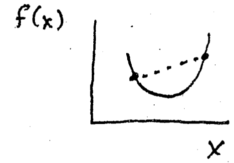
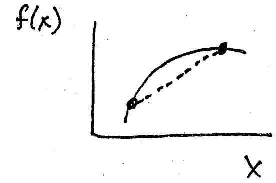
- <u>Set</u>: convex or not convex

Functions

Convex function



Concave function



Convex/concave function: Pick 2 points on the fnctn.; draw a line between them; is the line above/below the function?

Convex/not convex set: ___ " ___ in " set; ______ " ______; ___ " ___ in " set?

<u>quasiconcave function</u>

upper contour set is a convex <u>set</u>

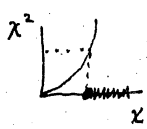
$\{\underset{\sim}{x} : f(\underset{\sim}{x}) \geq k\}$

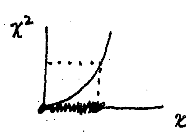
<u>quasiconvex function</u>

Lower contour set is a convex <u>set</u>

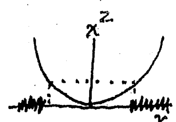
$\{\underset{\sim}{x} : f(\underset{\sim}{x}) \leq k\}$

Example 1:

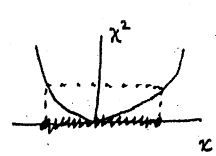




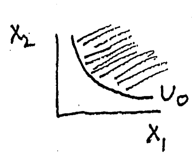
Example 2:

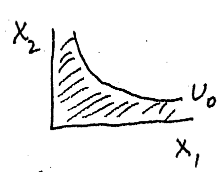


(not convex set)



Example 3:





(not convex set)

Concave ⇒ quasiconcave

Convex ⇒ quasiconvex

Example.

$$\underset{\text{prices of inputs 1 and 2}}{\mathrm{Min}\ w_1 \overset{\text{inputs 1 and 2}}{x_1} + w_2 x_2} \quad \text{s.t.} \quad \overbrace{x_1^{1/4} x_2^{3/4}}^{\text{production function}} \geq \underset{\text{output}}{y}. \quad \text{Find } \partial x_1 / \partial w_2.$$

Solution: $\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda \left[y - x_1^{1/4} x_2^{3/4} \right]$.

F.O.C.'s :

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = y - x_1^{1/4} x_2^{3/4}$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \frac{1}{4} \lambda x_1^{-3/4} x_2^{3/4}$$

$$0 = \frac{\partial \mathcal{L}}{\partial x_2} = w_2 - \frac{3}{4} \lambda x_1^{1/4} x_2^{-1/4}.$$

S.O.C. :

$$\nabla^2 \mathcal{L} = \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda x_1} & \mathcal{L}_{\lambda x_2} \\ \sim & \mathcal{L}_{x_1 x_1} & \mathcal{L}_{x_1 x_2} \\ \sim & \sim & \mathcal{L}_{x_2 x_2} \end{bmatrix}.$$

$\overbrace{\phantom{D_3 \text{ of }}}^{\text{The sign of}} D_3$ of $\nabla^2 \mathcal{L}$ should be $(-1)^m = (-1)^1 < 0$.

For comparative statics, totally differentiate the F.O.C.'s :

$$d\lambda \quad dx_1 \quad dx_2 \ ; \ \underbrace{dy \quad dw_1}_{=0 \text{ in this problem}} \quad dw_2$$

$$0 = \mathcal{L}_{\lambda\lambda}\, d\lambda + \mathcal{L}_{\lambda x_1}\, dx_1 + \mathcal{L}_{\lambda x_2}\, dx_2 + \mathcal{L}_{\lambda y}\, dy + \mathcal{L}_{\lambda w_1}\, dw_1 + \mathcal{L}_{\lambda w_2}\, dw_2$$

$$0 = \mathcal{L}_{x_1 \lambda}\, d\lambda + \mathcal{L}_{x_1 x_1}\, dx_1 + \mathcal{L}_{x_1 x_2}\, dx_2 + \mathcal{L}_{x_1 y}\, dy + \mathcal{L}_{x_1 w_1}\, dw_1 + \mathcal{L}_{x_1 w_2}\, dw_2$$

$$0 = \mathcal{L}_{x_2 \lambda}\, d\lambda + \mathcal{L}_{x_2 x_1}\, dx_1 + \mathcal{L}_{x_2 x_2}\, dx_2 + \mathcal{L}_{x_2 y}\, dy + \mathcal{L}_{x_2 w_1}\, dw_1 + \mathcal{L}_{x_2 w_2}\, dw_2.$$

