac{s_1}{p} = \frac{1}{2} (b_2 - b_1).$$

Equate  $X_2$ 's:  $\frac{\alpha_2}{2} (b_1 + b_2) = b_2 \alpha_2 + \alpha_2 \frac{s_2}{p}$   
 $= b_2 \alpha_2 - \alpha_2 \frac{s_1}{p}$

$$\frac{1}{2} b_1 \alpha_2 + \frac{1}{2} \alpha_2 b_2 = b_2 \alpha_2 - \alpha_2 \frac{s_1}{p}$$

$$\frac{1}{2} \alpha_2 b_1 = \frac{1}{2} \alpha_2 b_2 - \alpha_2 \frac{s_1}{p}$$

$$\frac{1}{2} b_1 = \frac{1}{2} b_2 - \frac{s_1}{p} \Rightarrow -\frac{s_1}{p} = \frac{1}{2} (b_1 - b_2) \Rightarrow \frac{s_1}{p} = \frac{1}{2} (b_2 - b_1).$$

same as before

Equate  $L_1$ 's:  $\frac{b_1 - b_2 + \alpha_1 (b_1 + b_2)}{2b_1} = \alpha_1 - \frac{s_1}{pb_1} (1 - \alpha_1)$

$$b_1 - b_2 + \alpha_1 (b_1 + b_2) = 2\alpha_1 b_1 - 2 \frac{s_1}{p} (1 - \alpha_1)$$

$$b_1 - b_2 + \alpha_1 b_1 + \alpha_1 b_2 = 2\alpha_1 b_1 - 2 \frac{s_1}{p} + 2\alpha_1 \frac{s_1}{p}$$

$$b_1 - b_2 + \alpha_1 b_2 = \alpha_1 b_1 + 2 \frac{s_1}{p} (\alpha_1 - 1)$$

$$b_1 - b_2 + \alpha_1 b_2 - \alpha_1 b_1 = 2 \frac{s_1}{p} (\alpha_1 - 1)$$

$$\frac{b_1 - b_2 + \alpha_1 (b_2 - b_1)}{2(\alpha_1 - 1)} = \frac{s_1}{p}$$

$$\frac{S_1}{P} = \frac{-(b_2 - b_1) + \alpha_1 (b_2 - b_1)}{2(\alpha_1 - 1)} = \frac{(b_2 - b_1)(-1 + \alpha_1)}{2(\alpha_1 - 1)} = \frac{b_2 - b_1}{2}$$

same as before

(Equation  $L_2$ 's is omitted.)

Look at the budget constraint. It is  $S_i + p b_i L_i - p X_i = 0$  or

$$1 + \left(\frac{P}{S_i}\right) b_i L_i - \left(\frac{P}{S_i}\right) X_i = 0.$$

So all that matters is the ratio  $P/S_i$ , not  $p$  or  $S_i$  individually.

Hence any  $p$  and  $S_1 = -S_2$  is OK as long as  $\frac{S_1}{P} = \frac{b_2 - b_1}{2}$ .

For example, if  $p$  were  $\frac{2}{b_1 + b_2}$ , then

$$S_1 = p * \frac{b_2 - b_1}{2}$$

$$= \frac{2}{b_1 + b_2} * \frac{b_2 - b_1}{2} = \frac{b_2 - b_1}{b_1 + b_2}$$

$$\text{and } S_2 = \frac{b_1 - b_2}{b_1 + b_2}$$

as obtained in the homework answers. However, this shows that there are many other possibilities for  $p$  and  $S_1 = -S_2$  beyond what the homework set obtained. Another example is  $p = 1$ ,  $S_1 = \frac{b_2 - b_1}{2}$ ,  $S_2 = \frac{b_1 - b_2}{2}$ .

[End of Problem 3b.]

Problem 3(c) In part (a) the budget constraint was

$$\lambda^* x_i = \lambda^* b_i L_i + S_i.$$

Here it is  $p_x x_i = p_x b_i L_i - t$

$$= p_x b_i L_i - [-a + m(p_x b_i L_i)]$$

$$= p_x b_i L_i + a - m p_x b_i L_i$$

$$x_i = b_i L_i + \frac{a}{p_x} - m b_i L_i = (1-m) b_i L_i + \frac{a}{p_x}.$$

Letting  $\hat{m} = 1-m$  and  $\hat{a} = a/p_x$ , we have

$$x_i = \hat{m} b_i L_i + \hat{a}.$$

The Lagrangian for the consumer's problem is <sup>Lagrange multiplier</sup>

$$\mathcal{L} = \alpha_i \ln x_i + (1-\alpha_i) \ln(1-L_i) + \beta (\hat{m} b_i L_i + \hat{a} - x_i)$$

$$0 = \partial \mathcal{L} / \partial \beta = \hat{m} b_i L_i + \hat{a} - x_i$$

$$0 = \partial \mathcal{L} / \partial x_i = \frac{\alpha_i}{x_i} - \beta \Rightarrow \beta = \alpha_i / x_i \Rightarrow x_i = \alpha_i / \beta$$

$$0 = \partial \mathcal{L} / \partial L_i = -\frac{1-\alpha_i}{1-L_i} + \beta \hat{m} b_i \Rightarrow \frac{1-\alpha_i}{1-L_i} = \beta \hat{m} b_i$$

$$1-L_i = \frac{1-\alpha_i}{\beta \hat{m} b_i}$$

$$L_i = 1 - \frac{1-\alpha_i}{\beta \hat{m} b_i} = 1 - \frac{x_i}{\alpha_i} \frac{1-\alpha_i}{\hat{m} b_i}$$

$$\begin{aligned} \text{Budget constraint} \Rightarrow x_i &= \hat{m} b_i \left( 1 - \frac{x_i}{\alpha_i} \frac{1-\alpha_i}{\hat{m} b_i} \right) + \hat{a} \\ &= \hat{m} b_i - \frac{1-\alpha_i}{\alpha_i} x_i + \hat{a} \end{aligned}$$

$$\left( 1 + \frac{1-\alpha_i}{\alpha_i} \right) x_i = \hat{m} b_i + \hat{a}$$

$$\frac{1}{\alpha_i} x_i = \hat{m} b_i + \hat{a}$$

$$x_i^* = \alpha_i (\hat{m} b_i + \hat{a}).$$

$$L_i^* = 1 - \frac{1-\alpha_i}{\alpha_i} \frac{1}{\hat{m} b_i} x_i$$

$$= 1 - \frac{1-\alpha_i}{\alpha_i} \frac{1}{\hat{m} b_i} \alpha_i (\hat{m} b_i + \hat{a})$$

$$= 1 - (1-\alpha_i) - \frac{1-\alpha_i}{\alpha_i} \frac{\alpha_i \hat{a}}{\hat{m} b_i}$$

$$= \alpha_i - \frac{1-\alpha_i}{b_i} \frac{\hat{a}}{\hat{m}} \quad \text{For later, } 1-L_i^* = (1-\alpha_i) + \frac{1-\alpha_i}{\alpha_i} \frac{\alpha_i \hat{a}}{\hat{m} b_i}$$

$$= (1-\alpha_i) \left[ 1 + \frac{\hat{a}}{\hat{m} b_i} \right]$$

$$0 = t_1 + t_2 = -a + m(p_x b_1 L_1) - a + m(p_x b_2 L_2)$$

$$\frac{2a}{p_x} = m(b_1 L_1 + b_2 L_2)$$

$$\frac{2\hat{a}}{m} = b_1 L_1 + b_2 L_2 = \sum_i b_i L_i = \sum_i \left[ \alpha_i b_i - (1-\alpha_i) \frac{\hat{a}}{\hat{m}} \right]$$

$$= \alpha_1 b_1 + \alpha_2 b_2 - (1-\alpha_1) \frac{\hat{a}}{\hat{m}} - (1-\alpha_2) \frac{\hat{a}}{\hat{m}}$$

$$\frac{2\hat{a}}{m} + (1-\alpha_1) \frac{\hat{a}}{\hat{m}} + (1-\alpha_2) \frac{\hat{a}}{\hat{m}} = \alpha_1 b_1 + \alpha_2 b_2$$

$$\frac{\hat{a}}{m \hat{m}} \left[ 2\hat{m} + (1-\alpha_1)m + (1-\alpha_2)m \right] = \alpha_1 b_1 + \alpha_2 b_2$$

$$\frac{\hat{a}}{m \hat{m}} \left[ 2 - 2m + m - \alpha_1 m + m - \alpha_2 m \right] = \alpha_1 b_1 + \alpha_2 b_2$$

$$\frac{\hat{a}}{m \hat{m}} (2 - \alpha_1 m - \alpha_2 m) = \alpha_1 b_1 + \alpha_2 b_2$$

$$\hat{a} = \frac{m(1-m)}{2 - \alpha_1 m - \alpha_2 m} (\alpha_1 b_1 + \alpha_2 b_2)$$

Social Planner:

$$\begin{aligned} & \max \sum \alpha_i \ln x_i + (1-\alpha_i) \ln(1-L_i) \\ & = \max \sum_i \alpha_i \ln(\alpha_i(\hat{m}b_i + \hat{a})) + (1-\alpha_i) \ln\left[(1-\alpha_i)\left(1 + \frac{\hat{a}}{\hat{m}b_i}\right)\right] \\ & \text{s.t. } \hat{a} = m(1-m) \frac{\alpha_1 b_1 + \alpha_2 b_2}{2 - \alpha_1 m - \alpha_2 m} \quad \text{and} \\ & \hat{m} = 1-m. \end{aligned}$$

Substituting in for  $\hat{a}$  and  $\hat{m}$ , this becomes a one-dimensional problem of maximizing with respect to  $m$  only.

