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NOTES ON FISHERIES ECONOMICS

Section 1. Optimal Control Theory

The core of the material in this handout is based on *Mathematical Bioeconomics: The Optimal Management of Renewable Resources* by Colin W. Clark (Second Edition, 1990).

Consider the problem of how to find

$$J(x_0) = \max_u \int_0^T f(x, u, t)\, dt \tag{1}$$

such that

$$\dot{x}_t = g(x, u, t) \tag{2}$$

with x_0 given and $u_t \in U$ for all t. Here x is the "state variable" and u is the "control variable." For simplicity, consider them scalars instead of vectors. Often we will use a final time $T = \infty$. Form the "Hamiltonian":

$$\mathscr{H} = f + \lambda g$$

where λ is a function of time called the "adjoint variable" or the "costate variable."[1] The "Maximum Principle" gives two necessary conditions for optimality (using asterisks to denote optimal values):

$$\boxed{\max_{u_t \in U} \mathscr{H}_t \ \forall\, t} \tag{3}$$

$$\boxed{\dot{\lambda}^*_t = -\frac{\partial \mathscr{H}^*_t}{\partial x}\,.} \tag{4}$$

The Maximum Principle also states that λ^*_t is continuous and is piecewise continuously differentiable (where "continuously differentiable" means that the derivative exists and the derivative is itself a continuous function, though the derivative may not be differentiable).

[1] In cases of no economic importance, it is mathematically possible that the correct Hamiltonian is actually $0f + \lambda g = \lambda g$.

There are infinitely many maximizations implied by (3), one for every t, but each one is a problem in standard calculus. Usually (3) implies

$$\partial \mathcal{H}^* / \partial u = 0\,, \tag{5}$$

but if there are constraints on u then the standard Kuhn-Tucker complementary slackness conditions would hold. If $U = [\underline{u}\,, \overline{u}]$, the best way to write these is

$$\left.\begin{array}{ll} \partial \mathcal{H}^* / \partial u \le 0 & \text{if } u^* = \underline{u} \\ \partial \mathcal{H}^* / \partial u = 0 & \text{if } u^* \in (\underline{u}\,, \overline{u}) \\ \partial \mathcal{H}^* / \partial u \ge 0 & \text{if } u^* = \overline{u}. \end{array}\right\} \tag{6}$$

When we discuss fisheries below, $\underline{u}$ will be zero, and we will not mention an upper bound, which implicitly means taking $\overline{u} = \infty$.

A solution u_t^* is said to be "interior" if it is an element of the interior of U. For example, if $U = [\underline{u}\,, \overline{u}]$, then u_t^* would be "interior" if it was an element of $(\underline{u}\,, \overline{u})$.

An interesting special case occurs when $\mathcal{H}$ is linear in u: say, $\mathcal{H} = \sigma u + z$ or, more explicitly,

$$\mathcal{H} = \sigma(x, \lambda, t)\, u + z(x, \lambda, t) \tag{7}$$

where σ and z are some functions which do not depend on u. (A mathematician would object to this use of the word "linear" because of the presence of the "$+\, z(x, \lambda, t)$" term; a mathematician would call such a function "affine" not "linear." In these notes, when we say "linear" we technically usually mean "affine" instead.) In economics, this often occurs when firms are competitive and have constant returns to scale.[2] In this case, $\partial \mathcal{H} / \partial u = \sigma$, so (6) implies

$$\left.\begin{array}{ll} \sigma \le 0 & \text{if } u^* = \underline{u} \\ \sigma = 0 & \text{if } u^* \in (\underline{u}\,, \overline{u}) \\ \sigma \ge 0 & \text{if } u^* = \overline{u} \end{array}\right\} \tag{8}$$

or, seen from another perspective,

$$\left. u^* \begin{cases} = \underline{u} & \text{if } \sigma < 0 \\ \in [\underline{u}\,, \overline{u}] & \text{if } \sigma = 0 \\ = \overline{u} & \text{if } \sigma > 0. \end{cases} \right\} \tag{9}$$

[2] A firm's profit is total revenue (price times output) minus total cost (average cost times output). Let "output" be the control, u. If the firm is competitive then price "p" is not a function of u. If the firm has constant returns to scale then average cost "AC" is not a function of u. Thus profit would be $pu - AC\,u$, which would be linear in u since neither p nor AC would depend on u. Then $\mathcal{H} = (p - AC)\,u + \lambda\, g$, whose the first term is linear in u. If g is linear in u as well, then $\mathcal{H}$ would be linear in u, as in (7).

The $\sigma \neq 0$ solutions are called "bang-bang" solutions because in many problems, as time goes on, the optimal control changes discontinuously from $\underline{u}$ to $\overline{u}$ or vice versa as the "switching function" σ changes sign. The $\sigma = 0$ solutions are called "singular" solutions. Since (7) implies that $\partial\mathcal{H}/\partial u = \sigma$, and since $\sigma = 0$ on a singular solution:

> (5) characterizes optimality both for interior solutions to nonlinear problems and for singular solutions to linear problems.

We could almost say:

> (5) characterizes optimality for interior solutions to all problems.

The only error in this formulation is that it ignores the fact that singular solutions to linear problems could have u^* being equal to $\underline{u}$ or $\overline{u}$. In the fisheries problems we study in these Notes, the second formulation is good enough.

Defining J as in (1), it can be shown that

$$\frac{\partial J}{\partial x_0} = \lambda_0 \tag{10}$$

as long as J is differentiable. (In economics, the left-hand side is how much the present discounted value J of the initial resource stock x_0 would increase if the initial stock rose by one unit; so λ_0 is the "shadow value" or "shadow price" of the resource.) Although (2) does not allow x_t to make discontinuous jumps, discontinuous jumps would be permitted if (2) only held "almost everywhere" ("a.e.," synonymous with "virtually everywhere," "v.e.," which all mean "except on a set of measure zero"). If we were to allow x_t to make a discontinuous jump at time $\tau > 0$, and define J_τ as $\max_u \int_\tau^T f(x,u,t)\,dt$ subject to (2) and $u_t \in U \ \forall\, t > \tau$ and the new x_τ, then

$$\frac{\partial J_\tau}{\partial x_\tau} = \lambda_\tau \tag{11}$$

(see p. 323 of Sydsæter, Hammond, Seierstad, and Strøm, *Further Mathematics for Economic Analysis*, 2005). A result which can be helpful in signing the costate variable is due to Caputo (*Foundations of Dynamic Economics Analysis*, 2005, (13) p. 57): there is a function $\beta_t > 0 \,\forall\, t$ (whose definition you do not need to know) such that

$$\lambda_t^* = \frac{1}{\beta_t}\int_t^T \beta_\tau f_x'(x_\tau^*, u_\tau^*, \tau)\,d\tau\,. \tag{12}$$

This can be helpful because often one knows the sign which f_x' will have in the future.

