Initial Yield as a Forecast of Constant-Duration or Constant-Maturity Bond Portfolio Return near Twice Duration

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Outline

1. Leibowitz' Framework
2. Extensions
3. Empirical Results

Background

- Bond duration, sense # 1: number of years until return is guaranteed to equal initial yield
- Constant-maturity bond portfolios
- *There is a silver lining to rising [interest] rates. If your time horizon is longer than the duration of the bond funds you are invested in, you actually want interest rates to rise. [McNabb 2014]*
- Langeteig, Leibowitz and Kogelman (1990)

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Basic Assumptions

- Yield in period $t > 1$, denoted Y_t, evolves as $Y_t = Y_{t-1} + \Delta Y_{t-1}$.
- Suppose that yield changes occur at the end of every period, when the bond's duration has shrunken to $D - 1$, and use the "Babcock Approximation" that

$$\text{return in period } t \approx R_t = Y_t - (D-1)\,\Delta Y_t \quad (1)$$

for $t = 1, 2, \ldots, N$, where D needs no subscript because it is assumed constant. (Bond duration, sense #2.)

- Assume $Y_t = Y_1 + (t-1)\,\Delta Y$, the "linear path assumption," for $t \in [1, N]$, with $\Delta Y = (Y_N - Y_1)/N$.
- This is: not a first-order Taylor Series approximation; cannot hold in the long run; is implausible in the very short run.
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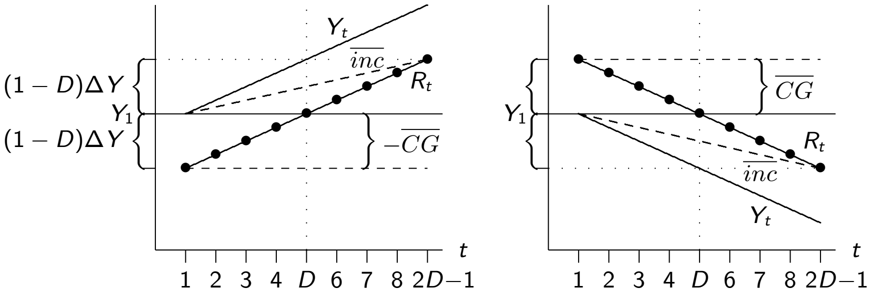
Figure 1: when will Avg. Return equal Y_1?



Average return up to time t, $\bar{R}_a$, is not shown; it would be a line joining the left-most bullet on the R_t line with the point $(2D-1, Y_1)$. (Ignore $\overline{inc}$ and $\overline{CG}$ for the moment.)

Proposition 1

Proposition 1. *If yields are linear in time, returns are approximated by (1), and twice duration is an integer, then the number of periods "N_a" which will make the arithmetic mean return equal to the initial yield is*

$$N_a = 2D - 1\,. \tag{3}$$

This yield path satisfies $R_t > -1$ for all $t \in [1, N_a]$ if

$$(D-1)\cdot|\Delta Y| - Y_1 < 1\,. \tag{4}$$

This yield path satisfies $Y_t > 0$ for all $t \in [1, N_a]$ if

$$Y_1 > 0 \quad \text{when } \Delta Y > 0 \text{ and} \tag{5}$$

$$Y_1 + 2(D-1)\Delta Y > 0 \quad \text{when } \Delta Y < 0. \tag{6}$$

Leibowitz's proof

The arithmetic mean return is

$$\bar{R}_a = \frac{1}{N_a}\sum_{t=1}^{N_a} R_t = \frac{1}{N_a}\sum_{t=1}^{N_a}\Big(Y_1 + (t-D)\,\Delta Y\Big)\,. \tag{7}$$

$$1 = \frac{1}{N_a}\sum_{t=1}^{N_a}\left(1 + (t-D)\frac{\Delta Y}{Y_1}\right) \tag{8}$$

$$= \frac{1}{N_a}\sum_{t=1}^{N_a}\left(1 - D\,\frac{\Delta Y}{Y_1}\right) + \frac{1}{N_a}\,\frac{\Delta Y}{Y_1}\sum_{t=1}^{N_a} t$$

$$= 1 - D\,\frac{\Delta Y}{Y_1} + \frac{1}{N_a}\,\frac{\Delta Y}{Y_1}\cdot\frac{N_a}{2}(N_a+1) \tag{9}$$

$$= 1 + \left(\frac{N_a+1}{2} - D\right)\frac{\Delta Y}{Y_1}\,,$$

so $(N_a+1)/2 = D$ and (3) follows.

A More Intuitive Proof

If D is an integer then $\sum_{t=1}^{2D-1} f(t)$ for an arbitrary function f can be reordered as

$$f(D) + \sum_{s=1}^{D-1} [f(D-s) + f(D+s)] ;$$

applying this reordering to the right-hand side of (7) (after substituting $2D-1$ for N_a) turns it into

$$\frac{[Y_1 + (D-D)\Delta Y] + \sum_{s=1}^{D-1} \{[Y_1 + (D-s-D)\Delta Y] + [Y_1 + (D+s-D)\Delta Y]\}}{2D-1}$$

$$= \frac{Y_1 + \sum_{s=1}^{D-1} \{Y_1 + Y_1\}}{2D-1} = \frac{1 + \sum_{s=1}^{D-1} 2}{2D-1} \cdot Y_1 = \frac{2D-1}{2D-1} Y_1 = Y_1 ,$$

confirming the conjecture when D is an integer.