It also has to be true that

$$X_1^* + X_2^* = b_1 L_1^* + b_2 L_2^*$$

$$\sum_i \alpha_i (\hat{m}b_i + \hat{a}) = \sum_i \left[ \alpha_i b_i - (1-\alpha_i) \frac{\hat{a}}{\hat{m}} \right].$$

$$0 = \sum_i \left[ \alpha_i \hat{m} b_i + \alpha_i \hat{a} - \alpha_i b_i + (1-\alpha_i) \frac{\hat{a}}{\hat{m}} \right]$$

$$\sum_i (-\alpha_i \hat{m} b_i + \alpha_i b_i) = \sum_i \left[ \alpha_i \hat{a} + (1-\alpha_i) \frac{\hat{a}}{\hat{m}} \right]$$

$$\frac{\sum_i \alpha_i b_i (1-\hat{m})}{\sum_i (\alpha_i + (1-\alpha_i) \frac{1}{\hat{m}})} = \hat{a}$$

$$\begin{aligned}
 \hat{a} &= \frac{(1-\hat{m}) \sum_i \alpha_i b_i}{\sum_i \frac{\alpha_i \hat{m} + (1-\alpha_i)}{\hat{m}}} = \frac{\hat{m} (1-\hat{m}) \sum_i \alpha_i b_i}{\sum_i [\alpha_i \hat{m} + (1-\alpha_i)]} \\
 &= \frac{(1-m) m \sum_i \alpha_i b_i}{\sum_i (\alpha_i (1-m) + 1 - \alpha_i)} = \frac{(1-m) m \sum_i \alpha_i b_i}{\sum_i (\alpha_i - \alpha_i m + 1 - \alpha_i)} \\
 &= \frac{(1-m) m \sum_i \alpha_i b_i}{\sum_i (1 - \alpha_i m)} = \frac{m(1-m) (\alpha_1 b_1 + \alpha_2 b_2)}{1 - \alpha_1 m + 1 - \alpha_2 m}
 \end{aligned}$$

but this is exactly the same condition that was arrived at by imposing  $0 = t_1 + t_2$ .

Even though the social planner's problem is one-dimensional, it is very tedious to find the optimal  $m$ , so it's not done here.

The core is

- feasible
- Pareto optimal
- impossible to block (no one wants to veto movement to it)

Find the Pareto optimal allocations:

$$\max U^1 \text{ s.t. } U^2 = \text{constant \& feasibility}$$

$$\max x_{11}^{1/2} + x_{21}^{1/2} \text{ s.t. } x_{12}^{1/2} + x_{22}^{1/2} = \bar{U}^2 \text{ and}$$

$$(1-x_{11})^{1/2} + (2-x_{21})^{1/2} = \bar{U}^2$$

$\uparrow$  total amount of goods 1 & 2 in the economy

$$\mathcal{L} = x_{11}^{1/2} + x_{21}^{1/2} + \lambda (\bar{U}^2 - (1-x_{11})^{1/2} - (2-x_{21})^{1/2})$$

$$0 = \partial \mathcal{L} / \partial \lambda = \bar{U}^2 - (1-x_{11})^{1/2} - (2-x_{21})^{1/2}$$

$$0 = \partial \mathcal{L} / \partial x_{11} = \frac{1}{2} x_{11}^{-1/2} + \frac{1}{2} \lambda (1-x_{11})^{-1/2}$$

$$0 = \partial \mathcal{L} / \partial x_{21} = \frac{1}{2} x_{21}^{-1/2} + \frac{1}{2} \lambda (2-x_{21})^{-1/2}$$

$$\left. \begin{array}{l} 0 = \partial \mathcal{L} / \partial x_{11} = \frac{1}{2} x_{11}^{-1/2} + \frac{1}{2} \lambda (1-x_{11})^{-1/2} \\ 0 = \partial \mathcal{L} / \partial x_{21} = \frac{1}{2} x_{21}^{-1/2} + \frac{1}{2} \lambda (2-x_{21})^{-1/2} \end{array} \right\} \frac{x_{11}^{-1/2}}{x_{21}^{-1/2}} = \frac{(1-x_{11})^{-1/2}}{(2-x_{21})^{-1/2}}$$

$$\frac{x_{21}}{x_{11}} = \frac{2-x_{21}}{1-x_{11}}$$

$$x_{21} - x_{21}x_{11} = 2x_{11} - x_{11}x_{21}$$

$$x_{21} = 2x_{11}, \text{ the contract curve.}$$

Can't be blocked  $\Rightarrow$

$$U_1^1 \geq U^1(\omega_1)$$

$$x_{11}^{1/2} + x_{21}^{1/2} \geq 1$$

$$x_{11}^{1/2} + (2x_{11})^{1/2} \geq 1$$

$$x_{11}^{1/2} \geq \frac{1}{1+\sqrt{2}}$$

$\Downarrow$

$$U^2 \geq U^2(\omega_2)$$

$$x_{12}^{1/2} + x_{22}^{1/2} \geq \sqrt{2}$$

$$(1-x_{11})^{1/2} + (2-x_{21})^{1/2} \geq \sqrt{2}$$

$$(1-x_{11})^{1/2} + (2-2x_{11})^{1/2} \geq \sqrt{2}$$

$\Downarrow$



$$x_{11} \geq \frac{1}{(1+\sqrt{2})^2} = \frac{1}{1+2\sqrt{2}+2} = \frac{1}{3+2\sqrt{2}} \cdot \frac{3-2\sqrt{2}}{3-2\sqrt{2}} = \frac{3-2\sqrt{2}}{9-8} = 3-2\sqrt{2}$$

and  $U^2 \geq U^2(\omega_{22}) \Rightarrow$

$$(1-x_{11})^{1/2} + 2^{1/2}(1-x_{11})^{1/2} \geq \sqrt{2}$$

$$(1-x_{11})^{1/2} [1+2^{1/2}] \geq \sqrt{2}$$

$$(1-x_{11})^{1/2} \geq \frac{\sqrt{2}}{1+\sqrt{2}}$$

$$1-x_{11} \geq \left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)^2 = \frac{2}{1+2\sqrt{2}+2} = \frac{2}{3+2\sqrt{2}} = 2(3-2\sqrt{2})$$

from above

$$= 6-4\sqrt{2}$$

$$-x_{11} \geq 5-4\sqrt{2}$$

$$x_{11} \leq 4\sqrt{2}-5$$

Combining these conditions on  $x_{11}$ :

$$3-2\sqrt{2} \leq x_{11} \leq 4\sqrt{2}-5 \Leftrightarrow 0.172 \leq x_{11} \leq 0.657 \quad \text{No Blocking}$$

$$x_{21} = 2x_{11} \quad \text{Pareto optimality}$$

$$x_{11} + x_{12} = 1 \quad \& \quad x_{21} + x_{22} = 2 \quad \text{feasibility.}$$

Competitive equilibrium:

$$\text{Consumer 1: } \max x_{11}^{1/2} + x_{21}^{1/2} \quad \text{s.t.} \quad p_1 x_{11} + p_2 x_{21} = p_1(1) + p_2(0)$$

$\Rightarrow$

$$\mathcal{L} = x_{11}^{1/2} + x_{21}^{1/2} + \lambda (p_1 - p_1 x_{11} - p_2 x_{21})$$

$$0 = \partial \mathcal{L} / \partial \lambda = p_1 - p_1 x_{11} - p_2 x_{21}$$

$$\left. \begin{aligned} 0 = \partial \mathcal{L} / \partial x_{11} &= \frac{1}{2} x_{11}^{-1/2} - \lambda p_1 \\ 0 = \partial \mathcal{L} / \partial x_{21} &= \frac{1}{2} x_{21}^{-1/2} - \lambda p_2 \end{aligned} \right\} \frac{p_1}{p_2} = \sqrt{\frac{x_{21}}{x_{11}}} \quad \text{Taking } p_2 = 1 \text{ as the numeraire,}$$

$$p_1^2 = \frac{x_{21}}{x_{11}} \Rightarrow x_{21} = p_1^2 x_{11}$$

Now we'll take a shortcut. Competitive equilibria are Pareto optimal, so in competitive equilibrium,  $x_{21} = 2x_{11}$  from our previous work.