If the final time T in (1) is finite, then (since we put no conditions on x_T except nonnegativity, which I don't go into here), the following "transversality condition" is necessary for an extremum (that is, a maximum or a minimum):

$$\lambda_T = 0\,. \tag{13}$$

However, if $T = \infty$, the situation requires special treatment. If neither f nor g depend explicitly on t, the optimal control problem is called "autonomous." We will not study any autonomous problems. However, if f takes the special form $\hat{f}(x,u)\,e^{-\delta t}$ and g does not depend explicitly on t, we say that the problem is "autonomous except for geometric discounting." Some the problems we study are such problems (namely, the monopolist's problem and the social planner's problem—but *not* the problem of the competitive firm). In such problems, if $T = \infty$ the following "transversality condition" is necessary for an extremum:

$$\lim_{t\to\infty} \mathscr{H}^* = 0 \tag{14}$$

(see Caputo, op. cit., Theorem 14.9).

Conditions (3) and (4), together with the appropriate transversality condition, are necessary conditions for optimality. Sometimes it is useful to know sufficient conditions for optimality. For a maximum, the sufficient conditions are the necessary conditions plus either:

Mangasarian Sufficient Condition: $\mathscr{H}(x,u,\lambda^*,t)$ is concave in (x,u) for all $t \in [0,T]$ and for all admissible (x,u). If $\mathscr{H}$ is strictly concave, the optimal solution is unique.

Arrow Sufficient Condition: $\mathscr{H}(x,u^*,\lambda^*,t)$ is concave in x at x^* for all $t \in [0,T]$. If $\mathscr{H}$ is strictly concave, the optimal path of x is unique but the optimal path of u is not necessarily unique.

(See for example Caputo, op. cit., pp. 53 and 60–61.) For a minimum, change "concave" to "convex." For a refresher on how to do the concavity check called for in the Mangasarian sufficient condition (concavity of a function of more than one variable), see the Econ. 7005 mathematical prerequisites notes.

Section 2. Private-Property Fishery: The General Formula

The problem of each firm is to

$$\max_{\langle h_t \rangle} \int_0^\infty \pi(x_t, h_t, t)\, e^{-\delta t}\, dt \tag{15}$$

subject to

$$\dot{x}_t = F(x_t) - h_t \tag{16}$$

and

$$h_t \geq 0 \,\forall\, t \tag{17}$$

where h is the amount of fish harvested, x is the amount of fish alive, π is profit, δ is the discount rate, F is the excess of births over natural deaths, and raised dots denote derivatives with respect to time.

To start, form

$$\mathcal{H} = e^{-\delta t}\pi(x_t, h_t, t) + \lambda_t [F(x_t) - h_t]\,. \tag{18}$$

Let $M\Pi$ denote marginal profit $\partial\pi/\partial h$.

In general, π is nonlinear in h. However, if the production function has constant returns to scale (and there are no fixed costs), and the firm is competitive, then π is linear in h (see footnote 2). In that case, (18) implies that $\mathcal{H}$ is linear in h, which is the case described by (7).

In this paragraph, assume either that h_t^* is interior or, if π is linear in h, then assume the solution is singular. Then as explained in Section 1, (5) characterizes the optimal solution. We have:

Lemma 1. *Assume h_t^* is interior or, if π is linear in h, assume the solution is singular. Then the solution to (15) subject to (16) and (17) is*

$$\boxed{\delta M\Pi_t = M\Pi_t\, F_t' + \dot{M\Pi}_t + \tfrac{\partial \pi_t}{\partial x}} \tag{19}$$

which can be rewritten as

$$\boxed{\delta = F'(x_t) + \frac{\dot{M\Pi}_t}{M\Pi_t} + \frac{\partial\pi(x_t, h_t, t)/\partial x}{M\Pi_t} \quad \text{if } M\Pi \neq 0.} \tag{20}$$

This implicitly defines h_t^* as a function of δ, x_t, and t.

Proof. This proof rests on using (5). To begin, (5) and (18) imply

$$0 = \frac{\partial \mathscr{H}^*}{\partial h} = e^{-\delta t}\,\frac{\partial \pi_t}{\partial h} - \lambda_t = e^{-\delta t} M\Pi_t - \lambda_t\,,$$

so

$$\lambda_t = e^{-\delta t} M\Pi_t\,. \tag{21}$$

On the other hand, (4) implies

$$\dot{\lambda}_t = -e^{-\delta t}(\partial \pi_t/\partial x) - \lambda_t F_t'(x)\,. \tag{22}$$

Using (21), this is equal to $-e^{-\delta t}(\partial \pi_t/\partial x) - e^{-\delta t} M\Pi_t F'(x_t)$, or

$$\dot{\lambda}_t = -e^{-\delta t}\left[\tfrac{\partial \pi_t}{\partial x} + M\Pi_t F_t'\right]\,. \tag{23}$$

But differentiating (21) with respect to time gives $\dot{\lambda}_t = -\delta e^{-\delta t} M\Pi_t + e^{-\delta t} M\dot{\Pi}_t = e^{-\delta t}(M\dot{\Pi}_t - \delta M\Pi_t)$. Equating this with $\dot{\lambda}_t$ from (23) gives:

$$\begin{aligned} -e^{-\delta t}\left[\tfrac{\partial \pi_t}{\partial x} + M\Pi_t\, F_t'\right] &= e^{-\delta t}(M\dot{\Pi}_t - \delta M\Pi_t) \\ -\tfrac{\partial \pi_t}{\partial x} - M\Pi_t\, F_t' &= M\dot{\Pi}_t - \delta M\Pi_t\,. \end{aligned}$$

This leads directly to (19). ∎

Lemma 2. *Suppose that there are no fixed costs, i.e., that if $h = 0$ then $\pi = 0$ (formally: $\pi(x_t, 0, t) = 0$ for all t and all $x \geq 0$). Then if $h_t^* > 0$ and $\lambda_t^* \geq 0$, one has $\pi_t^* \geq 0$.*

Proof. Suppose not; then $h_t^* > 0$ and $\pi_t^* < 0$. From this and (18) and the assumption that $\lambda_t^* \geq 0$,

$$\mathscr{H} = e^{-\delta t}\pi(x_t, h_t, t) + \lambda_t F(x_t) - \lambda_t h_t < \lambda_t F(x_t)\,. \qquad (24)\,.$$

However, if instead one set $h_t = 0$, then the Hamiltonian would be $\mathscr{H} = e^{-\delta t}\pi(x_t, 0, t) + \lambda_t[F(x_t) - 0] = \lambda_t F(x_t)$, which is larger than the Hamiltonian in (24). Hence setting $h_t^* > 0$ does not maximize the Hamiltonian and is not actually optimal. This is a contradiction. ∎

For dates when the solution is not interior (which happens for problems linear in the control for dates when the solution is not singular), $h_t^* = 0$. On those dates the Kuhn-Tucker conditions (6) imply that instead of $\partial\mathscr{H}/\partial h = e^{-\delta t} M\Pi_t - \lambda_t$ being equal to zero, it would only have to be less than or equal to zero on those dates; so on those dates, (21) is replaced by

$$\lambda_t \geq e^{-\delta t} M\Pi_t \tag{25}$$

and (20) does not hold.

