F. The Technology of Production

1. Sketch the graph of the total product of water curve, and from it the average product of water curve and the marginal product of water curve, for a case in which the Law of Diminishing Returns is *violated.* Explain your answer.

2. Suppose a "Superburger" consists of 3 hamburger patties and 2 slices of bread. What is the production function for Superburgers?

3. A firm produces wheat, Q, using water, W, and fertilizer, F, as inputs, according to the production function

$$Q = \sqrt{WF}\,.$$

   In the short run, the firm has committed itself to buy $W = 4$ units of water.

   (a) What is the average product of fertilizer in the short run?

   (b) Sketch the short-run relationship between total product Q and F.

   (c) Sketch the short-run graph of the average product of fertilizer and the marginal product of fertilizer.

4. Show that the production function $Q = \sqrt{WF}$ has constant returns to scale but that it also has diminishing marginal product of W and diminishing marginal product of F.

   Hint for the last two parts of this problem: Set $F = 1$ and graph the relationship between Q and W; then set $W = 1$ and graph the relationship between Q and F. Then interpret these graphs in terms of the marginal products.

5. Suppose corn Q is produced from water W and fertilizer F according to the production function

$$Q = W^{1/2}F^{3/4}.$$

   (a) If fertilizer is fixed at $F_0 = 1$ lb., sketch the total product of water curve. Label two points on this curve with numbers.

   (b) What is the algebraic equation for the average product of water curve? Sketch this curve.

(c) Sketch the marginal product of water curve (draw this on the graph you made in part (b)). Describe its shape (rising, falling, constant, U-shaped, or something else) and tell whether it is below, above, or crosses the average product of water curve. Explain your answer, as always!

(d) Does the production function in this question have diminishing returns to water? Explain.

(e) Does the production function in this question have decreasing returns to scale? Explain.

6. Suppose a firm produces corn Q from water W and fertilizer F according to the following production function:

$$Q = W^{1/2} F^{2/3}.$$

   (a) Does this production function show increasing, constant, or decreasing returns to scale? (Show all your work!)

   (b) Graph the relationship between Q and W holding F constant at $F = 1$. Then answer the following question, explaining how the graph helps in your answer: When $F = 1$, does the marginal product of water always rise, always stay constant, always fall, or none of the above? (Remember to define the marginal product of water.)

   (c) [This part of the question pertains to material in the next chapter, not this chapter.] Is the long-run average cost curve for this firm sloping upward, downward, or is it flat? Why? What is on the horizontal axis of such a graph? What is on the vertical axis?