Hence  $2x_{11} = p_1^2 x_{11} \Rightarrow p_1 = \sqrt{2}$ . Budget constraint  $\Rightarrow$

$$0 = \sqrt{2} - \sqrt{2} x_{11} - (1)(2x_{11}) = \sqrt{2} - \sqrt{2} x_{11} - 2x_{11}$$

$$x_{11} (2 + \sqrt{2}) = \sqrt{2} \Rightarrow x_{11} = \frac{\sqrt{2}}{2 + \sqrt{2}} \frac{2 - \sqrt{2}}{2 - \sqrt{2}} = \frac{2\sqrt{2} - 2}{2} = \sqrt{2} - 1 \approx 0.41.$$

$$x_{21} = 2(\sqrt{2} - 1)$$

$$x_{12} = 1 - x_{11} = 1 - (\sqrt{2} - 1) = 2 - \sqrt{2}$$

$$\begin{aligned} x_{22} &= 2 - x_{21} = 2 - 2\sqrt{2} + 2 = 4 - 2\sqrt{2} \\ &= 2(2 - \sqrt{2}). \end{aligned}$$

Following page 30, an equally good alternative approach is used to solve this problem.

⑤

Feasibility  $x_{11} + x_{12} = 3$        $x_{21} + x_{22} = 3$

Pareto Optimal

$\max U^2$  s.t.  $U'$  fixed

$\max x_{12} x_{22}$  s.t.  $(3 - x_{12})^{1/2} (3 - x_{22})^{1/2} = U'$

$\mathcal{L} = x_{12} x_{22} + \lambda [(3 - x_{12})^{1/2} (3 - x_{22})^{1/2} - U']$

$0 = \partial \mathcal{L} / \partial \lambda = (3 - x_{12})^{1/2} (3 - x_{22})^{1/2} - U'$

$0 = \partial \mathcal{L} / \partial x_{12} = x_{22} + \frac{1}{2} \lambda (3 - x_{12})^{-1/2} (-1) (3 - x_{22})^{1/2}$

$0 = \partial \mathcal{L} / \partial x_{22} = x_{12} + \frac{1}{2} \lambda (3 - x_{22})^{-1/2} (-1) (3 - x_{12})^{1/2}$

$\frac{x_{22}}{x_{12}} = \frac{(3 - x_{12})^{-1/2} (3 - x_{22})^{1/2}}{(3 - x_{22})^{-1/2} (3 - x_{12})^{1/2}} = \frac{3 - x_{22}}{3 - x_{12}}$

$3x_{22} - x_{22}x_{12} = 3x_{12} - x_{12}x_{22}$

$x_{22} = x_{12}$

Cannot be Blocked

$U^2 \geq U_0^2 \Leftrightarrow$

$U' \geq U'_0 \Leftrightarrow$

$x_{12} x_{12} \geq 2$

$\sqrt{x_{11} x_{21}} \geq \sqrt{2}$

$x_{12} \geq \sqrt{2}$

$x_{11} x_{21} \geq 2$  ; use feasibility  $\Rightarrow$

$(3 - x_{12})(3 - x_{22}) \geq 2$  ; use Pareto optimality  $\Rightarrow$

$(3 - x_{12})(3 - x_{12}) \geq 2$

$3 - x_{12} \geq \sqrt{2}$

$x_{12} \leq 3 - \sqrt{2}$

Verify that if you max  $U'$  s.t.  $U^2$  fixed, you get the same answer as if you max  $U^2$  s.t.  $U'$  fixed. This is always true for Pareto Optimality.

So the cone is  $\sqrt{2} \leq x_{12} \leq 3 - \sqrt{2} \Leftrightarrow \text{approx. } 1.41 \leq x_{12} \leq 1.586$

$$x_{22} = x_{12}$$

$$x_{11} = 3 - x_{12}$$

$$x_{21} = 3 - x_{22} = 3 - x_{12}$$

### Four Consumers

Coalitions like  $\{1, 1\}$  or  $\{2, 2\}$  have no trading.

Coalitions like  $\{1, 2\}$  or  $\{1, 1, 2, 2\}$  are like the  $\{1, 2\}$  economy studied in the first part of this question.

Coalitions like  $\{1, 1, 2\}$  and  $\{1, 2, 2\}$  require further analysis.

• Consider the  $\{1, 1, 2\}$  coalition.

$$\begin{aligned} \text{Feasibility } 2x_{11} + x_{12} &= 5 \\ 2x_{21} + x_{22} &= 4 \end{aligned}$$

$$\text{Pareto Optimal } \max 2U^1 \text{ s.t. } \bar{U}^2 \text{ fixed}$$

$$\max 2x_{11}^{1/2} x_{21}^{1/2} \text{ s.t. } \bar{U}^2 = x_{12} x_{22} \text{ fixed}$$

or  $\max U^1 \text{ s.t. } U^2 \text{ fixed}$   
 or  $\max U^2 \text{ s.t. } 2U^1 \text{ fixed}$   
 or  $\max U^2 \text{ s.t. } U^1 \text{ fixed}$  - all these yield the same answer

$$= (5 - 2x_{11})(4 - 2x_{21})$$

$$\mathcal{L} = 2x_{11}^{1/2} x_{21}^{1/2} + \lambda \left[ \overset{\text{fixed constant}}{\bar{U}^2} - (5 - 2x_{11})(4 - 2x_{21}) \right]$$

$$0 = \partial \mathcal{L} / \partial \lambda = \bar{U}^2 - (5 - 2x_{11})(4 - 2x_{21})$$

$$\begin{cases} 0 = \partial \mathcal{L} / \partial x_{11} = x_{11}^{-1/2} x_{21}^{1/2} + 2\lambda (4 - 2x_{21}) \\ 0 = \partial \mathcal{L} / \partial x_{21} = x_{11}^{1/2} x_{21}^{-1/2} + 2\lambda (5 - 2x_{11}) \end{cases}$$

$$\frac{x_{21}^{1/2} x_{21}^{1/2}}{x_{11}^{1/2} x_{11}^{1/2}} = \frac{4 - 2x_{21}}{5 - 2x_{11}}$$

$$\frac{x_{21}}{x_{11}} = \frac{4 - 2x_{21}}{5 - 2x_{11}}$$

$$5x_{21} - 2x_{11}x_{21} = 4x_{11} - 2x_{11}x_{21} \Rightarrow 5x_{21} = 4x_{11} \Rightarrow x_{21} = \frac{4}{5}x_{11},$$

$$x_{22} = 4 - 2x_{21} = 4 - 2\left(\frac{4}{5}x_{11}\right) = 4 - \frac{8}{5}x_{11} \quad x_{11} = \frac{5}{4}x_{21}$$

$$x_{11} = \frac{1}{2}(5 - x_{12})$$

$$\therefore x_{21} = \frac{4}{5}x_{11} = \frac{4}{5} \cdot \frac{1}{2}(5 - x_{12}) = \frac{2}{5}(5 - x_{12})$$

$$x_{22} = 4 - \frac{8}{5} \cdot \frac{1}{2}(5 - x_{12}) = 4 - \frac{4}{5}(5 - x_{12}) = 4 - 4 + \frac{4}{5}x_{12} = \frac{4}{5}x_{12}$$

Now using  $0 = \partial \mathcal{L} / \partial \lambda$ :

$$\begin{aligned} \bar{U}^2 &= (5 - 2x_{11})(4 - 2x_{21}) \\ &= (5 - (5 - x_{12})) \left(4 - 2 \cdot \frac{2}{5}(5 - x_{12})\right) \\ &= x_{12} \left(4 - \frac{4}{5}(5 - x_{12})\right) = x_{12} \left(4 - 4 + \frac{4}{5}x_{12}\right) \\ &= \frac{4}{5}x_{12}^2 \Rightarrow x_{12} = \left(\frac{5}{4}\bar{U}^2\right)^{1/2}. \end{aligned}$$

$$\begin{aligned} \text{Optimal } U' &= x_{11}^{1/2} x_{21}^{1/2} = \sqrt{\frac{1}{2}(5 - x_{12}) \left(\frac{2}{5}\right)(5 - x_{12})} \\ &= \sqrt{\frac{1}{5}(5 - x_{12})^2} = (5 - x_{12}) / \sqrt{5} \end{aligned}$$

$$\begin{aligned} &= \left(5 - \sqrt{\frac{5}{4}\bar{U}^2}\right) / \sqrt{5} = \sqrt{5} - \sqrt{\frac{1}{4}\bar{U}^2} \\ &= \sqrt{5} - \frac{1}{2}\sqrt{\bar{U}^2}. \end{aligned}$$