Since the transversality condition (14) is only applicable to problems that are autonomous (except possibly for geometric discounting), it only applies when $\pi(x_t, h_t, t)$ in (18) does not depend explicitly on t. Hence it does not apply when studying competitive firms. When it does apply, it requires

$$\lim_{t\to\infty}[e^{-\delta t}\pi + \lambda\,(F - h)] = 0\,. \tag{26}$$

If the solution is interior (or singular), (21) holds; substituting it into (26) yields

$$\lim_{t\to\infty} e^{-\delta t}[\pi + M\Pi\,(F - h)] = 0\,. \tag{27}$$

This will hold as long as the term in brackets, $[\pi + M\Pi(F - h)]$, grows with time more slowly than $e^{\delta t}$. This will be the case if the system approaches a steady state or a limit cycle, where a "steady state" is defined to be a situation where all time derivatives are zero, and a "limit cycle" is an "isolated closed trajectory." (Mathematicians sometimes call a steady state an "equilibrium," but we will not do that, reserving the term "equilibrium" to mean "quantity supplied equals quantity demanded," which may happen outside of a steady state.)

This completes listing the necessary conditions for solving (15) subject to (16) and (17), both for the nonlinear and the linear case. Traditionally, however, the linear case has been analyzed in a different way, and I explain that traditional way for the rest of this paragraph. When π is linear in h (a mathematician would say "when π is affine in h" as explained after (7)), it can be written as

$$\pi(x_t, h_t, t) = \pi_1(x_t, t)\, h_t + \pi_2(x_t, t)$$

for some functions π_1 and π_2. To rule out fixed costs, which give rise to nonconvexities which could imperil existence of an optimal solution, it is necessary to require that $\pi(x_t, 0, t)$ be identically zero; this means $\pi_2 \equiv 0$. That means that π_1 is equal to average profit and is equal to marginal profit; accordingly, I will rename π_1 to "$AM\Pi$." Using "$\triangleq$" to mean "is defined to be," the Hamiltonian then becomes

$$\begin{aligned}\mathcal{H} &= e^{-\delta t} AM\Pi(x_t, t)\, h_t + \lambda_t [F(x_t) - h_t] \\ &= [e^{-\delta t} AM\Pi(x_t, t) - \lambda_t]\, h_t + \lambda_t\, F(x_t) \\ &\triangleq \sigma(x_t, \lambda_t, t)\, h_t + \lambda_t\, F(x_t)\end{aligned} \tag{28}$$

defining the "switching function" σ as $e^{-\delta t} AM\Pi - \lambda$. This has the form of (7), so from (9) the optimal solution is:

$$h_t^* = \begin{cases} 0 & \text{if } e^{-\delta t} AM\Pi_t - \lambda_t < 0 \\ \in [0, \infty) & \text{if } e^{-\delta t} AM\Pi_t - \lambda_t = 0 \\ +\infty & \text{if } e^{-\delta t} AM\Pi_t - \lambda_t > 0. \end{cases} \tag{29}$$

In the singular solution, $e^{-\delta t} AM\Pi_t - \lambda_t = 0$, which means that $e^{-\delta t} M\Pi - \lambda_t = 0$, which is the same as (21). This is to be expected because (21) came from (5), which is valid both for interior solutions to nonlinear problems and for singular solutions to linear problems, as discussed just after (9). The nonsingular solution $h_t^* = 0$ has $e^{-\delta t} AM\Pi_t - \lambda_t \leq 0$, which means that $\lambda_t \geq e^{-\delta t} M\Pi_t$, just as concluded in (25). Note that if $\lambda_t > 0$ and $AM\Pi_t = p_t - c(x_t) < 0$ then the first line of (29) implies that $h_t^* = 0$; this loosely resembles a converse of Lemma 2.

Section 3. Private-Property Competition: First Pair of Examples (Search Fisheries & Constant Returns to Scale)

We will now consider two special cases:

$$\pi(x, h) = [\, p - c(x)]\, h \quad \text{and} \tag{30s}$$

$$\pi(x_t, h_t, t) = [\, p_t - c(x_t)]\, h_t. \tag{30d}$$

In both (30s) and (30d), price is not a function of h; therefore both (30s) and (30d) describe competitive behavior. (30s) is the special case of (30d) in which all the variables are constant. Therefore, if (30d) ever reaches a steady state, the steady state will be the solution to (30s).

In general, the average cost function is written as $c(x_t, h_t)$. If, as in (30s) and (30d), the average cost function c does not depend on h, then

- marginal cost is equal to average cost (where "marginal" means the derivative with respect to output h, not with respect to x, and "average" means divided by output not x, so "marginal" and "average" mean in these Notes what they mean in the rest of economics);
- we say that there is "constant average cost," even though $c(x)$, which is both average and marginal cost, may vary with x, because "constant average cost" in the rest of economics means that average cost *does not vary with output* and we want to use the same terminology in these Notes;

- the production function has constant returns to scale;
- π is linear in h; and (assuming perfect competition, as (30s) and (30d) do),
- average profit is equal to marginal profit; and finally, as explained in Section 2 in the paragraph after (18),
- $\mathscr{H}$ is linear in h.

In order for harvest to be neither zero nor infinity, (29) requires that the solution be singular. Therefore we will use (20) for studying both (30s) and (30d).

If c actually does not depend on x, the fishery is said to be a "schooling" fishery. By contrast if, as intended in (30s) and (30d), c does depend on x, the fishery is said to be a "search" fishery (see Philip A. Neher, *Natural Resource Economics: Conservation and Exploitation*, Cambridge University Press 1990, p. 177, p. 195). In a search fishery, $c'(x) < 0$: as the stock declines, average costs go up, the "stock effect." Throughout this section, we will make the additional assumption that

$$c''(x) > 0\,. \tag{31}$$

The first reason to make this assumption is that it is difficult to make the opposite assumption, $c'' < 0$, and draw a $c(x)$ function with $c' < 0$ that still obeys $c(x) > 0$ for all x. On the other hand, "difficult" does not mean impossible, and one could in addition argue that c'' could be negative for small and medium x and positive for x's so large as to be economically irrelevant. Another reason to assume (31) is that it is plausible that the stock effect is largest for very small x, where it becomes difficult to find any fish at all.