The More Intuitive Proof, continued

If D is not an integer but $2D$ is an integer then $\sum_{t=1}^{2D-1} f(t)$ can be reordered as

$$\sum_{s=1}^{D-1/2} \left[f(D+{}^1\!/\!_2-s) + f(D-{}^1\!/\!_2+s) \right];$$

applying this reordering to the right-hand side of (7) (after substituting $2D-1$ for N_a) turns it into

$$\frac{\sum_{s=1}^{D-1/2}\{[Y_1 + (D+\frac{1}{2}-s-D)\Delta Y] + [Y_1 + (D-\frac{1}{2}+s-D)\Delta Y]\}}{2D-1}$$

$$= \frac{\sum_{s=1}^{D-1/2}\{Y_1 + Y_1\}}{2D-1} = \frac{\sum_{s=1}^{D-1/2} 2}{2D-1} \cdot Y_1 = \frac{2D-1}{2D-1} Y_1 = Y_1 \,,$$

confirming the conjecture and finishing this proof of (3).

Expressions for $\bar{R}_a(N)$

Proposition 2. *Assuming returns are approximated by (1),*

$$\bar{R}_a(N) = \frac{(Y_1 - Y_{N+1}) \cdot (D-1)}{N} + \frac{1}{N}\sum_{t=1}^{N} Y_t \qquad (10)$$

at date N.

Corollary 1. *Assuming returns are approximated by (1) and yields are linear in time,*

$$\bar{R}_a(N) = \Delta Y(1-D) + \left(Y_1 + \frac{N-1}{2}\,\Delta Y\right) \qquad (11)$$

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Corollary 2. *Assuming returns are approximated by (1),*

$$\lim_{N\to\infty} \bar{R}_a(N) = \lim_{N\to\infty} \frac{\sum_{t=1}^{N} Y_t}{N} \quad \text{if and only if} \quad \lim_{N\to\infty} \frac{Y_{N+1}}{N} = 0\,. \qquad (12)$$

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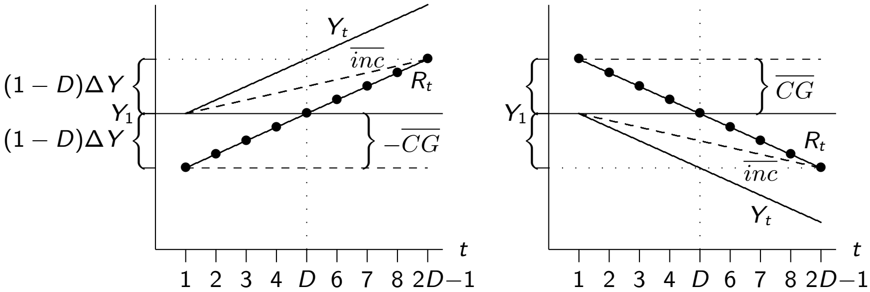

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Quadratic Paths

Give intuition, then. . .

Proposition 3. *If yields follow the quadratic function* $at^2 + bt + c$ *where* t *is time, if returns are approximated by (1), and if twice duration is an integer, then over a time period of length* $N_a = 2D - 1$ *the "forecast error"*

$$\bar{R}_a - Y_1 = \frac{2a}{3}(1 - D)\,D\,. \tag{13}$$

Proof of Proposition 3

Proof.

- $\bar{R}_a = (1/N_a)\sum_{t=1}^{N_a}[Y_t - (D-1)\Delta Y_t]$.
- Writing Y_t as $at^2 + bt + c$, one has:
- Y_t is convex if $a > 0$ and concave if $a < 0$;
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- $\Delta Y_t = Y_{t+1} - Y_t = 2at + a + b$, so
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- It can then be shown, either by tedious calculations using at one point $\sum_{t=1}^{T} t^2 = T^3/3 + T^2/2 + T/6$ and, as in (9), $\sum_{t=1}^{T} t = T(T+1)/2$,
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∎

Convexity of the Yield Path

Corollary. *Under the conditions of Proposition 3, forecast error* $\bar{R}_a - Y_1$ *is negative if the yield path is convex and positive if the yield path is concave.*

Proof. Since $D > 1$, the right-hand side of (13) has the opposite sign of a. ∎

Table 1

	A	G
1. $\Delta Y > 0$ and $2D$ an integer	$N_a = 2D - 1$	$2D - 1 < N_g^+ < \infty$
2. $\Delta Y > 0$ and $2D$ not an integer	$N_a^+ = \lceil 2D - 1 \rceil$	$2\lfloor D \rfloor - 1 < N_g^+ < \infty$
3. $\Delta Y < 0$ and $2D$ an integer	$N_a = 2D - 1$	$D < N_g^+ \leq 2D - 1$
4. $\Delta Y < 0$ and $2D$ not an integer	$N_a^+ = \lceil 2D - 1 \rceil$	$D < N_g^+ \leq 2\lceil D \rceil - 1$

Table: 1: Theoretical Results for Linear Yield Paths

Finish Column A.