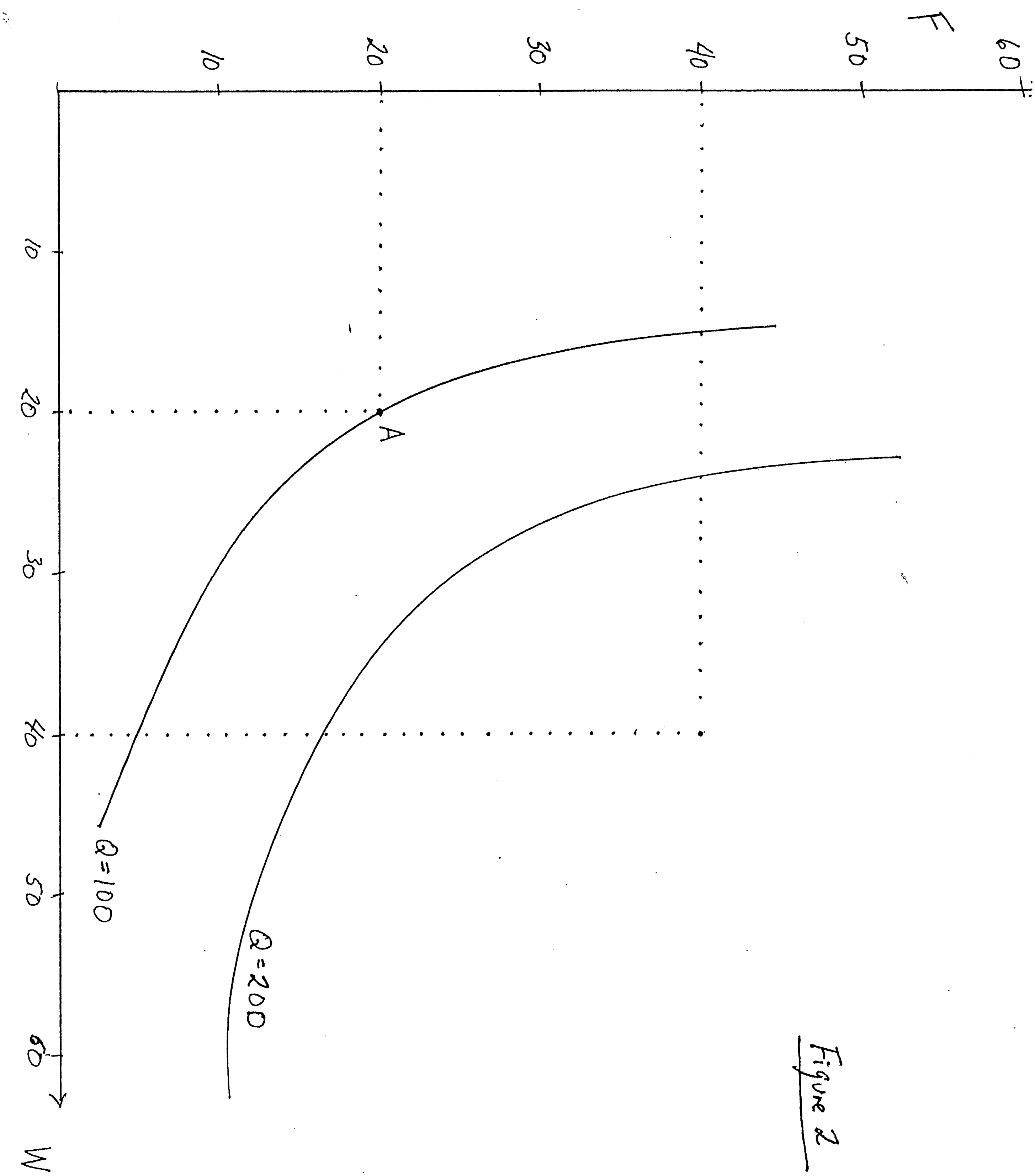
7. Give an example of a production function which always has increasing returns to scale. (Hint: the most convenient form is probably $Q = W^{\alpha} F^{\beta}$; what should α and β be?)

8. Refer to Figure 2.

   (a) Starting at point A, determine if these isoquants depict decreasing returns to scale, constant returns to scale, increasing returns to scale, or none of these.

   (b) Suppose I tell you that starting at point A (that is, starting with 20 gallons of water and 20 points of fertilizer), the extra amount of corn produced when you add one extra gallon of water is 10 bushels. If you start at point A again, but now you add one extra pound of fertilizer instead of one extra gallon of water, how much extra corn will be produced? (Hint: Think about the relationship between Rate of Technical Substitution, on the one hand, and marginal products, on the other.)

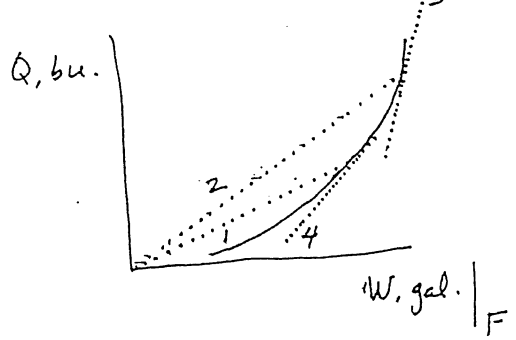
9. The example your textbook uses for a production function is $Q = f(L, K)$ where Q is output, L is labor, and K is capital. Such a formulation implies aggregation of capital. Discuss the problems associated with this aggregation.