First consider (30s). It implies that $\partial\pi/\partial x = -c'(x)\,h$ and $M\Pi = p - c(x)$, so (19) leads to the following, where $F'(x)$ means $F'_x(x)$ not $\partial F/\partial t$:

$$\begin{aligned} \delta\,[p - c(x)] &= [p - c(x)]\,F'(x) + 0 - c'(x)\,h \\ [\delta - F'(x)]\,[p - c(x)] &= -c'(x)\,h\,. \end{aligned} \tag{32}$$

In a search fishery the right-hand side is not zero, so $p - c(x) \neq 0$ and

$$\begin{aligned} \delta - F'(x) &= \frac{-hc'}{p - c} \\ p - c &= \frac{-hc'}{\delta - F'} \end{aligned}$$

$$p = c(x) - \frac{hc'}{\delta - F'} \,. \tag{33}$$

(Time subscripts have been omitted because (30s) pertains to the steady state.) In a sense, (33) is a steady-state supply curve for fish—it can be rewritten as

$$\begin{aligned} h &= -\frac{\delta - F'(x)}{c'(x)}\, p + \frac{[c(x)]\,[\delta - F'(x)]}{c'(x)} \\ &= \frac{\delta - F'(x)}{c'(x)}\,[c(x) - p]\,, \end{aligned} \tag{34}$$

which explicitly shows h depending on p—but (34) contains x, which is tied to the value of h in the steady state in a way not captured by (34).[3] To get a complete "supply relationship" (I would not strictly call it a "supply curve"), substitute the steady-state condition $\dot{x} = 0$ into (16), obtaining

$$h = F(x), \tag{35}$$

then use (33) to obtain

$$p = c(x) - \frac{c'(x)\,F(x)}{\delta - F'(x)} \,. \tag{36}$$

(36) and (35) work together to yield a complete supply relationship: each value of x will give a value for price p from (36), and it will give a value for quantity h from (35), so running through values of x will give rise to price–quantity combinations; the graph of these combinations is the steady-state supply relationship of this firm. For details, see the Example 1 below or see Figure 5.12, p. 136 of Clark, reproduced as Figure 1 here. In that figure, Quadrant IV shows (35) and Quadrant II shows (36) in a special case which is analyzed in the paragraph after next; Clark denotes the right-hand side of (36) by "$H_\delta(x^*)$."

We know from Lemma 2 that for harvest in the steady state to strictly positive, profit must be nonnegative. In (30), $\pi = M\Pi\, h$, so for nonnegative steady-state profit, we need $M\Pi \geq 0$. Imposing that on (36) implies

$$0 \leq M\Pi = p - c = -\frac{c'(x)\,F(x)}{\delta - F'(x)} \,,$$

[3]From (32), another not-particularly-useful expression is $[\delta - F'(x)]\,[\phi(F(x)) - c(x)] = -c'(x)\,F(x)$ where ϕ is the inverse demand curve, as defined just before Proposition 1.

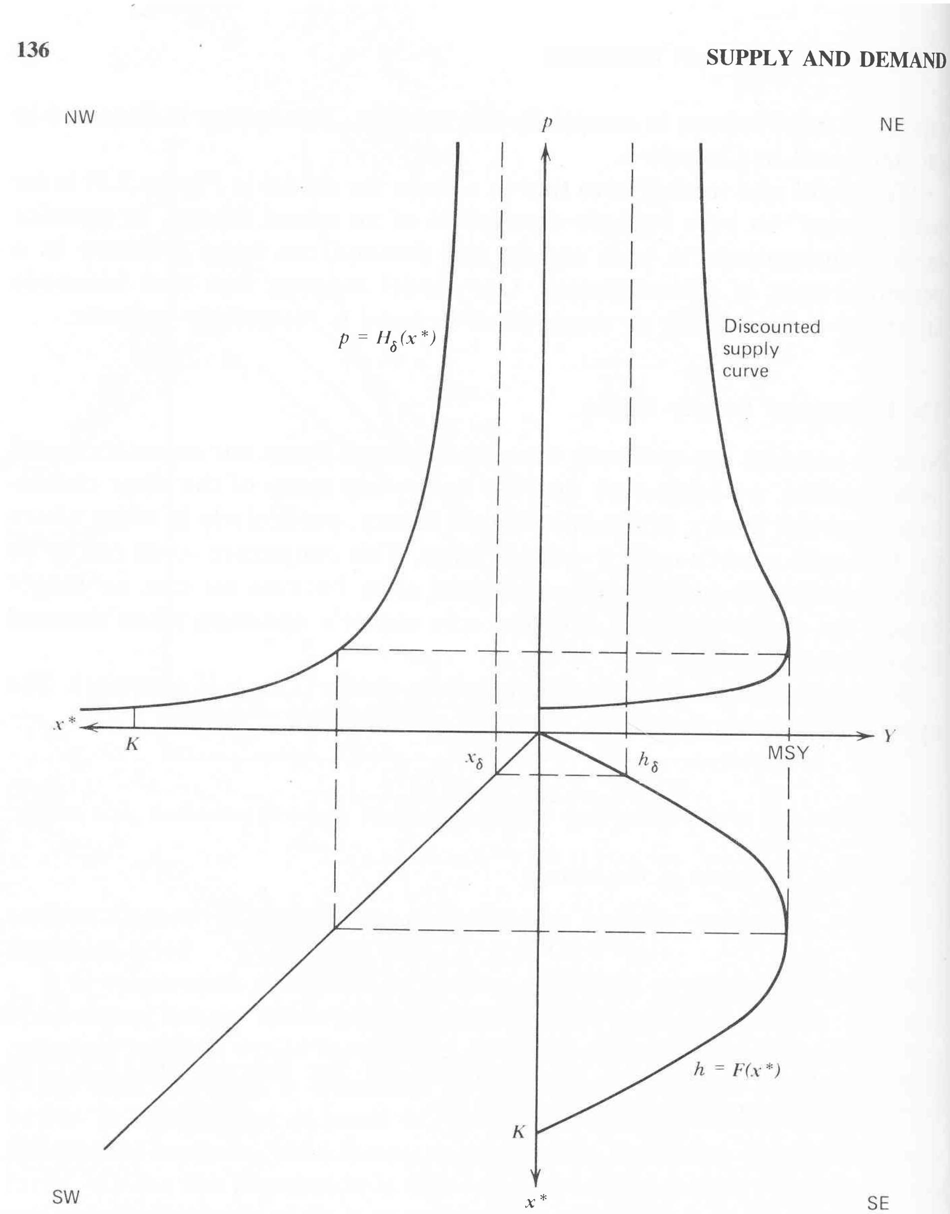

Figure 5.12. Discounted supply curve (Schaefer model).

