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Econ. 7250
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NOTES ON TIMBER ECONOMICS

Assume the volume of wood in a tree expressed as a function of tree age t (for "time") is $V(t)$ and the total cost of cutting down a tree is c. Suppose the price of a unit volume of wood (such as 1 liter of wood) is fixed at one dollar. (This ignores issues of demand and supply at various dates, and so does not address questions of competitive equilibrium.) Let T be the age at which the tree is cut down, noting that

$$\frac{d}{dT} = \frac{d}{dt}\frac{dt}{dT} = \frac{d}{dt} \cdot 1 = \frac{d}{dt}\,. \tag{1}$$

The firm wishes to

$$\max_T e^{-\delta T}[V(T) - c]$$

for which the first-order condition is

$$\begin{aligned} 0 &= -\delta e^{-\delta T}[V(T) - c] + e^{-\delta T}V'(T), \text{implying} \\ V'(T) &= \delta\,[V(T) - c] \\ \delta &= \frac{V'(T)}{V(T) - c}, \end{aligned} \tag{2}$$

which implicitly defines the optimal value of T. Note that this can be rewritten as

$$\delta = \frac{V'(T)}{V(T) - c} = \frac{\frac{d}{dT}[V(T) - c]}{V(T) - c} = \frac{d}{dT}\ln[V(T) - c], \tag{3}$$

which is the elasticity of $V(T) - c$ with respect to T. In the graph of $\ln[V(T) - c]$ shown in the bottom part of Figure 1, the condition (3) is satisfied at the T (there labeled "I_1^* for reasons explained later) at which the slope of $\ln[V(T) - c]$ is equal to δ.

Next, suppose the tree is replanted after being harvested. Subsume the planting costs into c, since replanting occurs just after cutting down (we model them as occurring at the same date). If T_i denotes the time at which the i^{th} generation of trees is cut down, then the present value of the plot of land is

$$e^{-\delta T_1}[V(T_1) - c] + e^{-\delta T_2}[V(T_2 - T_1) - c] + e^{-\delta T_3}[V(T_3 - T_2) - c] + \cdots\,.$$

Given that neither costs c nor price (which is defined to be equal to one) depends on time, each generation of trees will be cut down at the same age. Call this age