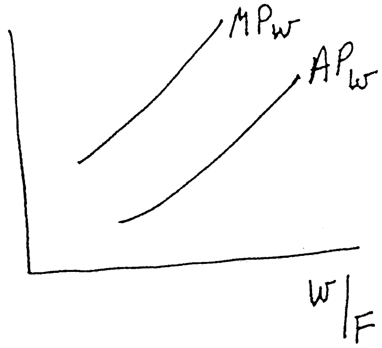


Question 8's Fig. 2

Answers

1





When the Law of Diminishing Returns is violated, Q increases at an increasing rate as W is increased while F is held constant. Thus the Q vs. $W|_F$ graph is convex (or linear). AP_W is rising because line 2 is steeper than line 1; MP_W is rising because line 3 is steeper than line 4; and for a given amount of W, $MP_W \geq AP_W$ because line 4 is steeper than line 1, or because line 3 is steeper than line 2. (You can also use the rule, if the average is rising then the marginal must be above it.)

The Law of Diminishing Returns states that $MP_W|_F$ eventually decreases; in words, the "marginal product of water holding fertilizer fixed" eventually falls.

2 The ratio is 3 patties : 2 slices or $\frac{\text{patties}}{\text{slices}} = \frac{2}{3}$. So,

quantity is proportional to min (2 · patties, 3 · slices). (5 points

For 3 patties and 2 slices, Quantity = 1; hence

$$Q = 1 = (\text{constant})\ \text{Min.}(2 \cdot 3, 3 \cdot 2)$$

$$= (\text{constant})\ 6,$$

$$\frac{1}{6} = \text{constant}.$$

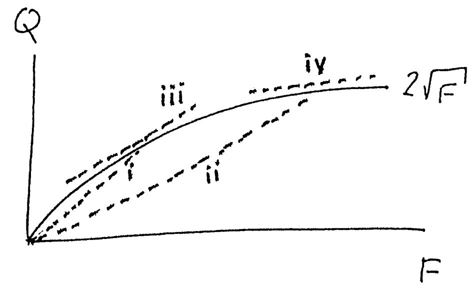
So $Q = \frac{1}{6} \min(2 \cdot \#\text{patties}, 3 \cdot \#\text{slices})$, or (5 points)

$$= \min\left(\tfrac{1}{3}\#\text{patties}, \tfrac{1}{2} \cdot \#\text{slices}\right).$$

(3) $Q = \sqrt{WF} = \sqrt{4F} = 2\sqrt{F}$

(a) $AP_F = \frac{Q}{F} = \frac{2\sqrt{F}}{F} = \frac{2}{\sqrt{F}}$

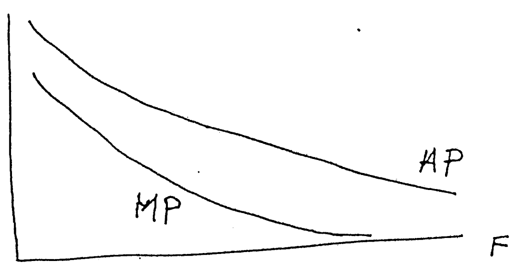
(b)



(5 points for each part)

Some data points would be (0,0), (1,2), (4,4), (9,6), (16,8), (25,10), (36,12), (49,14).

(c)



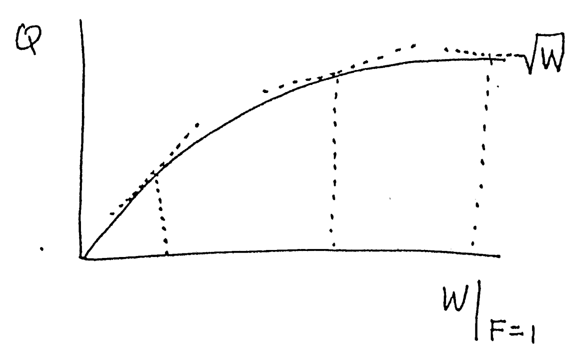
(i) and (ii) in (b) show that AP is falling. (iii) and (iv) show that MP is falling. (i) and (iii), and (iii) and (iv), show that MP < AP.

