A. Mathematics

Note: The graphs for Questions 1, 2, and 3 appear grouped together on only two pages in order to save space, but you will probably want to re-copy each one of them onto the top of its own individual blank page before answering the questions. Question 3 assumes you have already done this.

1. Suppose $f(x)$ is as depicted in Figure 2. Below Figure 2, sketch the graph of the average value of $f(x)$ and sketch a graph of the marginal value of $f(x)$. As always, explain your work.

2. (a) Sketch the average value of the function shown in Figure 1.
   (b) Sketch the marginal value of the function shown in Figure 1.
   (c) Sketch the average value of the function shown in Figure 2.
   (d) Sketch the marginal value of the function shown in Figure 2.

3. In answering the following parts of this question, be as precise as you can be. In particular, use numbers on the X axis of my diagrams in your own answers to help explain what is going on.
   (a) Figure 1 depicts the graph of Y as a function of X. Over what range is this function convex? Over what range is it concave? (If over some range the function is linear, then it is both convex and concave there.)
   (b) Refer again to Figure 1. Sketch the marginal value of Y as a function of X. Also, sketch the average value of Y as a function of X. You may want to make these sketches directly below Figure 1, on the same sheet of paper.
   (c) Refer to Figure 2. For this figure, sketch the average value of Y as a function of X. As in part b, you may want to make this sketch directly below Figure 2.

4. Suppose the average value of $f(x)$ and the marginal value of $f(x)$ appear as in Figure 1 below. Sketch the graph of $f(x)$; remember to explain your answer.

5. Suppose the average value of $f(x)$ and the marginal value of $f(x)$ appear as in Figure 1. Sketch the graph of $f(x)$; a brief explanation will suffice.

6. (a) Sketch the graph of $f(x)$ if the graph of the average of $f(x)$ and the marginal of $f(x)$ look as in Figure 1.

(b) Sketch the graph of $f(x)$ if the graph of the average of $f(x)$ and the marginal of $f(x)$ look as in Figure 2.

(c) On one graph, sketch the average of $f(x)$ and the marginal of $f(x)$ if $f(x)$ looks as in Figure 3.

Do not forget to explain your answers, at least briefly.

7. US federal income taxes in 1990 were calculated in the following way:

Income Range	Marginal Tax Rate
$0 - I_1$	0
$I_1 - I_2$	15%
$I_2 - I_3$	28%
$I_3 - I_4$	33%
$> I_4$	28% (also 28% average tax rate)

In the fall of 1990 Congress changed this to (approximately):

Income Range	Marginal Tax Rate
$0 - I_1$	0
$I_1 - I_2$	15%
$I_2 - I_3$	28%
$> I_3$	31%

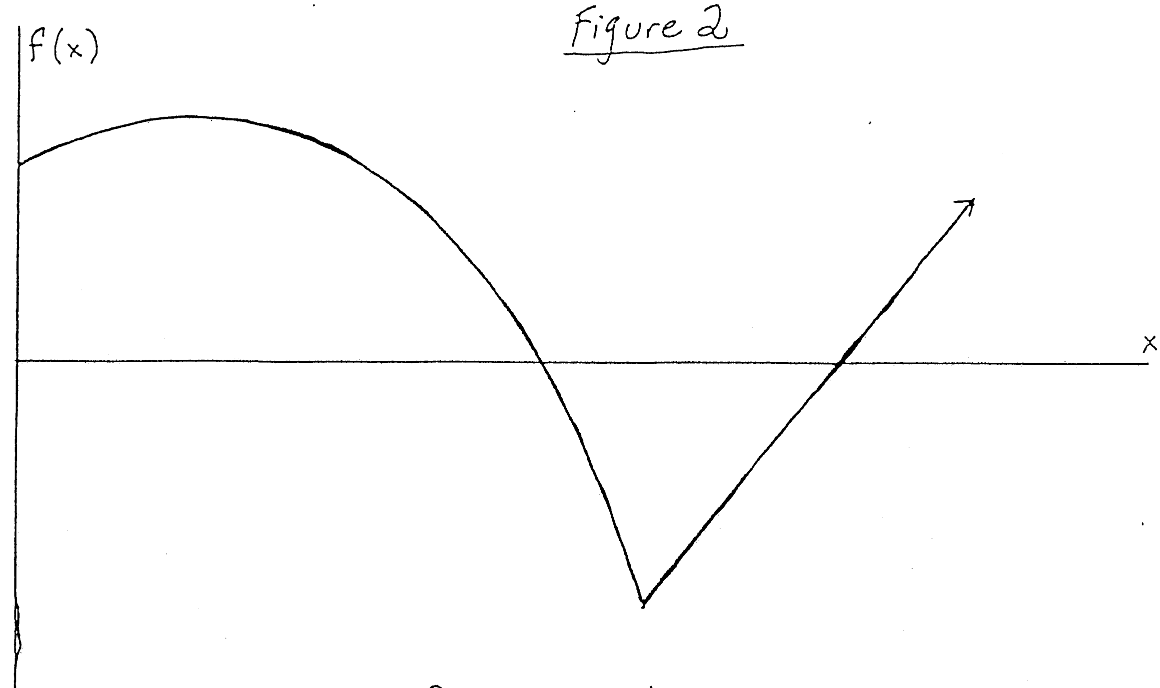
On one graph (not two), sketch the older and newer *total* tax functions (put tax on the vertical axis and income on the horizontal axis). Show on your graph whose taxes go up and whose go down under the newer law.

8. Suppose there are two persons in an economy. One has an income of \$1 and the other has an income of \$2. Let I stand for income. Also, suppose each consumer enjoys utility U where $U = \sqrt{I'}$ and I' is after-tax income.

(a) Suppose the government assesses a "flat tax" of 50% on income, so that $T = 0.5I$ where T is the tax bill. Is the tax progressive, regressive, neither, or can you not tell? Explain your answer with a quick sketch of average tax rates. Also answer: How much money does the government collect? What is the sum of the utilities of the two people?

(b) Suppose the government tax bill is given by $T = 0.3I^2$. (Just ignore income levels higher than $3\frac{1}{3}$.) Is the tax progressive, regressive, neither, or can you not tell? Explain your answer with a quick sketch of average tax rates. Also answer: How much money does the government collect? What is the sum of the utilities of the two people?