Arithmetic and (Financial) Geometric Means

Returns: -50%, $+50\%$	Arithmetic Mean of returns	Geometric Mean of returns
continuously compounded	Mean Ret $= -.5 + .5 = 0$ Final Wealth/Initial Wealth $=$ $e^{-.5}e^{+.5} = 1$	Mean Ret $\sqrt{(1-.5)(1+.5)} - 1$ $= \sqrt{1-.5^2} - 1 < 0$ Final Wealth/Initial Wealth $=$ $e^{-.5}e^{+.5} = 1$
periodically compounded	Mean Ret $= -.5 + .5 = 0$ Final Wealth/Initial Wealth $=$ $(1-.5)(1+.5) = 1 - .5^2 < 1$	Mean Ret $\sqrt{(1-.5)(1+.5)} - 1$ $= \sqrt{1-.5^2} - 1 < 0$ Final Wealth/Initial Wealth $=$ $(1-.5)(1+.5) = 1 - .5^2 < 1$

The Geometric Mean

Proposition 4. *Assuming yields are linear in time and returns are approximated by (1), the number of periods "N_g" which will make the geometric mean return equal to the initial yield satisfies*

$$1 = \prod_{t=1}^{N_g} \left(1 + (t - D)\frac{\Delta Y}{1 + Y_1}\right) \tag{14}$$

$$= \begin{cases} \left(\frac{\Delta Y}{1+Y_1}\right)^{N_g} \frac{\Gamma\left(\frac{1+Y_1}{\Delta Y} - D + N_g + 1\right)}{\Gamma\left(\frac{1+Y_1}{\Delta Y} - D + 1\right)} & \text{if } \Delta Y > 0 \\ \left(\frac{-\Delta Y}{1+Y_1}\right)^{N_g} \frac{\Gamma\left(-\frac{1+Y_1}{\Delta Y} + D\right)}{\Gamma\left(-\frac{1+Y_1}{\Delta Y} + D - N_g\right)} & \text{if } \Delta Y < 0 \end{cases} \tag{15}$$

(where Γ means the gamma function of mathematics, not to be confused with the Gamma of option price theory).

Needed for the proof

Lemma. *If a and b are positive real numbers and b − a is a positive integer then*

$$\sum_{j=a}^{b} \ln j = \ln a + \ln(a+1) + \cdots + \ln(b) = \ln \Gamma(b+1) - \ln \Gamma(a)\,.$$

Proof. Since $\Gamma(n) = (n-1)!$ when n is a positive integer, if a and b are both positive integers then this is merely the claim that $\sum_{j=a}^{b} \ln j = \ln[b!] - \ln[(a-1)!]$, which can be proven by writing out $b!$. To construct a proof for non-integer a and b, recall that a basic property of the gamma function is $\Gamma(x+1) = x\,\Gamma(x)$. Letting Δ denote the difference operator in this paragraph only, $\Delta \ln \Gamma(x) = \ln \Gamma(x+1) - \ln \Gamma(x) = \ln(x\,\Gamma(x)) - \ln \Gamma(x) = \ln x$. We know from the definition of the discrete antiderivative, the definition of the discrete definite integral, and the Fundamental Theorem of Finite Calculus (pages 7–8 of Gleich (2005)) that if $\Delta f(x) = g(x)$ then $\sum_{x=a}^{b} g(x) = f(b+1) - f(a)$. Identify $\ln \Gamma(x)$ with f and $\ln x$ with g. ∎

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Proof of Proposition 4, Part 1.

Define $A = 1 - D\Delta Y/(1+Y_1)$ and $B = \Delta Y/(1+Y_1)$ [...] so either

$$\tfrac{A}{B} + t > 0 \text{ for all } t \text{ and } B > 0, \text{ i.e., } \Delta Y > 0, \text{ or}$$
$$\tfrac{A}{B} + t < 0 \text{ for all } t \text{ and } B < 0 \text{ , i.e., } \Delta Y < 0.$$

For the $\Delta Y > 0$ case,

$$\exp \ln \prod_{t=1}^{N_g} (A + Bt) = \exp \sum_{t=1}^{N_g} \ln(A + Bt) = \exp \sum_{t=1}^{N_g} \left(\ln B + \ln(\tfrac{A}{B} + t)\right);$$

using the identity $\sum_{t=1}^{T} f(t+C) = \sum_{j=1+C}^{T+C} f(j)$ and setting its C equal to $\frac{A}{B}$ and its f equal to ln,

$$= \exp\{N_g \ln B + \sum_{j=\frac{A}{B}+1}^{\frac{A}{B}+N_g} \ln j\}; \quad (16)$$

$$= \exp\{N_g \ln B + \ln\Gamma(\tfrac{A}{B} + N_g + 1) - \ln\Gamma(\tfrac{A}{B} + 1)\} \quad (17)$$

$$= \exp \ln\left[B^{N_g} \Gamma(\tfrac{A}{B} + N_g + 1)/\Gamma(\tfrac{A}{B} + 1)\right]$$

$$= B^{N_g} \Gamma(\tfrac{A}{B} + N_g + 1)/\Gamma(\tfrac{A}{B} + 1). \quad (18)$$