(4) Double inputs: $\sqrt{(2W)(2F)} = \sqrt{4WF} = 2\sqrt{WF}$.

(13 points)

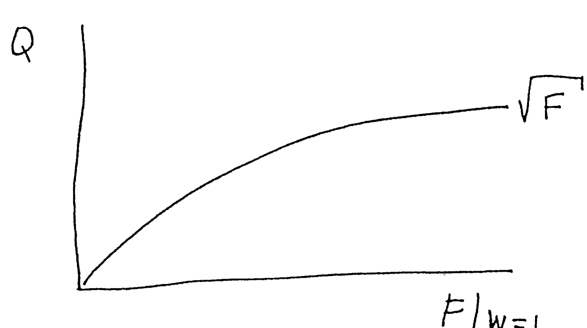
So doubling inputs results in doubling outputs. This is constant returns to scale.

(6 points)



The slope is the $\overbrace{MPP}^{\text{also just called the } MP_W}_W$. It is diminishing, since the curve is getting flatter and flatter.

(6 points)

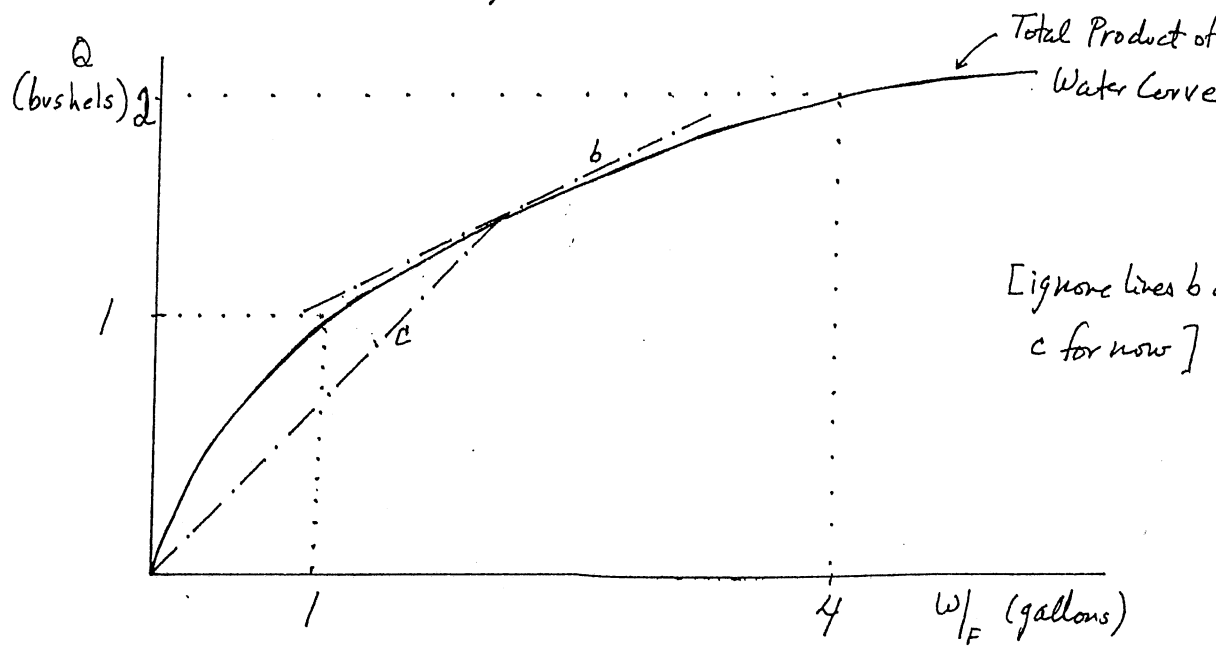


Same as above, exchanging W and F.