Figure 2



Question 1's Fig. 2

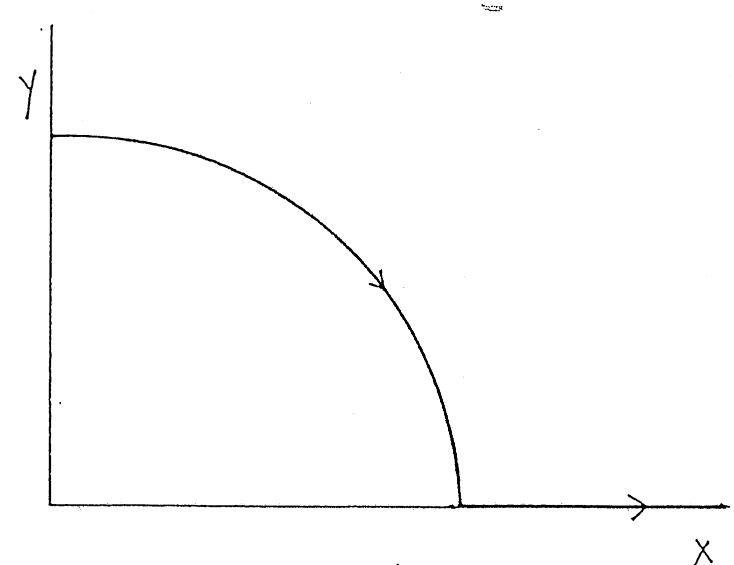


Figure 1

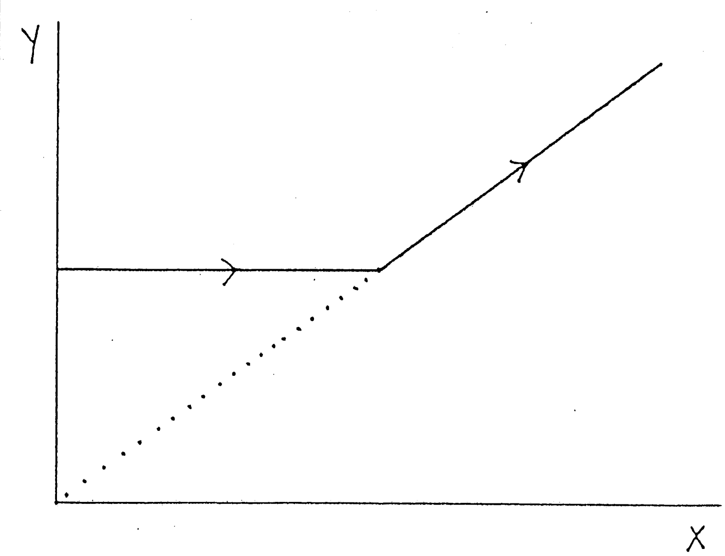
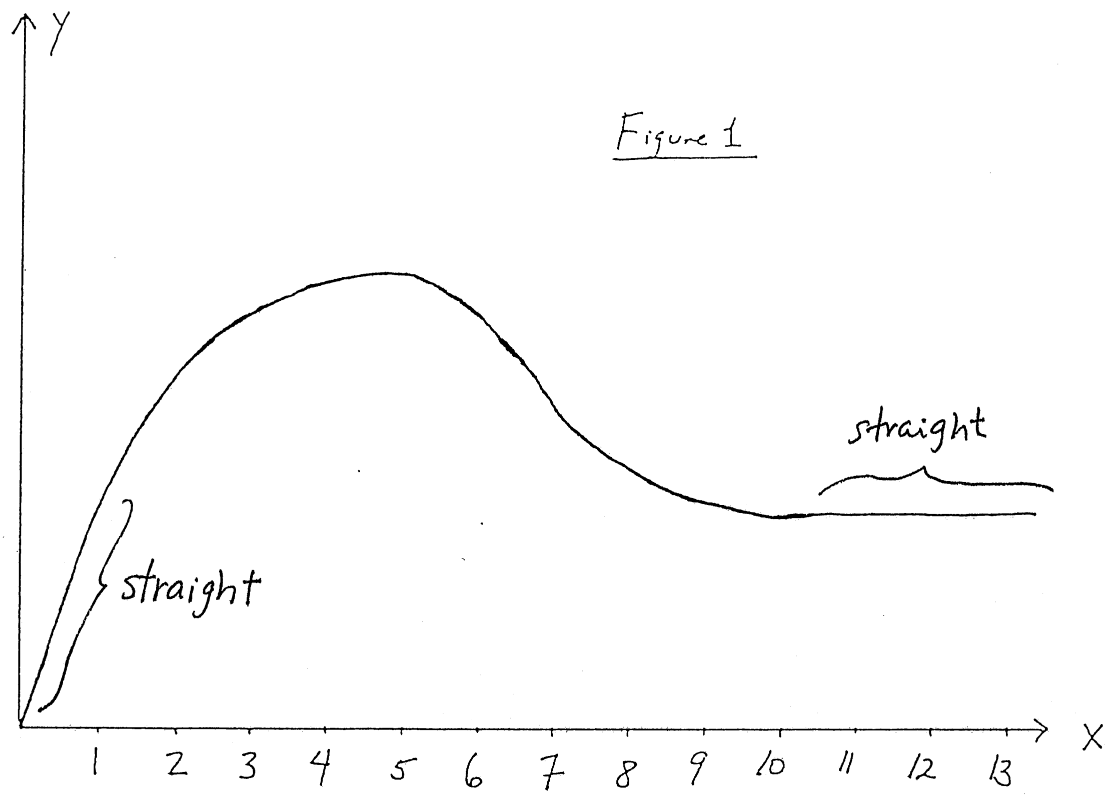
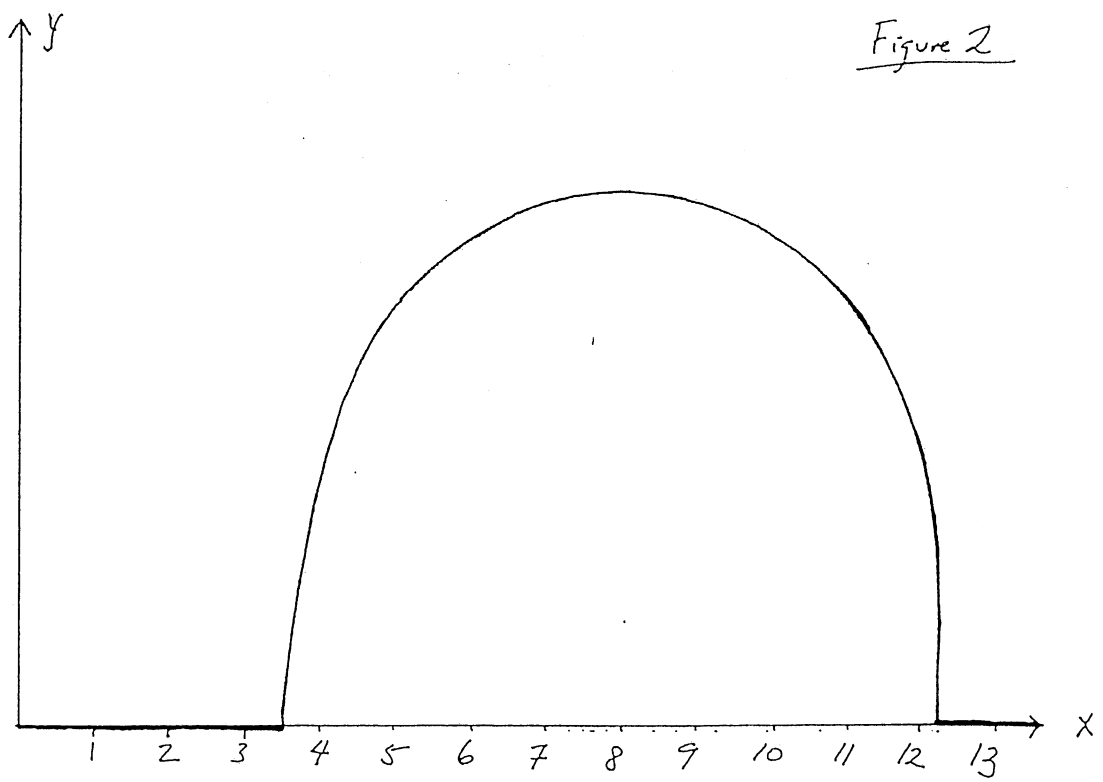