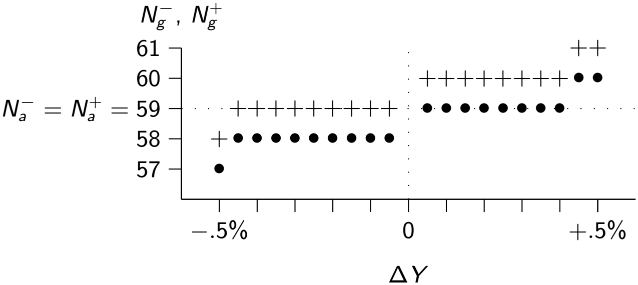
Proof of Proposition 4, Part 2.

For the $\Delta Y < 0$ case, using the identity $\sum_{t=1}^{T} f(-t+C) = \sum_{j=-T+C}^{-1+C} f(j)$ one has

$$\begin{aligned}
\exp \ln \prod_{t=1}^{N_g}(A+Bt) &= \exp \sum_{t=1}^{N_g} \ln(A+Bt) = \exp \sum_{t=1}^{N_g} \left(\ln(-B) + \ln(-\tfrac{A}{B} - t)\right) \\
&= \exp\{N_g \ln(-B) + \sum_{j=-\frac{A}{B}-N_g}^{-\frac{A}{B}-1} \ln j\} \\
&= \exp\{N_g \ln(-B) + \ln\Gamma(-\tfrac{A}{B}) - \ln\Gamma(-\tfrac{A}{B} - N_g)\} \\
&= \exp \ln\left[(-B)^{N_g}\Gamma(-\tfrac{A}{B})/\Gamma(-\tfrac{A}{B} - N_g)\right] \\
&= (-B)^{N_g}\Gamma(-\tfrac{A}{B})/\Gamma(-\tfrac{A}{B} - N_g)\,.
\end{aligned}$$

∎

Figure 2: an Asymmetry



For $D = 30$, $Y_1 = 30\%$, and linear yield paths with different values of ΔY, the plus signs denote N_g^+ and the bullets denote N_g^-.

Table 1

	A	G
1. $\Delta Y > 0$ and $2D$ an integer	$N_a = 2D - 1$	$2D - 1 < N_g^+ < \infty$
2. $\Delta Y > 0$ and $2D$ not an integer	$N_a^+ = \lceil 2D - 1 \rceil$	$2\lfloor D \rfloor - 1 < N_g^+ < \infty$
3. $\Delta Y < 0$ and $2D$ an integer	$N_a = 2D - 1$	$D < N_g^+ \leq 2D - 1$
4. $\Delta Y < 0$ and $2D$ not an integer	$N_a^+ = \lceil 2D - 1 \rceil$	$D < N_g^+ \leq 2\lceil D \rceil - 1$

Table: 1: Theoretical Results for Linear Yield Paths

Column G shows this asymmetry (proof starts next slide)

Catch-up Date for Geometric Mean

Proposition 5. *Assuming yields are linear in time and returns are approximated by (1), N_g^+ satisfies 1G, 2G, 3G, and 4G of Table 1.*

Proof. If D is an integer then $\prod_{t=1}^{2D-1} f(t)$ for an arbitrary function f can be reordered as $f(D)\prod_{s=1}^{D-1} f(D-s)\cdot f(D+s)$. Defining the return up to time T as $G(T) = \prod_{t=1}^{T}[1+Y_1+(t-D)\Delta Y]$, one has

$$\begin{aligned} G(2D-1) &= (1+Y_1)\prod_{s=1}^{D-1}[1+Y_1+((D-s)-D)\Delta Y]\cdot[1+Y_1+((D+s)-D)\Delta Y] \\ &= (1+Y_1)\prod_{s=1}^{D-1}[1+Y_1-s\Delta Y]\cdot[1+Y_1+s\Delta Y] \\ &= (1+Y_1)\prod_{s=1}^{D-1}[(1+Y_1)^2-(s\Delta Y)^2] \\ &< (1+Y_1)\prod_{s=1}^{D-1}(1+Y_1)^2 = (1+Y_1)^{2D-1}\,. \end{aligned} \tag{19}$$

Proof of Proposition 5, p. 2

Next suppose that D is not an integer but $2D$ is an integer. Since for an arbitrary function f in this case one has the reordering $\prod_{t=1}^{2D-1} f(t) = \prod_{s=1}^{D-1/2} f(D+1/2-s) \cdot f(D-1/2+s)$,

$$G(2D-1) = \prod_{s=1}^{D-1/2} [1+Y_1+((D+\tfrac{1}{2}-s)-D)\Delta Y][1+Y_1+((D-\tfrac{1}{2}+s)-D)\Delta Y]$$

