Retirement Saving: Which are the Important Years?

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Abstract. In a theoretical model, suppose retirement savers increase savings each working year, then withdraw a constant real income each year during retirement. We calculate the derivative of retirement income with respect to investment returns for each year and show that it peaks near the date of retirement. The concavity and convexity of the relationship shows that the most important dates lie just before, not after, retirement. An empirical study using U.S. data illuminates the theoretical predictions. The best response to these cohort-specific risks would involve intergenerational risk pooling.

"Sequencing risk" or "sequence of return risk" refers to the observation that the amount of sustainable retirement income depends strongly not only on the average investment returns earned during retirement but also on the order in which they are earned: poor returns at the beginning of retirement followed by good returns later result in a low sustainable retirement income, while good returns at the beginning of retirement followed by poor returns later result in a high sustainable retirement income.[1] This has given rise to papers trying to define or identify the "retirement risk zone," which are the years during which bad investment outcomes will lead to low retirement income. This literature is completely empirical, except for some work by Milevsky using hypothetical investment returns (see for example Milevsky and Macqueen 2010 pp. 42–46).

In this paper we present a simple analytical model of saving for retirement and then living on those savings while retired. We measure the "importance" of the investment results in a given year in a way which is natural but apparently novel: the derivative of "sustainable retirement income" with respect to that investment return. We will prove that a graph of this measure of importance rises until the date of retirement, then begins to fall, which proves theoretically what these empirical studies find: the level of sustainable retirement income is mostly determined by investment results around the date of retirement. We also prove that this graph is concave before retirement and convex after retirement, which shows that most of the "important" years occur just *before* retirement; only a few of them occur just after retirement, despite the fact that many papers in this area do not even consider the pre-retirement period. The only paper which has a similar result is that of Doran, Drew and Walk (2012), which observes the phenomenon empirically but has no analytical analysis.

1. The Model and Basic Results

A consumer starts saving in period $t = 1$. In that period, he saves amount "c" (for "retirement savings *c*ontribution"). In each successive period until he retires, he saves more, so that his contributions go up as

$$\{c, ce^{\gamma}, ce^{2\gamma}, ce^{3\gamma}, \ldots\} .$$

(All variables are stated in real terms.) Assume his contributions happen at the end of every period. His retirement account balance obeys the recursion (or difference equation)

$$B_{t+1} = B_t \, e^{r_{t+1}} + ce^{t\gamma}$$

where r_{t+1} is the investment return during period t+1 and B_{t+1} is the balance at the end of period t+1.

[1]See for example Bierwirth 1994, p. 18, "Importance of Timing."

The agent retires in period "R." During retirement, he withdraws a constant real amount "w" (for "$w$ithdrawal") at the beginning of each period. During retirement,

$$B_{t+1} = (B_t - w)\, e^{r_{t+1}} .$$

At the end of period "D," the consumer dies. We suppose he dies with zero wealth, so $B_D = 0$. This obviously abstracts from the important uncertainty about the value of D, but by making that part of this model much simpler, it allows us to focus on the novel issues raised in this paper. In the same spirit, we suppose that dates R and D and all the returns $r_1, r_2, \ldots, r_D$ are known in advance (or that we are doing an *ex post* analysis). Then w is the largest real constant consumption possible during retirement.

We wish to prove that the most important returns for this consumer are the ones near $t = R$; or, in particular, that

$$\frac{\partial w}{\partial r_t} < \frac{\partial w}{\partial r_{t+1}} \quad \text{for } t < R \text{ and}$$
$$\frac{\partial w}{\partial r_t} > \frac{\partial w}{\partial r_{t+1}} \quad \text{for } t > R.$$

We have:

Lemma 1. *For $t \le R - 1$,*

$$B_t = c \sum_{j=1}^{t} e^{(t-j)\gamma + \sum_{k=1}^{j-1} r_{t-k+1}} . \tag{1}$$

Proof. We proceed by induction. For $t = 1$, (1) asserts that

$$B_1 = c \sum_{j=1}^{1} e^{(1-j)\gamma + \sum_{k=1}^{j-1} r_{1-k+1}}$$
$$= c e^{(1-1)\gamma + \sum_{k=1}^{0} r_{1-k+1}} = c .$$

This is correct.

Next we assume (1) is true for t, then prove it is true for t+1. The difference equation which (1) claims to solve is

$$B_{t+1} = B_t\, e^{r_{t+1}} + c\, e^{t\gamma} . \tag{2}$$

Substituting (1) into (2),

$$c \sum_{j=1}^{t+1} e^{(t+1-j)\gamma + \sum_{k=1}^{j-1} r_{t+1-k+1}} \stackrel{?}{=} \left[c \sum_{j=1}^{t} e^{(t-j)\gamma + \sum_{k=1}^{j-1} r_{t-k+1}} \right] e^{r_{t+1}} + c e^{t\gamma}$$

which is equivalent to

$$\sum_{j=1}^{t+1} e^{(t+1-j)\gamma+\sum_{k=1}^{j-1} r_{t-k+2}} \stackrel{?}{=} e^{r_{t+1}} \sum_{j=1}^{t} e^{(t-j)\gamma+\sum_{k=1}^{j-1} r_{t-k+1}} + e^{t\gamma} . \tag{3}$$

Write the left-hand side of (3) as the sum of its $j = 1$ term and all its other terms:

$$e^{t\gamma} + \sum_{j=2}^{t+1} e^{(t+1-j)\gamma+\sum_{k=1}^{j-1} r_{t-k+2}} . \tag{4}$$

Let $m = j - 1$. Then $j = m + 1$ and (4) is

$$e^{t\gamma} + \sum_{m=1}^{t} e^{[t+1-(m+1)]\gamma+\sum_{k=1}^{(m+1)-1} r_{t-k+2}} = e^{t\gamma} + \sum_{m=1}^{t} e^{(t-m)\gamma+\sum_{k=1}^{m} r_{t-k+2}} ,$$

which can be rewritten (since j and m are merely dummy indexes) as

$$e^{t\gamma} + \sum_{j=1}^{t} e^{(t-j)\gamma+\sum_{k=1}^{j} r_{t-k+2}} . \tag{5}$$

The right-hand side of (3) is

$$\sum_{j=1}^{t} e^{(t-j)\gamma+r_{t+1}+\sum_{k=1}^{j-1} r_{t-k+1}} + e^{t\gamma} . \tag{6}$$