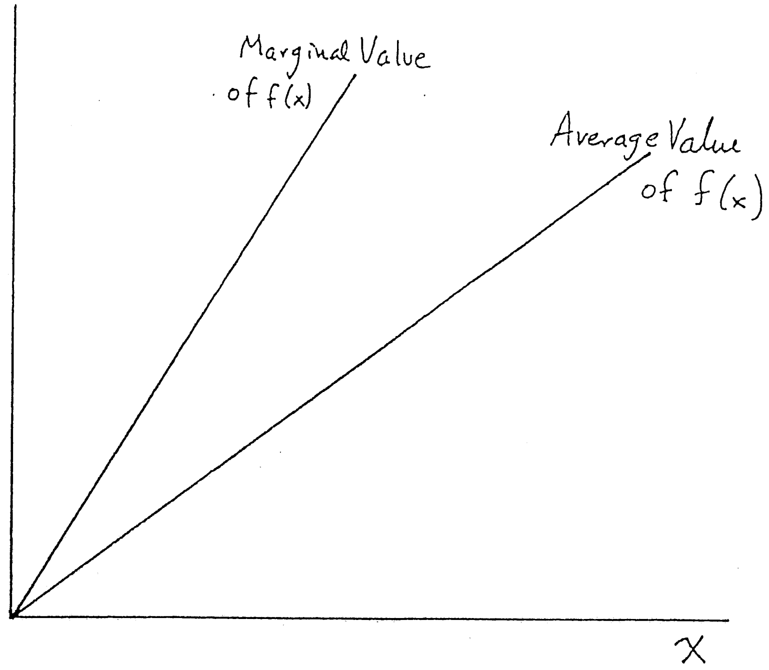
Figure 2

Question 2's Fig. 1, Fig. 2

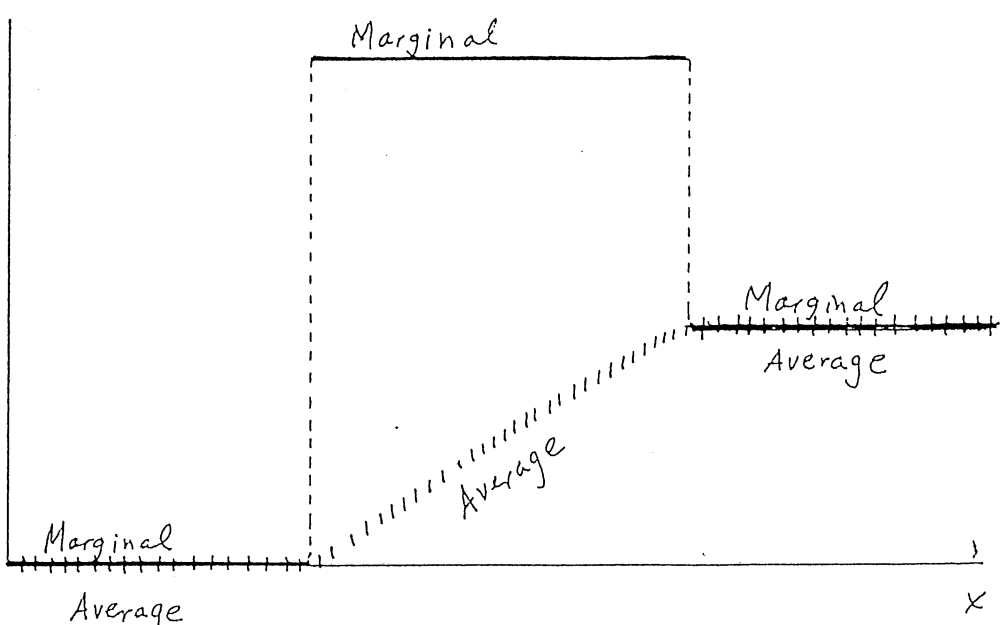




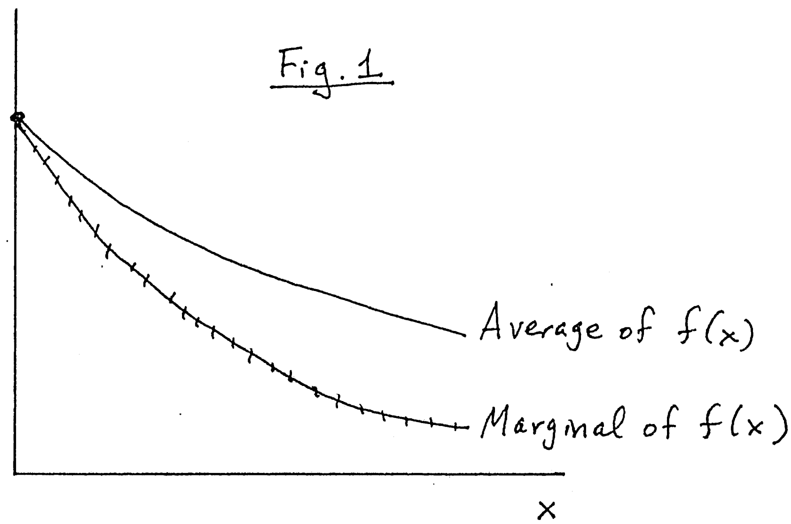
Question 3's Fig. 1 and Fig. 2

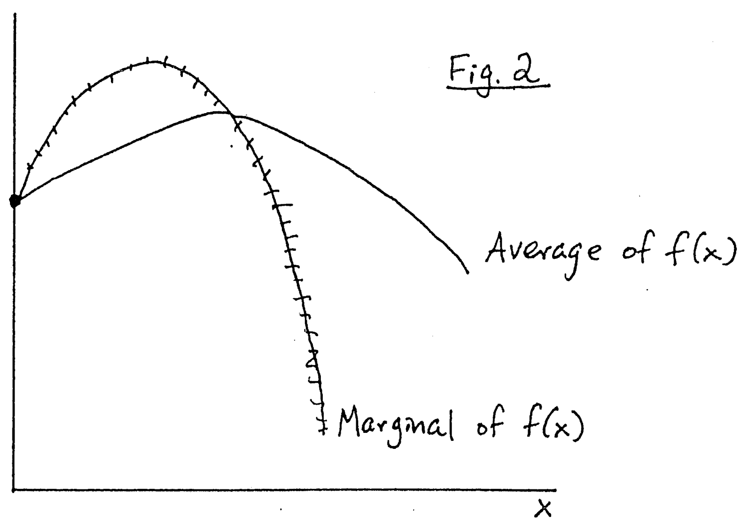


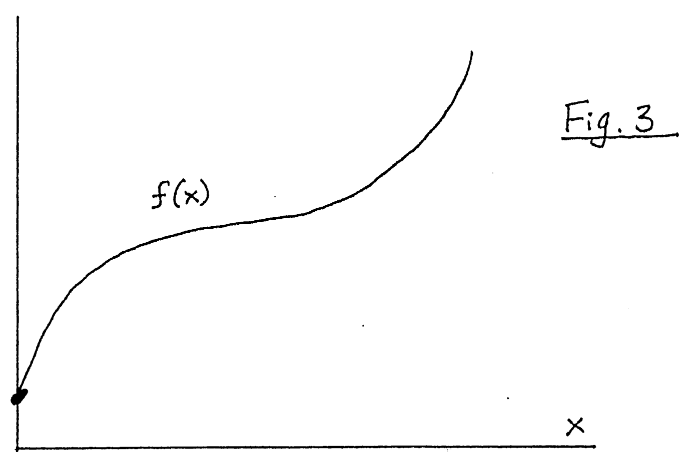
Question 4's Fig 1



Question 5's Figure 1







Question 6's Figs. 1, 2, 3

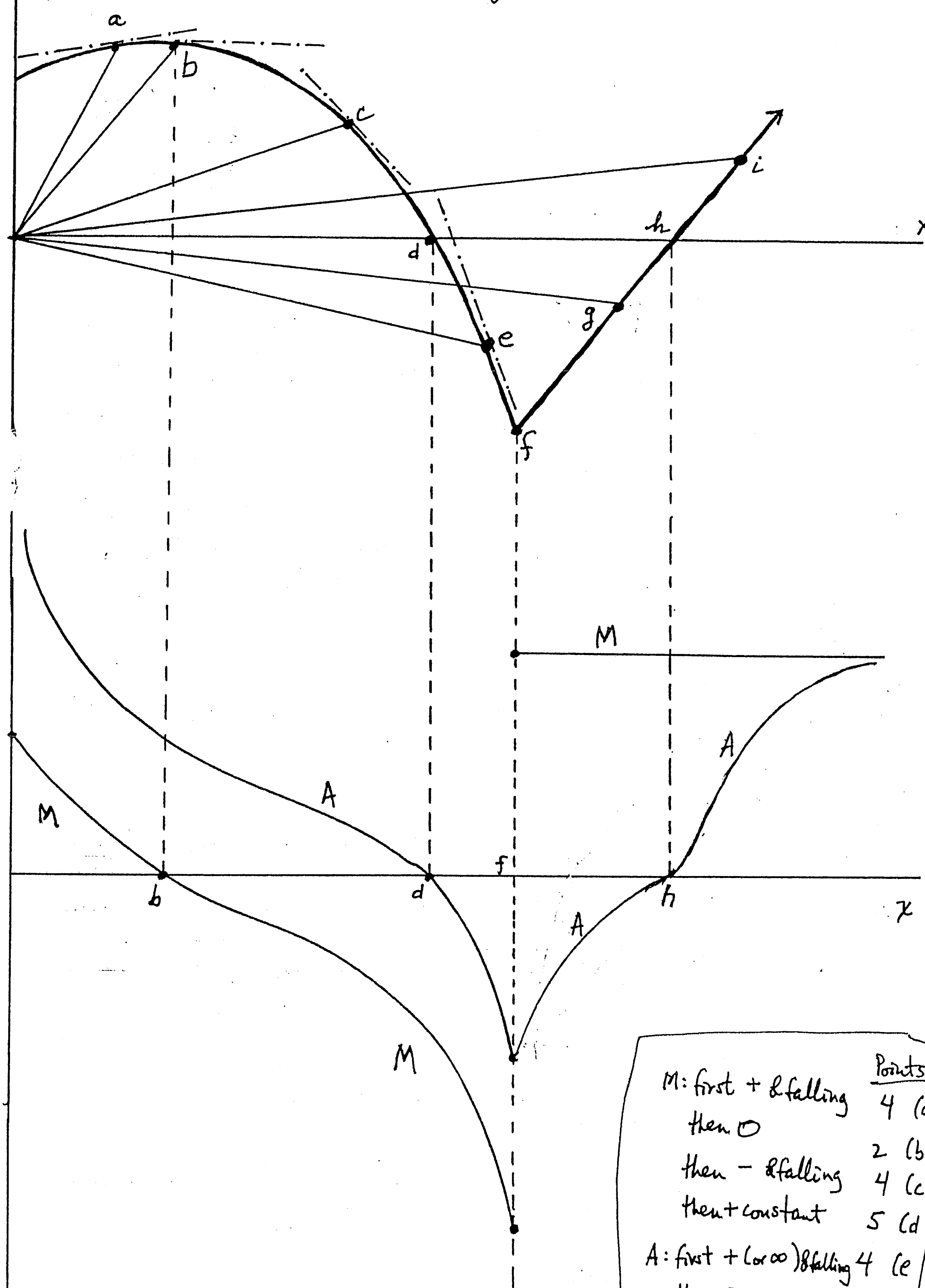
Answers

① Refer to the graph on the next page. "M" refers to the marginal value of f. This is determined by the slope of the tangent lines, denoted —·—·—·, in the top graph. From point a, M is positive (the tangent line is upward-sloping); however, by point b, the tangent line's slope is zero, so M is zero. After that, at points like c and e, M is negative and getting more negative. After point f, however, the function f(x) is a straight, upward-sloping line, so M is a positive constant.

"A" refers to the average value of f. This is indicated by the slope of the solid lines from the origin in the top graph. Going from point a to the left, you can see that this line becomes vertical, so its slope (which is A) becomes infinite. Going from point a to the right, the line from the origin rotates clockwise to b, c, d, e, and finally f; hence on this range, A is falling. At point d, A is zero, and at points e and f it is negative. Beyond point f, A slowly rises again; it is still negative at point g, it is zero at point h, and it is positive at point i.

The question did not ask you to compare M and A, but here is how to do it. At every point to the left of point f, the tangent lines are more negative than the rays from the origin, meaning that M < A. Looking at points g and i, though, the rays from the origin are not as steep as f(x) itself (indeed at g, the slope of the ray from the origin is still negative). So beyond point f, A < M.

