Answers to Microeconomics Qualifying Exam

Questions of Prof. Lozada, Summer 2011

Section 1 Question 1.

[This question is very closely related to Fall 2008 Final Exam Question 1.]

a) We know that $v(\underset{\sim}{p}, e(\underset{\sim}{p}, u)) = u$, or, abbreviating the expenditure function by "e",

$$v(\underset{\sim}{p}, e) = u \quad \text{so}$$

$$u = v(\underset{\sim}{p}, e) = v(\bar{p}_x, \bar{p}_y, \underset{\substack{\uparrow \\ e \text{ in} \\ \text{our case}}}{m}) = \ln \frac{\alpha^\alpha \beta^\beta e^{\alpha+\beta}}{\bar{p}_x^\alpha \bar{p}_y^\beta (\alpha+\beta)^{\alpha+\beta}}.$$

We now have to solve for e, remembering that e is the expenditure function, not the base of the natural logarithms.

$$\exp u = \frac{\alpha^\alpha \beta^\beta e^{\alpha+\beta}}{\bar{p}_x^\alpha \bar{p}_y^\beta (\alpha+\beta)^{\alpha+\beta}} \Rightarrow e^{\alpha+\beta} = (\alpha+\beta)^{\alpha+\beta} \frac{\bar{p}_x^\alpha \bar{p}_y^\beta}{\alpha^\alpha \beta^\beta} (\exp u)$$

$$\Rightarrow e = (\alpha+\beta) \left(\frac{\bar{p}_x^\alpha \bar{p}_y^\beta}{\alpha^\alpha \beta^\beta} \right)^{\frac{1}{\alpha+\beta}} \exp\left(\frac{u}{\alpha+\beta}\right).$$

b) Since $e(\hat{p}_x, \hat{p}_y, v) = (\alpha+\beta) \left[\frac{\hat{p}_x^\alpha \hat{p}_y^\beta}{\alpha^\alpha \beta^\beta} \right]^{\frac{1}{\alpha+\beta}} \exp\left(\frac{v}{\alpha+\beta}\right)$,

$$e(\hat{p}_x, \hat{p}_y, v(\bar{p}_x, \bar{p}_y, m)) = (\alpha+\beta) \left[\frac{\hat{p}_x^\alpha \hat{p}_y^\beta}{\alpha^\alpha \beta^\beta} \right]^{\frac{1}{\alpha+\beta}} \exp\left[\frac{v(\bar{p}_x, \bar{p}_y, m)}{\alpha+\beta} \right]$$

$$= (\alpha+\beta) \left[\frac{\hat{p}_x^\alpha \hat{p}_y^\beta}{\alpha^\alpha \beta^\beta} \right]^{\frac{1}{\alpha+\beta}} \left[\exp v(\bar{p}_x, \bar{p}_y, m) \right]^{\frac{1}{\alpha+\beta}} =$$

$$(\alpha+\beta)\left[\frac{\hat{p}_x^{\alpha}\ \hat{p}_y^{\beta}}{\alpha^{\alpha}\beta^{\beta}}\right]^{\frac{1}{\alpha+\beta}}\left[\frac{\alpha^{\alpha}\beta^{\beta}m^{\alpha+\beta}}{\bar{p}_x^{\alpha}\ \bar{p}_y^{\beta}(\alpha+\beta)^{\alpha+\beta}}\right]^{\frac{1}{\alpha+\beta}}$$

. This simplifies to $m\left[\hat{p}_x^{\alpha}\ \hat{p}_y^{\beta}/\bar{p}_x^{\alpha}\ \bar{p}_y^{\beta}\right]^{\frac{1}{\alpha+\beta}}$.