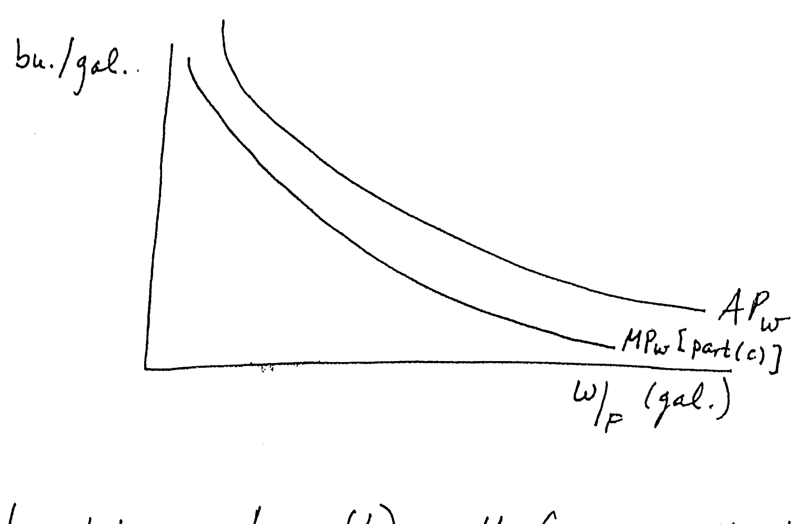
⑤ a) $Q = W^{1/2} F^{3/4}$

7 pts

$F = 1 \Rightarrow Q = W^{1/2}$, whose graph looks like this:



7 pts b) $AP_W = \left.\frac{Q}{W}\right|_F = \frac{W^{1/2} F_0^{3/4}}{W} = \frac{F_0^{3/4}}{W^{1/2}} = \frac{1}{W^{1/2}}$ for $F_0 = 1$.



7 pts c) Drawing tangent lines such as (b) in the figure at the top of this page, it will be clear that the slopes of these lines get flatter as W increases.

Hence MP_w falls. Comparing line (b) and line (c), (b) is flatter, so $MP_w < AP_w$ there. This is true at all points on the Total Product of Water curve. So MP_w is <u>falling</u>, and it is <u>below</u> AP_w.

4pts d) Yes, it has diminishing returns to water, because MP_w is falling everywhere.

4pts e) old $Q = W^{1/2} F^{3/4}$. Now double inputs; does output double, more than double, or less than double?

new $Q = (2W)^{1/2} (2F)^{3/4}$

$= 2^{1/2} W^{1/2} 2^{3/4} F^{3/4} = 2^{\frac{1}{2}+\frac{3}{4}} W^{1/2} F^{3/4}$

$= 2^{5/4} W^{1/2} F^{3/4}$

$> 2 W^{1/2} F^{3/4} = 2$ (old Q). So "new Q" > 2 "old Q" : <u>increasing</u> returns to scale

6

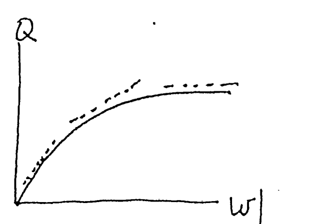
a) $Q(W, F) = W^{1/2} F^{2/3}$

Double Inputs : $\underbrace{Q(2W, 2F)}_{\text{new output}} = (2W)^{1/2} (2F)^{2/3} = 2^{\frac{1}{2}} W^{\frac{1}{2}} 2^{\frac{2}{3}} F^{\frac{2}{3}}$

$= 2^{7/6} W^{1/2} F^{2/3} > \underbrace{2 W^{1/2} F^{2/3}}_{\text{old output}}$

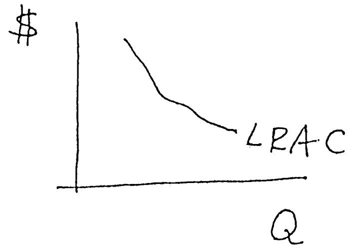
So output has more than doubled : ↑RS.

b) $F = 1 \Rightarrow Q = W^{1/2}$



Marginal product of water is the slope of this curve. This slope falls going from left to right, so the MP_w is falling.

c)



This is downward-sloping because, from part a, there are increasing returns to scale : ↑ in Q ⇒ less than proportional ↑ in costs.