Figure 2



		Points
M: first + & falling	4	(a
then 0	2	(b
then − & falling	4	(c
then + constant	5	(d
A: first + (or ∞) & falling	4	(e
then 0	1	(f
then − & falling	4	(g
then − & rising	4	(h
then 0	1	(i
then + & rising	4	(j

(2) a) At point 1, the chord from the origin to the point is vertical, so its slope is infinity and so is the average. At point 2, the slope of the chord is roughly $+1$. At point 3, it is close to zero, and at point four it is zero.

b) At point 1 the tangent line is flat, so the marginal value is zero. At point 2 the tangent line has slope ≈ -1. At point 3 the tangent line is almost vertical; so its slope is almost $-\infty$, as must be the marginal value of the function. At point 4 and beyond, the slope of the tangent line is flat, so the marginal value is zero.

c) Between points 1 and 4 the average decreases from $+\infty$ to some number such as $+1$. This is because chords drawn from points 1, 2, 3, and 4 to the origin start out to be vertical, but then have ever shallower slopes. At points 4, 5, and beyond, the slope of chords from the origin is the same as the slope from the origin to point 4, so the average does not change.

d) The graph is made up of two linear portions, and linear portions always have a constant slope. From points 1 to 4, that slope is zero (since the function is flat); beyond point 4, the slope is constant but strictly positive.

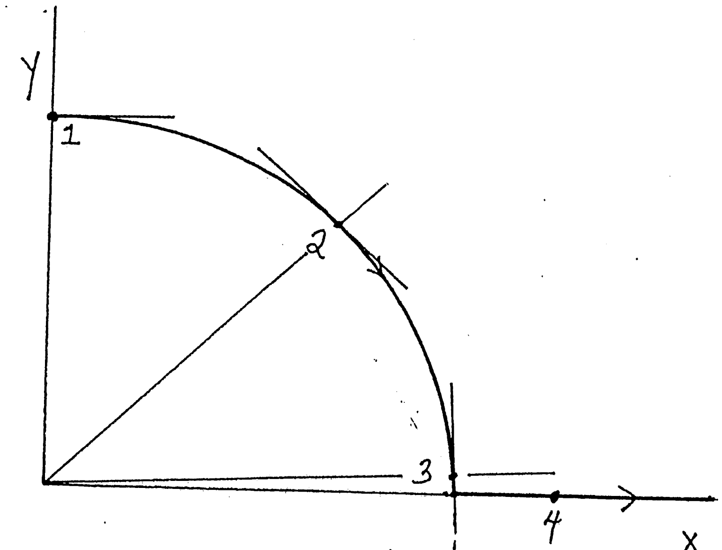
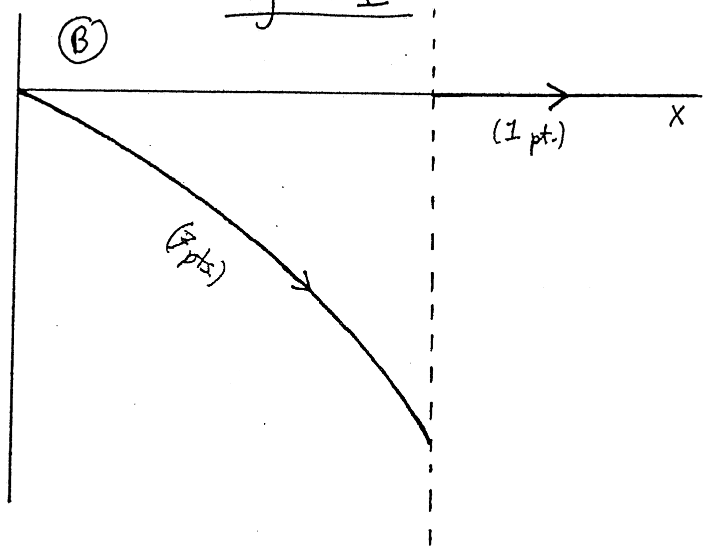
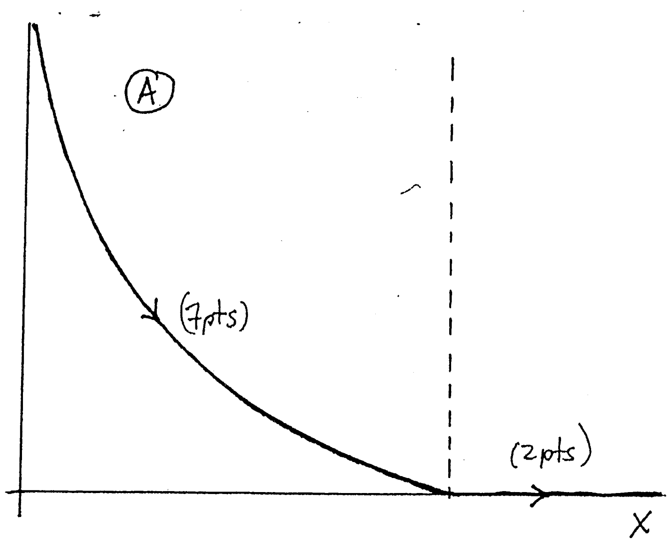