Figure 1. Clark's Figure 5.12.

and since $c'(x) \leq 0$ and $F(x) \geq 0$ in the steady state because $F(x) = h \geq 0$ there, this implies that $\delta - F'(x) \geq 0$, so the requirement is that

$$F'(x) \leq \delta \tag{37}$$

at the steady-state value of x. (In (34), this would make h an increasing function of p, which we expect for a supply curve.) In the Examples and Exercises of these Notes we will consider the important special case where F is the "logistic" growth function

$$F(x) = rx\left(1 - \frac{x}{K}\right), \tag{38}$$

where r is the "intrinsic growth rate" and K is the "carrying capacity." In the absence of harvesting, (16) applied to (38) implies $\dot{x}_t = F(x_t) = rx - (r/K)x^2$, so the species' growth rate is

$$\begin{aligned} \frac{\dot{x}}{x} &= r - \frac{r}{K}x < r \quad \text{and} \\ \lim_{x \to 0}\left(\frac{\dot{x}}{x}\right) &= r, \end{aligned}$$

explaining why "r" is called the "intrinsic growth rate." For logistic growth,

$$\begin{aligned} F'(x) &= r - \frac{2r}{K}x \quad \text{so} \\ F'(0) &= r, \end{aligned}$$

another interpretation of r. For logistic growth, the requirement for postive profit, (37), is equivalent to

$$r - \frac{2rx}{K} \leq \delta. \tag{39}$$

Solving (39) for x yields

$$x \geq \frac{K}{2} \cdot \frac{r - \delta}{r}. \tag{40}$$

If we implicitly defined x_δ by

$$F'(x_\delta) = \delta \tag{41}$$

then $\delta = F'(x_\delta) = r - 2rx_\delta/K$ and

$$x_\delta = \frac{K}{2} \cdot \frac{r - \delta}{r}, \tag{42}$$

which would be of little use when $r - \delta < 0$. A more useful definition of x_δ is

$$\begin{aligned} F'(x_\delta) &= \delta \quad \text{if } r - \delta \geq 0 \text{ and} \\ x_\delta &= 0 \quad \text{otherwise.} \end{aligned} \tag{43}$$

Adopting this definition,

$$x_\delta = \begin{cases} \dfrac{K}{2} \cdot \dfrac{r-\delta}{r} & \text{if } r \geq \delta \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

The combination of "the positive-profit condition (40) and the obvious constraint that $x \geq 0$" can then be rewritten as

$$x \geq x_\delta\,. \tag{45}$$

If $\delta \geq r$, this imposes no restriction on steady-state x besides the obvious one that it be nonnegative; if $\delta = 0$, it imposes $x \geq K/2$; and if $\delta < r$ (that is, $\delta \in [0, r)$), it requires steady-state x to be greater than or equal to a number between zero and $K/2$.

Turning attention now to Quadrant II of Figure 1, first note that in general, one cannot show that $p = H_\delta(x)$ (the right-hand side of (36)) is monotonic in x: we have $c'(x) < 0$, $\delta - F'(x) > 0$ from (37), $F(x) > 0$, and $c''(x) > 0$ from (31), but $F'(x)$, and $F''(x)$ ambiguous (although in the special case of (38), $F'' < 0$ and F' is positive or negative as x is less than or greater than $K/2$) in:

$$\begin{aligned} \frac{dp}{dx} &= c'(x) - \frac{c''(x)F(x)}{\delta - F'(x)} - \frac{c'(x)F'(x)}{\delta - F'(x)} - \frac{F''(x)}{(\delta - F'(x))^2} c'(x)F(x) \\ &= (-) - \frac{(+)(+)}{(+)} - \frac{(-)F'(x)}{(+)} - \frac{(-)\text{if logistic}}{(+)}(-)(+) \text{ ; if logistic,} \\ &= (-) - (+) + \{(+) \text{ for small } x, (-) \text{ for large } x\} + (-)\,. \end{aligned}$$

For the logistic case (Quadrant IV shows this case), for large x this can be signed, and it is negative, as drawn in Quadrant II of Figure 1, but for small x it is ambiguous even in the logistic case. If we impose not only (38) but also

$$c(x) = \gamma / x \tag{46}$$

for some constant γ then we have what Clark (p. 45) calls "the Schaefer model." Figure 2 uses *Mathematica* to analyze it. The basic conclusions are as follows. (1) Although in Quadrant II of Clark's figure the relationship between p and x is monotonic, it does not seem possible to guarantee monotonicity, because dp/dx in 'Out[7]' is difficult to sign. (2) 'Out[6]' confirms Clark's figure's property that $\lim_{x\to\infty} p(x) = 0$. (3) The results after 'In[12]' are that $\lim_{p\to\infty} x(p)$ is zero if $\delta > r$ and it is $K\,(r-\delta)/(2r)$ if $\delta \leq r$. From (44) this means that

$$\lim_{p\to\infty} x(p) = x_\delta \tag{47}$$

as shown in Clark's graph, meaning that our definition of x_δ is the same as his, and that (45) and (47) are connected to requiring that profit be positive.

Equations (35) and (36) characterize the steady-state supply curve in this section. In this paragraph I first slightly extend an interesting result of Clark (pp. 60–61) to show that if δ is sufficiently large and if, unlike in (46), $c(0)$ is sufficiently small, then if (35) holds and if $x > 0$ then (36) cannot hold. To start, let the market inverse demand curve be denoted

$$p = \phi(Nh_t) \tag{48}$$

where p is price and N is the number of (identical) firms. (The most general form would instead be $p = \phi_t(N_t\, h_t)$, but I will suppose that neither N nor the demand curve are time-varying, so $\phi_t(Nh_t)$ can just be written as $\phi(Nh_t)$.) For notational simplicity, I will always assume $N = 1$. (The firm does not know that $N = 1$ so it still thinks it is in a competitive industry, not that it is a monopolist.)

Proposition 1. *Let* $\max_x F(x) \triangleq MSY$ *for "maximum sustainable yield." Suppose that (35) holds, that* $x > 0$*, and that:*

$$\begin{aligned}
&c''(x) \geq 0 \text{ (this is (31))},\\
&c(0) < \phi(MSY)\,,\\
&\delta > 2F'(0)\,,\\
&F''(x) < 0\,\forall\, x > 0, \text{ and}\\
&F(0) = 0.
\end{aligned}$$

Then (36) cannot hold.

Remark. Since $h \leq MSY$, in the right-hand side of the second equation $\phi(MSY) < \phi(h)$, that is, $\phi(MSY)$ is the smallest market equilibrium price possible. Since c is a decreasing function of x, $c(x) < c(0)$, that is, $c(0)$ is the largest marginal cost possible. The second equation thus guarantees that marginal cost is less than price for all possible equilibrium x and h.

```
In[1]:= (* Schaefer model, p. 45 of Clark 1976 *)
        F[x_] := r x (1 - x / K)
        c[x_] := gamma / x

In[3]:= (* Clark p. 157 H_delta *)
        c[x] - D[c[x], x] F[x] / (delta - D[F[x], x])
```

Out[3]= $$\frac{\text{gamma}}{x} + \frac{\text{gamma}\ r\left(1 - \frac{x}{K}\right)}{x\left(\text{delta} + \frac{r\,x}{K} - r\left(1 - \frac{x}{K}\right)\right)}$$

```
In[4]:= Simplify[%]
```

Out[4]= $$\frac{\text{gamma}\ (\text{delta}\ K + r\,x)}{x\ (\text{delta}\ K - K\,r + 2\,r\,x)}$$

```
In[5]:= p[x_] = Simplify[%%]
```