We want this  $\{1, 1, 2\}$  coalition to be blocked by the  $\{1, 2\} =$

$\{1, 1, 2, 2\}$  coalition. Otherwise  $\{1, 1, 2\}$  would block the  $\{1, 1, 2, 2\}$

allocation. So we want  $\bar{U}^2$  of  $\{1, 1, 2\} \leq U^2$  of  $\{1, 2\}$

$$U' \text{ of } \{1, 1, 2\} \leq U' \text{ of } \{1, 2\}$$

$$U^2 \text{ of } \{1, 2\} = x_{12} x_{22} = x_{12} x_{12} = x_{12}^2$$

$$= x_{12}^2 \text{ call this } x_{120}^2$$

$$U^1 \text{ of } \{1, 2\} = x_{11}^{1/2} x_{21}^{1/2} = [(3 - x_{120})(3 - x_{120})]^{1/2}$$

$$= 3 - x_{120}$$

Set  $\bar{U}^2$  of  $\{1, 1, 2\}$  to be equal to  $U^2$  of  $\{1, 2\} = x_{120}^2$ ; then

$$U^1 \text{ of } \{1, 1, 2\} \leq U^1 \text{ of } \{1, 2\} \iff$$

$$\sqrt{5} - \frac{1}{2} \sqrt{\bar{U}^2 \text{ of } \{1, 1, 2\}} \leq 3 - x_{120}$$

$$\sqrt{5} - \frac{1}{2} \sqrt{x_{120}^2} \leq 3 - x_{120}$$

$$\sqrt{5} - \frac{1}{2} x_{120} \leq 3 - x_{120}$$

$$\frac{1}{2} x_{120} \leq 3 - \sqrt{5}$$

$$x_{120} \leq 6 - 2\sqrt{5}$$

• Consider the  $\{1, 2, 2\}$  coalition.

$$\text{Feasibility } x_{11} + 2x_{12} = 4$$

$$x_{21} + 2x_{22} = 5$$

Pareto optimal  $\max U^1$  s.t.  $2U^2$  fixed

$$\iff U^2 \text{ fixed}$$

$$\max x_{11}^{1/2} x_{21}^{1/2} \text{ s.t. } U^2 = x_{12} x_{22} \text{ fixed}$$

$$= (2 - \frac{1}{2} x_{11}) (\frac{5}{2} - \frac{1}{2} x_{21}) \text{ fixed}$$

$$\mathcal{L} = x_{11}^{\frac{1}{2}} x_{21}^{\frac{1}{2}} + \lambda \left( \bar{U}^2 - \left(2 - \frac{1}{2} x_{11}\right) \left(\frac{5}{2} - \frac{1}{2} x_{21}\right) \right)$$

$$0 = \partial \mathcal{L} / \partial \lambda = \bar{U}^2 - \left(2 - \frac{1}{2} x_{11}\right) \left(\frac{5}{2} - \frac{1}{2} x_{21}\right)$$

$$\begin{cases} 0 = \partial \mathcal{L} / \partial x_{11} = \frac{1}{2} x_{11}^{-\frac{1}{2}} x_{21}^{\frac{1}{2}} + \frac{1}{2} \lambda \left(\frac{5}{2} - \frac{1}{2} x_{21}\right) \\ 0 = \partial \mathcal{L} / \partial x_{21} = \frac{1}{2} x_{11}^{\frac{1}{2}} x_{21}^{-\frac{1}{2}} + \frac{1}{2} \lambda \left(2 - \frac{1}{2} x_{11}\right) \end{cases}$$

$$\rightarrow \frac{x_{11}^{-\frac{1}{2}} x_{21}^{\frac{1}{2}}}{x_{11}^{\frac{1}{2}} x_{21}^{-\frac{1}{2}}} = \frac{\frac{5}{2} - \frac{1}{2} x_{21}}{2 - \frac{1}{2} x_{11}} \Leftrightarrow \frac{x_{21}}{x_{11}} = \frac{5 - x_{21}}{4 - x_{11}} \Leftrightarrow 4x_{21} - x_{11}x_{21} = 5x_{11} - x_{11}x_{21}$$

Therefore  $4x_{21} = 5x_{11}$ ,  $x_{11} = \frac{4}{5} x_{21}$ .

$$x_{22} = \frac{1}{2} (5 - x_{21})$$

$$x_{12} = \frac{1}{2} (4 - x_{11}) = \frac{1}{2} \left(4 - \frac{4}{5} x_{21}\right) = 2 - \frac{2}{5} x_{21}.$$

Now using  $0 = \partial \mathcal{L} / \partial \lambda$ :

$$\bar{U}^2 = \left(2 - \frac{1}{2} x_{11}\right) \left(\frac{5}{2} - \frac{1}{2} x_{21}\right) = \left(2 - \frac{1}{2} \cdot \frac{4}{5} x_{21}\right) \left(\frac{5}{2} - \frac{1}{2} x_{21}\right)$$

$$= \left(2 - \frac{2}{5} x_{21}\right) \frac{1}{2} (5 - x_{21}) = \left(1 - \frac{1}{5} x_{21}\right) (5 - x_{21})$$

$$= 5 - x_{21} - x_{21} + \frac{1}{5} x_{21}^2 = \frac{1}{5} x_{21}^2 - 2x_{21} + 5$$

$$0 = x_{21}^2 - 10x_{21} + 25 - 5\bar{U}^2$$

$$x_{21} = \frac{10 \pm \sqrt{100 - 4(-5\bar{U}^2)}}{2} = \frac{10 \pm \sqrt{100 + 20\bar{U}^2}}{2} = \frac{10 \pm 2\sqrt{25 + 5\bar{U}^2}}{2}$$

$$= 5 \pm \sqrt{25 + 5\bar{U}^2}, \text{ must take "+" sign to ensure } x_{21} > 0.$$

$$\text{Optimal } U' = x_{11}^{\frac{1}{2}} x_{21}^{\frac{1}{2}} = \left[ \frac{4}{5} x_{21} \cdot x_{21} \right]^{\frac{1}{2}} = \frac{2}{\sqrt{5}} x_{21} = \frac{2}{\sqrt{5}} \left(5 + \sqrt{25 + 5\bar{U}^2}\right).$$

We want this  $\{1, 2, 2\}$  coalition to be blocked by the  $\{1, 2\} = \{1, 1, 2, 2\}$

coalition. So we want

$$\begin{aligned} x_{21} &= \frac{10 \pm \sqrt{100 - 4(25 - 5\bar{U}^2)}}{2} = \frac{10 \pm \sqrt{100 - 100 + 20\bar{U}^2}}{2} \\ &= \frac{10 \pm \sqrt{20\bar{U}^2}}{2} = \frac{10 \pm 2\sqrt{5\bar{U}^2}}{2} = 5 \pm \sqrt{5\bar{U}^2} \end{aligned}$$

$$\begin{aligned} \text{Optimal } U' &= x_{11}^{\frac{1}{2}} x_{21}^{\frac{1}{2}} = \left[ \frac{4}{5} x_{21} \cdot x_{21} \right]^{\frac{1}{2}} = \frac{2}{\sqrt{5}} x_{21} = \frac{2}{\sqrt{5}} (5 \pm \sqrt{5\bar{U}^2}) \\ &= 2(\sqrt{5} \pm \sqrt{\bar{U}^2}) \end{aligned}$$

We want this  $\{1, 2, 2\}$  coalition to be blocked by the  $\{1, 2\} = \{1, 1, 2, 2\}$  coalition. So we want

$$\bar{U}^2 \text{ of } \{1, 2, 2\} \leq U^2 \text{ of } \{1, 2\} = x_{120}^2 \text{ from before}$$

$$U' \text{ of } \{1, 2, 2\} \leq U' \text{ of } \{1, 2\} = 3 - x_{120} \text{ from before}$$

Set  $\bar{U}^2$  of  $\{1, 2, 2\}$  to be equal to  $U^2$  of  $\{1, 2\} = x_{120}^2$ .

Then  $U' \text{ of } \{1, 2, 2\} \leq U' \text{ of } \{1, 2\} \Leftrightarrow$

$$2(\sqrt{5} \pm \sqrt{\bar{U}^2 \text{ of } \{1, 2, 2\}}) \leq 3 - x_{120}$$

$$2(\sqrt{5} \pm \sqrt{x_{120}^2}) \leq 3 - x_{120}$$

$$2(\sqrt{5} \pm x_{120}) \leq 3 - x_{120}$$

$$2(\sqrt{5} - x_{120}) \leq 3 - x_{120}$$

$$2\sqrt{5} - 2x_{120} \leq 3 - x_{120}$$

$$2\sqrt{5} - 3 \leq x_{120}$$

$$x_{120} \geq 2\sqrt{5} - 3 \approx 1.472$$

$$2(\sqrt{5} + x_{120}) \leq 3 - x_{120}$$

$$2\sqrt{5} + 2x_{120} \leq 3 - x_{120}$$

$$3x_{120} \leq 3 - 2\sqrt{5}$$

$$x_{120} \leq 1 - \frac{2}{3}\sqrt{5} \approx -0.491$$

not possible



$$2\sqrt{5}-3 \leq \chi_{120} \leq 6-2\sqrt{5} \text{ or approximately}$$

↙ from  $\{1,2,2\}$  analysis      ↙ from  $\{1,1,2\}$  analysis

$1.472 \leq \chi_{120} \leq 1.528$  for the  $\{1,1,2,2\}$  case

versus  $1.414 \leq \chi_{12} \leq 1.586$  for the  $\{1,2\}$  case.