Figure 1





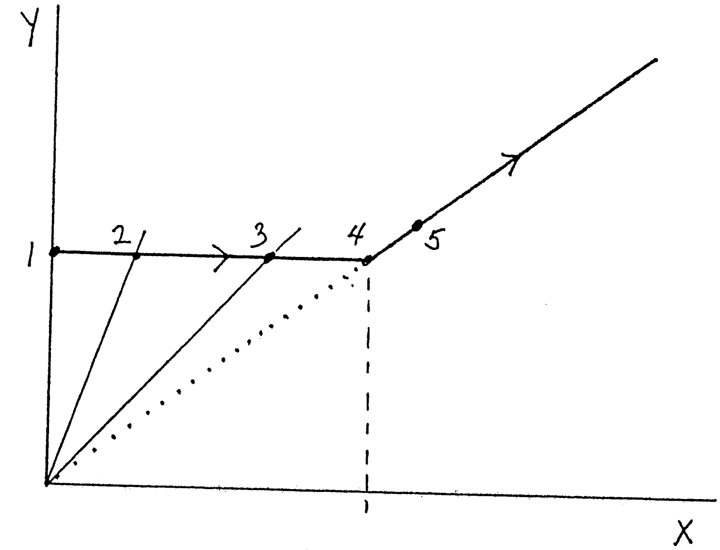
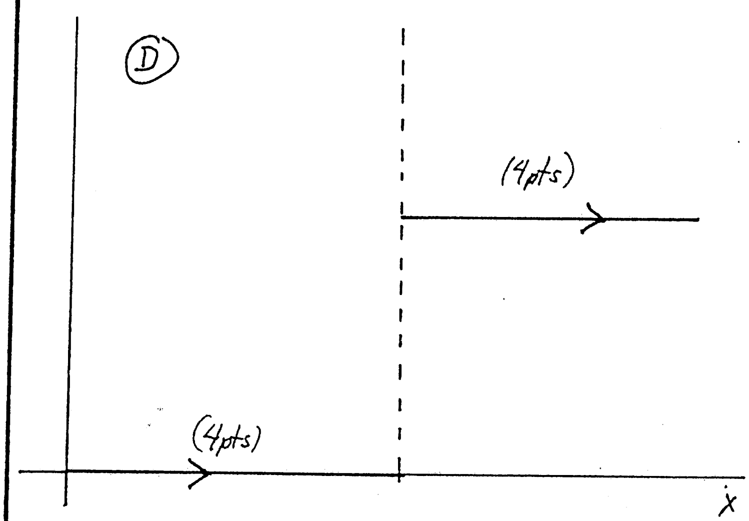
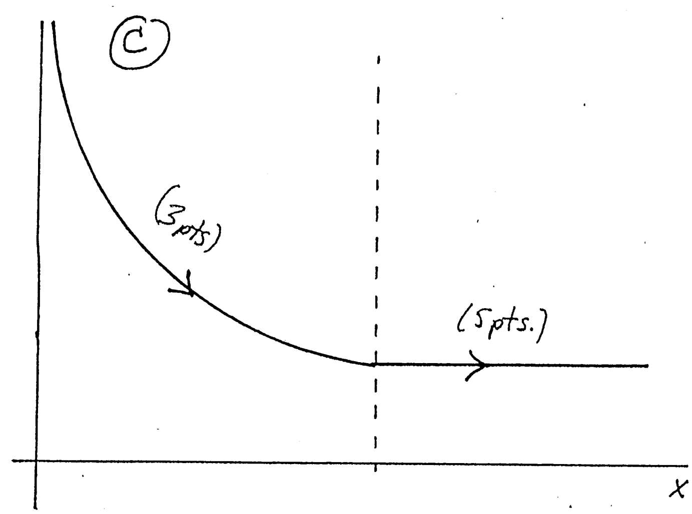


Figure 2





③ a. $0 \le x \le 1$: the function is linear, but this was hard to see, so it's worth 0 points

(2) $1 \le x \le 7$: the function is concave (like an upside-down "U"; negative second derivative)

(2) $7 \le x \le 10$: the function is convex (like a "U"; positive second derivative)

(2) $x > 10$: the function is linear

b. (2) $0 \le x \le 4\frac{1}{2}$: marginal value is positive since the slope of the tangent line (drawn at $x = 2$ for an example) is positive

(2) $x = 4\frac{1}{2}$: marginal value is zero since tangent line (drawn) is flat

(2) $4\frac{1}{2} < x \le 10$: marginal value is negative since the tangent line (drawn at $x = 7$ for an example) slopes downward

(2) $x > 10$: marginal value is zero since the tangent line, which coincides with the function itself here, is flat

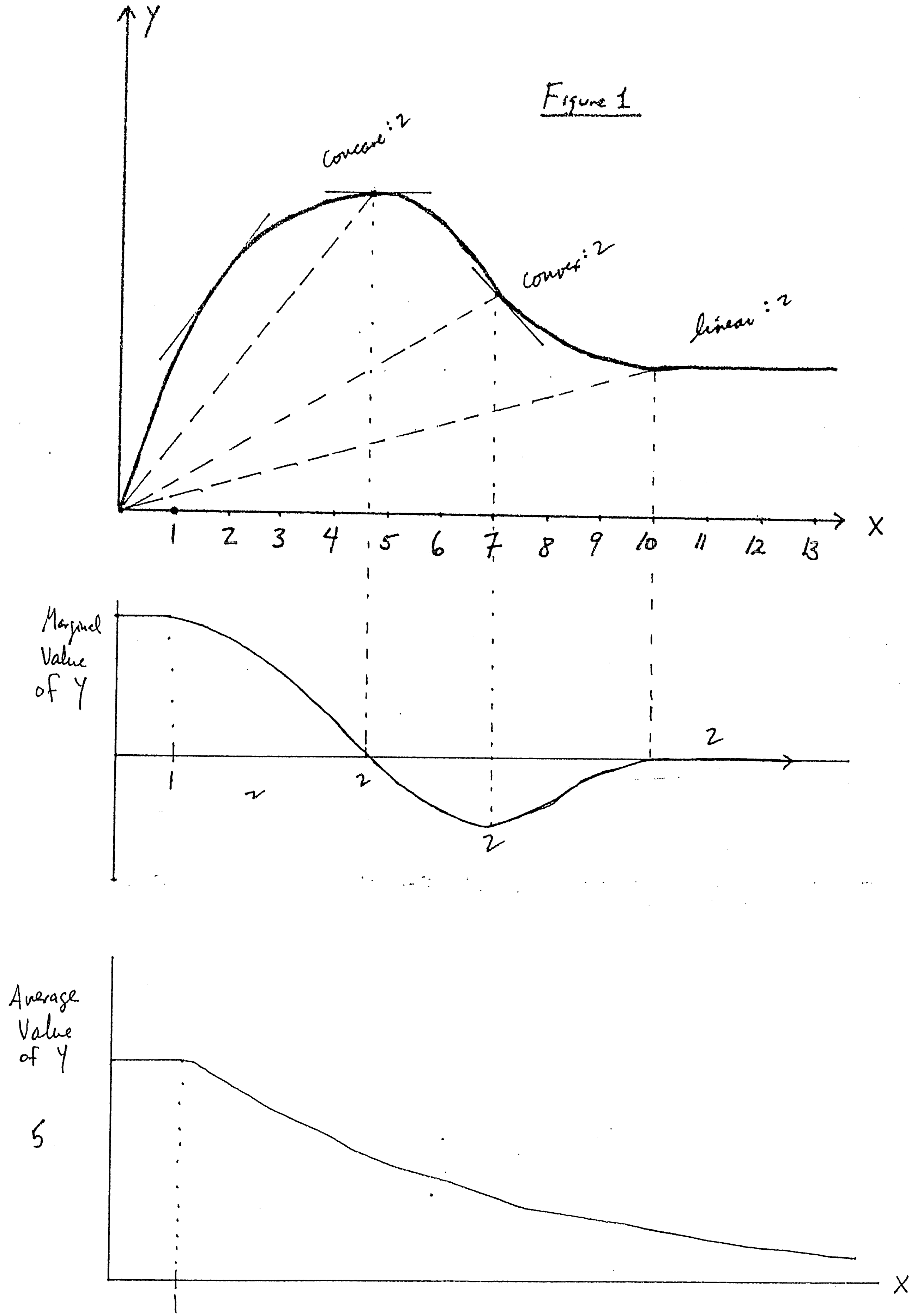
(5) Chords drawn from the origin to the function (at, for example, $x = 4\frac{1}{2}$, $x = 7$, and $x = 10$) get flatter as x increases, but they all have a positive slope. So the average (which is the slope of such chords) is positive but decreasing.

c. (1) $0 \le x < 3\frac{1}{2}$: Average $= 0$ since chord from origin to the function is flat

(2) $3\frac{1}{2} \le x < 5$: Average is positive and rising because the slope of chords like 1 and 2 is positive and increasing with x