Out[5]= $$\frac{\text{gamma}\ (\text{delta}\ K + r\,x)}{x\ (\text{delta}\ K - K\,r + 2\,r\,x)}$$

```
In[6]:= Limit[p[x], x → Infinity]
```

Out[6]= 0

```
In[7]:= D[p[x], x] // Simplify
```

Out[7]= $$-\frac{\text{gamma}\left(\text{delta}^2 K^2 - \text{delta}\ K\,r\ (K - 4\,x) + 2\,r^2 x^2\right)}{x^2\ (\text{delta}\ K - K\,r + 2\,r\,x)^2}$$

```
In[8]:= Solve[p == p[x], x]
```

Out[8]= $$\left\{\left\{x \to \frac{-\text{delta}\ K\,p + \text{gamma}\ r + K\,p\,r - \sqrt{8\ \text{delta}\ \text{gamma}\ K\,p\,r + (\text{delta}\ K\,p - \text{gamma}\ r - K\,p\,r)^2}}{4\,p\,r}\right\},\right.$$
$$\left.\left\{x \to \frac{-\text{delta}\ K\,p + \text{gamma}\ r + K\,p\,r + \sqrt{8\ \text{delta}\ \text{gamma}\ K\,p\,r + (\text{delta}\ K\,p - \text{gamma}\ r - K\,p\,r)^2}}{4\,p\,r}\right\}\right\}$$

```
In[9]:= x[p_] := x /. Part[Solve[p == p[x], x] , 2]

In[10]:= x[p]
```

Out[10]= $$\frac{-\text{delta}\ K\,p + \text{gamma}\ r + K\,p\,r + \sqrt{8\ \text{delta}\ \text{gamma}\ K\,p\,r + (\text{delta}\ K\,p - \text{gamma}\ r - K\,p\,r)^2}}{4\,p\,r}$$

```
In[11]:= Limit[x[p], p → Infinity]
```

Out[11]= $$\frac{-\text{delta}\ K + \sqrt{K^2\ (\text{delta} - r)^2} + K\,r}{4\,r}$$

```
In[12]:= FullSimplify[Limit[x[p], p → Infinity], Assumptions → {K > 0}] /. Sqrt[x_^2] → Abs[x]

         Simplify[%, Assumptions → {delta - r < 0}]
         Simplify[%%, Assumptions → {delta - r >= 0}]
```

Out[12]= $$\frac{K\ (-\text{delta} + r + \text{Abs}[\text{delta} - r])}{4\,r}$$

Out[13]= $$\frac{K\ (-\text{delta} + r)}{2\,r}$$

Out[14]= 0

Figure 2. A *Mathematica* analysis of Quadrant II of Figure 1 and of (36).

Proof. Rewrite (36) as

$$\delta - F'(x) = \frac{-c'(x)\, F(x)}{p - c(x)} .$$

In market equilibrium, this means steady-state x and h obey

$$\frac{-c'(x)\, F(x)}{\phi(h) - c(x)} = \delta - F'(x) . \tag{49}$$

Suppose ξ is in $(0, x)$. I will show below that

$$\begin{align}
\frac{-c'(x)\, F(x)}{\phi(h) - c(x)} &\leq \frac{-c'(x)\, F(x)}{\phi(MSY) - c(x)} \tag{50}\\
&< \frac{-c'(x)\, F(x)}{c(0) - c(x)} \tag{51}\\
&= \frac{c'(x)}{c'(\xi)}\, F'(\xi) \tag{52}\\
&\leq F'(\xi) \tag{53}\\
&< F'(0) \tag{54}\\
&< F'(0) + [F'(0) - F'(x)] \tag{55}\\
&= 2F'(0) - F'(x) \tag{56}\\
&< \delta - F'(x) . \tag{57}
\end{align}$$

Since the left-hand side ("LHS") of (50) is the LHS of (49), and the right-hand side ("RHS") of (57) is the RHS of (49), this will prove it is impossible for (49) to hold.

To prove (50): First note that $-c' \geq 0$. By definition, $F(x) \leq MSY$. By (35), this means $h \leq MSY$. Since ϕ is downward-sloping, $\phi(h) \geq \phi(MSY)$.

To prove (51): use the second assumption of the proposition.

To prove (52): By the Generalized Mean Value Theorem (sometimes known as the Cauchy Mean Value Theorem),[4]

$$\frac{F(0) - F(x)}{c(0) - c(x)} = \frac{F'(\xi)}{c'(\xi)} \quad \text{for some } \xi \in (0, x). \tag{58}$$

[4]The Mean Value Theorem itself states, for example, that $c(0) - c(x) = c'(\xi)(0 - x)$ for some $\xi \in (0, x)$. Bartle (*Elements of Real Analysis*, Second Edition, p. 197) writes, "In fact the Mean Value Theorem is a wolf in sheep's clothing and is *the* Fundamental Theorem of the Differential Calculus." One could divide $F'(\hat{\xi}) = (F(0) - F(x))/(0 - x)$ for $\hat{\xi} \in (0, x)$ by $c'(\xi) = (c(0) - c(x))/(0 - x)$ to obtain the left-hand side of (58), but the other side would be $F'(\hat{\xi})/c'(\xi)$ instead of the right-hand side of (58), so the Generalized Mean Value Theorem is not a trivial consequence of the Mean Value Theorem.

The last assumption of the proposition gives $F(0) = 0$; hence

$$\frac{-F(x)}{c(0) - c(x)} = \frac{F(0) - F(x)}{c(0) - c(x)} = \frac{F'(\xi)}{c'(\xi)} \quad \text{for some } \xi \in (0, x).$$

To prove (53): $\xi < x$, and by the proposition's first assumption, $c'' \geq 0$, so $c'(\xi) \leq c'(x)$. Dividing by $c'(\xi)$ and recalling that $c'(x) < 0$ for all x, we get $1 \geq c'(x)/c'(\xi) > 0$.

To prove (54): $0 < \xi$, and by the proposition's fourth assumption, $F'' < 0$, so $F'(0) > F'(\xi)$.

To prove (55): The proposition assumes that $0 < x$. Then as in the proof of (54), $F'(0) > F'(x)$; so the term in brackets in (55) is positive.

(56) is trivial, and (57) follows from the proposition's third assumption.

∎

Corollary. *Under the conditions of Proposition 1, the only possible steady state would have $x = 0$.*

The assumption $F(0) = 0$ is true in any fishery; and the assumption $F'' < 0$, called "pure compensation," means $F(x)$ is larger for small x than in the cases of "depensation" (namely F is convex for small x, then becomes concave, while $F > 0 \, \forall x \in (0, K)$) or "critical depensation" (namely $F < 0$ for some small x). The economically important assumptions therefore are the second and third. The second states that the cost of driving the stock to extinction is not too high, and that the demand for fish is not too low. The third requires δ to be high, as mentioned above.[5] As stated in the corollary, under the conditions of the proposition the only possible steady-state outcome would be extinction. Exercise 5 below suggests a graphical method by which one might be able to show that what happens is indeed a steady state with extinction. However the proposition does not rule out non-steady-state outcomes (such as limit cycles), so if a graphical analysis such as that suggested by Exercise 5 does not show that the outcome is a steady state with extinction, then a dynamic analysis would be needed to determine whether or not extinction is the inevitable outcome of the situation in the proposition. (Clark p. 61 by contrast says that '(35) holds but (36) does not hold' "clearly implies that extinction is optimal." This seems hasty.[6])

[5] The proposition goes through even if either the third or the fourth strict inequality in its assumptions is turned into a weak inequality.