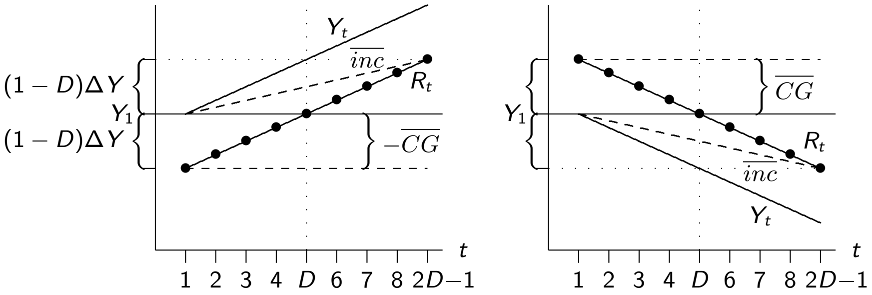
$$= \prod_{s=1}^{D-1/2} [1+Y_1-(s-\tfrac{1}{2})\Delta Y]\cdot[1+Y_1+(s-\tfrac{1}{2})\Delta Y]$$

$$= \prod_{s=1}^{D-1/2} [(1+Y_1)^2-(s-\tfrac{1}{2})^2(\Delta Y)^2]$$

$$< \prod_{s=1}^{D-1/2} (1+Y_1)^2 = (1+Y_1)^{2D-1}$$

as in (19).

Figure 1: when will Avg. Return equal Y_1?



Average return up to time t, $\bar{R}_a$, is not shown; it would be a line joining the left-most bullet on the R_t line with the point $(2D-1, Y_1)$. The geometric mean, $\bar{R}_g$, is lower than this.

Proof of Proposition 5, p. 3

- For $\Delta Y > 0$, the interpretation of $G(2D-1) < (1+Y_1)^{2D-1}$ is that $2D-1$ is not enough time to compensate for the initial less-than-Y_1 returns, so $N_g^+ > 2D-1$, proving one of the inequalities of 1G.
- For $\Delta Y < 0$, the interpretation is that $2D-1$ is so large that it over-compensates for the initial greater-than-Y_1 returns, so $N_g^+ \leq 2D-1$, proving one of the inequalities of 3G.

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Proof of Proposition 5, p. 4

- To prove 2G, note that D cannot be an integer when $2D$ is not an integer. Thus

$$\prod_{t=1}^{2\lfloor D\rfloor-1}[1+Y_1+(t-D)\Delta Y] < \prod_{t=1}^{2\lfloor D\rfloor-1}[1+Y_1+(t-\lfloor D\rfloor)\Delta Y] < (1+Y_1)^{2\lfloor D\rfloor-1}$$

using (19).

- Similarly for 4G:

$$\prod_{t=1}^{2\lceil D\rceil-1}[1+Y_1+(t-D)\Delta Y] < \prod_{t=1}^{2\lceil D\rceil-1}[1+Y_1+(t-\lceil D\rceil)\Delta Y] < (1+Y_1)^{2\lceil D\rceil-1}.$$

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∎

Outline

1. Leibowitz' Framework
2. Extensions
3. Empirical Results

Data

- All yields and returns will be continuously compounded.
- If yields in the past followed linear paths through time and the Babcock Approximation were completely accurate, then returns should have equaled initial yields after the passage of an amount of time equal to duration times a factor "F" where $F = 2 - 1/D \approx 2$.
- However, yields historically were not linear and the Babcock Approximation was not completely accurate, and therefore historical evidence on performance of different F's is useful in deciding on what F to use when forecasting.
- I study fifteen different F's between 0.75 and 2.5 and six types of bonds to see how well initial yield predicted realized return over those periods.
- It is of interest to study constant-maturity portfolios.

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    - 3-Year Treasury Constant Maturity Rate (GS3)
    - 5-Year Treasury Constant Maturity Rate (GS5)
    - 10-Year Treasury Constant Maturity Rate (GS10)
    - 20-Year Treasury Constant Maturity Rate (GS20) [missing data]
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- Since for each month and each Treasury series we can calculate duration, it is possible to synthesize constant-duration time series. I synthesized an 80-month constant-duration Treasury yield series by linear interpolation using the duration and yield data for the 5-year and 10-year series when 80 months fell between their durations, and for the 10-year and 20-year series when 80 months fell between their durations. Eighty months was the longest-duration series that could be synthesized by interpolation.

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Table 3, Entire Period

	3 year Treasury	5 year Treasury	80 month duration Treasury	10 year Treasury	20 year Treasury	Long-Term Corporate
Entire Period (3.59% avg. inflation)						
Change in yields	−1.6%	−0.9%	−0.5%	−0.1%	+0.2%	+1.2%
Avg. annual yield	5.4%	5.6%	5.8%	5.9%	6.1%	7.6%
Nominal Avg. annual return	5.3%	5.5%	5.6%	5.7%	5.9%	7.1%
Real Avg. annual return	1.8%	2.0%	2.1%	2.1%	2.3%	3.5%

Table: 3: All rates, including inflation, are continuously-compounded annual percentages. Inflation is calculated from the FRED database "Consumer Price Index for All Urban Consumers: All Items, Index 1982–84 = 100, Monthly, Not Seasonally Adjusted" (CPIAUCNS).