But

$$\begin{aligned}
\sum_{k=1}^{j} r_{t-k+2} &= r_{t-1+2} + \sum_{k=2}^{j} r_{t-k+2} \text{ ; letting } n = k-1, \\
&= r_{t-1+2} + \sum_{n=1}^{j-1} r_{t-(n+1)+2} \\
&= r_{t+1} + \sum_{n=1}^{j-1} r_{t-n+1} \text{ ; changing dummy indexes,} \\
&= r_{t+1} + \sum_{k=1}^{j-1} r_{t-k+1} . \qquad (7)
\end{aligned}$$

Comparing this with (6) shows that (6) can be rewritten as

$$\sum_{j=1}^{t} e^{(t-j)\gamma+\sum_{k=1}^{j} r_{t-k+2}} + e^{t\gamma} , \tag{8}$$

and since (8) is equal to (5), the proposition is proven. ▮

Lemma 2. *The solution to the difference equation*

$$B_t = (B_{t-1} - w)\, e^{r_t} \quad \text{for } t \geq R \tag{9}$$

is

$$B_t = B_{R-1} e^{\sum_{j=R}^{t} r_j} - w \sum_{j=R}^{t} e^{\sum_{k=R}^{j} r_{t-k+R}}\,, \tag{10}$$

which is the value of B_t when $t \geq R$.

Proof. Again we prove by induction.

The proof for the first period, $t = R$: from (10),

$$\begin{aligned} B_R &= B_{R-1} e^{r_R} - w e^{r_{R-R+R}} \\ &= B_{R-1} e^{r_R} - w e^{r_R} = (B_{R-1} - w)\, e^{r_R}\,, \end{aligned}$$

which satisfies (9). This completes the proof for the first period.

Assume true for t, prove true for $t+1$: does (10), evaluated at t and $t+1$, satisfy (9) evaluated there, i.e., $B_{t+1} = (B_t - w)\, e^{r_{t+1}}$?

$$\begin{aligned} & B_{R-1} e^{\sum_{j=R}^{t+1} r_j} - w \sum_{j=R}^{t+1} e^{\sum_{k=R}^{j} r_{t+1-k+R}} \\ & \qquad \stackrel{?}{=} \left[B_{R-1} e^{\sum_{j=R}^{t} r_j} - w \sum_{j=R}^{t} e^{\sum_{k=R}^{j} r_{t-k+R}} - w \right] e^{r_{t+1}} \end{aligned} \tag{11}$$

is equivalent to

$$\begin{aligned} & B_{R-1} e^{\sum_{j=R}^{t+1} r_j} - w \sum_{j=R}^{t+1} e^{\sum_{k=R}^{j} r_{t+1-k+R}} \\ & \qquad \stackrel{?}{=} B_{R-1} e^{r_{t+1} + \sum_{j=R}^{t} r_j} - w \sum_{j=R}^{t} e^{r_{t+1} + \sum_{k=R}^{j} r_{t-k+R}} - w\, e^{r_{t+1}}\,. \end{aligned} \tag{12}$$

The first term on the left-hand side of (12) is equal to the first term on the right-hand side of (12). So (12) is equivalent to

$$\sum_{j=R}^{t+1} e^{\sum_{k=R}^{j} r_{t+1-k+R}} \stackrel{?}{=} \sum_{j=R}^{t} e^{r_{t+1} + \sum_{k=R}^{j} r_{t-k+R}} + e^{r_{t+1}}\,. \tag{13}$$

The left-hand side of (13) can be broken up into the sum of the R^{th} term and the other terms:

$$
\begin{aligned}
e^{\sum_{k=R}^{R} r_{t+1-k+R}} + \sum_{j=R+1}^{t+1} e^{\sum_{k=R}^{j} r_{t+1-k+R}} \\
&= e^{r_{t+1}} + \sum_{j=R+1}^{t+1} e^{\sum_{k=R}^{j} r_{t+1-k+R}} \\
&= e^{r_{t+1}} + \sum_{m=R}^{t} e^{\sum_{k=R}^{m+1} r_{t+1-k+R}} \\
&= e^{r_{t+1}} + \sum_{j=R}^{t} e^{\sum_{k=R}^{j+1} r_{t+1-k+R}} \\
&= e^{r_{t+1}} + \sum_{j=R}^{t} e^{r_{t+1-R+R}+\sum_{k=R+1}^{j+1} r_{t+1-k+R}} \\
&= e^{r_{t+1}} + \sum_{j=R}^{t} e^{r_{t+1}+\sum_{k=R+1}^{j+1} r_{t+1-k+R}} \text{; let } m = k-1\text{:} \\
&= e^{r_{t+1}} + \sum_{j=R}^{t} e^{r_{t+1}+\sum_{m=R}^{j} r_{t-m+R}} .
\end{aligned}
$$

This equals the right-hand side of (13). ▮

Lemma 3. *The sustainable level of annual real retirement spending is*

$$
w = \frac{c \sum_{j=1}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}} .
$$

Proof. We impose $0 = B_D$. From Lemma 2,

$$
\begin{aligned}
0 = B_D &= B_{R-1} e^{\sum_{j=R}^{D} r_j} - w \sum_{j=R}^{D} e^{\sum_{k=R}^{j} r_{D-k+R}} \\
&= B_{R-1} e^{\sum_{m=R}^{D} r_m} - w \sum_{j=R}^{D} e^{\sum_{k=R}^{j} r_{D-k+R}} \\
&= B_{R-1} - w \sum_{j=R}^{D} e^{-\sum_{m=R}^{D} r_m + \sum_{k=R}^{j} r_{D-k+R}} . \qquad (14)
\end{aligned}
$$

In this paragraph, we prove the following claim:

$$-\sum_{m=R}^{D} r_m + \sum_{k=R}^{j} r_{D-k+R} = -\sum_{n=R}^{D-1+R-j} r_n \tag{15}$$

for $j \in [R, D]$. Is this true for $j = R$?