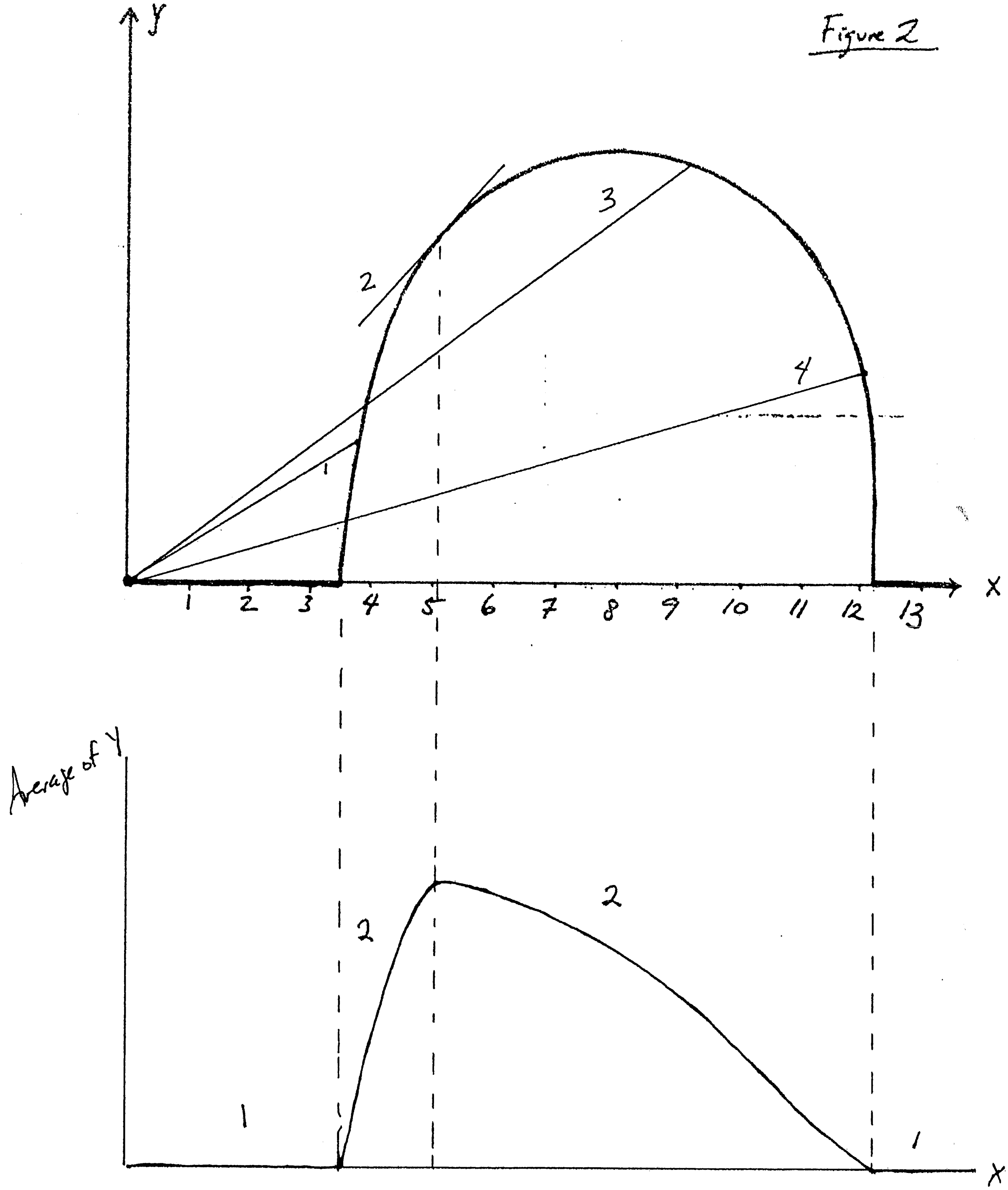
(2) $5 \le x \le 12\frac{1}{2}$: From chords like 3 and 4, the average is positive but falling (the chords are getting flatter).

(1) $x > 12\frac{1}{2}$: Average $= 0$ since chords are flat.

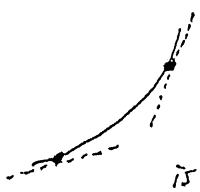


Y
Figure 1
concave : 2
convex : 2
linear : 2
X
1 2 3 4 5 6 7 8 9 10 11 12 13
Marginal Value of Y
2
2
2
2
1
Average Value of Y
5
X

Figure 2

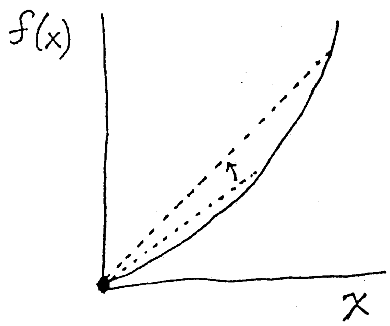


(4) The rising marginal value of $f(x)$ indicates that the slope of $f(x)$ is rising, and the fact that the marginal value is positive means that the slope of $f(x)$ is positive (that is, that $f(x)$ is an increasing function). Hence $f(x)$ must be shaped like



so that the slope is positive and rising. 10 pts The only question remaining is whether f is positive or negative or sometimes is zero. This is solved by looking at the average of $f(x)$, which is zero at $x = 0$ and is positive for $x > 0$. If the average of $f(x)$ is positive and x is positive then $f(x)$ must be positive, since $\text{Avg.} = \frac{f(x)}{x}$. 7 pts

Therefore one gets a graph like the one below. The fact that the dotted lines in this graph

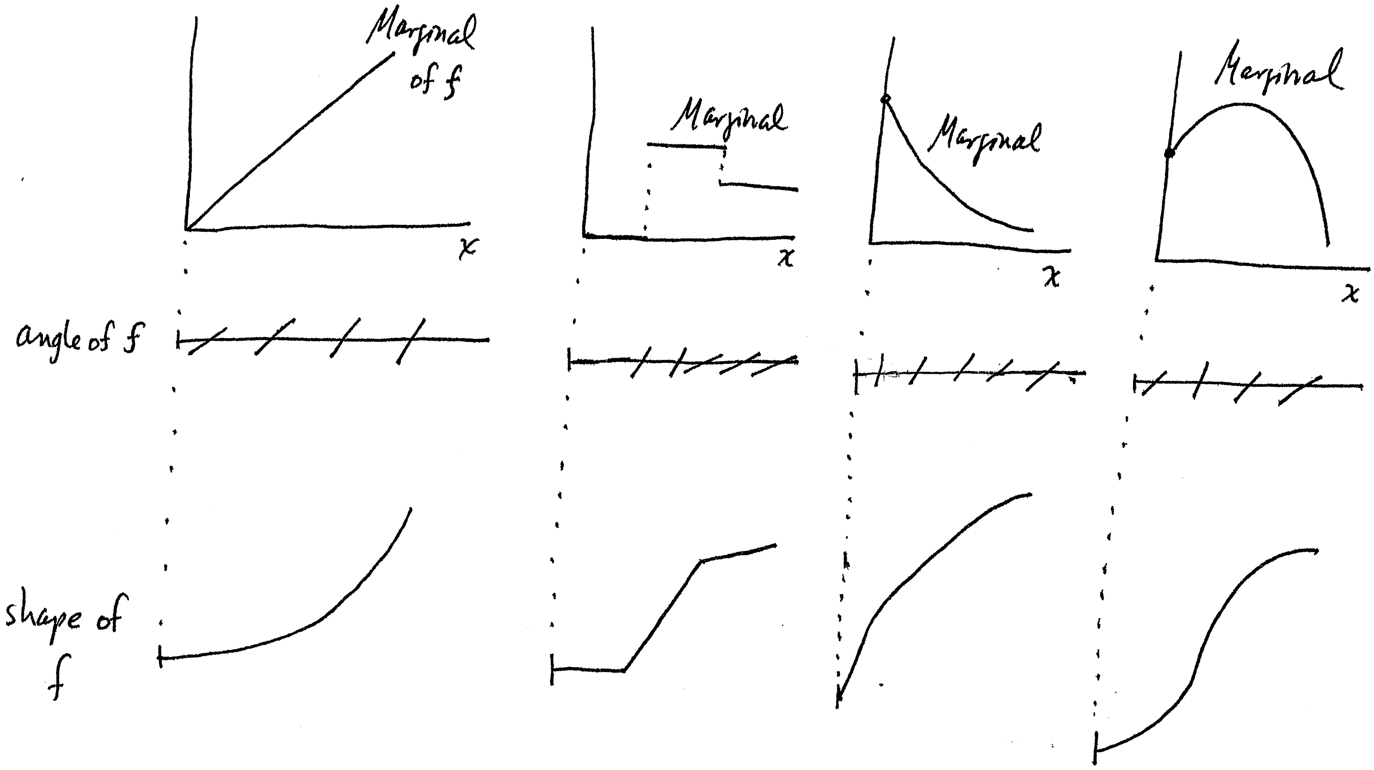


pivot counterclockwise as x increases shows that the average of $f(x)$ is increasing. 7 pts In fact, one can deduce the basic shape of $f(x)$ just by having the average of $f(x)$.