So the cone did shrink.

### Alternative Approach.

$$\text{Pareto Optimal: } \max \alpha U_1 + (1-\alpha) U_2 \quad (\text{changing superscripts to subscripts}) \quad (1)$$

$$\max \alpha x_{11}^{1/2} x_{21}^{1/2} + (1-\alpha) x_{12} x_{22} \quad (2)$$

$$\{1, 2\} \text{ coalition feasibility } x_{11} + x_{12} = 3 \quad (3)$$

$$x_{21} + x_{22} = 3 \quad (4)$$

To avoid having six first-order conditions (w.r.t.  $x_{11}$ ,  $x_{12}$ ,  $x_{21}$ ,  $x_{22}$ , and  $\lambda_1$  and  $\lambda_2$  for these two feasibility conditions), use the first feasibility condition to eliminate either  $x_{11}$  or  $x_{12}$  (it doesn't matter which), and use the second feasibility condition to eliminate either  $x_{21}$  or  $x_{22}$  (it doesn't matter which). I'll use

$$x_{12} = 3 - x_{11} \quad (5)$$

$$x_{22} = 3 - x_{21} \quad (6)$$

$$(2), (5), \text{ and } (6) \Rightarrow \max \alpha x_{11}^{1/2} x_{21}^{1/2} + (1-\alpha) (3-x_{11}) (3-x_{21}) \quad (7)$$

$$0 = \frac{\partial (7)}{\partial x_{11}} = \frac{1}{2} \alpha x_{11}^{-1/2} x_{21}^{1/2} + (1-\alpha) (-1) (3-x_{21}) \quad (8)$$

$$0 = \frac{\partial (7)}{\partial x_{21}} = \frac{1}{2} \alpha x_{11}^{1/2} x_{21}^{-1/2} + (1-\alpha) (3-x_{11}) (-1) \quad (9)$$

$$(8) \Rightarrow (1-\alpha) (x_{21}-3) = \frac{\alpha}{2} \frac{x_{21}^{1/2}}{x_{11}^{1/2}} \quad (10)$$

$$(9) \Rightarrow (1-\alpha) (x_{11}-3) = \frac{\alpha}{2} \frac{x_{11}^{1/2}}{x_{21}^{1/2}} \quad (11)$$

$$(10) \text{ and } (11) \Rightarrow \frac{(1-\alpha) (x_{21}-3)}{(1-\alpha) (x_{11}-3)} = \frac{\frac{\alpha}{2} \frac{x_{21}^{1/2}}{x_{11}^{1/2}}}{\frac{\alpha}{2} \frac{x_{11}^{1/2}}{x_{21}^{1/2}}} \quad (12)$$

$$\frac{x_{21}-3}{x_{11}-3} = \frac{x_{21}^{1/2}}{x_{11}^{1/2}} \cdot \frac{x_{21}^{1/2}}{x_{11}^{1/2}} \quad (13)$$

$$= \frac{x_{21}}{x_{11}} \quad (14)$$

$$x_{11}(x_{21}-3) = x_{21}(x_{11}-3) \quad (15)$$

$$x_{11}x_{21} - 3x_{11} = x_{11}x_{21} - 3x_{21} \quad (16)$$

$$x_{11} = x_{21} \quad (17)$$

(17) can be used with (2) and (3) to express three of the four variables in terms of any one of the other variables. In terms of  $x_{11}$  we have

$$x_{12} = 3 - x_{11} \text{ from (3)} \quad (18)$$

$$x_{21} = x_{11} \text{ from (17)} \quad (19)$$

$$x_{22} = 3 - x_{21} \text{ from (4)} \quad (20)$$

$$= 3 - x_{11} \text{ from (19)}. \quad (21)$$

Or in terms of  $x_{12}$  we'd have

$$x_{11} = 3 - x_{12} \text{ from (3)} \quad (22)$$

$$x_{21} = x_{11} \text{ from (17)} \quad (23)$$

$$= 3 - x_{12} \text{ from (22)} \quad (24)$$

$$x_{22} = 3 - x_{21} \text{ from (4)}$$

$$= 3 - (3 - x_{12}) = x_{12} \text{ from (23)}. \quad (25)$$

Or in terms of  $x_{21}$  we'd have

$$x_{11} = x_{21} \text{ from (17)} \quad (26)$$

$$x_{12} = 3 - x_{11} = 3 - x_{21} \text{ from (3) and (26)} \quad (27)$$

$$x_{22} = 3 - x_{21} \text{ from (4)}. \quad (28)$$

Or in terms of  $x_{22}$ :

$$x_{21} = 3 - x_{22} \text{ from (4)} \quad (29)$$

$$x_{11} = x_{21} = 3 - x_{22} \text{ from (17) and (29)} \quad (30)$$

$$x_{12} = 3 - x_{11} = 3 - (3 - x_{22}) = x_{22} \text{ from (3) and (30).} \quad (31)$$

Arbitrarily, I'll express things in terms of  $x_{11}$ , using (18), (19), and (21), though I could have chosen to express things in terms of any of the other variables.

Cannot be blocked

$$U_2 \geq U_{2,0}$$

$$x_{12} x_{22} \geq x_{12,0} x_{22,0} = 1 \cdot 2$$

$\Leftrightarrow$

$$(3 - x_{11})(3 - x_{11}) \geq 2 \text{ from (18) and (21)}$$

$$(3 - x_{11})^2 \geq 2.$$

Since  $\sqrt{x}$  is a monotonically increasing function of  $x$ , this is equivalent to

$$\sqrt{(3 - x_{11})^2} \geq \sqrt{2}.$$

(Note that applying a monotonically decreasing function, such as the reciprocal, to both sides of an inequality reverses the inequality:  $2 < 4$   
 $1/2 > 1/4$ , for example.)

Next,  $\sqrt{a^2} = |a|$ . In our case,  $a = 3 - x_{11}$ , which we know from (3) is  $x_{12} \geq 0$ , so we don't need the absolute value sign:  $a \geq 0 \Rightarrow \sqrt{a^2} = a$ :

$$3 - x_{11} \geq \sqrt{2}$$

$$3 - \sqrt{2} \geq x_{11} \quad (33)$$

$\uparrow$   
about 1.59

$$U_1 \geq U_{1,0}$$

$$x_{11}^{1/2} x_{21}^{1/2} \geq x_{11,0}^{1/2} x_{21,0}^{1/2} = 2^{1/2} \cdot 1^{1/2}$$

$$x_{11}^{1/2} x_{11}^{1/2} \geq \sqrt{2} \text{ from (19)}$$

$$x_{11} \geq \sqrt{2} \quad (32)$$

$\uparrow$   
about 1.41

So the core is  $\sqrt{2} \leq x_{11} \leq 3 - \sqrt{2} \Leftrightarrow \text{approx. } 1.41 \leq x_{11} \leq 1.59.$  (34)

Optional: The core also has

(34) implies  $\sqrt{2} \leq x_{21} \leq 3 - \sqrt{2}$  from (17) (35)

(34) and (18) imply  $\sqrt{2} \leq 3 - x_{12} \leq 3 - \sqrt{2}$

$$\sqrt{2} - 3 \leq -x_{12} \leq -\sqrt{2}$$

↓                      ↓

$$3 - \sqrt{2} \geq x_{12} \quad x_{12} \geq \sqrt{2}$$

↓                      ↓

$$\sqrt{2} \leq x_{12} \leq 3 - \sqrt{2} \quad (36)$$

(18) and (21) imply  $x_{22} = x_{12}$ , so with (36),

$$\sqrt{2} \leq x_{22} \leq 3 - \sqrt{2}. \quad (37)$$

Useful for the next steps:

$$U_1 = x_{11}^{1/2} x_{21}^{1/2} = x_{11}^{1/2} x_{11}^{1/2} \text{ from (19) } = x_{11} \text{ in the core, subject to (34)} \quad (38)$$

$$\begin{aligned} U_2 = x_{12} x_{22} &= (3 - x_{11})(3 - x_{11}) \text{ from (18) and (21)} \\ &= (3 - x_{11})^2 \text{ in the core, subject to (34)}. \end{aligned} \quad (39)$$

### Four Consumers (the 2-replica of the economy)

As before, only the  $\{1, 1, 2\}$  and  $\{1, 2, 2\}$  coalitions require further analysis.