$$-\sum_{m=R}^{D} r_m + \sum_{k=R}^{R} r_{D-k+R} \stackrel{?}{=} -\sum_{n=R}^{D-1+R-R} r_n$$

$$-\sum_{m=R}^{D} r_m + r_D \stackrel{?}{=} -\sum_{m=R}^{D-1} r_m$$

$$-\sum_{m=R}^{D-1} r_m - r_D + r_D \stackrel{?}{=} -\sum_{m=R}^{D-1} r_m$$

which is true. Now assuming (15) is true for j, we need to prove it is true for $j+1$. At $j+1$ it is

$$-\sum_{m=R}^{D} r_m + \sum_{k=R}^{j+1} r_{D-k+R} \stackrel{?}{=} -\sum_{n=R}^{D-1+R-(j+1)} r_n$$

$$-\sum_{m=R}^{D} r_m + \sum_{k=R}^{j} r_{D-k+R} + r_{D-(j+1)+R} \stackrel{?}{=} -\sum_{n=R}^{D-1+R-j-1} r_n$$

$$-\sum_{m=R}^{D} r_m + \sum_{k=R}^{j} r_{D-k+R} \stackrel{?}{=} \left(-\sum_{n=R}^{D-1+R-j-1} r_n\right) - r_{D-j-1+R}$$

$$-\sum_{m=R}^{D} r_m + \sum_{k=R}^{j} r_{D-k+R} \stackrel{?}{=} -\sum_{n=R}^{D-1+R-j} r_n\,,$$

which is true because it is the same as (15), which was assumed to be true for j by the induction hypothesis.

Having proven (15), we can substitute it into (14), obtaining

$$0 = B_{R-1} - w \sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}$$

and

$$w = \frac{B_{R-1}}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}}\,. \tag{16}$$

Replacing B_{R-1} in this equation with, from Lemma 1,

$$B_{R-1} = c \sum_{j=1}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}$$

completes the proof of Lemma 3. ∎

Proposition 1. *For $t \le R-1$, one has $\partial w / \partial r_{t+1} > \partial w / \partial r_t$.*

Proof. Using w from Lemma 3, for $t \le R-1$,

$$\begin{aligned}
\frac{\partial w}{\partial r_t} &= \frac{c}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}} \frac{\partial}{\partial r_t} \sum_{j=1}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}} \\
&= \frac{c}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}} \sum_{j=1}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}} \frac{\partial}{\partial r_t} \left[(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k} \right].
\end{aligned}$$

We have

$$\begin{aligned}
\frac{\partial}{\partial r_t} \left[(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k} \right] &= \frac{\partial}{\partial r_t} \sum_{k=1}^{j-1} r_{R-k} \\
&= \frac{\partial}{\partial r_t} [r_{R-1} + r_{R-2} + \cdots + r_{R-(j-1)}].
\end{aligned}$$

This is equal to one if t is between $R-1$ and $R-(j-1)$, and it is equal to zero elsewhere. Since Proposition 1 already imposes $t \le R-1$, this derivative is one if $t \ge R-(j-1)$, i.e., if $j \ge R+1-t$. It is zero for all other j. Hence

$$\frac{\partial w}{\partial r_t} = \frac{c}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}} \sum_{j=R+1-t}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}. \tag{17}$$

Next,

$$\begin{aligned}
\frac{\partial w}{\partial r_{t+1}} &= \frac{c}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}} \sum_{j=R+1-(t+1)}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}} \\
&= \frac{c}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}} \Bigg[e^{(R-1-(R+1-(t+1)))\gamma + \sum_{k=1}^{(R+1-(t+1))-1} r_{R-k}} \\
&\qquad + \sum_{j=R+1-t}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}} \Bigg] \\
&= \frac{c}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}} e^{(t-1)\gamma + \sum_{k=1}^{R-1-t} r_{R-k}} + \frac{\partial w}{\partial r_t} > \frac{\partial w}{\partial r_t},
\end{aligned}$$

completing the proof. ▮

Proposition 2. *For $t \geq R$, one has $\partial w/\partial r_t > \partial w/\partial r_{t+1}$.*

Proof. If, from Lemma 3, we express $w = c \cdot$ numerator/denominator, then for $t \geq R$,

$$\begin{aligned}\frac{\partial w}{\partial r_t} &= \frac{\partial}{\partial r_t}\frac{c\,\text{numerator}}{\text{denominator}} = \frac{-c\,\text{numerator}}{\text{denominator}^2}\frac{\partial}{\partial r_t}\text{denominator}\\ &= \frac{-w}{\text{denominator}}\frac{\partial}{\partial r_t}\text{denominator}\\ &= \frac{-w}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}}\frac{\partial}{\partial r_t}\Big[\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}\Big]\\ &= \frac{-w}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}}\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}\frac{\partial}{\partial r_t}\Big[-\sum_{k=R}^{D-1+R-j} r_k\Big]\end{aligned}$$

In order for this last sum to have at least one term, $R \leq D - 1 + R - j$, so $j \leq D - 1$.

$$\frac{\partial w}{\partial r_t} = \frac{+w}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}}\sum_{j=R}^{D-1} e^{-\sum_{k=R}^{D-1+R-j} r_k}\frac{\partial}{\partial r_t}\left[r_R + r_{R+1} + \cdots + r_{D-1+R-j}\right].$$

If t is between R and $D - 1 + R - j$ then the derivative is one, otherwise it is zero. Since Proposition 2 already imposes $t \geq R$, this derivative is one if $t \leq D - 1 + R - j$, i.e., if $j \leq D - 1 + R - t$. It is zero for all other j. Note that $D - 1 + R - t \leq D - 1$ when $t \geq R$, which is always the case in this proof. Therefore

$$\frac{\partial w}{\partial r_t} = \frac{w}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}}\sum_{j=R}^{D-1+R-t} e^{-\sum_{k=R}^{D-1+R-j} r_k}.$$

Using (16),

$$\frac{\partial w}{\partial r_t} = \frac{w^2}{B_{R-1}}\sum_{j=R}^{D-1+R-t} e^{-\sum_{k=R}^{D-1+R-j} r_k}. \tag{18}$$

Next,

$$\frac{\partial w}{\partial r_{t+1}} = \frac{w^2}{B_{R-1}}\sum_{j=R}^{D-1+R-(t+1)} e^{-\sum_{k=R}^{D-1+R-j} r_k}.$$

But from (18),

$$\frac{\partial w}{\partial r_t} = \frac{w^2}{B_{R-1}} \sum_{j=R}^{D-1+R-(t+1)} e^{-\sum_{k=R}^{D-1+R-j} r_k} + \frac{w^2}{B_{R-1}} e^{-\sum_{k=R}^{D-1+R-(D-1+R-t)} r_k}$$
$$= \frac{\partial w}{\partial r_{t+1}} + \frac{w^2}{B_{R-1}} e^{-\sum_{k=R}^{t} r_k} > \frac{\partial w}{\partial r_{t+1}},$$

completing the proof. ▮

Proposition 3. *For $t \leq R-1$, the function $\partial w(t)/\partial r$ is concave in t if and only if $r_{t+1} > \gamma$.*