$$\underset{\sim}{0} = \underbrace{\begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda x_1} & \mathcal{L}_{\lambda x_2} \\ \mathcal{L}_{x_1\lambda} & \mathcal{L}_{x_1 x_1} & \mathcal{L}_{x_1 x_2} \\ \mathcal{L}_{x_2\lambda} & \mathcal{L}_{x_2 x_1} & \mathcal{L}_{x_2 x_2} \end{bmatrix}}_{= \nabla^2\mathcal{L}} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \mathcal{L}_{\lambda y} \\ \mathcal{L}_{x_1 y} \\ \mathcal{L}_{x_2 y} \end{bmatrix} dy}_{= 0} + \underbrace{\begin{bmatrix} \mathcal{L}_{\lambda w_1} \\ \mathcal{L}_{x_1 w} \\ \mathcal{L}_{x_2 w_1} \end{bmatrix} dw_1}_{= 0} + \begin{bmatrix} \mathcal{L}_{\lambda w_2} \\ \mathcal{L}_{x_1 w_2} \\ \mathcal{L}_{x_2 w_2} \end{bmatrix} dw_2$$

"Divide" by dw_2:

$$\underset{\sim}{0} = [\nabla^2\mathcal{L}] \begin{bmatrix} \partial\lambda/\partial w_2 \\ \partial x_1/\partial w_2 \\ \partial x_2/\partial w_2 \end{bmatrix} + \begin{bmatrix} \mathcal{L}_{\lambda w_2} \\ \mathcal{L}_{x_1 w_2} \\ \mathcal{L}_{x_2 w_2} \end{bmatrix} \Rightarrow$$

$$[\nabla^2\mathcal{L}] \begin{bmatrix} \partial\lambda/\partial w_2 \\ \partial x_1/\partial w_2 \\ \partial x_2/\partial w_2 \end{bmatrix} = \begin{bmatrix} -\mathcal{L}_{\lambda w_2} \\ -\mathcal{L}_{x_1 w_2} \\ -\mathcal{L}_{x_2 w_2} \end{bmatrix}$$ and by Cramer's Rule,

$$\frac{\partial x_1}{\partial w_2} = \underbrace{\frac{1}{|\nabla^2\mathcal{L}|}}_{\text{negative}} \begin{vmatrix} \mathcal{L}_{\lambda\lambda} & -\mathcal{L}_{\lambda w_2} & \mathcal{L}_{\lambda x_2} \\ \mathcal{L}_{x_1\lambda} & -\mathcal{L}_{x_1 w_2} & \mathcal{L}_{x_1 x_2} \\ \mathcal{L}_{x_2\lambda} & -\mathcal{L}_{x_2 w_2} & \mathcal{L}_{x_2 x_2} \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & -\frac{3}{4} x_1^{1/4} x_2^{-1/4} \text{ (symmetry)} \\ -\frac{1}{4} x_1^{-3/4} x_2^{3/4} & 0 & -\frac{3}{16} \lambda x_1^{-3/4} x_2^{-1/4} \\ -\frac{3}{4} x_1^{1/4} x_2^{-1/4} & -1 & \frac{3}{16} \lambda x_1^{1/4} x_2^{-5/4} \end{vmatrix} = (-1)(-1)^{3+2} \text{ times } -\left(-\frac{1}{4} x_1^{-3/4} x_2^{3/4} \text{ times } -\frac{3}{4} x_1^{1/4} x_2^{-1/4}\right) < 0.$$

$\Rightarrow \partial x_1/\partial w_2 > 0.$

The other way to solve this is to find x_1^* explicitly. The F.O.C.'s can be rewritten

$$0 = y - x_1^{1/4} x_2^{3/4}$$

$$0 = w_1 - \frac{1}{4}\frac{y}{x_1} \Rightarrow w_1 = \frac{1}{4}\frac{y}{x_1}$$

$$0 = w_2 - \frac{3}{4}\frac{y}{x_2} \Rightarrow w_2 = \frac{3}{4}\frac{y}{x_2}$$

$$\left.\right\} \frac{w_1}{w_2} = \frac{1}{\cancel{4}}\frac{\cancel{y}}{x_1}\cdot\frac{\cancel{4}}{3}\frac{x_2}{\cancel{y}} = \frac{x_2}{3x_1} \Rightarrow$$

$$x_1 = \frac{1}{3}\frac{w_2}{w_1}x_2.$$ Substitute this into the first F.O.C.:

$$y = x_1^{1/4}x_2^{3/4} = \left(\frac{1}{3}\frac{w_2}{w_1}x_2\right)^{1/4}x_2^{3/4} = 3^{-1/4}\left(\frac{w_2}{w_1}\right)^{1/4}x_2$$