Points: 3 each for a & b and 2 for c

(4) There are many possible answers to this question. <u>What is necessary is to show that doubling the inputs into the production function leads to a more than doubling of output.</u> For example, take $Q = W^{\alpha} F^{\beta}$. Doubling W and F gives a new quantity of $(2W)^{\alpha}(2F)^{\beta} = 2^{\alpha} W^{\alpha} 2^{\beta} F^{\beta} = 2^{\alpha+\beta} W^{\alpha} F^{\beta}$. In order for this to be greater than twice the old quantity, we must have $2^{\alpha+\beta} > 2$. Therefore any α and β satisfying $\alpha + \beta > 1$ will work (though $\alpha > 0$ and $\beta > 0$ are also required in most cases).

13 pts: What's required is doubling all inputs ⇒ more than doubling output
9 pts: Having a correct function – but no credit for a guess; some points here must be made
11 pts: Correct proof that the function has increasing returns to scale

(8)

a) Point A has $W = 20$, $F = 20$, and $Q = 100$. If you double W and double F, you get to $W = 40$, $F = 40$ (indicated by point B on the graph attached here). At point B, $Q > 200$ (because the $Q = 200$ isoquant lies below and to the left of point B). So doubling inputs caused output to more than double : the production function has increasing returns to scale.

9 points

9 points

b) You are given that, at point A, $MP_W = 10$. (This is because $MP_W = \left.\frac{\Delta Q}{\Delta W}\right|_F$, and you are told that if $\Delta W = 1$ gallon then $\Delta Q = 10$ bushels.) You need to find MP_F.

You are reminded in the first hint that RTS of W for F $= \frac{MP_W}{MP_F}$ (you should know this already). The key is to remember that, if W is on the horizontal axis and F is on the vertical axis, then

RTS of W for F = - slope of the isoquant. ← 8 points

So the RTS of W for F at point A is the opposite of the slope of the $Q = 100$ isoquant at A. The tangent line at A is drawn in the accompanying figure; its slope is approximately $\frac{-60}{30} = -2$. So the RTS of W for F at A is 2. We then have

3 points

$$2 = \text{RTS of } W \text{ for } F = \frac{MP_W}{MP_F} = \frac{10}{MP_F}$$

$$\Rightarrow MP_F = 5. \qquad \leftarrow 1 \text{ point}$$

So if you start at point A and hold W constant, adding one more pound of fertilizer will increase output by 5 bushels.