Table 3, Early Period

	3 year Treasury	5 year Treasury	80 month duration Treasury	10 year Treasury	20 year Treasury	Long-Term Corporate
Early Period, April 1953 to September 1981 (4.41% avg. inflation)						
Change in yields	+12.6%	+12.2%	+11.4%	+11.5%	+11.0%	+12.0%
Avg. annual yield	5.5%	5.5%	5.6%	5.6%	5.6%	6.9%
Nominal Avg. annual return	4.2%	3.7%	2.8%	2.6%	1.5%	2.3%
Real Avg. annual return	−0.2%	−0.7%	−1.6%	−1.8%	−3.0%	−2.1%

Table: All rates, including inflation, are continuously-compounded annual percentages. Inflation is calculated from the FRED database "Consumer Price Index for All Urban Consumers: All Items, Index 1982–84 = 100, Monthly, Not Seasonally Adjusted" (CPIAUCNS).

Table 3, Late Period

	3 year Treasury	5 year Treasury	80 month duration Treasury	10 year Treasury	20 year Treasury	Long-Term Corporate
Late Period, October 1981 to April 2014 (2.87% avg. inflation)						
Change in yields	−13.5%	−12.6%	−11.9%	−11.4%	−10.9%	−11.0%
Avg. annual yield	5.3%	5.6%	5.9%	6.1%	6.6%	8.3%
Nominal Avg. annual return	6.3%	7.2%	8.2%	8.4%	9.8%	11.4%
Real Avg. annual return	3.5%	4.3%	5.3%	5.6%	6.9%	8.5%

Table: All rates, including inflation, are continuously-compounded annual percentages. Inflation is calculated from the FRED database "Consumer Price Index for All Urban Consumers: All Items, Index 1982–84 = 100, Monthly, Not Seasonally Adjusted" (CPIAUCNS).

Comparisons: *same* bonds, different F's

- The "early period" data will only include bonds whose "purchase date plus 2.5 times their initial duration" occurred on or before the September 1981 (2.5 being the largest F used).
- Since bonds whose purchase dates are before 9/1/81 but whose "purchase date plus 2.5 $*$ initial duration" are after 9/1/81 are excluded from the early and late periods but are included in the full period, the full period has more purchase dates, and thus has more observations, than the union of the early period and the late period.
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Calculations

- For each month, the capital gain was the difference between 100 and the price of coupon bond with par value of 100, coupon interest rate equal to the semiannually-compounded constant-maturity yield at the beginning of the month, maturity of one month less than it had at the beginning of the month, and current yield equal to that of the first day of the next month, as calculated not by using the Babcock Approximation but instead by Excel's "price" function.
- (This assumes that the yield curve is flat between 35 and 36 months for 3-year bonds; flat between 59 and 60 months for 5-year bonds; and so on.)
- The capital gain was converted to a monthly percent, then to a monthly continuously-compounded percent; the yield was also converted to a monthly continuously-compounded percent; then the capital gain was combined with the yield to obtain the monthly continuously-compounded total return $(\exp(\text{total return}) - 1 = (\exp(\text{interest income}) - 1) + (\exp(\text{capital gains}) - 1))$.

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- From this a monthly "growth of $10,000" series was generated.
- The initial modified duration of each month's bond was calculated using Excel's "MDuration" function, multiplied by the factor F, rounded to the nearest integer, and then the annual continuously-compounded return was calculated for that forward time span.
- "Forecast Error" was defined as this value minus the annual continuously-compounded initial yield.
- Formally: if $\bar{R}_{amFt}$ denotes the arithmetic mean realized annual continuously-compounded return for a bond purchased at date t with maturity (or duration) m over the forward period of length "$F *$ initial duration," then since its initial yield Y_{mt} is its predicted annual return, its forecast error is $\bar{R}_{amFt} - Y_{mt}$.

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Judging the Forecast Errors

- With fifteen values of F and sixteen series of bonds (six each for the full and late periods and four for the early period) there were 240 time series of forecast errors.
- Summarizing criteria chosen were:
  - Centered R^2 with slope 1 and intercept 0.
  - Root mean square ('RMS') forecast error: Given n purchase dates for bonds of a fixed maturity or duration, and a fixed choice of F, this is $(\sum_t (\bar{R}_{amFt} - Y_{mt})^2/n)^{1/2}$ over the relevant purchase dates.
  - Average forecast error: $\sum_t (\bar{R}_{amFt} - Y_{mt})/n$. While all the other measures of goodness of fit would rank forecast errors of $\{-2, +2, -2, +2\}$ worse than $\{+1/2, +1/2, +1/2, +1/2\}$, this one will rank the latter worse than the former, and it is possible investors would have such a preference (for example, that they would care about some moving average of the errors).
  - Frequency of Absolute Value of Forecast Error less than $x\%$.