$$\Rightarrow x_2^* = 3^{+1/4}y\left(\frac{w_1}{w_2}\right)^{1/4}.$$ Then

$$x_1^* = \frac{1}{3}\left(\frac{w_2}{w_1}\right)3^{1/4}y\left(\frac{w_1}{w_2}\right)^{1/4} = 3^{-1}3^{\frac{1}{4}}\left(\frac{w_2}{w_1}\right)\left(\frac{w_2}{w_1}\right)^{-1/4}y$$

$$= 3^{-3/4}\left(\frac{w_2}{w_1}\right)^{3/4}y = 3^{-3/4}w_1^{-3/4}w_2^{3/4}y$$ and

$$\frac{\partial x_1^*}{\partial w_2} = \frac{3}{4}\cdot 3^{-3/4}w_1^{-3/4}w_2^{-1/4}y > 0.$$

1) Find the second-order conditions for the problem

$$\text{minimize } f(x_1, x_2) \text{ s.t. } \quad c_1 x_1 + c_2 x_2 = m \qquad (1)$$

where c_1, c_2 and m are constants. By using information from the first-order conditions, express your answer without using c_1 or c_2.

2) Find the conditions for $f(x_1, x_2)$ to be quasicon~~cave~~vex.

3) Compare your answers to problem 1 and problem 2.

4) Rework problem (1), replacing $f(x_1, x_2)$ with $f(\underset{\sim}{x})$ for $\underset{\sim}{x} \in \mathbb{R}^n$ and replacing the constraint with $\underset{\sim}{c} \cdot \underset{\sim}{x} = m$.

5) Rework problem (2), replacing $f(x_1, x_2)$ with $f(\underset{\sim}{x})$ for $\underset{\sim}{x} \in \mathbb{R}^n$.

6) Compare your answers to problem 4 and problem 5.

Answer to (4): $\min f(\underset{\sim}{x})$ s.t. $\underset{\sim}{c} \cdot \underset{\sim}{x} = m$

$$\mathcal{L} = f(\underset{\sim}{x}) + \lambda(m - \underset{\sim}{c} \cdot \underset{\sim}{x})$$

F.O.C. $$0 = \frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x_i} - \lambda c_i \Rightarrow -c_i = \frac{-1}{\lambda} \frac{\partial f}{\partial x_i}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = m - \underset{\sim}{c} \cdot \underset{\sim}{x}$$

$$\left.\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} &= 0 \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_i} &= -c_i \\ \frac{\partial^2 \mathcal{L}}{\partial x_i^2} &= \frac{\partial^2 f}{\partial x_i^2} \\ \frac{\partial^2 \mathcal{L}}{\partial x_i \partial x_j} &= \frac{\partial^2 f}{\partial x_i \partial x_j} \end{aligned}\right\} \Rightarrow \nabla^2 \mathcal{L} = \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda 1} & \mathcal{L}_{\lambda 2} & \cdots \\ \mathcal{L}_{1\lambda} & \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots \\ \mathcal{L}_{2\lambda} & \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} 0 & -c_1 & -c_2 & \cdots \\ -c_1 & f_{11} & f_{12} & \cdots \\ -c_2 & f_{21} & f_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

and from the F.O.C.'s, this is equal to

$$\nabla^2 \mathcal{L} = \begin{bmatrix} 0 & \frac{-1}{\lambda} f_1 & \frac{-1}{\lambda} f_2 & \cdots \\ \frac{-1}{\lambda} f_1 & f_{11} & f_{12} & \cdots \\ \frac{-1}{\lambda} f_2 & f_{21} & f_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \Rightarrow |\nabla^2 \mathcal{L}| = \left(\frac{-1}{\lambda}\right)^2 \begin{vmatrix} 0 & f_1 & f_2 & \cdots \\ f_1 & f_{11} & f_{12} & \cdots \\ f_2 & f & f_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

Now $\left(\frac{-1}{\lambda}\right)^2$ is positive. So it can be ignored in stating the S.O.C's as follows:

$(-1)^m = (-1)^1 = \ominus$ and $2m+1 = 2(1)+1 = 3$.

$$D_3 = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} \quad \text{should be negative}$$

$$D_4 = \begin{vmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{21} & f_{22} & f_{23} \\ f_3 & f_{31} & f_{32} & f_{33} \end{vmatrix} \quad \text{should be negative} \qquad \text{(etc.)}$$

Answer to (5):

$\delta_2 = D_3$ above should be negative

$\delta_3 = D_4$ above should be negative

$\vdots$ $\qquad\qquad$ $\vdots$

Answer to (6):

The conditions are the same. In other words, the S.O.C.'s for the problem "min $f(x)$ s.t. $\underset{\sim}{c} \cdot \underset{\sim}{x} = m$" are the same as the sufficient conditions for "$f(x)$ is quasiconvex."