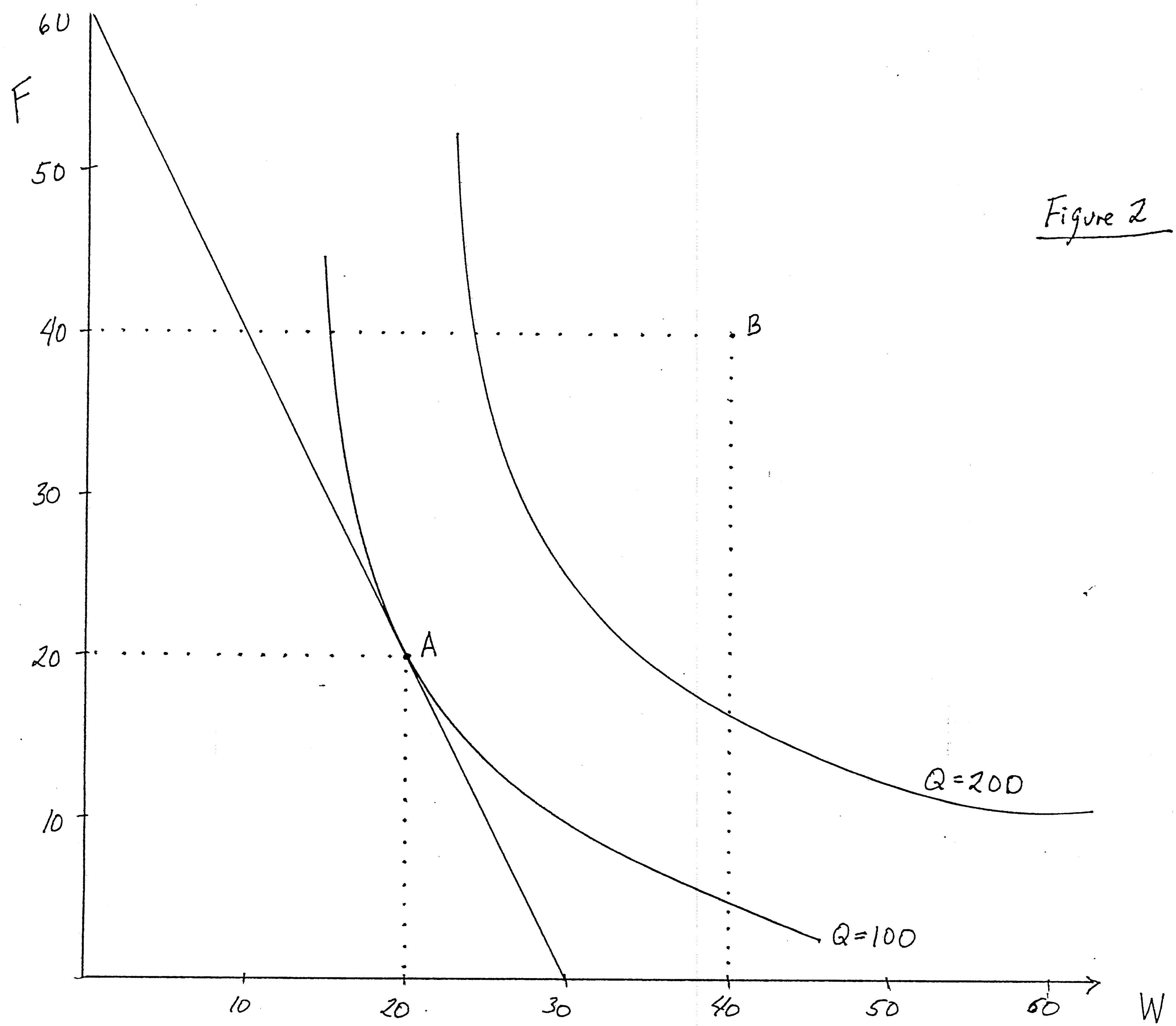


Figure 2

(9) Suppose the firm uses two types of machines, m_1 and m_2, and suppose the prices of these machines are p_1 and p_2 respectively. The only natural, widely-accepted method of "aggregating" (i.e. putting together) capital goods would be

$$K = p_1 m_1 + p_2 m_2 \; ; \qquad (1)$$

in other words, the amount of capital the firm has is equal to the dollar value of its type 1 machines plus the dollar value of its type 2 machines. 10pts

9pts

The first problem with this aggregation scheme is that it assumes that \$1000 worth of capital 'K' contributes the same amount to production regardless of the composition of m_1 versus m_2 machines. For instance, if $p_1 = \$2$ and $p_2 = \$200$, using zero machines of type 1 and 5 machines of type 2 is supposed to yield the same output as using 400 machines of type 1 and 1 machine of type 2 (given the same labor input in both cases). This may be true, but it also may not be true. Essentially all aggregation schemes have this problem.

A more serious problem — and one that is unique to "aggregation by value" as shown in equation (1) — is shown by substituting (1) into the production function:

$$Q = f(L, K)$$
$$= f(L,\ p_1 m_1 + p_2 m_2). \qquad (2)$$

15 pts

(2) shows that price enters into the production function! This violates the whole idea of the production function as coming from technological considerations (engineering considerations) without regard to price or other economic values. In (2), if p_1 or p_2 were to change then the isoquants would move; this contradicts the standard theory that if input prices (like p_1 and p_2) change then the isoquants remain the same and the isocost lines move. So (2) would lead to a completely different theory of the firm than the neoclassical theory presented in class.