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Table 6

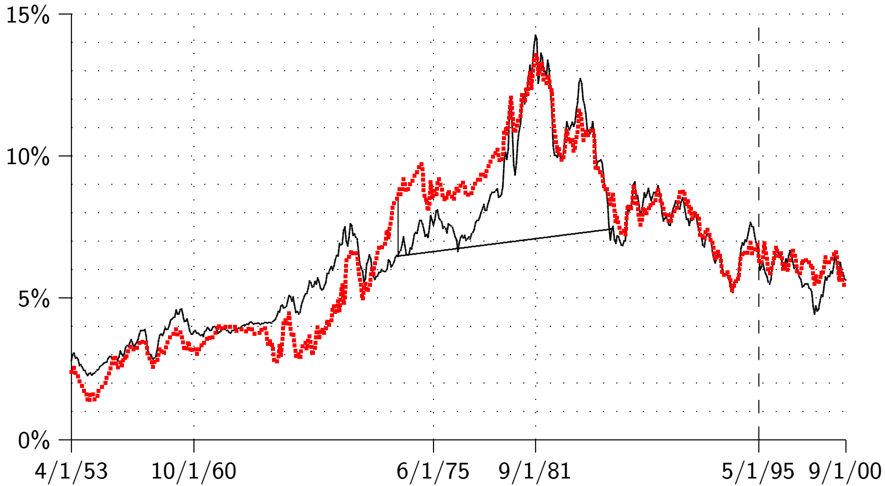
F	0.75	1	1.25	1.5	1.6	1.7	1.75	1.8	
3 YEAR									
Full									
RMS FE	1.74%	1.29%	1.06%	0.92%$\star$	0.92%$\star$	0.93%$\star$	0.93%$\star$	0.93%$\star$	0
Avg FE	-0.01%$\star$	0.02%	0.03%	0.04%	0.04%	0.06%	0.06%	0.06%	0
Cent R^2 **G**	0.78	0.85	0.88$\star$	0.90$\star$	0.90$\star$	0.89$\star$	0.89$\star$	0.89$\star$	
FFE $<$ .5%**B**	24%	33%	36%	40%$\star$	40%$\star$	41%$\star$	42%$\star$	41%$\star$	
FFE $<$ 1%	44%	58%	67%	73%$\star$	73%$\star$	71%$\star$	72%$\star$	71%$\star$	
FFE $<$ 2%	78%	88%	94%	97%	97%	97%	97%	97%	
FFE $<$ 3%	92%	97%	99%	100%	100%	100%	100%	100%	
FFE $<$ 4%	97%	100%	100%	100%	100%	100%	100%	100%	
FFE $<$ 5%	99%	100%	100%	100%	100%	100%	100%	100%	

Table: Excerpt of results. Other F's are 1.8, 1.9, 2, 2.1, 2.2, 2.3, 2.4, and 2.5. Other series are 5 year, 80 month, 10 year, 20 year, and Long-term Corporate, for Full, Early, and Late Periods.

Summary of Table 6

F	Stars	F	Stars
0.75	0	1.9	39
1	1	2	19
1.25	9	2.1	14
1.5	17	2.2	12
1.6	27	2.3	14
1.7	37	2.4	9
1.75	29	2.5	6
1.8	34		

Figure 3



The solid line is the 10-year Constant-Maturity Treasury yield and the dotted red line is the 10-year Constant-Maturity Treasury's forward realized return over a period $1\frac{3}{4}$ times its initial duration.

Measure of Nonlinearity

- There are three sources of forecast errors: nonlinear yield paths, the Babcock Approximation, and (for all series except for one) nonconstant duration.
- There are several potential ways of measuring the first source of forecast errors.
- If $Y_t^{\alpha\beta l}$ is the linear path between Y_α and Y_β, the measure of nonlinearity "NL" we will use is $NL = \sum_{t=\alpha}^{\beta}(Y_t - Y_t^{\alpha\beta l})/(\beta - \alpha + 1)$.
- By allowing positive and negative deviations from linearity to cancel, this measure ensures that "nonlinearity" will be furthest from zero when yields mostly deviate from linearity in a single direction.

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Convexity and the Measure of Nonlinearity

- If the yield path is convex, the yield path will be below the linear path and $NL < 0$; if the yield path is concave, $NL > 0$.
- According to Proposition 3's corollary, if the yield path is convex and quadratic, forecast error is negative, and if the yield path is concave and quadratic, forecast error is positive, so if the actual yield paths are sufficiently close to being quadratic, NL will have the same sign as forecast error.
- I do not use the right-hand side of Proposition 3's (13), $\bar{R}_a - Y_1 = \frac{2a}{3}(1-D)\,D$, as a measure of nonlinearity because it is only a measure of concavity.

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- According to Proposition 3's corollary, if the yield path is convex and quadratic, forecast error is negative, and if the yield path is concave and quadratic, forecast error is positive, so if the actual yield paths are sufficiently close to being quadratic, NL will have the same sign as forecast error.
- I do not use the right-hand side of Proposition 3's (13), $\bar{R}_a - Y_1 = \frac{2a}{3}(1 - D)\,D$, as a measure of nonlinearity because it is only a measure of concavity.