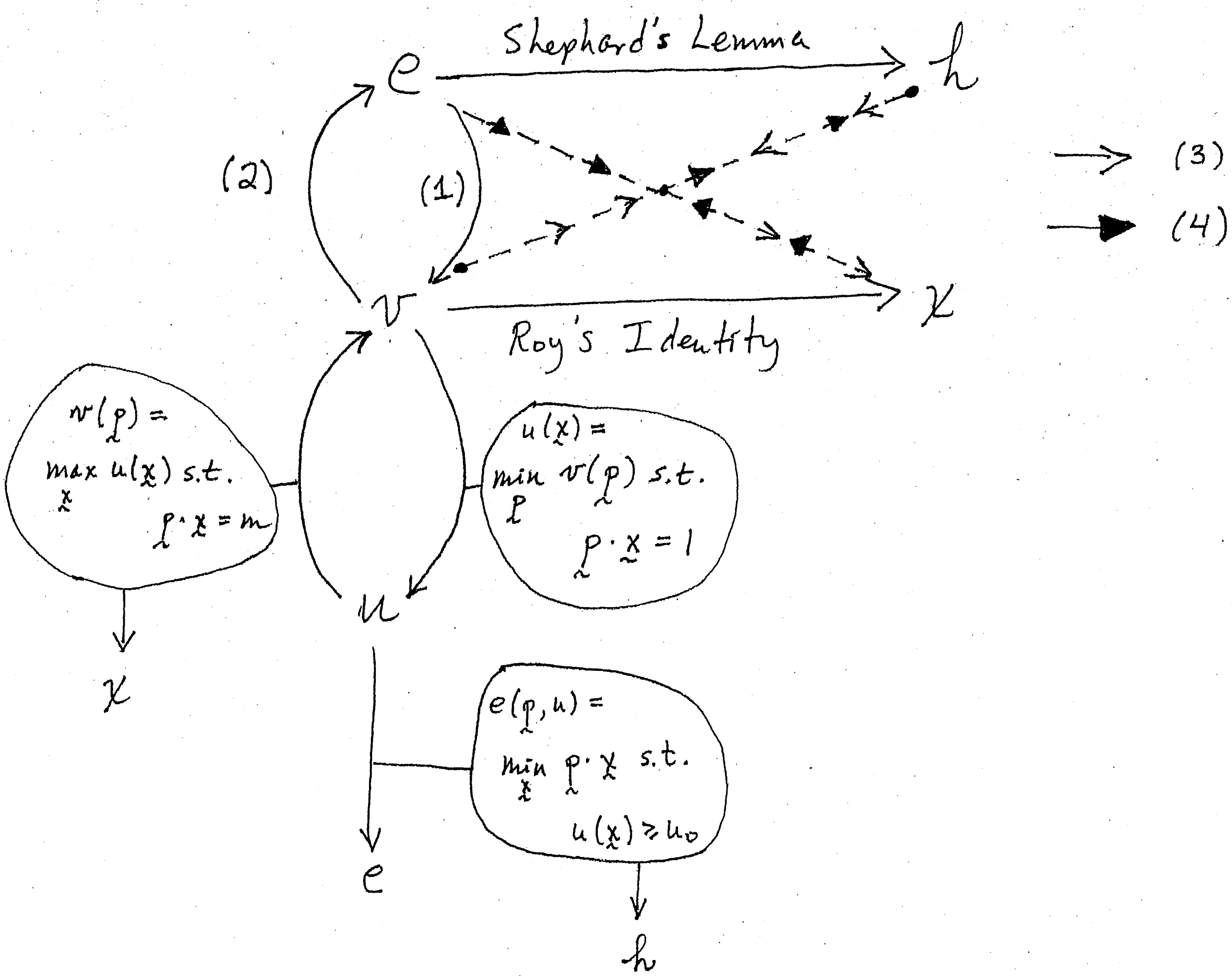
Similarly, it can be shown that the S.O.C's for the problem "max $f(x)$ s.t. $\underset{\sim}{c} \cdot \underset{\sim}{x} = m$" are the same as the sufficient conditions for "$f(x)$ is quasiconcave."

$$(1)\quad e(\underset{\sim}{p}, v(\underset{\sim}{p}, m)) \equiv m$$

$$(2)\quad v(\underset{\sim}{p}, e(\underset{\sim}{p}, u)) \equiv u$$

$$(3)\quad x_i(\underset{\sim}{p}, m) \equiv h_i(\underset{\sim}{p}, v(\underset{\sim}{p}, m))$$

$$(4)\quad h_i(\underset{\sim}{p}, u) \equiv x_i(\underset{\sim}{p}, e(\underset{\sim}{p}, u))$$