- Consider the  $\{1, 1, 2\}$  coalition. Use  $\hat{\cdot}$  to denote values in the  $\{1, 1, 2\}$  coalition. The feasibility conditions are

$$2 \hat{x}_{11} + \hat{x}_{12} = 5 \quad (40)$$

$$2 \hat{x}_{21} + \hat{x}_{22} = 4 \quad (41)$$

Pareto Optimal :  $\max 2\alpha U_1 + (1-\alpha)U_2$  s.t. (40) and (41) (42)

$$\max 2\alpha \hat{x}_{11}^{1/2} \hat{x}_{21}^{1/2} + (1-\alpha) \hat{x}_{12} \hat{x}_{22}$$

$\underbrace{\hspace{10em}}_{\substack{\uparrow \\ (4-\hat{x}_{22})/2 \\ \text{from (41)}}} \quad \underbrace{\hspace{10em}}_{\substack{\uparrow \\ 5-2\hat{x}_{11} \\ \text{from (40)}}$ 
(43)

Could also keep  $\hat{x}_{21}$   
and use (41) to  
eliminate  $\hat{x}_{22}$

Could also keep  $\hat{x}_{12}$   
and use (40) to  
eliminate  $\hat{x}_{11}$

$$\max 2\alpha \hat{x}_{11}^{1/2} \left( \frac{4-\hat{x}_{22}}{2} \right)^{1/2} + (1-\alpha) (5-2\hat{x}_{11}) \hat{x}_{22}$$
(44)

F.O.C.

$$0 = \frac{\partial (44)^\uparrow}{\partial \hat{x}_{11}} = \alpha \hat{x}_{11}^{-1/2} \left( 2 - \frac{1}{2} \hat{x}_{22} \right)^{1/2} + (1-\alpha) (-2) \hat{x}_{22}$$
(50)

$$0 = \frac{\partial (44)^\uparrow}{\partial \hat{x}_{22}} = \alpha \hat{x}_{11}^{1/2} \left( 2 - \frac{1}{2} \hat{x}_{22} \right)^{-1/2} \left( -\frac{1}{2} \right) + (1-\alpha) (5-2\hat{x}_{11})$$
(51)

$$(50) \Rightarrow 2(1-\alpha) \hat{x}_{22} = \alpha \hat{x}_{11}^{-1/2} \left( 2 - \frac{1}{2} \hat{x}_{22} \right)^{1/2}$$
(52)

$$(51) \Rightarrow (1-\alpha)(2\hat{x}_{11} - 5) = \frac{-\alpha}{2} \hat{x}_{11}^{1/2} \left( 2 - \frac{1}{2} \hat{x}_{22} \right)^{-1/2}$$
(53)

(52) and (53)  $\Rightarrow$

$$\frac{2(1-\alpha) \hat{x}_{22}}{(1-\alpha)(2\hat{x}_{11} - 5)} = \frac{\alpha \hat{x}_{11}^{-1/2} \left( 2 - \frac{1}{2} \hat{x}_{22} \right)^{1/2}}{-\frac{\alpha}{2} \hat{x}_{11}^{1/2} \left( 2 - \frac{1}{2} \hat{x}_{22} \right)^{-1/2}}$$
(54)

$$\frac{2\hat{x}_{22}}{5-2\hat{x}_{11}} = +2 \frac{1}{\hat{x}_{11}} \left( 2 - \frac{1}{2} \hat{x}_{22} \right)$$
(55)

$$\hat{x}_{11} \hat{x}_{22} = (5-2\hat{x}_{11}) \left( 2 - \frac{1}{2} \hat{x}_{22} \right)$$
(56)

$$= 10 - \frac{5}{2} \hat{x}_{22} - 4\hat{x}_{11} + \hat{x}_{11} \hat{x}_{22}$$
(57)

$$\frac{5}{2} \hat{x}_{22} = 10 - 4 \hat{x}_{11} \quad (58)$$

$$\hat{x}_{22} = \frac{2}{5} \cdot 10 - \frac{2}{5} \cdot 4 \hat{x}_{11} = 4 - \frac{8}{5} \hat{x}_{11}. \quad (59)$$

$$\text{From (40), } \hat{x}_{12} = 5 - 2 \hat{x}_{11}. \quad (60)$$

$$\text{From (41) and (59), } \hat{x}_{21} = \frac{4 - \hat{x}_{22}}{2} = 2 - \frac{1}{2} \hat{x}_{22}$$

$$= 2 - \frac{1}{2} \left( 4 - \frac{8}{5} \hat{x}_{11} \right) = 2 - 2 + \frac{4}{5} \hat{x}_{11} = \frac{4}{5} \hat{x}_{11}. \quad (61)$$

Useful non-negativity conditions:

$$\hat{x}_{22} \geq 0 \text{ and (59)} \Rightarrow 4 - \frac{8}{5} \hat{x}_{11} \geq 0$$

$$4 \geq \frac{8}{5} \hat{x}_{11}$$

$$\frac{20}{8} \geq \hat{x}_{11} \Rightarrow \hat{x}_{11} \leq \frac{5}{2} \quad (62)$$

$$\hat{x}_{12} \geq 0 \text{ and (60)} \Rightarrow 5 - 2 \hat{x}_{11} \geq 0 \Rightarrow \hat{x}_{11} \leq \frac{5}{2}. \quad (63)$$

Also,

$$\hat{U}_1 = \hat{x}_{11}^{1/2} \hat{x}_{21}^{1/2} = \hat{x}_{11}^{1/2} \cdot \left( \frac{4}{5} \hat{x}_{11} \right)^{1/2} \text{ from (61)} \quad (64)$$

$$= \frac{2}{\sqrt{5}} \hat{x}_{11} \text{ and} \quad (65)$$

$$\hat{U}_2 = \hat{x}_{12} \hat{x}_{22} = (5 - 2 \hat{x}_{11}) \left( 4 - \frac{8}{5} \hat{x}_{11} \right) \text{ from (60) and (59)} \quad (66)$$

$$= (5 - 2 \hat{x}_{11}) \cdot \frac{4}{5} (5 - 2 \hat{x}_{11}) = \frac{4}{5} (5 - 2 \hat{x}_{11})^2. \quad (67)$$

We want this  $\{1, 1, 2\}$  coalition to be blocked by the  $\{1, 2\} = \{1, 1, 2, 2\}$  coalition. Otherwise,  $\{1, 1, 2\}$  would block the  $\{1, 1, 2, 2\}$  allocation.

So we want, using the notation of (38), (39), (65), and (67),

$\hat{U}_1 \leq U_1$  and  $\hat{U}_2 \leq U_2$ . One way to set  $\hat{U}_2 = U_2$  and impose  $\hat{U}_1 \leq U_1$ . Another to ensure this is

way to ensure this is to set  $\hat{U}_1 = U_1$  and impose  $\hat{U}_2 \leq U_2$ . We did the former in the original solution to this problem, so we do the latter here, to show that the two approaches are equivalent.

$$\hat{U}_1 = U_1 \Rightarrow \frac{2}{\sqrt{5}} \hat{x}_{11} = x_{11} \text{ from (65) and (38)} \quad (68)$$

$$\hat{U}_2 \leq U_2 \Leftrightarrow \frac{4}{5} (5 - 2\hat{x}_{11})^2 \leq (3 - x_{11})^2 \text{ from (67) and (39)} \quad (69)$$

$$\Leftrightarrow \frac{4}{5} (5 - \sqrt{5} x_{11})^2 \leq (3 - x_{11})^2 \text{ from (68)}. \quad (70)$$

Method 1 to solve (70):

$$\text{From (62), } \frac{5}{2} \geq \hat{x}_{11} = \frac{\sqrt{5}}{2} x_{11} \text{ from (68)} \quad (71)$$

$$\sqrt{5} \geq x_{11} ; \text{ multiply by } \sqrt{5} \quad (72)$$

$$5 \geq \sqrt{5} x_{11} \Rightarrow x_{11} \leq \sqrt{5} \approx 2.24 \Rightarrow 3 - x_{11} \geq 0 \quad (72a)$$

$$5 - \sqrt{5} x_{11} \geq 0 \text{ so on the left hand side ("LHS"), } 5 - \sqrt{5} x_{11} \geq 0. \quad (73)$$

$$\text{So (70) } \Rightarrow \frac{2}{\sqrt{5}} |5 - \sqrt{5} x_{11}| \leq |3 - x_{11}| \text{ taking the square root of} \quad (74)$$

both sides of (70) (the inequality stays the same because  $\sqrt{x}$  is increasing in  $x$ )

$$\Leftrightarrow \frac{2}{\sqrt{5}} (5 - \sqrt{5} x_{11}) \leq 3 - x_{11} \text{ from (70a) and (73)} \quad (75)$$

$$2\sqrt{5} - 2x_{11} \leq 3 - x_{11} \quad (76)$$

$$2\sqrt{5} - 3 \leq x_{11}. \quad (77)$$

$$\text{Method 2 to solve (70): } \frac{4}{5} (25 - 10\sqrt{5} x_{11} + 5x_{11}^2) \leq 9 - 6x_{11} + x_{11}^2 \quad (78)$$

$$100 - 40\sqrt{5} x_{11} + 20x_{11}^2 \leq 45 - 30x_{11} + 5x_{11}^2 \quad (79)$$

$$20 - 8\sqrt{5} x_{11} + 4x_{11}^2 \leq 9 - 6x_{11} + x_{11}^2 \quad (80)$$



$$3x_{11}^2 + (6 - 8\sqrt{5})x_{11} + 11 \leq 0. \quad (81)$$

The roots are at  $\frac{-6 + 8\sqrt{5} \pm \sqrt{(6 - 8\sqrt{5})^2 - 4(3)(11)}}{2 \cdot 3}$

$$= -1 + \frac{4}{3}\sqrt{5} \pm \frac{1}{6} \sqrt{36 - 96\sqrt{5} + 64 \cdot 5 - 12 \cdot 11}$$

$$= -1 + \frac{4}{3}\sqrt{5} \pm \frac{1}{6} \sqrt{36 - 96\sqrt{5} + 320 - 132}$$

$$= -1 + \frac{4}{3}\sqrt{5} \pm \frac{1}{6} \sqrt{224 - 96\sqrt{5}} = -1 + \frac{4}{3}\sqrt{5} \pm \frac{1}{6} \sqrt{16(14 - 6\sqrt{5})}$$

$$= -1 + \frac{4}{3}\sqrt{5} \pm \frac{4}{6} \sqrt{14 - 6\sqrt{5}}, \text{ This looks hopeless but by coincidence,}$$