Proof. From the last step in the proof of Proposition 1,

$$\frac{\partial w}{\partial r_{t+1}} - \frac{\partial w}{\partial r_t} = \frac{c}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}} e^{(t-1)\gamma + \sum_{k=1}^{R-1-t} r_{R-k}}$$

from which it follows that

$$\frac{\partial w}{\partial r_{t+2}} - \frac{\partial w}{\partial r_{t+1}} = \frac{c}{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}} e^{t\gamma + \sum_{k=1}^{R-1-(t+1)} r_{R-k}}$$

and

$$\frac{\frac{\partial w}{\partial r_{t+1}} - \frac{\partial w}{\partial r_t}}{\frac{\partial w}{\partial r_{t+2}} - \frac{\partial w}{\partial r_{t+1}}} = \frac{e^{(t-1)\gamma + \sum_{k=1}^{R-1-t} r_{R-k}}}{e^{t\gamma + \sum_{k=1}^{R-1-(t+1)} r_{R-k}}} = \frac{e^{t\gamma - \gamma + \sum_{k=1}^{R-1-(t-1)} r_{R-k} + r_{R-(R-1-t)}}}{e^{t\gamma + \sum_{k=1}^{R-1-(t+1)} r_{R-k}}}$$
$$= e^{-\gamma + r_{t+1}}$$

If $r_{t+1} > \gamma$ then $e^{-\gamma + r_{t+1}} > 1$ and

$$\frac{\partial w}{\partial r_{t+1}} - \frac{\partial w}{\partial r_t} > \frac{\partial w}{\partial r_{t+2}} - \frac{\partial w}{\partial r_{t+1}}.$$

▮

Proposition 4. *For $t \geq R$, the function $\partial w(t)/\partial r$ is convex in t if and only if $r_{t+1} > 0$.*

Proof. From the last step of the proof of Proposition 2,

$$\frac{\partial w}{\partial r_{t+1}} - \frac{\partial w}{\partial r_t} = \frac{-w^2}{B_{R-1}} e^{-\sum_{k=R}^{t} r_k},$$

from which it follows that

$$\frac{\partial w}{\partial r_{t+2}} - \frac{\partial w}{\partial r_{t+1}} = \frac{-w^2}{B_{R-1}} e^{-\sum_{k=R}^{t+1} r_k} .$$

Hence

$$\frac{\frac{\partial w}{\partial r_{t+1}} - \frac{\partial w}{\partial r_t}}{\frac{\partial w}{\partial r_{t+2}} - \frac{\partial w}{\partial r_{t+1}}} = \frac{e^{-\sum_{k=R}^{t} r_k}}{e^{-\sum_{k=R}^{t+1} r_k}} = \frac{1}{e^{-r_{t+1}}} = e^{r_{t+1}} .$$

If $r_{t+1} > 0$ then

$$\frac{\frac{\partial w}{\partial r_{t+1}} - \frac{\partial w}{\partial r_t}}{\frac{\partial w}{\partial r_{t+2}} - \frac{\partial w}{\partial r_{t+1}}} > 1 .$$

From Proposition 2, the denominator is negative, so multiplying through by it yields

$$\frac{\partial w}{\partial r_{t+1}} - \frac{\partial w}{\partial r_t} < \frac{\partial w}{\partial r_{t+2}} - \frac{\partial w}{\partial r_{t+1}} \quad \text{or}$$
$$\frac{\partial w}{\partial r_t} - \frac{\partial w}{\partial r_{t+1}} > \frac{\partial w}{\partial r_{t+1}} - \frac{\partial w}{\partial r_{t+2}} .$$

Both sides of this inequality are positive. Since the right-hand side is smaller, the conclusion follows. ▮

2. How does the Graph behave at the Retirement Date?

Section 1 has conducted two largely separate analyses, one for before the date of retirement and one after, joined only by Lemma 3 and its implications. If the graph of dw/dr_t were continuous and smooth at dates before and after R, we would be interested in trying to prove that its two parts match up at R—that is, that the graph was continuous at R. (Smoothness at R is unlikely and would not be important anyway.) However, we are of course using discrete time, so the graph of dw/dr_t is nowhere continuous: it only exists at the discrete instants $\{1, 2, 3, \ldots, D\}$. It is therefore somewhat unclear what it even means to "match up" at R. We will investigate this issue three ways, in this order: using the discrete-time model of Section 1; using continuous time; and showing the behavior of dw/dr_t empirically, using U.S. data.

Proposition 5. *In the model of Section 1,*

$$\frac{\partial w/\partial r_R}{\partial w/\partial r_{R+1}} = \left[1 + \frac{1}{\sum_{j=R}^{D-1} e^{-\sum_{k=R}^{D-1+R-j} r_k}}\right] \left[1 + \frac{e^{(R-2)\gamma}}{\sum_{j=2}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}}\right]^{-1} .$$

Corollary. *For large D and R, the ratio in Proposition 5 is approximately equal to one.*

Proof of Proposition 5. (17), which holds for $t \leq R-1$, can be rewritten, using (16), as

$$\frac{dw}{dr_t} = c \cdot \frac{w}{B_{R-1}} \cdot \sum_{j=R+1-t}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}$$

so

$$\frac{dw}{dr_{R-1}} = \frac{cw}{B_{R-1}} \sum_{j=2}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}.$$

(18), which holds for $t \geq R$, implies

$$\frac{\partial w}{\partial r_R} = \frac{w^2}{B_{R-1}} \sum_{j=R}^{D-1} e^{-\sum_{k=R}^{D-1+R-j} r_k}.$$