Expenditure Minimization (consumer)	Profit Maximization (competitive firm)	Cost Minimization (firm competitive in input mkts.)	
• $\mathbf{h} = \nabla_{\mathbf{p}}\, e$ [§7.3]	• $\mathbf{y} = \nabla_{\mathbf{p}}\, \pi$ [Hotelling's Lemma, §3.2]	• $\mathbf{x} = \nabla_{\mathbf{w}}\, c$ [Shephard's Lemma, §5.4]	Envelope Theorem results
$\Rightarrow \nabla_{\mathbf{p}}\, \mathbf{h} = \nabla^2_{\mathbf{p}}\, e$	$\Rightarrow \nabla_{\mathbf{p}}\, \mathbf{y} = \nabla^2_{\mathbf{p}}\, \pi$	$\Rightarrow \nabla_{\mathbf{w}}\, \mathbf{x} = \nabla^2_{\mathbf{w}}\, c$	
• e is concave [§7.3]	• π is convex [§3.1]	• c is concave [§5.4]	Intuition: Fig. 3.1, Fig. 5.4
$\Rightarrow \nabla^2_{\mathbf{p}}\, e$ and $\nabla_{\mathbf{p}}\, \mathbf{h}$ are negative semidefinite symmetric and $\partial h_i / \partial p_i \leq 0$ [§8.3]	$\Rightarrow \nabla^2_{\mathbf{p}}\, \pi$ and $\nabla_{\mathbf{p}}\, \mathbf{y}$ are positive semidefinite symmetric and $\partial y_i / \partial p_i \geq 0$ [§3.4]	$\Rightarrow \nabla^2_{\mathbf{w}}\, c$ and $\nabla_{\mathbf{w}}\, \mathbf{x}$ are negative semidefinite symmetric and $\partial x_i / \partial w_i \leq 0$ [§5.6]	

Note:
The Envelope Theorem result for consumer utility maximization is Roy's Identity [§7.4].

Profit Maximization		Cost Minimization
§ 2.4	§ 3.4	§ 5.6
$\frac{\partial x_i}{\partial w_i} < 0$	$\frac{\partial y_i}{\partial p_i} \geq 0$	$\frac{\partial x_i}{\partial w_i} \leq 0$
[because $\nabla_{\underset{\sim}{w}} \underset{\sim}{x} = (\nabla^2 f)^{-1}$ concave from S.O.C.]	[because $\nabla_{\underset{\sim}{p}} \underset{\sim}{y} = \nabla^2_{\underset{\sim}{p}} \pi$ convex]	[because $\nabla_{\underset{\sim}{w}} \underset{\sim}{x} = \nabla^2_{\underset{\sim}{w}} c$ concave]
$d\underset{\sim}{w} \cdot d\underset{\sim}{x} \leq 0$	$d\underset{\sim}{p} \cdot d\underset{\sim}{y} \geq 0$ is a missing result: $d\underset{\sim}{y} = \nabla_{\underset{\sim}{p}} \underset{\sim}{y} \, d\underset{\sim}{p}$ $d\underset{\sim}{p} \cdot d\underset{\sim}{y} = d\underset{\sim}{p} \cdot \nabla_{\underset{\sim}{p}} \underset{\sim}{y} \, d\underset{\sim}{p}$ $= d\underset{\sim}{p}^T (\nabla^2 \pi) d\underset{\sim}{p}$ $\geq 0.$	$d\underset{\sim}{w} \cdot d\underset{\sim}{x} \leq 0$
[because the LHS is $d\underset{\sim}{w}^T (\nabla^2 f)^{-1} d\underset{\sim}{w} \leq 0$ concave from S.O.C.]		[because the LHS is $d\underset{\sim}{w}^T (\nabla^2 c) \, d\underset{\sim}{w} \leq 0$ concave]
		(See also § 4.4, which considers exogenous changes in y.)

Summary of Firms' Comparative Statics Results

§ 2.5 (profit-maximization):

WAPM $\Rightarrow \Delta \underset{\sim}{p} \cdot \Delta \underset{\sim}{y} \geq 0$

§ 4.5 (cost-minimization):

WACM $\Rightarrow \Delta \underset{\sim}{w} \cdot \Delta \underset{\sim}{x} \leq 0$

Suppose a competitive firm transforms a single input (z) into two outputs (q_1 and q_2) according to a well-behaved, fully differentiable, production function.

(i) What are the first-order and second-order conditions determining the profit maximizing levels of q_1 and q_2?

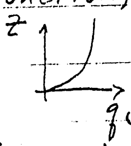
(ii) A tax is now introduced on each unit of q_1 sold. How will this tax change (1) the firm's demand for the input z, (2) the supply of the taxed commodity q_1, and (3) the supply of the untaxed commodity q_2?