$14 - 6\sqrt{5}$  is a perfect square:  $14 = 5 + 9$  and

$$= -1 + \frac{4}{3}\sqrt{5} \pm \frac{2}{3} \sqrt{5 - 6\sqrt{5} + 9}$$

$$= -1 + \frac{4}{3}\sqrt{5} \pm \frac{2}{3} \sqrt{(\sqrt{5} - 3)^2} = -1 + \frac{4}{3}\sqrt{5} \pm 4 \left| \frac{\sqrt{5} - 3}{\sqrt{5} - 3} \right| \quad (82)$$

$$= -1 + \frac{4}{3}\sqrt{5} \pm \frac{2}{3} (-1)(\sqrt{5} - 3) \text{ since } \sqrt{5} - 3 < 0$$

$$= -1 + \frac{4}{3}\sqrt{5} \mp \frac{2}{3}(\sqrt{5} - 3)$$

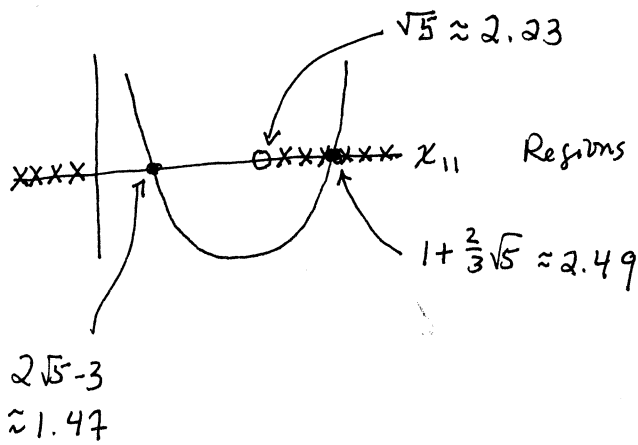
$$= \begin{cases} -1 + \frac{4}{3}\sqrt{5} - \frac{2}{3}(\sqrt{5} - 3) = -1 + \frac{4}{3}\sqrt{5} - \frac{2}{3}\sqrt{5} + 2 = 1 + \frac{2}{3}\sqrt{5} \approx 2.49 & (83) \\ -1 + \frac{4}{3}\sqrt{5} + \frac{2}{3}(\sqrt{5} - 3) = -1 + \frac{4}{3}\sqrt{5} + \frac{2}{3}\sqrt{5} - 2 = 2\sqrt{5} - 3 \approx 1.47. & (84) \end{cases}$$

(84) agrees with (77). The root of (83) is  $x_{11} = 1 + \frac{2}{3}\sqrt{5} \approx 2.49$ , but

from (63) and (68),  $\frac{5}{2} \geq \hat{x}_{11} = \frac{\sqrt{5}}{2} x_{11} \Leftrightarrow \sqrt{5} \geq x_{11} \Leftrightarrow x_{11} \leq \text{approx. } 2.23$ ,

so the root at  $1 + \frac{2}{3}\sqrt{5}$  would violate the non-negativity condition that gave rise to (63), so the root at  $1 + \frac{2}{3}\sqrt{5}$  is economically irrelevant.

The graph of (81) is



Regions marked by ~~XXXX~~ violate a non-negativity condition.

So the values of  $x_{11}$  that satisfy both (81) and all the non-negativity conditions are  $[2\sqrt{5}-3, \sqrt{5}]$ .

Method 1 is subject to the

non-negativity condition  $x_{11} \leq \sqrt{5}$  just as Method 2 is, so (77) would in Method 1 have to be supplemented with  $x_{11} \leq \sqrt{5}$ . Hence Methods 1 and 2 are equivalent.

— End of discussion of how to solve (70) —

We have  $x_{11} \in [2\sqrt{5}-3, \sqrt{5}]$  as being unblocked by the  $\{1, 1, 2\}$  coalition. (85)

• Consider the  $\{1, 2, 2\}$  coalition. Use  $\tilde{\cdot}$  to denote values in the  $\{1, 2, 2\}$

coalition. The feasibility conditions are

$$\tilde{x}_{11} + 2\tilde{x}_{12} = 2 + 2 \cdot 1 = 4 \quad (86)$$

$$\tilde{x}_{21} + 2\tilde{x}_{22} = 1 + 2 \cdot 2 = 5 \quad (87)$$

Pareto Optimal :  $\max \alpha U_1 + 2(1-\alpha) U_2$  s.t. (86) and (87)

$$\max \alpha \tilde{x}_{11}^{1/2} \tilde{x}_{21}^{1/2} + 2(1-\alpha) \tilde{x}_{12} \tilde{x}_{22} \quad (88)$$

$$\underbrace{\frac{4 - \tilde{x}_{11}}{2}}_{\text{from (86)}} \quad \underbrace{\frac{5 - \tilde{x}_{21}}{2}}_{\text{from (87)}}$$

from (86) from (87) There are many

other ways of using (86) and (87) to eliminate two of the four variables. It

is arbitrary which <sup>two</sup> variables you keep and which you <sup>two</sup> eliminate.

$$\max \alpha \tilde{x}_{11}^{\frac{1}{2}} \tilde{x}_{21}^{\frac{1}{2}} + 2(1-\alpha) \frac{4-\tilde{x}_{11}}{2} \cdot \frac{5-\tilde{x}_{21}}{2} \quad (89)$$

$$\max \alpha \tilde{x}_{11}^{\frac{1}{2}} \tilde{x}_{21}^{\frac{1}{2}} + \frac{1-\alpha}{2} (20 - 4\tilde{x}_{21} - 5\tilde{x}_{11} + \tilde{x}_{11}\tilde{x}_{21}) \quad (90)$$

F.O.C.

$$0 = \frac{\partial (90)}{\partial \tilde{x}_{11}} = \frac{\alpha}{2} \tilde{x}_{11}^{-\frac{1}{2}} \tilde{x}_{21}^{\frac{1}{2}} + \frac{1-\alpha}{2} (-5 + \tilde{x}_{21}) \quad (91)$$

$$0 = \frac{\partial (90)}{\partial \tilde{x}_{21}} = \frac{\alpha}{2} \tilde{x}_{11}^{\frac{1}{2}} \tilde{x}_{21}^{-\frac{1}{2}} + \frac{1-\alpha}{2} (-4 + \tilde{x}_{11}) \quad (92)$$

$$\frac{\frac{\alpha}{2} \tilde{x}_{11}^{-\frac{1}{2}} \tilde{x}_{21}^{\frac{1}{2}}}{\frac{\alpha}{2} \tilde{x}_{11}^{\frac{1}{2}} \tilde{x}_{21}^{-\frac{1}{2}}} = \frac{\frac{1-\alpha}{2} (\tilde{x}_{21} - 5)}{\frac{1-\alpha}{2} (\tilde{x}_{11} - 4)} \Rightarrow \frac{\tilde{x}_{21}}{\tilde{x}_{11}} = \frac{\tilde{x}_{21} - 5}{\tilde{x}_{11} - 4} \quad (93)$$

$$\tilde{x}_{11} \tilde{x}_{21} - 4\tilde{x}_{21} = \tilde{x}_{11} \tilde{x}_{21} - 5\tilde{x}_{11} \quad (94)$$

$$\tilde{x}_{21} = \frac{5}{4} \tilde{x}_{11} \quad (95)$$

$$\tilde{x}_{12} = 2 - \frac{1}{2} \tilde{x}_{11} \text{ from (86)} \quad (96)$$

$$\tilde{x}_{22} = \frac{5}{2} - \frac{1}{2} \tilde{x}_{21} \text{ from (88)}$$

$$= \frac{5}{2} - \frac{1}{2} \cdot \frac{5}{4} \tilde{x}_{11} \text{ from (95)}$$

$$= \frac{20}{8} - \frac{5}{8} \tilde{x}_{11}$$

$$= \frac{5}{2} - \frac{5}{8} \tilde{x}_{11} \quad (97)$$

Some non-negativity conditions:

$$(96) \text{ and } \tilde{x}_{12} \geq 0 \Rightarrow 2 - \frac{1}{2} \tilde{x}_{11} \geq 0 \Rightarrow 2 \geq \frac{1}{2} \tilde{x}_{11} \Rightarrow \tilde{x}_{11} \leq 4 \quad (98)$$

$$(97) \text{ and } \tilde{x}_{22} \geq 0 \Rightarrow \frac{5}{2} - \frac{5}{8} \tilde{x}_{11} \geq 0 \Rightarrow \frac{1}{2} \geq \frac{1}{8} \tilde{x}_{11} \Rightarrow \tilde{x}_{11} \leq 4. \quad (99)$$