So

$$\begin{aligned}
\frac{\partial w/\partial r_{R-1}}{\partial w/\partial r_R} &= \frac{c}{w} \frac{\sum_{j=2}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}}{\sum_{j=R}^{D-1} e^{-\sum_{k=R}^{D-1+R-j} r_k}} \text{; from Lemma 3,} \\
&= \frac{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}}{\sum_{j=1}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}} \frac{\sum_{j=2}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}}{\sum_{j=R}^{D-1} e^{-\sum_{k=R}^{D-1+R-j} r_k}} \\
&= \frac{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}}{\sum_{j=R}^{D-1} e^{-\sum_{k=R}^{D-1+R-j} r_k}} \frac{\sum_{j=2}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}}{\sum_{j=1}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}} \\
&= \frac{\sum_{j=R}^{D} e^{-\sum_{k=R}^{D-1+R-j} r_k}}{\sum_{j=R}^{D-1} e^{-\sum_{k=R}^{D-1+R-j} r_k}} \left[\frac{\sum_{j=1}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}}{\sum_{j=2}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}} \right]^{-1} \\
&= \left[1 + \frac{e^{-\sum_{k=R}^{R-1} r_k}}{\sum_{j=R}^{D-1} e^{-\sum_{k=R}^{D-1+R-j} r_k}} \right] \left[1 + \frac{e^{(R-2)\gamma}}{\sum_{j=2}^{R-1} e^{(R-1-j)\gamma + \sum_{k=1}^{j-1} r_{R-k}}} \right]^{-1}.
\end{aligned}$$

Proposition 5 follows by simplifying the first term. ▮

Proof of Corollary. First term: if both D and R are large numbers, there are many terms in the denominator, each of which is positive, so the term tends toward one. Second term: if both D and R are large numbers, there are many terms in the denominator, each of which is positive, so the term tends toward one. ▮

A difficulty with the Corollary is that if, for example, D and R become large numbers because time is measured in months instead of in years, then the r's would simultaneously shrink.

To determine if using continuous time could help, let $\dot{B}$ be dB/dt. During the periods of accumulation and decumulation, respectively,

$$\dot{B} = r(t)\,B + c_0\,e^{\gamma t}$$
$$\dot{B} = r(t)\,B - w\,.$$

For some $\phi(t)$, both of these have the form $\dot{B} - r(t)\,B = \phi(t)$, which is a linear differential equation in B and $\dot{B}$. Multiplying both sides by the integrating factor $e^{-\int r(\lambda)\,d\lambda}$ leads to

$$\begin{aligned} \frac{dB}{dt}e^{-\int r(\lambda)\,d\lambda} - r(t)\,B(t)\,e^{-\int r(\lambda)\,d\lambda} &= \phi(t)\,e^{-\int r(\lambda)\,d\lambda} \\ \frac{d}{dt}\left[Be^{-\int r(\lambda)\,d\lambda}\right] &= \phi(t)\,e^{-\int r(t)\,dt} \\ B\,e^{-\int r(\lambda)\,d\lambda} &= \int \phi(t)\,e^{-\int r(\lambda)\,d\lambda}dt + (\text{const.}) \end{aligned}$$

but better notation would be

$$B\,e^{-\int r(\lambda)\,d\lambda} = \int \phi(\tau)\,e^{-\int r(\lambda)\,d\lambda}d\tau + (\text{const.})\,.$$

During accumulation,

$$0 = B_0 = e^{\int_?^0 r(\lambda)\,d\lambda} \int c_0\,e^{\gamma t}e^{-\int_?^\tau r(\lambda)\,d\lambda}d\tau + (\text{const.})\,,$$

which leads to "?" and const. both being zero:

$$\begin{aligned} B_t &= e^{\int_0^t r(\lambda)\,d\lambda} \int_0^t c_0\,e^{\gamma t - \int_0^\tau r(\lambda)\,d\lambda}d\tau \\ &= \int_0^t c_0\,e^{\int_0^t r(\lambda)\,d\lambda + \gamma\tau - \int_0^\tau r(\lambda)\,d\lambda}d\tau \\ &= c_0 \int_0^t e^{\gamma\tau + \int_\tau^t r(\lambda)\,d\lambda}d\tau\,. \end{aligned}$$

Then

$$B_R = c_0 \int_0^R e^{\gamma\tau + \int_\tau^R r(\lambda)\,d\lambda}d\tau\,.$$

During decumulation,

$$0 = B_D = e^{\int_?^0 r(\lambda)\,d\lambda} \int w e^{-\int_?^\tau r(\lambda)\,d\lambda} d\tau + (\text{const.}),$$

which leads to "?" being D and const. being zero:

$$\begin{aligned} B_t &= e^{\int_D^t r(\lambda)\,d\lambda} \int_t^D w\, e^{-\int_D^\tau r(\lambda)\,d\lambda} d\tau \\ &= w \int_t^D e^{-\int_t^D r(\lambda)\,d\lambda + \int_\tau^D r(\lambda)\,d\lambda} d\tau \\ &= w \int_t^D e^{-\int_t^\tau r(\lambda)\,d\lambda} d\tau\,. \end{aligned}$$

Then w is chosen so that B_R is equal to that given by the accumulation phase:

$$\begin{aligned} & c_0 \int_0^R e^{\gamma\tau + \int_\tau^R r(\lambda)\,d\lambda} d\tau = B_R = w \int_R^D e^{-\int_R^\tau r(\lambda)\,d\lambda} d\tau \implies \\ & w = c_0 \frac{\int_0^R e^{\gamma\tau + \int_\tau^R r(\lambda)\,d\lambda} d\tau}{\int_R^D e^{-\int_R^\tau r(\lambda)\,d\lambda} d\tau}\,. \end{aligned}$$

The problem is that it is not clear how to proceed from here, because there seems to be no continuous-time analog to the discrete-time notion of $\partial w / \partial r_t$. Perhaps progress could be made by using impulse functions, which would be similar in some ways to the empirical technique used by Doran, Drew and Walk (2012), but this is speculative.

3. Empirical Results

The final way we used to investigate how the accumulation and decumulation periods match up was to calculate dw/dr_t graphs for U.S. investors holding 60% stocks and 40% bonds, working for 40 years and living in retirement for 20 years, increasing their initial $100/year savings by $1/2$ of 1% per year during the accumulation period, with the earliest cohort starting to work and save for retirement in 1927. With the last year of data being 2015, there are 29 cohorts.

Figure 1 superimposes the dw/dr_t graphs of each of the 29 cohorts; "$t = 1$" here means the first year for each cohort, "$t = 2$" means the second year for each cohort, and so forth. It can be hard in that figure to see the graph for each cohort, and it is impossible to see there the individual data points, so Figure 2 separately presents the graphs of Cohorts 9–26.