[6] Clark p. 61 also says "We show that in this case Eq. (2.42) has no solution $x \geq 0$," but his proof only goes through if it is assumed that $x > 0$.

Example 1, steady-state: (30s) with: $c(x) = \frac{\gamma}{qx}$, $q = 1$, $\gamma = 50$; $F(x) = rx(1 - \frac{x}{K})$ with $r = 0.1$, $K = 100$; and $\delta = 0.2$. (The function $c(x)$ has two constants instead of one by tradition.)

Aside: in this example (36) leads to

$$p = \frac{\gamma}{qx} \frac{K\delta + rx}{K(\delta - r) + 2rx} \tag{59}$$

because $c'(x) = -\gamma/(qx^2)$ and $F'(x) = r - (2rx/K)$, so substituting into (36),

$$\begin{aligned} p &= \frac{\gamma}{qx} - \frac{\frac{-\gamma}{qx^2} rx(1 - \frac{x}{K})}{\delta - r + \frac{2rx}{K}} \\ &= \frac{\gamma}{qx} - \frac{\frac{-\gamma}{qx} r(K - x)}{K(\delta - r) + 2rx} \\ &= \frac{\gamma}{qx} + \frac{\gamma r(K - x)}{qK(\delta - r) + 2qrx} \\ &= \frac{\gamma}{qx} \left[1 + \frac{r(K - x)}{K(\delta - r) + 2rx} \right] \\ &= \frac{\gamma}{qx} \frac{K\delta - Kr + 2rx + rK - rx}{K(\delta - r) + 2rx} \end{aligned}$$

which simplifies to the above expression.

Figure 3 is a steady-state supply curve derived using *Mathematica* in the following way.

```
(* For Steady-State Supply Curve *)

(* Definitions:

        x: stock size (of fish)
     c[x]: average & marginal cost (decreases in x, constant in h)
     F[x]: excess of natural births over deaths
    delta: interest rate
        r: intrinsic growth rate
        K: carrying capacity
        h: harvest size
   phi[h]: inverse demand curve, not used below
xdot[x,h]: derivative of x with respect to time
hdot[x,h]: derivative of h with respect to time
*)


p[x_] := c[x] - (D[c[x], x]*F[x])/(delta - D[F[x], x])
   (* steady-state price as a function of x; see above *)
h[x_] := F[x]
   (* steady-state yield as a function of x; see above. *)

(* One example *)
```

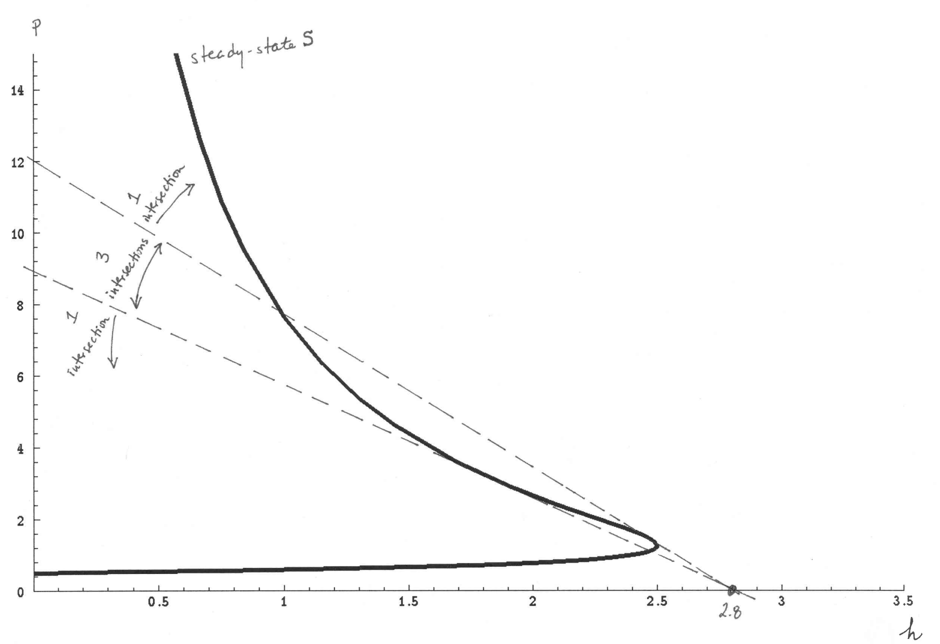

Figure 3. Dynamic analysis of the upper-most “1 intersection” region is given in Figure 5A based on (72); of the middle “3 intersections” region is given in Figure 7A based on (74); and of the lower-most “1 intersection” region is given in Figure 9A based on (76).

```
c[x_] := gamma/(q*x)
F[x_] := r*x*(1 - x/K) (* logistic growth *)

gamma = 50
q = 1
r = 0.1  (* intrinsic growth rate *)
K = 100  (* carrying capacity *)
delta = 0.2 (* interest rate *)

(* Steady-State Supply Curve *)
ParametricPlot[Evaluate[{h[x],p[x]}], {x,1,100}];
Show[%, PlotRange->{{0,3.5},{0,15}}];
```

Figures 3 and 4 show the steady-state supply curve. Intersections of it with the market demand curve will determine the steady-state equilibrium values of h and p.

For concreteness, suppose the demand curve is linear and pivots around the point $h = 2.8$, as sketched in Figure 3. As the slope gets progressively steeper, demand intersects supply: once; twice (but only for a single slope); three times; twice (but only for a single slope); and once. So, depending on the slope of the demand curve, there may be one, two, or three steady-state equilibria.

On the other hand, if the demand curve is linear and pivots around the point $h = 3.5$, as sketched in Figure 4, then demand always intersects supply exactly once.

Now consider (30d). As before (though now with time subscripts), $M\Pi_t = p_t - c(x_t)$ and $\partial\pi/\partial x = -c'(x_t)\, h_t$. We also have $\dot{M\Pi}_t = \dot{p}_t - c'(x_t)\,\dot{x}_t$. Substituting (30d) into (19) therefore gives

$$\delta\,(p-c) = (p-c)F' + \dot{p} - c'\dot{x} - c'h$$

$$(\delta - F')(p-c) = \dot{p} - c'\dot{x} - c'h \tag{60}$$

$$= \dot{p} - c'F \tag{61}$$

recalling (16).