c) $u(\underset{\sim}{x}) = \min\limits_{\underset{\sim}{p}} v(\underset{\sim}{p})$ s.t. $\underset{\sim}{p}\cdot\underset{\sim}{x} = 1$

setting $m=1$

$$= \min \ln\left[\bar{p}_x^{-\alpha}\bar{p}_y^{-\beta}\frac{\alpha^{\alpha}\beta^{\beta}1^{\alpha+\beta}}{(\alpha+\beta)^{\alpha+\beta}}\right] \text{ s.t. } \underset{\sim}{p}\cdot\underset{\sim}{x} = 1$$

$$= \min\ -\alpha\ln\bar{p}_x - \beta\ln\bar{p}_y + \ln\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}}$$

$$\mathcal{L} = -\alpha\ln\bar{p}_x - \beta\ln\bar{p}_y + \ln\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} + \lambda(\bar{p}_x x + \bar{p}_y y - 1)$$

F.O.C.'s

$$0 = \frac{\partial\mathcal{L}}{\partial\bar{p}_x} = \frac{-\alpha}{\bar{p}_x} + \lambda x$$

$$0 = \frac{\partial\mathcal{L}}{\partial\bar{p}_y} = \frac{-\beta}{\bar{p}_y} + \lambda y$$

$$\Rightarrow \lambda = \frac{\alpha}{x\bar{p}_x} = \frac{\beta}{y\bar{p}_y}$$

$$\Rightarrow \bar{p}_x = \frac{\alpha}{x}\frac{y}{\beta}\bar{p}_y$$

$$0 = \frac{\partial\mathcal{L}}{\partial\lambda} = \bar{p}_x x + \bar{p}_y y - 1 \longrightarrow$$

$$1 = \left(\frac{\alpha}{x}\frac{y}{\beta}\bar{p}_y\right)x + \bar{p}_y y$$

$$= \frac{\alpha y}{\beta}\bar{p}_y + \bar{p}_y y = y\bar{p}_y\left(\frac{\alpha}{\beta}+1\right)$$

$$= y\bar{p}_y\frac{\alpha+\beta}{\beta} \Rightarrow \bar{p}_y = \frac{\beta}{\alpha+\beta}\frac{1}{y} \text{ and}$$

therefore $\bar{p}_x = \frac{\alpha}{x}\frac{y}{\beta}\left[\frac{\beta}{\alpha+\beta}\frac{1}{y}\right] = \frac{\alpha}{\alpha+\beta}\frac{1}{x}$. Substituting these minimizing $\bar{p}_x$ and $\bar{p}_y$ into v results in

$$u = \ln\left[\frac{\alpha}{\alpha+\beta}\frac{1}{x}\right]^{-\alpha}\left[\frac{\beta}{\alpha+\beta}\frac{1}{y}\right]^{-\beta}\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} = \ln\frac{(\alpha+\beta)^{\alpha}}{\alpha^{\alpha}}x^{\alpha}\frac{(\alpha+\beta)^{\beta}}{\beta^{\beta}}y^{\beta}\frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}}$$

$$= \ln x^{\alpha} y^{\beta} = \alpha \ln x + \beta \ln y .$$

Note that since $\ln x$ is increasing in x, one can, instead of minimizing

$$\ln \left[\bar{p}_x^{-\alpha} \bar{p}_y^{-\beta} \frac{\alpha^{\alpha} \beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} \right] \qquad \text{s.t. } \underset{\sim}{p} \cdot \underset{\sim}{x} = 1 ,$$

minimize instead

$$\left[\bar{p}_x^{-\alpha} \bar{p}_y^{-\beta} \frac{\alpha^{\alpha} \beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}} \right] \quad \text{s.t. } \underset{\sim}{p} \cdot \underset{\sim}{x} = 1 .$$

If you do this, the result is $u = x^{\alpha} y^{\beta}$, which represent the same preferences as $\ln x^{\alpha} y^{\beta} = \alpha \ln x + \beta \ln y$.

Section 1 Question 2.

a) Both kinds of income increase (indirect) utility. This captures the reason some super-rich people did unethical things. However, the square root function means that \$1 of unethically-earned income contributes less to (indirect) utility than \$1 of ethically-earned income: so the super-rich do have some moral misgivings about unethically-earned income.

b) The utility gained from pre-tax income, minus taxes. An objective function for the super-rich.

c) W_h wage rate of honest work

W_d " " " dishonest "

ℓ_h hours worked doing honest labor

ℓ_d " " " dishonest "

Working time constraint: $\ell_h + \ell_d = 1$ (the "1" stands for "one working day"; instead of "1" you could use "18 hours" or "8 hours" or "24 hours").

honest income $= W_h \ell_h$

dishonest income $= W_d \ell_d = W_d(1-\ell_h)$.