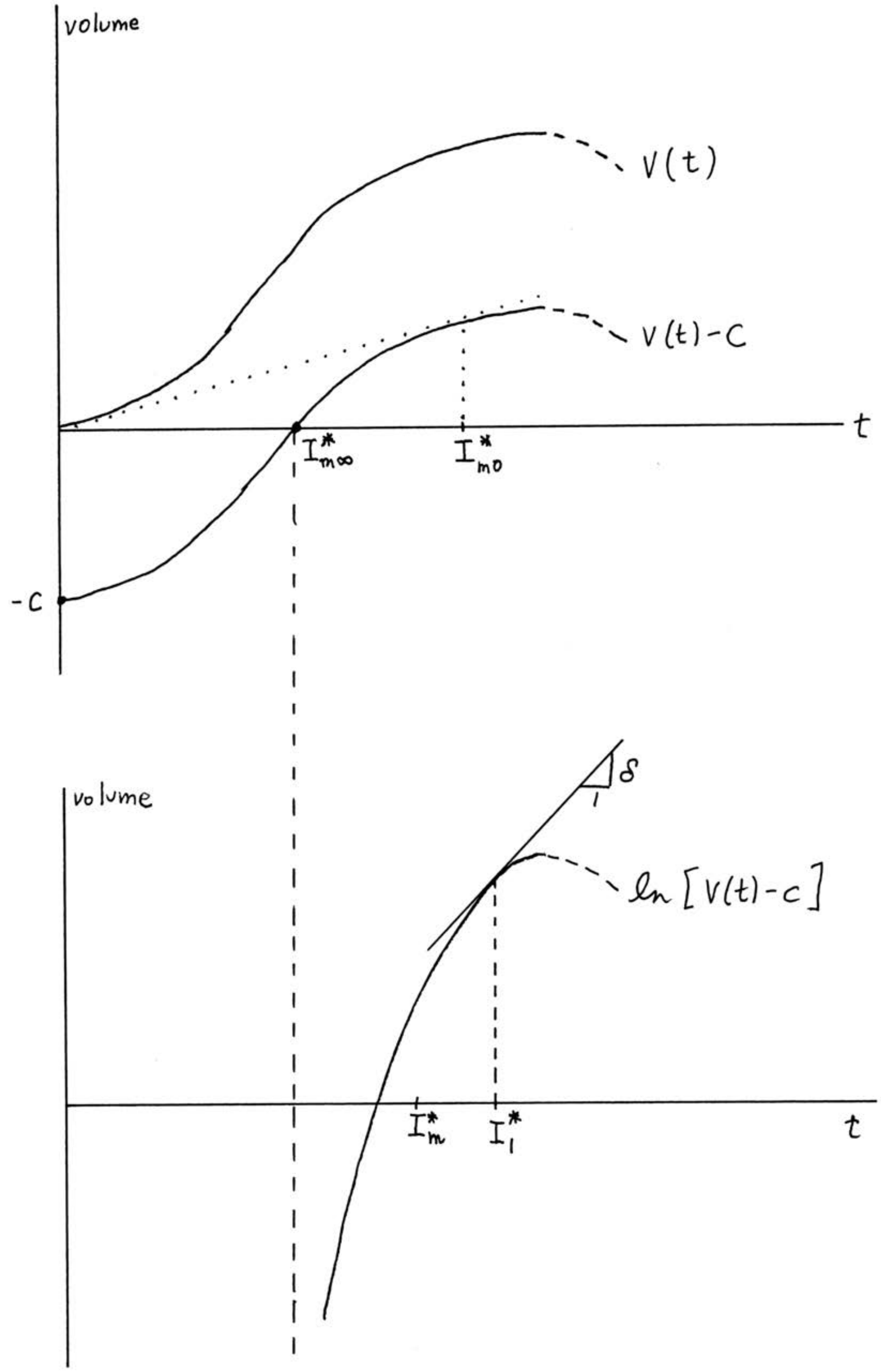

Figure 1. Top diagram: volume of wood as a function of age (with the dotted curve representing average shrinking of volume as very old trees die); and volume of wood minus cutting costs. Bottom diagram: the logarithm of the latter.

"I" for "rotation interval": $I = T_1 = T_2 - T_1 = T_3 - T_2 = \cdots$. Then the present value is

$$\begin{aligned} &e^{-\delta I}[V(I) - c] + e^{-\delta \cdot 2I}[V(I) - c] + e^{-\delta \cdot 3I}[V(I) - c] + \cdots \\ &= [V(I) - c] \sum_{k=1}^{\infty} e^{-k\delta I} = \frac{V(I) - c}{e^{\delta I} - 1} \end{aligned}$$

and maximizing the present value with respect to I yields the first-order condition

$$\begin{aligned} 0 &= \frac{V'(I)}{e^{\delta I} - 1} - \frac{V(I) - c}{(e^{\delta I} - 1)^2} \delta e^{\delta I} \\ \frac{V(I) - c}{e^{\delta I} - 1} \delta e^{\delta I} &= V'(I) \\ \frac{V'(I)}{V(I) - c} &= \frac{\delta e^{\delta I}}{e^{\delta I} - 1} = \frac{\delta}{1 - e^{-\delta I}} . \end{aligned} \tag{4}$$

Let I_1^* be the optimal age to cut down the tree in the single-crop case and let I_m^* be the optimal age to cut down the tree in the multiple-cropping case. Note that, using (1),

$$\frac{d}{dI} = \frac{d}{dt} \frac{dt}{dI} = \frac{d}{dt} \cdot 1 = \frac{d}{dt} = \frac{d}{dT} . \tag{5}$$

Therefore, as in (3), (4) can be rewritten as

$$\frac{d}{dI} \ln[V(I_m^*) - c] = \frac{\delta}{1 - e^{-\delta I}} . \tag{6}$$

Equation (3) is

$$\frac{d}{dI} \ln[V(I_1^*) - c] = \delta . \tag{7}$$

Since the right-hand side of (6) is larger than the right-hand side of (7), in the multiple-cropping case

$$\frac{d}{dI} \ln[V(I) - c]$$

is larger than it is in the single-crop case. As can be seen from our graph of $\ln[V(I) - c]$, this means that the multi-cropping case's I must become smaller, so $I_m^* < I_1^*$.