(iii) Now suppose that all competitive firms jointly producing q_1 and q_2 are subject to this tax on q_1. On the assumption that q_1 and q_2 are independent in demand, derive an expression for the effect of this tax on the equilibrium price of q_1. What is the sign of this expression? (By "independent in demand," we mean that the demand for q_1 depends upon p_1 but not upon p_2 and the demand for q_2 depends upon p_2 but not upon p_1.)

(iv) Let each and every firm's production function be $z = Aq_1^{\alpha}q_2^{\beta}$ where $A > 0$, $\alpha > 0$, $\beta > 0$, and $\alpha + \beta < 1$ are fixed parameters. With this information, evaluate the expression derived in part (iii) of this question. Also consider the case when $\alpha+\beta > 1$.

ra +.

i) let z be the transformation function, $z = z(q_1, q_2)$



convex $\longleftrightarrow$ $z_{ii} > 0$ and $z_{ii} z_{jj} - z_{ij}^2 > 0$

$$\max_{q_1, q_2} \pi_0 = p_1 q_1 + p_2 q_2 - w\, z(q_1, q_2)$$

$$\frac{\partial \pi_0}{\partial q_1} = 0 = p_1 - w \frac{\partial z}{\partial q_1} \qquad p_i = w_i\, \partial z / \partial q_i \quad (FOC)$$

$$\frac{\partial \pi_0}{\partial q_2} = 0 = p_2 - w \frac{\partial z}{\partial q_2}$$

$$\begin{bmatrix} \pi^0_{11} & \pi^0_{12} \\ \pi^0_{21} & \pi^0_{22} \end{bmatrix} = \begin{bmatrix} -w z_{11} & -w z_{12} \\ -w z_{12} & -w z_{22} \end{bmatrix} = (-w) \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

is negative definite

$-w z_{11} < 0$ or $z_{11} > 0$ which is true

$$\Delta = \begin{vmatrix} \pi^0_{11} & \pi^0_{12} \\ \pi^0_{21} & \pi^0_{22} \end{vmatrix} = w^2 z_{11} z_{22} - w^2 z_{12}^2$$

$$= w^2 (z_{11} z_{22} - z_{12}^2) > 0$$

or $z_{11} z_{22} - z_{12}^2 > 0$, which is true

ii) $$\max_{q_1, q_2} \pi = p_1 (1-t) q_1 + p_2 q_2 - w\, z(q_1, q_2) \qquad \text{exo}: p_1, p_2, t, w$$

$$\partial \pi / \partial q_1 = 0 = p_1 (1-t) - w \frac{\partial z}{\partial q_1}$$

$$\partial \pi / \partial q_2 = 0 = p_2 - w\, \partial z / \partial q_2$$

$$\begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -w z_{11} & -w z_{12} \\ -w z_{12} & -w z_{22} \end{bmatrix} = \begin{bmatrix} \pi^0_{11} & \pi^0_{12} \\ \pi^0_{21} & \pi^0_{22} \end{bmatrix}$$

so all the above relationships hold

$$\begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} = - d_{p_1, p_2, t, w} \begin{bmatrix} p_1 (1-t) - w z_1 \\ p_2 - w z_2 \end{bmatrix}$$

$$= - \begin{bmatrix} 1-t \\ 0 \end{bmatrix} dp_1 - \begin{bmatrix} 0 \\ 1 \end{bmatrix} dp_2 - \begin{bmatrix} -p_1 \\ 0 \end{bmatrix} dt - \begin{bmatrix} -z_1 \\ -z_2 \end{bmatrix} d$$

$$= \begin{bmatrix} t-1 \\ 0 \end{bmatrix} dp_1 + \begin{bmatrix} 0 \\ -1 \end{bmatrix} dp_2 + \begin{bmatrix} p_1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} dw$$

$$\begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} D_{11} & D_{21} \\ D_{12} & D_{22} \end{bmatrix}\left\{ \begin{bmatrix} t-1 \\ 0 \end{bmatrix} dp_1 + \begin{bmatrix} 0 \\ -1 \end{bmatrix} dp_2 + \begin{bmatrix} p_1 \\ 0 \end{bmatrix} dt + \cdot \right\}$$

calculating the inverse matrix instead of using Cramer's Rule

$$= \frac{1}{\Delta}\begin{bmatrix} -wz_{22} & wz_{12} \\ wz_{12} & -wz_{11} \end{bmatrix}\left\{ \begin{bmatrix} t-1 \\ 0 \end{bmatrix} dp_1 + \begin{bmatrix} 0 \\ -1 \end{bmatrix} dp_2 + \begin{bmatrix} p_1 \\ 0 \end{bmatrix} dt + \cdot \right\}$$