Objective: max $w_h \ell_h + \sqrt{w_d(1-\ell_h)} - t\left[w_h \ell_h + w_d(1-\ell_h)\right]$

over ℓ_h :

$$0 = \frac{d\,(\text{objective})}{d\ell_h} = w_h + \frac{1}{2}\,\frac{-w_d}{\sqrt{w_d(1-\ell_h)}} - t\left[w_h - w_d\right]. \quad (1)$$

Solution Method 1 : No need to solve for ℓ_h.

$$0 = dt\left[-w_h + w_d\right] + d\ell_h\left[\frac{-1}{4}\,\frac{-w_d}{\left(w_d(1-\ell_h)\right)^{+3/2}}\,(-w_d)\right]$$

$$(w_h - w_d)\,dt = \left[\frac{-1}{4}\,w_d^{2}\,w_d^{-3/2}\,\frac{1}{(1-\ell_h)^{3/2}}\right] d\ell_h$$

$$= \frac{-\sqrt{w_d}}{4(1-\ell_h)^{3/2}}\,d\ell_h$$

$$\Rightarrow \quad \frac{d\ell_h}{dt} = \frac{w_h - w_d}{-\sqrt{w_d}}\,4\,(1-\ell_h)^{3/2} = \frac{w_h - w_d}{-\sqrt{w_d}}\,4\,\ell_d^{3/2}$$

$$= \frac{4\,\ell_d^{3/2}}{\sqrt{w_d}}\,(w_d - w_h).$$

So if dishonest labor pays more than honest labor (if it didn't, then in this model no one would do dishonest labor, which contradicts the article's opinion), one has $w_d - w_h > 0$, so $d\ell_h / dt > 0$, therefore (due to the working hour constraint) we obtain $d\ell_d / dt < 0$, supporting the author's hypothesis.

Solution Method 2: Solving for ℓ_h.

From (1),

$$t(w_h - w_d) = w_h - \frac{1}{2}\frac{w_d}{\sqrt{w_d}\sqrt{1-\ell_h}} = w_h - \frac{1}{2}\frac{\sqrt{w_d}}{\sqrt{1-\ell_h}} \Rightarrow$$

$$\frac{1}{2}\frac{\sqrt{w_d}}{\sqrt{1-\ell_h}} = w_h - t(w_h - w_d)$$

$$\frac{\sqrt{w_d}}{2[w_h - t(w_h-w_d)]} = \sqrt{1-\ell_h} \Rightarrow$$

$$1-\ell_h = \frac{w_d}{4[w_h - t(w_h-w_d)]^2}.$$

It would be trivial to solve this for ℓ_h, but it's even easier to solve it for ℓ_d

$$\ell_d = \frac{w_d}{4[w_h - t(w_h-w_d)]^2}$$

and then calculate

$$\frac{d\ell_d}{dt} = \frac{w_d}{4}\frac{-2}{[w_h - t(w_h-w_d)]^3}[-(w_h-w_d)]$$

$$= \frac{w_d}{4}\frac{1}{[w_h - t(w_h-w_d)]^2}\frac{-2}{[w_h - t(w_h-w_d)]}[-(w_h-w_d)]$$

$$= \ell_d \frac{-2}{w_h - t(w_h-w_d)}[-(w_h-w_d)] = \ell_d \frac{2(w_h-w_d)}{w_h - t(w_h-w_d)}$$

$$= \frac{2\, l_d\, (w_h - w_d)}{w_h + t(w_d - w_h)} = \frac{-2\, l_d\, (w_d - w_h)}{w_h + t\,(w_d - w_h)} < 0$$

since $w_d - w_h > 0$. This is the same sign obtained using Method 1.