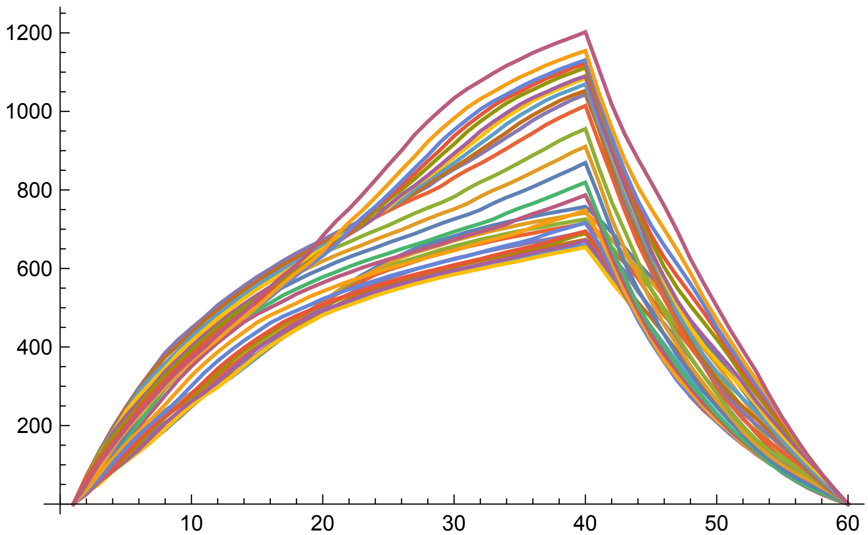

Figure 1. The value of dw/dr_t versus t for each of the years spent working or in retirement, for each of the 29 cohorts described in the text.

The before-retirement (before $t = 40$) and after-retirement periods match closely in these figures. The value of "dw/dr_{R-1} divided by dw/dr_R" varies from 1.04 to 1.12.

In addition, the graphs clearly show a generally concave relationship in the accumulation period and an essentially universally convex relationship in the decumulation period. This pattern is to be expected from our theoretical results: the accumulation period's concavity is less pervasive than the decumulation period's convexity because Proposition 3 holds under stricter conditions, $r_{t+1} > \gamma$, than Proposition 4, $r_{t+1} > 0$ (since we assumed $\gamma > 0$.)

The average over all the cohorts of the lines in Figure 1 is shown in Figure 3.

Table 1 lists each year from Figure 3 ranked from highest to lowest in terms of dw/dr_t. For example, from the table we read that the five years with the highest dw/dr_t were Years $\{40, 39, 38, 37, 36\}$ (listed in order of their individual dw/dr, which is not given in the table but is taken from Figure 3); that five years out of 60 total constitute $5/60 \approx 8.3\%$ of years; and that these 8.3% of years together constituted 15.2% of the total sum of dw/dr_t over all the years. Interpreting dw/dr_t as the importance of each year's returns in determining sustainable retirement income, this means that Years $\{40, 39, 38, 37, 36\}$ had a

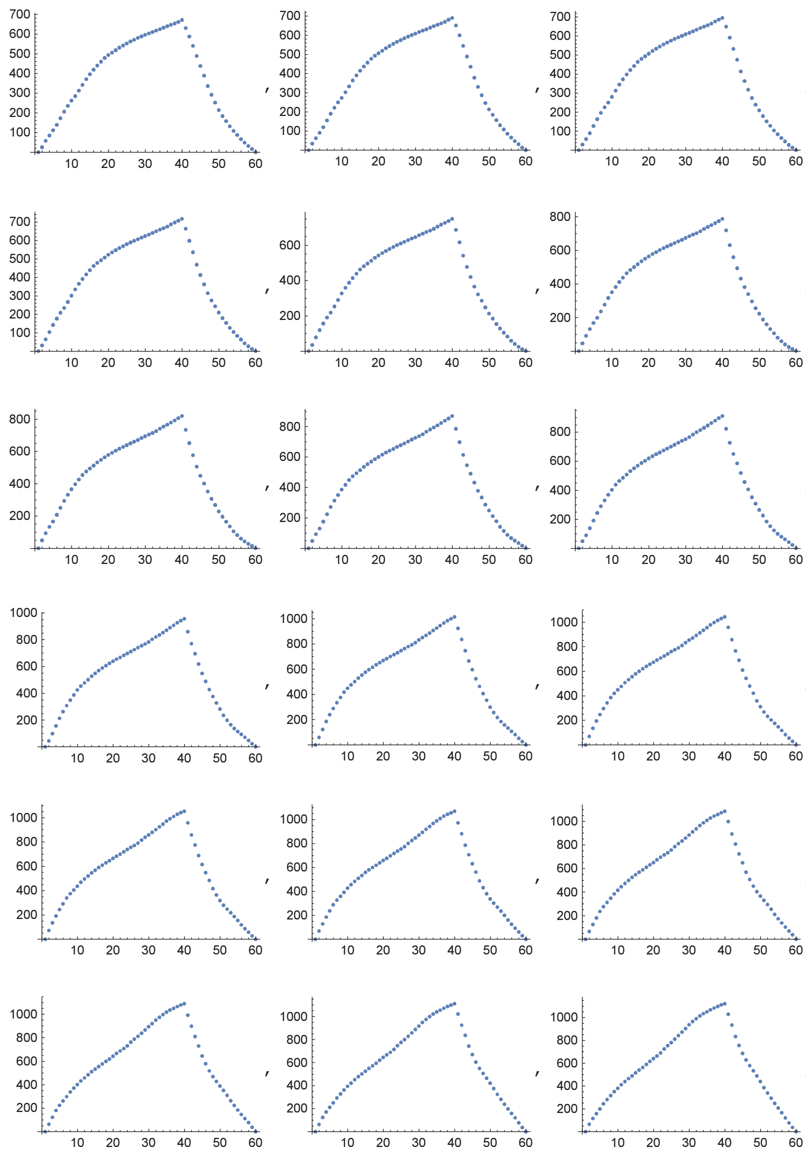

Figure 2. The graph of dw/dr_t versus t for Cohorts 9–29.