Basic Shape: 9 pts

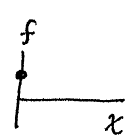
Note on Answers to Qu. 4, 5, and 6

An alternative approach is



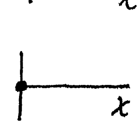
	Question 4	Question 5	Question 6	Question 6
Average at X = 0 is :	zero	zero	positive	positive

Then use : $f(0) > 0$



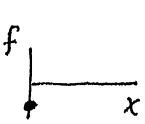
$\Rightarrow$ average at $X = 0$ is $+\infty$

$f(0) = 0$



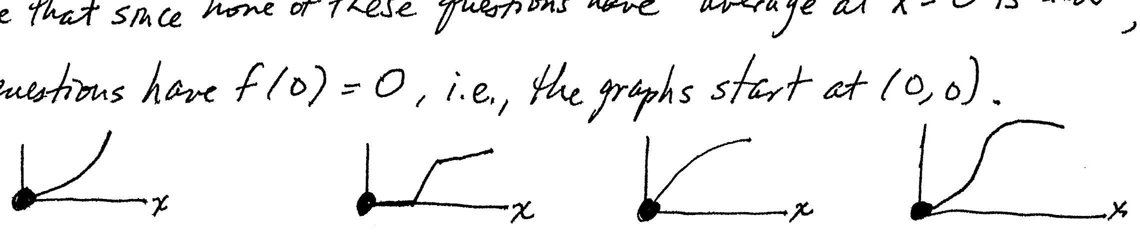
$\Rightarrow$ average at $X = 0$ is not obvious

$f(0) < 0$



$\Rightarrow$ average at $X = 0$ is $-\infty$

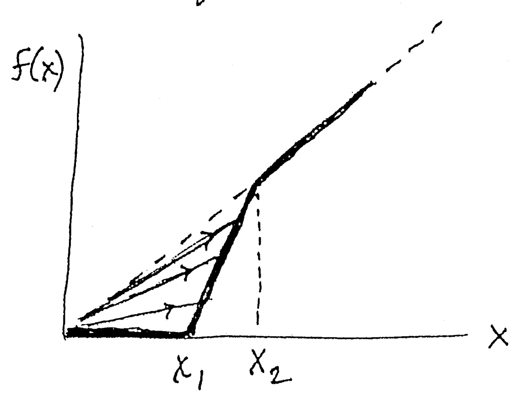
to conclude that since none of these questions have "average at $X = 0$ is $\pm\infty$", all these questions have $f(0) = 0$, i.e., the graphs start at $(0, 0)$.



(5) Let the point at which the marginal jumps up from zero be x_1, and let the point where it drops down (but remains positive) be x_2.

First look at the marginal curve. Whenever it is constant (flat), that means that the slope of $f(x)$ is constant; hence $f(x)$ is linear then. One can see that $f(x)$ must be composed of three line segments. The first line segment must be flat (marginal = 0), the second must be steeply sloped (high marginal value), and the last must have a more shallow (but still positive) slope. So the segments are shaped —, /, and ⁄.

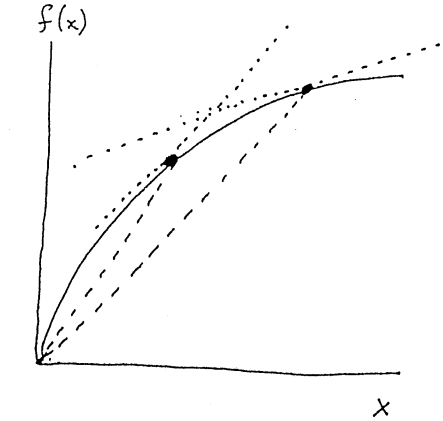
Now looking at the average, one can see that the average curve has no discontinuities (gaps). Therefore $f(x)$ has no gaps either [think about this a little], so $f(x)$ looks like . Furthermore, the average is zero first, then becomes positive and rises slowly, but then becomes constant. Remember that the average is constant on a line drawn from the origin. So we have:



(The solid lines with arrows have slopes equal to the average of $f(x)$ for x where the arrows touch $f(x)$.)

Correct Interpretation of:		$(0, x_1)$	(x_1, x_2)	(x_2, ∞)
	Marginal	5 pts.	5 pts.	5 pts.
	Average	5 "	5 "	5 "

⑥ a)



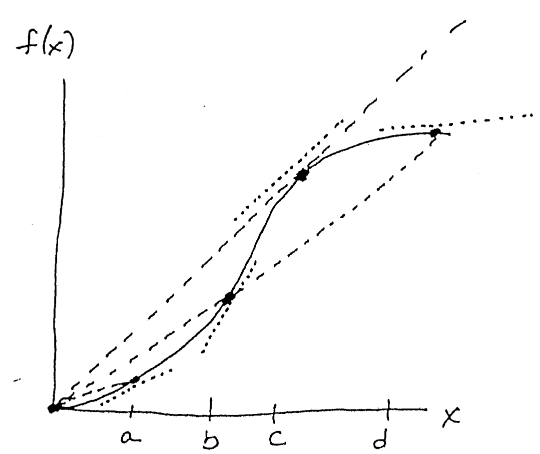
slope of --- lines is the average; as x↑ the second --- line is flatter than the first, so average falls, as in Fig 1.

slope of ..... lines is the marginal; as x↑ the second ..... line is flatter than the first, so marginal falls, as in Fig. 1.

Also, at each black dot, the --- line is steeper than the ..... line, so average > marginal, as in Fig. 1.

Note: $f(0) = 0$, else the average value at zero $= \frac{f(0)}{0} = \frac{\text{something not zero}}{0} = \pm\infty$; in Fig. 1, the average value at zero is not $\pm\infty$.

b)

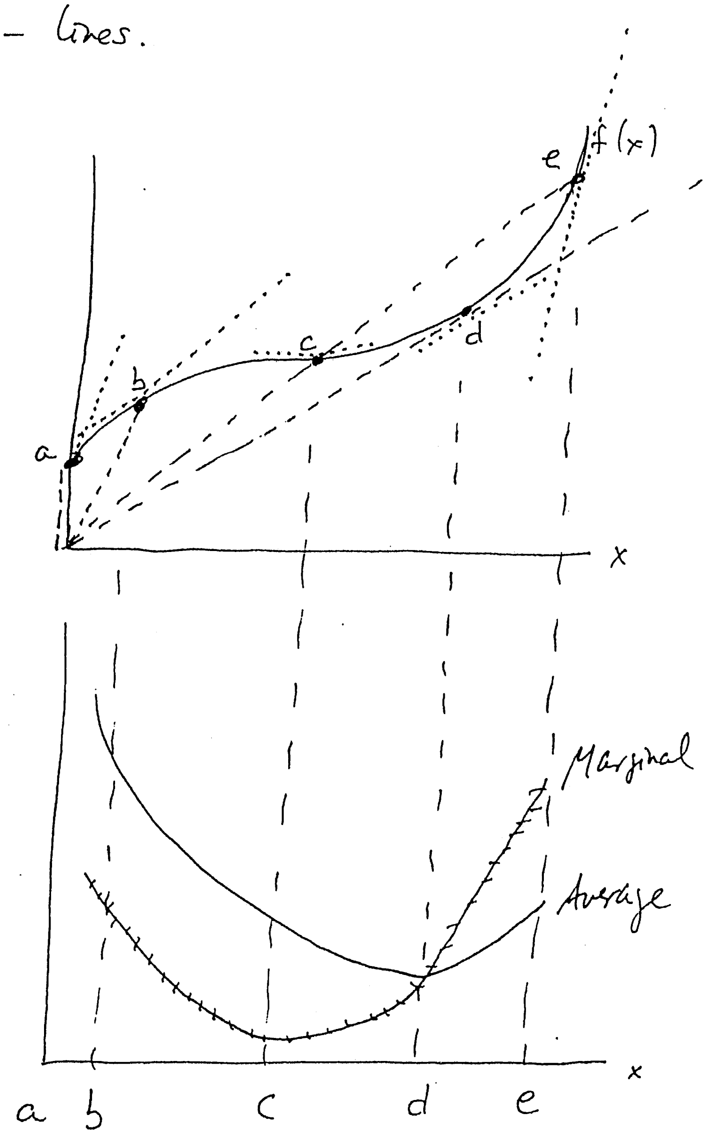


Average (slope of --- lines):
a→b it ↑,
b→c it ↑,
c→d it ↓.