Section 2 Question 1

production possibilities set Y

input requirement set $V(y)$

production function $f(x)$

a) Y being a convex set means that if

$$\left.\begin{array}{l} (y, -\underset{\sim}{x}) \in Y \\ \text{and } (y, -\underset{\sim}{x}') \in Y \end{array}\right\} \quad (1)$$

then $t(y, -\underset{\sim}{x}) + (1-t)(y, -\underset{\sim}{x}') \in Y. \quad (2)$

(2) implies $(ty + (1-t)y, \; -t\underset{\sim}{x} - (1-t)\underset{\sim}{x}') \in Y$

$$(y, \; -t\underset{\sim}{x} - (1-t)\underset{\sim}{x}') \in Y$$

$$\Longleftrightarrow \quad t\underset{\sim}{x} + (1-t)\underset{\sim}{x}' \in V(y). \quad (3)$$

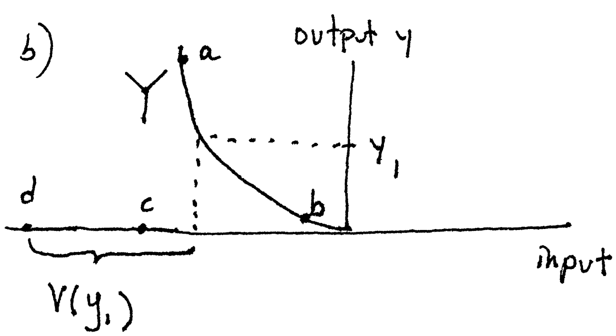
(1) implies
$$\left.\begin{array}{l} \underset{\sim}{x} \in V(y) \\ \underset{\sim}{x}' \in V(y). \end{array}\right\} \quad (4)$$

In particular,

$$\begin{array}{c} (1) \Rightarrow (2) \Rightarrow (3) \\ \Updownarrow \\ (4) \end{array}$$

so $(4) \Rightarrow (3)$, meaning that $V(y)$ is a convex set.

b)



Here Y is not a convex set (for example, $a \in Y$ and $b \in Y$ but points on a line between a and b are not in Y), but $V(y)$ for a typical y such as y_1 is a convex set (since points on a line between c and d are in $V(y_1)$).

c) $V(y)$ is $f(x)$'s upper contour set.

Quasiconcavity of f is defined to mean that f's upper contour sets are convex sets.

So f is quasiconcave if and only if its upper contour sets, $V(y)$, are convex sets.

Section 2 Question 2.

a) $\pi(\underset{\sim}{p}) = \max_{\underset{\sim}{y} \in Y} \underset{\sim}{p} \cdot \underset{\sim}{y}$ by definition

$$\frac{\partial \pi}{\partial p_i} = \frac{\partial \mathcal{L}^*}{\partial p_i} \quad \text{by the Envelope Theorem}$$

$$= \frac{\partial}{\partial p_i} (\underset{\sim}{p} \cdot \underset{\sim}{y})^*$$

$= y_i^*$, which is Hotelling's Lemma;

$$= y_i^* \begin{cases} < 0 & \text{if } i \text{ is an input} \\ > 0 & \text{if } i \text{ is an output.} \end{cases}$$

b) $\pi(\lambda \underset{\sim}{p}) = \max_{y \in Y} \lambda \underset{\sim}{p} \cdot \underset{\sim}{y} = \lambda \max_{y \in Y} \underset{\sim}{p} \cdot \underset{\sim}{y} = \lambda \pi(\underset{\sim}{p})$.

c) $\nabla_{\underset{\sim}{p}} \pi(\underset{\sim}{p}) = \underset{\sim}{y}$ as shown in part (a).

If f is homogeneous of degree k, its derivative is homogeneous of degree $k-1$.

$\pi(\underset{\sim}{p})$ is homogeneous of degree 1 from part (b).

Hence its derivative, $\nabla \pi = \underset{\sim}{y}$, is homogeneous of degree zero.

d) If Y reflects increasing returns to scale, then the optimal $\underset{\sim}{y}$ for any fixed $\underset{\sim}{p}$ is $\underset{\sim}{y}^* = \infty$, so $\pi^* = \infty$ for all $\underset{\sim}{p}$ and the properties do not follow.

↳ (competitive firms think $\underset{\sim}{p}$ is fixed)