rank	cum. % years	year	cum. % dw/dr_t	rank	cum. % years	year	cum. % dw/dr_t
1	1.7	40	3.1	31	51.7	16	76.4
2	3.3	39	6.2	32	53.3	15	78.
3	5.	38	9.2	33	55.	14	79.6
4	6.7	37	12.2	34	56.7	47	81.2
5	8.3	36	15.2	35	58.3	13	82.7
6	10.	35	18.1	36	60.	12	84.1
7	11.7	41	21.	37	61.7	48	85.5
8	13.3	34	23.8	38	63.3	11	86.8
9	15.	33	26.6	39	65.	49	88.1
10	16.7	32	29.4	40	66.7	10	89.3
11	18.3	31	32.1	41	68.3	9	90.4
12	20.	30	34.7	42	70.	50	91.5
13	21.7	42	37.4	43	71.7	8	92.5
14	23.3	29	40.	44	73.3	51	93.4
15	25.	28	42.5	45	75.	7	94.3
16	26.7	27	45.	46	76.7	52	95.1
17	28.3	26	47.4	47	78.3	6	95.8
18	30.	43	49.8	48	80.	53	96.5
19	31.7	25	52.1	49	81.7	5	97.1
20	33.3	24	54.5	50	83.3	54	97.7
21	35.	23	56.7	51	85.	4	98.2
22	36.7	22	58.9	52	86.7	55	98.6
23	38.3	44	61.	53	88.3	56	99.
24	40.	21	63.1	54	90.	3	99.3
25	41.7	20	65.2	55	91.7	57	99.6
26	43.3	19	67.2	56	93.3	58	99.8
27	45.	45	69.1	57	95.	2	99.9
28	46.7	18	71.	58	96.7	59	100.
29	48.3	17	72.8	59	98.3	60	100.
30	50.	46	74.6	60	100.	1	100.

Table 1.

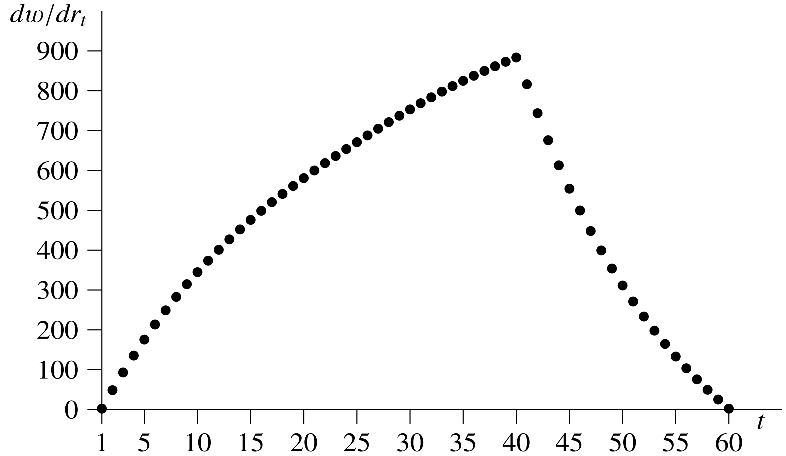

Figure 3. The average of the lines of Figure 1.

disproportionably large influence on the size of w: their influence was 15.2% of the total while they represent only 8.3% of the total time period.

The first post-retirement year to appear in Table 1 is Year 41, at position 7; the next post-retirement year is Year 42, at position 13. These years are, as we know from Proposition 2, more important than years later on in retirement, but clearly the most important years are the ones just before retirement, not just after. This reflects Propositions 3 and 4's concavity and convexity results.

Of the ten least important years, three are pre-retirement (Years 1, 2, and 3) and four are post-retirement (Years 57, 58, 59, and 60). These years constitute $100\%-85\% = 15\%$ of the number of years, but contribute only $100\%-98.2\% = 1.8\%$ of the total impact of returns on sustainable retirement income.

Roughly half (49.8%) of the total impact of returns on sustainable retirement income is contributed by only 18 of the 60 years. These are years 26–43: three post-retirement years and 15 pre-retirement years. (Another perspective: only $3/20 = 15\%$ of post-retirement years have this level of importance, while $15/40 = 37.5\%$ of pre-retirement years have this level of importance.) The other half of the total impact of returns on sustainable retirement income was contributed by Years 1–25 and Years 44–60: a total of 42 years, 25 pre-retirement and 17 post-retirement.

No small importance should be placed on the similarities between Figures 1, 2, and 3, on the one hand, and Figure 1 from Doran, Drew, and Walk (2012) (reproduced here as Figure 4), on the other hand. Our figures and their figure clearly have the same prominent concave/convex pattern, despite the fact that

Figure 1: Probability of ruin — negative sequencing event across the life cycle

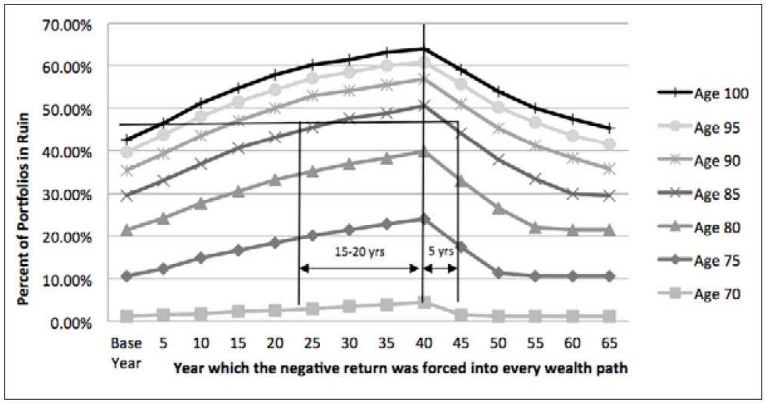

Figure 1 plots longevity risk estimates when investors are subjected to a -21.6 per cent return sequence at different five-year points in the investing life cycle; that is, in both the accumulation and decumulation phases.