$$\begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -\omega(t-1)z_{22} \\ \omega(t-1)z_{12} \end{bmatrix} dp_1$$

$$\frac{dq_1}{dp_1} = \frac{\omega(1-t)z_{22}}{\Delta} = \frac{(+)(+)(+)}{(+)} > 0$$

$$\frac{dq_2}{dp_1} = \frac{\omega(t-1)z_{12}}{\Delta} = ?$$

$$\begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -\omega z_{12} \\ \omega z_{11} \end{bmatrix} dp_2$$

$$\frac{dq_1}{dp_2} = \frac{-\omega z_{12}}{\Delta} = ?$$

$$\frac{dq_2}{dp_2} = \frac{\omega z_{11}}{\Delta} > 0$$

$$\begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -\omega z_{22} p_1 \\ \omega z_{12} p_1 \end{bmatrix} dt$$

$$\frac{dq_1}{dt} = \frac{-\omega z_{22} p_1}{\Delta} < 0 \quad (2)$$

$$\frac{dq_2}{dt} = \frac{\omega z_{12} p_1}{\Delta} = ? \quad (3)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial z}{\partial q_2}\frac{\partial q_2}{\partial t}$$
$$= (+)\,(-) + (+)\,(?)$$
$$= ? \quad (1)$$

Optional (used in part iii)

Summary (of additional results):

$\partial q_1/\partial p_1 = \frac{w(1-t)z_{22}}{\Delta} > 0$ $\quad \partial q_1/\partial p_2 = \frac{-w z_{12}}{\Delta} = ?$

$\partial q_2/\partial p_1 = \frac{-w(1-t)z_{12}}{\Delta} = ?$ $\quad \partial q_2/\partial p_2 = \frac{w z_{11}}{\Delta} > 0$

iii) $q_1^D = q_1^D(p_1)$ $\quad q_1^s = q_1^s(p_1, p_2, w, t)$

$q_2^D = q_2^D(p_2)$ $\quad q_2^s = q_2^s(p_1, p_2, w, t)$

$$q_1^D(p_1) = q_1^s(p_1, p_2, w, t)$$
$$q_2^D(p_2) = q_2^D(p_1, p_2, w, t)$$

totally differentiating (finding the total differentials),

$$\frac{dq_1^D}{dp_1}dp_1 = \frac{\partial q_1^s}{\partial p_1}dp_1 + \cancel{\frac{\partial q_1^s}{\partial p_2}}dp_2 + \frac{\partial q_1^s}{\partial w}dw + \frac{\partial q_1^s}{\partial t}dt$$

$$\frac{dq_2^D}{dp_2}dq_2 = \cancel{\frac{\partial q_2^s}{\partial p_1}}dp_1 + \frac{\partial q_2^s}{\partial p_2}dp_2 + \frac{\partial q_2^s}{\partial w}dw + \frac{\partial q_2^s}{\partial t}dt$$

(the terms with a slash are 0 due to the assumption of "independence in demand"), and hence

$$\frac{dq_1^D}{dp_1}\frac{\partial p_1}{\partial t} = \frac{\partial q_1^s}{\partial p_1}\frac{\partial p_1}{\partial t} + \cancel{\frac{\partial q_1^s}{\partial p_2}\frac{\partial p_2}{\partial t}} + \frac{\partial q_1^s}{\partial t}$$

$$\frac{dq_2^D}{dp_2}\frac{\partial p_2}{\partial t} = \cancel{\frac{\partial q_2^s}{\partial p_1}\frac{\partial p_1}{\partial t}} + \frac{\partial q_2^s}{\partial p_2}\frac{\partial p_2}{\partial t} + \frac{\partial q_2^s}{\partial t}$$

since $\partial w/\partial t = 0$ (the input price is constant)

$$\begin{bmatrix} \frac{\partial q_1^D}{\partial p_1} - \frac{\partial q_1^s}{\partial p_1} & -\cancel{\frac{\partial q_1^s}{\partial p_2}} \\ -\cancel{\frac{\partial q_2^s}{\partial p_1}} & \frac{\partial q_2^D}{\partial p_2} - \frac{\partial q_2^s}{\partial p_2} \end{bmatrix} \begin{bmatrix} \partial p_1/\partial t \\ \partial p_2/\partial t \end{bmatrix} = \begin{bmatrix} \partial q_1^s/\partial t \\ \partial q_2^s/\partial t \end{bmatrix}$$