Also,

$$\begin{aligned}\tilde{U}_1 &= \tilde{x}_{11}^{1/2} \tilde{x}_{21}^{1/2} = \tilde{x}_{11}^{1/2} \cdot \left(\frac{5}{4} \tilde{x}_{11}\right)^{1/2} \text{ from (95)} \\ &= \frac{\sqrt{5}}{2} \tilde{x}_{11} \text{ and}\end{aligned}\tag{100}$$

$$\begin{aligned}\tilde{U}_2 &= \tilde{x}_{12} \tilde{x}_{22} = \left(2 - \frac{1}{2} \tilde{x}_{11}\right) \left(\frac{5}{2} - \frac{5}{8} \tilde{x}_{11}\right) \text{ from (96) and (97)} \\ &= \frac{1}{2} (4 - \tilde{x}_{11}) \cdot \frac{5}{8} (4 - \tilde{x}_{11}) = \frac{5}{16} (4 - \tilde{x}_{11})^2.\end{aligned}\tag{101}$$

We want this  $\{1, 2, 2\}$  coalition to be blocked by the  $\{1, 2\} = \{1, 1, 2, 2\}$  coalition. One way to ensure this is to set  $\tilde{U}_1 = U_1$  and impose  $\tilde{U}_2 \leq U_2$ .

$$\tilde{U}_1 = U_1 \Rightarrow \frac{\sqrt{5}}{2} \tilde{x}_{11} = x_{11} \text{ from (100) and (38)}\tag{102}$$

$$\Rightarrow \tilde{x}_{11} = \frac{2}{\sqrt{5}} x_{11}\tag{103}$$

$$\tilde{U}_2 \leq U_2 \Rightarrow \frac{5}{16} (4 - \tilde{x}_{11})^2 \leq (3 - x_{11})^2 \text{ from (101) and (39)}\tag{104}$$

$$\frac{5}{16} \left(4 - \frac{2}{\sqrt{5}} x_{11}\right)^2 \leq (3 - x_{11})^2 \text{ from (103)}.\tag{105}$$

As with solving (70), there are two methods for solving (105).

Method 1 to solve (105): From (98), (99), and (103),  $\frac{2}{\sqrt{5}} x_{11} = \tilde{x}_{11} \leq 4$

$$x_{11} \leq 2\sqrt{5} \approx 4.47\tag{106}$$

If we were to take the square root of both sides of (105), the right-hand side would be  $\sqrt{(3 - x_{11})^2} = |3 - x_{11}|$ . However with (106), it seems that  $x_{11}$  could be  $> 3$ , so  $3 - x_{11}$  cannot be signed. On the other hand, from (5),  $x_{12} = 3 - x_{11}$ , so  $x_{12} \geq 0 \Rightarrow x_{11} \leq 3$ .

So on the right-hand side of (105),

$$3 - x_{11} \geq 0. \text{ On the left hand side of (105), } 4 - \frac{2}{\sqrt{5}} x_{11} \stackrel{?}{\geq} 0$$

$$4 \stackrel{?}{\geq} \frac{2}{\sqrt{5}} x_{11}$$

$$4.47 \approx 2\sqrt{5} \stackrel{?}{\geq} x_{11} \quad (107)$$

which is true by (106). Taking square roots in (105),

$$\frac{\sqrt{5}}{4} \left| 4 - \frac{2}{\sqrt{5}} x_{11} \right| \leq |3 - x_{11}| \quad (108)$$

and given  $x_{11} \leq 3$  and (107),

$$\frac{\sqrt{5}}{4} \left( 4 - \frac{2}{\sqrt{5}} x_{11} \right) \leq 3 - x_{11}$$

$$\sqrt{5} - \frac{1}{2} x_{11} \leq 3 - x_{11}$$

$$\frac{1}{2} x_{11} \leq 3 - \sqrt{5}$$

$$x_{11} \leq 6 - 2\sqrt{5} \approx 1.53$$

(109)

which is consistent with  $x_{11} \leq 3$  and with (107).

$$\text{From (3), } x_{11} = 3 - x_{12}, \text{ so } 3 - x_{12} \leq 6 - 2\sqrt{5}$$

$$x_{12} - 3 \geq 2\sqrt{5} - 6$$

$$x_{12} \geq 2\sqrt{5} - 3 \approx 1.47,$$

(110)

(110) agrees with the previous answer.

Method 2 to solve (105):

$$\frac{5}{16} \left( 16 - \frac{16}{\sqrt{5}} x_{11} + \frac{4}{5} x_{11}^2 \right) \leq 9 - 6x_{11} + x_{11}^2 \quad (111)$$

$$5 - \sqrt{5} x_{11} + \frac{1}{4} x_{11}^2 \leq 9 - 6x_{11} + x_{11}^2$$

$$0 \leq \frac{3}{4} x_{11}^2 + (\sqrt{5}-6)x_{11} + 4 \quad (112)$$

$$0 \leq 3x_{11}^2 + (4\sqrt{5}-24)x_{11} + 16 \quad (113)$$

$$\text{roots } \frac{24-4\sqrt{5} \pm \sqrt{(4\sqrt{5}-24)^2 - 4(3)(16)}}{2 \cdot 3} = \frac{24-4\sqrt{5} \pm \sqrt{80-192\sqrt{5}+576-192}}{6} \quad (114)$$

$$= 4 - \frac{2}{3}\sqrt{5} \pm \frac{1}{6}\sqrt{464-192\sqrt{5}} = 4 - \frac{2}{3}\sqrt{5} \pm \frac{1}{6}\sqrt{24 \cdot 29 - 2^6 \cdot 3\sqrt{5}} \quad (115)$$

$$= 4 - \frac{2}{3}\sqrt{5} \pm \frac{1}{6} \cdot 2^2 \sqrt{29 - 4 \cdot 3\sqrt{5}} = 4 - \frac{2}{3}\sqrt{5} \pm \frac{2}{3}\sqrt{29 - 12\sqrt{5}} \quad (116)$$

After some trial and error, replace 29 with 20+9:

$$= 4 - \frac{2}{3}\sqrt{5} \pm \frac{2}{3}\sqrt{20 - 12\sqrt{5} + 9} \quad (117)$$

$$= 4 - \frac{2}{3}\sqrt{5} \pm \frac{2}{3}\sqrt{(2\sqrt{5}-3)^2} = 4 - \frac{2}{3}\sqrt{5} \pm \frac{2}{3}|2\sqrt{5}-3| \quad (118)$$

$2\sqrt{5}-3 \approx 1.47 > 0$  and

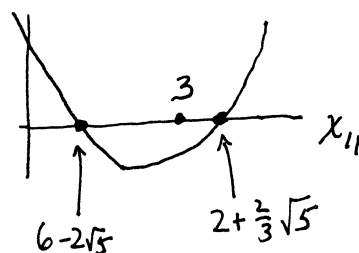
$$= 4 - \frac{2}{3}\sqrt{5} \pm \frac{2}{3}(2\sqrt{5}-3) = 4 - \frac{2}{3}\sqrt{5} \pm (\frac{4}{3}\sqrt{5}-2) \quad (119)$$

$$= \begin{cases} 4 - \frac{2}{3}\sqrt{5} + \frac{4}{3}\sqrt{5} - 2 = 2 + \frac{2}{3}\sqrt{5} \approx 3.49 & (120) \\ 4 - \frac{2}{3}\sqrt{5} - \frac{4}{3}\sqrt{5} + 2 = 6 - 2\sqrt{5} \approx 1.53 & (121) \end{cases}$$

Solution (120) can be ruled out because from (5),  $x_{12} = 3 - x_{11}$ , so values of  $x_{11}$  greater

than three would make  $x_{12}$  negative. Hence the only solution is (121), namely

$x_{11} = 6 - 2\sqrt{5}$ . A sketch of (113) is so the only values of  $x_{11}$  which are less than 3 and which satisfy (113) are  $x_{11} \in [0, 6 - 2\sqrt{5}]$ . This agrees with Method 1's (109).



Summarizing, then,

$$\underbrace{2\sqrt{5}-3}_{\approx 1.47} \leq x_{11} \leq \underbrace{6-2\sqrt{5}}_{\approx 1.53}$$

from the  $\{1, 1, 2\}$  analysis      from the  $\{1, 2, 2\}$  analysis

for the core of the 2-replica, versus the core of the  $\{1, 2\}$  economy, from (34),

$$1.41 = \sqrt{2} \leq x_{11} \leq 3 - \sqrt{2} = 1.59,$$

So the core did shrink.