Figure 4. Figure 1 from Doran, Drew, and Walk (2012).

their approach and ours are very different. To be sure, they and we are studying the same question: workers contribute for retirement a constantly growing amount, then each year after they retire they withdraw from those savings a constant real retirement income. Both analyses assume a 40-year working life. However, Doran, Drew and Walk ("DDW"), however, use a "70 per cent Australian equities, 20 per cent Australian bonds and 10 per cent Australian cash" data, while we use a 70% stock/30% bond U.S. portfolio; they simulate 10,000 75-year investment horizons, while we use no simulations. Also, DDW assume the date of death is unknown, set the annual retirement withdrawals equal to the Association of Superannuation Funds of Australia's 2011 "Comfortable Living Standard" (indexed for inflation), and study the "percent of portfolios in ruin" ("ruin" means zero balance) for different ages, which is obviously a completely different model than the one we used. Furthermore, DDW do not conceptualize the importance of a year's return as dw/dr_t, or as any mathematical derivative at all. Instead, to measure the importance of a year, "a 'forced' negative return was input into the same year of all 10,000 paths and final account balance and longevity risk were evaluated" (p. 11 fn. 5). In Figure 4, the horizontal axis represents the year in which the "forced" negative return was introduced. The same forced negative return had a greater effect on the probability of ruin if it was introduced near the date of retirement than if it was introduced at a much earlier or later date. Furthermore, while they do not use the terms "convex" and "concave," they clearly note what we call the concave/convex pattern and point out its importance:

> Figure 1 suggests an asymmetry in the impact of sequencing risk in the pre- and post-retirement phases. Consider the horizontal line which bisects the age 85 series pre- and post-retirement. This line identifies the shock timing, pre- and post-retirement, which results in equal probabilities of ruin at age 85. This asymmetry suggests that superannuants are exposed to the potentially negative consequences of sequencing risk earlier in the accumulation phase than conventional wisdom suggests (again, we reiterate the caveat that these are baseline estimates and will change given different member circumstances).

DDW's paper studies, in the empirical context of an unknown date of death, unknown market returns, and therefore the possibility of portfolio ruin, the same question that we study in the theoretical context of a known date of death, known market returns, and therefore a known level of sustainable retirement income: how important is each year in determining the fate of the saver? And they and we get the same fundamental result.

4. What is to be Done?

Kingston and Fisher (2014 p. 11) write:

> Basu et al. conclude with the claim that sequencing risk is a primitive one rather than purely derivative of underlying financial risks combined with a constant-mix allocation. Section 1 above disputed this claim.

The theoretical model we presented earlier in this paper assumed no particular portfolio allocation at all. We therefore side firmly with Basu et al.—Basu's coauthors are Doran and Drew, of the DDW paper—and against Kingston and Fisher on this point (although "sequencing risk" may not be the best term to describe the variation in dw/dr_t). Kingston and Fisher's main point, however, is very well-taken: given that dw/dr_t varies over the course of a retirement saver's lifetime, those savers need some advice about how to handle this situation. Kingston and Fisher give this advice: retirement portfolios should contain their lowest stock-to-bond ratios around the date of retirement. The percentage of stocks held can be high when workers are young, and high when retirees are old, but should not be high near R. This is reasonable advice.

Kenigsberg, Mazumdar and Feinschreiber (2014) suggest:

> Annuitization can stabilize a portfolio and be used to address SOR ["sequence of returns"] risk. In essence, an insurance company providing the annuity accepts the SOR risk, and then diversifies the risk over a time period longer than any single retirement horizon. Fundamentally, there are two ways in which annuities can be used to help mitigate SOR risk for the retiree: 1) through truncation of the retirement horizon, which reduces the probability that an unfavorable SOR will cause failure, and 2) by hedging part of the retirement income requirement, which reduces the impact of a failure if it happens.

This is also reasonable. We have clearly shown that each cohort suffers from acute exposure to investment market risk only over a limited set of ages; as long as people are generally risk-averse concerning their living standard during retirement, they will benefit if some kind of risk-pooling institutions or mechanisms are available that can alleviate each cohort's idiosyncratic risk.

As described in Kingston and Fisher (2014 p. 10), Milevsky and his coauthors suggest a new type of product sold by U.S. insurance companies called a "guaranteed minimum withdrawal benefit" or GMWB, but note that these have potential problems, including "transparency and costs," and note that "our public-sector substitute for GMWBs [that is, Australia's public pension system] limits their potential appeal to Australian retirees.... Why pay for insurance against market and longevity risk when the government provides it for free?"

Indeed it is important to discuss public options. They, by pooling risk, could result in large welfare improvements over purely individual retirement savings plans. If public options are politically unpalatable, there are new private (but not-for-profit) risk pooling schemes under investigation. Kenigsberg et al. write (2014 p. 92, emphasis mine):

> Collective defined contribution (CDC) schemes are one of a variety of strategies used to address SOR risk that are similar to annuities in that they diversify across time but do not require an insurance company. CDC schemes are a relatively recent invention that lies between traditional defined benefit (DB) and defined contribution (DC) plans. Both employee and employer make contributions to a pool rather than to an individual's account. On retirement, a targeted pension (not guaranteed) is paid directly from the scheme (potentially in the form of an indexed annuity). In a CDC plan, *risk is pooled between members and across generations, allowing the plan to diversify SOR risk over time (and reducing longevity risk as well).* However, although CDC plans already exist in countries

> like Denmark and the Netherlands, and are under consideration in the U.K.[footnote here in the original], they would currently not be possible under U.S. regulations.

In fact there is a growing literature on optimal pension system design, written by authors to whom it is obvious that purely-individual saving for retirement is a far from optimal arrangement for risk-averse individuals. Examples include Boes and Siegmann (2016), Gollier (2008), Beetsma, Romp, and Vos (2012), and Bams, Schotman, and Tyagi (2016). Most of these authors work in the Netherlands and none works in the U.S. It seems that in the U.S., at least beyond the Social Security system, few authors question the wisdom of saving for retirement via private, individual mechanisms, despite the fact that such mechanisms, by making it impossible to pool risks, generate large welfare losses.

Conclusion

Although a common way of thinking about how different years' investment results affect retirement income differently is using terms such as "sequencing risk," a more straightforward way is to observe that different years have different values of dw/dr_t, the derivative of sustainable retirement income with respect to each year's return. Making common assumptions on retirement accumulation and decumulation behavior, we have shown that most of the years having the highest of these derivatives lie just before the date of retirement; a few lie just after the retirement date.

This "retirement risk zone" is also a "retirement opportunity zone": low returns during these times will significantly hurt the individual but high returns during these times will correspondingly help the individual. Nevertheless, if people are risk-averse with respect to their standard of living in retirement, then the risk is more important than the opportunity.

The presence of these risks, which are idiosyncratic for each cohort, mean that *ex ante* welfare could be improved for all cohorts by sharing their risks. Some form of collective retirement savings is required if these risk are to be pooled. Purely individual retirement arrangements, exposing the person to the risks not only of uncertain market returns but also uncertain date of death, are nonoptimal (assuming risk aversion), and the spread of these private arrangements at the expense of pooling arrangements in the U.S. over the last few decades has certainly generated welfare losses.

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