$$\frac{\partial p_1}{\partial t} = \frac{\begin{vmatrix} \partial q_1^s/\partial t & -\cancel{\partial q_1^s/\partial p_2} \\ \partial q_2^s/\partial t & \frac{\partial q_2^D}{\partial p_2} - \frac{\partial q_2^s}{\partial p_2} \end{vmatrix}}{\begin{vmatrix} \frac{\partial q_1^D}{\partial p_1} - \frac{\partial q_1^s}{\partial p_1} & -\cancel{\partial q_1^s/\partial p_2} \\ -\cancel{\partial q_2^s/\partial p_1} & \frac{\partial q_2^D}{\partial p_2} - \frac{\partial q_2^s}{\partial p_2} \end{vmatrix}}$$

$$= \frac{\frac{\partial q_1^s}{\partial t}\left[\frac{\partial q_2^D}{\partial p_2} - \frac{\partial q_2^s}{\partial p_2}\right] + \cancel{\frac{\partial q_2^s}{\partial t}\frac{\partial q_1^s}{\partial p_2}}}{\left(\frac{\partial q_1^D}{\partial p_1} - \frac{\partial q_1^s}{\partial p_1}\right)\left(\frac{\partial q_2^D}{\partial p_2} - \frac{\partial q_2^s}{\partial p_2}\right) - \cancel{\frac{\partial q_1^s}{\partial p_2}\frac{\partial q_2^s}{\partial p_1}}}$$

and assuming non-Giffen goods,

$$= \frac{\overset{(-)}{\dfrac{-wZ_{22}p_1}{\Delta}}\left[\overset{(+)}{(-) - \dfrac{wZ_{11}}{\Delta}}\right] + \cancel{\overset{\ominus}{\left[\dfrac{wZ_{12}p_1}{\Delta} \quad \dfrac{-wZ_{12}}{\Delta}\right]}}}{\underset{(+)}{\left[(-) - \dfrac{w(1-t)Z_{22}}{\Delta}\right]}\underset{(+)}{\left[(-) - \dfrac{wZ_{11}}{\Delta}\right]} - \cancel{\underset{\oplus}{\left(\dfrac{-wZ_{12}}{\Delta}\right)\left(\dfrac{-w(1-t)Z_{12}}{\Delta}\right)}}}$$

$$= \frac{(-)(-) + \cancel{(+)}}{(-)(-) - \cancel{(+)}} = \frac{(+)}{(+)} > 0$$

which is, indeed, intuitive

iv) check for consistency

$$Z_1 = A\alpha q_1^{\alpha-1} q_2^{\beta} \qquad Z_2 = A\beta q_1^{\alpha} q_2^{\beta-1}$$

$$Z_{11} = A\alpha(\alpha-1) q_1^{\alpha-2} q_2^{\beta} = \frac{\alpha(\alpha-1)}{q_1^2} Z \qquad Z_{22} = A\beta(\beta-1) q_1^{\alpha} q_2^{\beta-2} = \frac{\beta(\beta-1)}{q_2^2} Z$$

but $Z_{ii} > 0$, so $\alpha+\beta > 1$ (not $<$). I.e., $\alpha > 0, \beta > 0, \alpha+\beta < 1$ from the question, so $0 < \alpha < 1$ and $0 < \beta < 1$, so $\alpha - 1 < 0$ and $\beta - 1 < 0$, so $Z_{11} < 0$ and $Z_{22} < 0$, contradicting part (i)

$$Z_{12} = A\alpha\beta q_1^{\alpha-1} q_2^{\beta-1} = \frac{Z\alpha\beta}{q_1 q_2}$$

, making $\partial q_1/\partial p_2 \neq 0$ and $\partial q_2/\partial p_1 \neq 0$, contradicting part (iii). So $\alpha+\beta<1$ won't work. How about $\alpha+\beta>1$? Continuing,

$$Z_{11}Z_{22} - Z_{12}^2 = \frac{Z^2\alpha(\alpha-1)\beta(\beta-1)}{q_1^2 q_2^2} - \frac{Z^2\alpha^2\beta^2}{(q_1 q_2)^2}$$

$$= \text{sign}\left[(\alpha-1)(\beta-1) - \alpha\beta\right]$$

$$= \text{sign}\left[\alpha\beta - \alpha - \beta + 1 - \alpha\beta\right]$$

$$= \text{sign}\left[1 - (\alpha+\beta)\right]$$

which should be positive from part (i) — but it won't be positive if $\alpha+\beta > 1$. So $\alpha+\beta>1$ won't work either.

~~The requirements thus are $\alpha > 1$, $\alpha+\beta < 1$, $\beta > 1$ which contradict.~~

There is no point in continuing. The function $Z = Aq_1^{\alpha}q_2^{\beta}$ with $\alpha > 0$ and $\beta > 0$ and $A > 0$ cannot be convex.