For another way of comparing the single- and multiple-cropping cases, rewrite (4) as

$$\frac{V'(I_m^*)}{V(I_m^*) - c} = \delta \frac{1 - e^{-\delta I_m^*} + e^{-\delta I_m^*}}{1 - e^{-\delta I_m^*}} = \delta \left[1 + \frac{e^{-\delta I_m^*}}{1 - e^{-\delta I_m^*}}\right] = \delta \left[1 + \frac{1}{e^{\delta I_m^*} - 1}\right]$$

so

$$V'(I_m^*) = \delta\,[V(I_m^*){-}c] + \delta \frac{V(I_m^*) - c}{e^{\delta I_m^*} - 1} \tag{8}$$

$$= \delta\,[V(I_m^*){-}c] + \delta \cdot \underbrace{[V(I_m^*){-}c] \sum_{k=1}^{\infty} e^{-k\delta I_m^*}}_{\text{“site value”}} \tag{9}$$

compared with (2)'s

$$V'(I_1^*) = \delta\,[V(I_1^*){-}c]\,.$$

Clark (first edition, section 8.1 p. 259) writes (reading I_m^* for T)[1]:

> "[(4)] for the optimal rotation period T is called the *Faustmann formula;* it was derived in 1849 by M. Faustmann, a German forester. . . . The third term [in (8)] reflects the rotation aspect of the problem. The expression
>
> $$\frac{V(T) - c}{e^{\delta T} - 1}$$
>
> is the present value of this stream of future revenues; in the forestry literature this expression is called the *site value.* The condition of Eq. [(8)] is that the forest be cut at age T, when the marginal increment to the value of the trees equals the sum of the opportunity costs of investment tied up in the standing trees and in the site."

Note that the right-hand side of (4) has the properties that

$$\lim_{\delta\to 0} \frac{\delta}{1 - e^{-\delta I}} = \text{“}\,\frac{0}{0}\,\text{”} = \lim_{\delta\to 0} \frac{1}{Ie^{-\delta I}} = \frac{1}{I}$$

using L'Hôpital's Rule, and

$$\lim_{\delta\to\infty} \frac{\delta}{1 - e^{-\delta I}} = \infty\,.$$

Denoting I_m^* in the limit as $\delta \to 0$ by I_{m0}^*, and denoting I_m^* in the limit as $\delta \to \infty$ by $I_{m\infty}^*$, it follows that in the limit as $\delta \to 0$, (4) implies

$$\frac{d}{dI}[V(I_{m0}^*) - c] = V'(I_{m0}^*) = \frac{V(I_{m0}^*) - c}{I_{m0}^*}\,,$$

so in this limit, the marginal value of $V(I) - c$ is equal to its average value. This is illustrated by the position of I_{m0}^* in the upper graph of Figure 1. In the limit as $\delta \to \infty$, (4) implies

$$\frac{V'(I_{m\infty}^*)}{V(I_{m\infty}^*) - c} = \infty \quad \Rightarrow \quad V(I_{m\infty}^*) = c\,,$$

[1] See also `https://en.wikipedia.org/wiki/Faustmann%27s_formula`.

so in this limit, the optimal value of I makes $V(I^*_{m\infty}) - c$ equal to zero. This is illustrated by the position of $I^*_{m\infty}$ in the upper graph of Figure 1. It follows that one can put bounds $(I^*_{m\infty}, I^*_{m0})$ on the location of I^*_m in the upper graph of Figure 1, as in Clark 1e Fig. 8.3.