Marginal (slope of ..... lines):
a→b it ↑,
b→c it ↓,
c→d it ↓.

At point c, average = marginal. Before then, marginal > average since ..... lines are steeper than --- lines. After (to the right of) point c, marginal < average since .... lines are flatter than --- lines.

c)



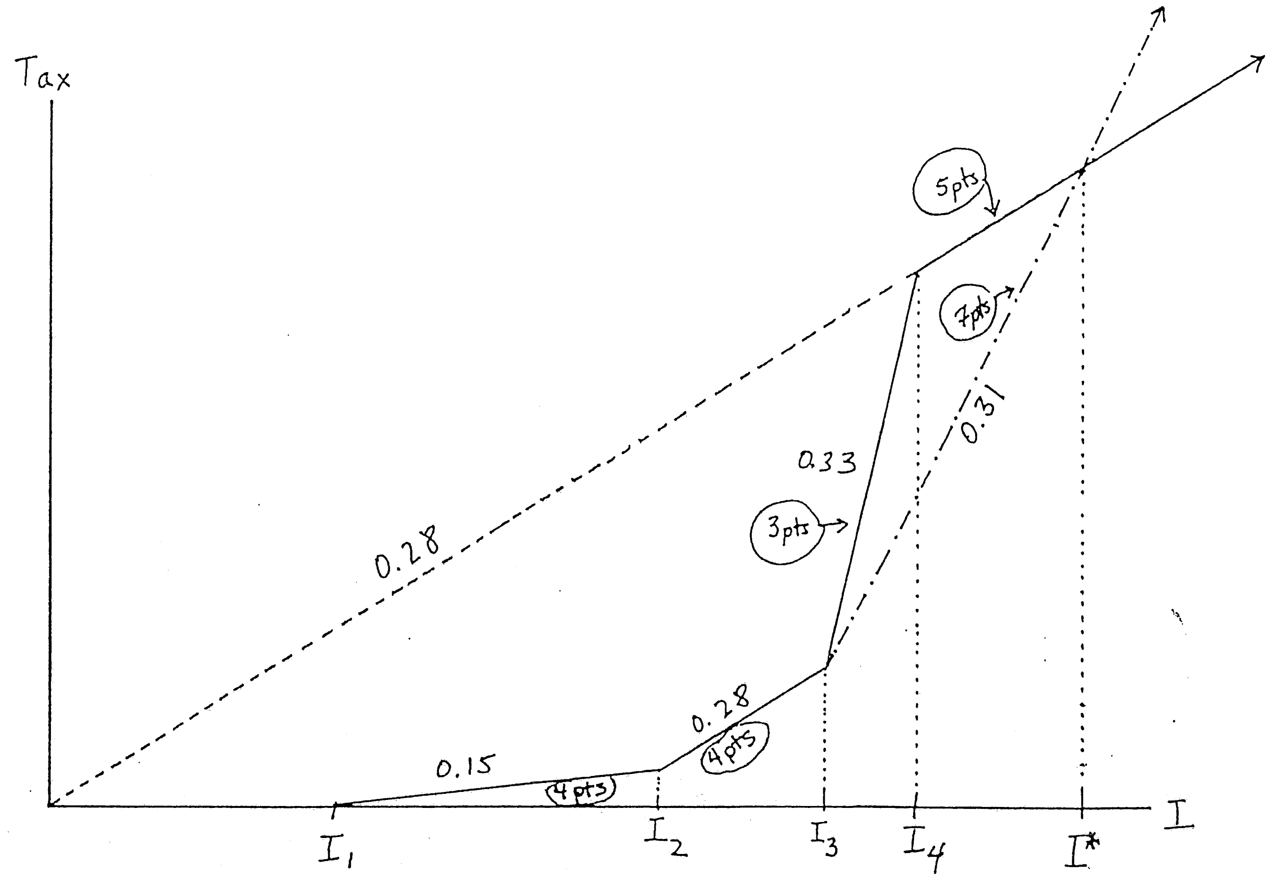
3 points each for a, b, and c

Average falls to point d, then rises to point e. At point a, average is ∞ since the dashed line is vertical.

Marginal (.... lines) falls from a to b and b to c, then rises to d (where it equals average) and rises more to e.

Before point d, marginal < average; afterwards it's the other way.

⑦

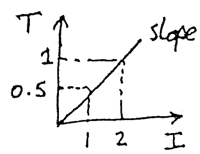


The solid line is the old tax structure. The graph has been drawn so that for $I > I_4$, both average and marginal taxes are 0.28. The solid line between I_2 and I_3 is parallel to the dotted line with the 0.28 slope.

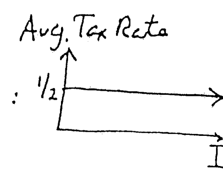
The new tax structure is shown by the —·—·—· line for $I > I_3$; for $I < I_3$, the new and old taxes are the (2pts) same. People with income (4pts) between I_3 and I^* pay less taxes under the new law; people with income (4pts) larger than I^* pay more taxes under the new law.

8

	Person 1	Person 2		
I	1	2		
Taxes	.5	1	→ sum = 1.5, govt. revenue; 1 point each (3 total)	50% tax
After-Tax Income	.5	1	2pts each	
Utility	$\sqrt{\frac{1}{2}}$	1	→ sum : $1 + \sqrt{\frac{1}{2}} \doteq 1.707$; 1pt	
Taxes	0.3	0.3(4) = 1.2	→ sum = 1.5, govt. revenue; 1pt each	$0.3I^2$ tax
After-Tax Income	0.7	0.8	2pts each	
Utility	$\sqrt{0.7}$	$\sqrt{0.8}$	sum : $\sqrt{0.7} + \sqrt{0.8} = 1.731$	

a) 

2pts

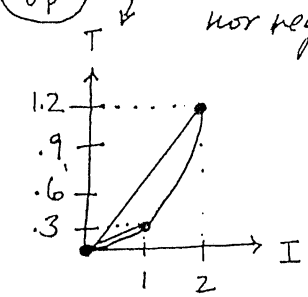
The average tax rate is constant at $\frac{1}{2}$: 

4pts

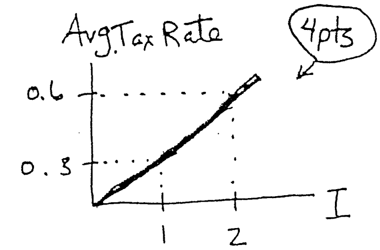
The reason is that any line drawn from the origin to any point on the tax function lies directly on the tax function; therefore, any such line has a slope of ½.

Since the average tax rate is constant, the tax is neither progressive nor regressive (you could say it is neutral).

2pts

b) 

The curved line is the tax function. The small straight line's slope is the average tax rate at I = 1 (you did not have to pick specific numbers like this). The long straight line's slope is the average tax rate at I = 2. The long straight line has a steeper slope than the short straight line, so the average tax rate must be larger at I = 2 than at $I = 1$. So:



(It is not important to have any numbers, nor to have a straight line, nor to start at (0,0).) Since the average tax rate is increasing, the tax is progressive.

Note that the government collects the same revenue under both schemes, but the sum of utilities is higher with the progressive tax than with the flat tax. (This does not necessarily mean that progressive taxes are better than flat taxes!)