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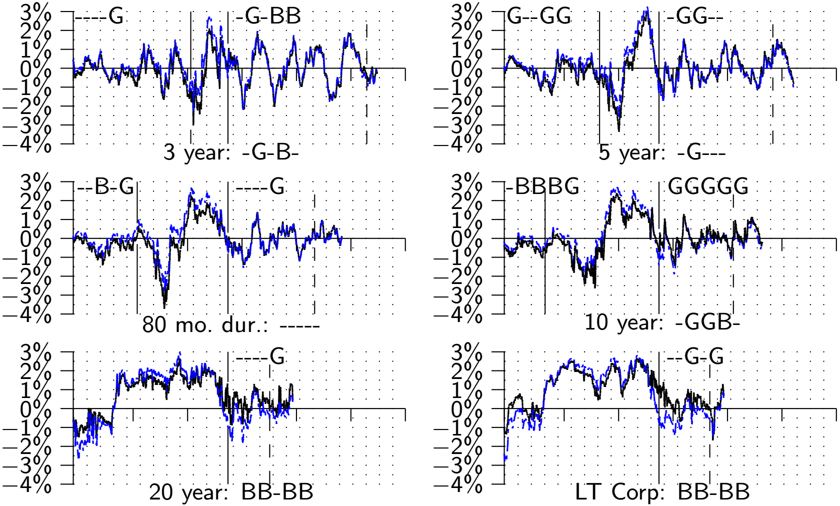
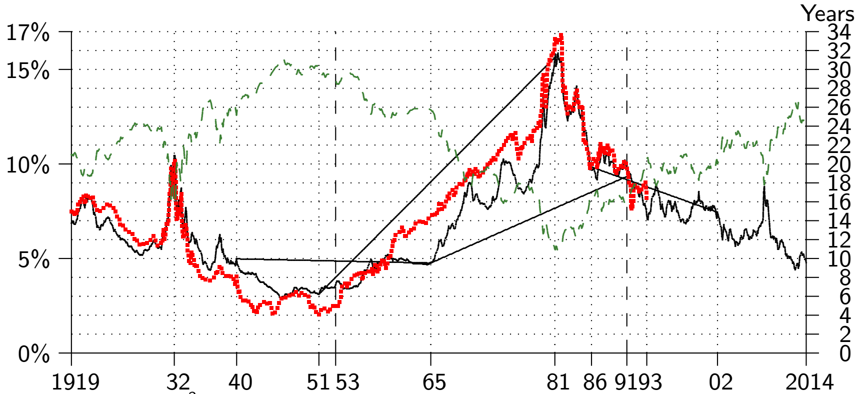
Figure 4



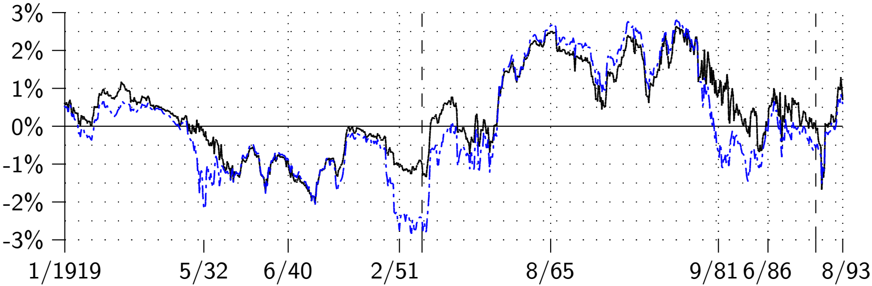
The solid lines are the forecast errors for each series when $F = 1\frac{3}{4}$. The dash-dot blue lines are the Nonlinearity Measure for each "$1\frac{3}{4}$ times initial duration" period.

Figure 5



Results for $F = 1\frac{3}{4}$ for the Long-term Corporate series. The solid line is the yield and the dotted red line is the forward realized return over a period $1\frac{3}{4}$ times its initial duration. The right-hand axis is in years and the dashed green line, which is read on this axis, represents 1.75 times duration.

Figure 6



The dash-dot blue line is the nonlinearity measure *NL*, and the solid line is the forecast error, of Figure 5.

Table 7 (excerpts)

	3 year Treasury	5 year Treasury	80-mo. dur. Treasury	10 year Treasury	20 year Treasury	Long-Term Corporate
FULL PERIOD						
RMS^2/St. Dev.^{2}	0.11	0.11	0.11	0.10	0.15	0.17
NL & Forec. Err. Corr. Coeff.	0.96	0.96	0.96	0.96	0.97	0.94
R^2 *NLFE*	0.88	0.88	0.83	0.88	0.75	0.57
EARLY PERIOD						
RMS^2/St. Dev.^{2}	0.15	0.15	0.65	0.69		
NL & Forec. Err. Corr. Coeff.	0.99	0.98	0.97	1.00		
R^2 *NLFE*	0.81	0.72	0.24	0.20		
LATE PERIOD						
RMS^2/St. Dev.^{2}	0.15	0.10	0.10	0.06	0.12	0.07
NL & Forec. Err. Corr. Coeff.	0.98	0.98	0.99	0.93	0.90	0.43
R^2 *NLFE*	0.97	0.95	0.98	0.58	-1.23	-3.27

Work for the far future

How well does the initial yield on inflation-indexed bonds predict their real return?