One could solve (60) for h to give

$$h = \frac{[-p + c(x)]\,[\delta - F'(x)]}{c'(x)} + \frac{\dot{p} - c'(x)\,\dot{x}}{c'(x)}\,. \tag{62}$$

If all these time derivatives are zero, (62) is the same as (34). However, (62) is too complicated to be particularly enlightening. Alternatively, one could solve (61) for p to get

$$p = c(x) + \frac{\dot{p} - c'(x)\,F(x)}{\delta - F'(x)}$$

$$= c(x) + \frac{\phi'(h)\,\dot{h} - c'(x)\,F(x)}{\delta - F'(x)} \tag{63}$$

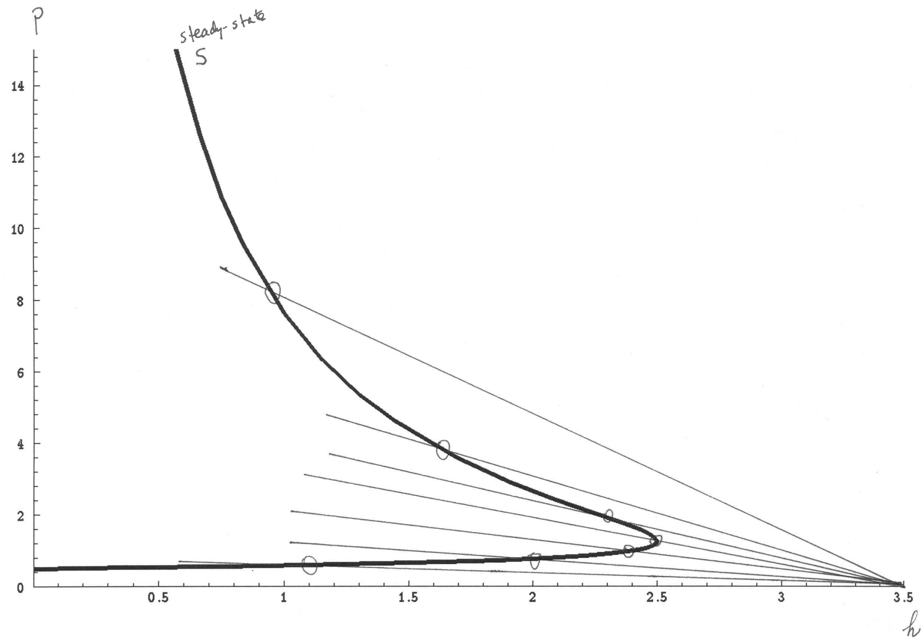

Figure 4.

then combine it with (from (16))

$$h = F(x_t) - \dot{x}_t \tag{64}$$

to get a dynamic version of the pair (36) and (35). However, the presence of time derivatives makes it impossible to proceed as before to sketch a (now dynamic) supply curve.

At this point, we abandon any attempt to further describe the firm's supply response, and instead proceed to combine: (i) what we know about the firm's supply response, with (ii) a market demand curve. This will enable us to derive the market equilibrium dynamic paths. Substituting $p_t = \phi(h_t)$ into (63) gives:

$$\phi = c + \frac{\phi' \dot{h} - c' F}{\delta - F'}$$
$$(\delta - F')(\phi - c) = \phi' \dot{h} - c' F\,.$$

This leads to

$$\dot{h}_t = \frac{[\delta - F'(x_t)][\phi(h_t) - c(x_t)] + c'(x_t)\,F(x_t)}{\phi'(h_t)}\,, \tag{65}$$

which along with

$$\dot{x}_t = F(x_t) - h_t \tag{16}$$

forms a dynamical system of the form

$$\begin{aligned} \dot{x}_t &= f_1(x_t, h_t) \\ \dot{h}_t &= f_2(x_t, h_t) \end{aligned} \tag{66}$$

where f_1 is the right-hand side of (16) and f_2 is the right-hand side of (65). To derive the phase-plane diagrams of the dynamics one finds the isoclines (more properly, the "nullclines"), which are the set of all (x, h) which make $\dot{x}$ or $\dot{h}$ equal to zero.

If one started on a point (x, h) which made $\dot{x} = 0$, then kept x the same but increased h, (16) implies that $\dot{x}$ would change from being zero to being negative. So the area of the (x, h) plane which lies above the $\dot{x} = 0$ isocline has $\dot{x} < 0$; similarly, the area of the (x, h) plane which lies below the $\dot{x} = 0$ isocline has $\dot{x} > 0$.

If one started on a point (x, h) which made $\dot{h} = 0$, then kept x the same but increased h, (65) implies that $\dot{h}$ would change from being zero. To see how it would change, abbreviate (65)'s numerator so that

$$\dot{h} = \frac{\alpha(x, h)}{\phi'(h)}\,.$$

Then on the $\dot{h} = 0$ curve,

$$
\begin{aligned}
\frac{\partial \dot{h}}{\partial h} &= \frac{\partial\alpha/\partial h}{\phi'} - \frac{\phi''\alpha}{(\phi')^2} = \frac{\partial\alpha/\partial h}{\phi'} - \frac{\phi''}{\phi'}\dot{h} \\
&= \frac{\partial\alpha/\partial h}{\phi'} - \frac{\phi''}{\phi'}\cdot 0 = \frac{\partial\alpha/\partial h}{\phi'} \\
&= \frac{(\delta - F')\phi'}{\phi'} = \delta - F'\,. \qquad (67)
\end{aligned}
$$

Hence if $\delta > F'(x)$, deviating from the $\dot{h} = 0$ isocline by raising h will raise $\dot{h}$ from zero to something positive—implying that the area of the (x, h) plane which lies above the $\dot{h} = 0$ isocline has $\dot{h} > 0$, and the area of the (x, h) plane which lies below the $\dot{h} = 0$ isocline has $\dot{h} < 0$. Conversely, if $\delta < F'(x)$, then deviating from the $\dot{h} = 0$ isocline by raising h will lower $\dot{h}$ from zero to something negative—implying that the area of the (x, h) plane which lies above the $\dot{h} = 0$ isocline has $\dot{h} < 0$, and the area of the (x, h) plane which lies below the $\dot{h} = 0$ isocline has $\dot{h} > 0$. The $\dot{x} = 0$ and $\dot{h} = 0$ isoclines thus divide the (x, h) phase plane into "isosectors"; inside one isosector, neither $\dot{x}$ nor $\dot{h}$ change signs.

Because neither f_1 nor f_2 in (66) depends explicitly on t, the dynamic system (66) is autonomous. It follows that paths in its phase space cannot cross, because every (x, h) generates a unique $(\dot{x}_t, \dot{h}_t)$, that is, a unique direction of motion in phase space.

In this paragraph we show that h_t^* will not take jumps in the interior of the (x, h) plane. From (18) and (30d), the Hamiltonian is

$$\mathscr{H} = [e^{-\delta t}p_t - e^{-\delta t}c(x_t) - \lambda_t]\cdot h_t + \lambda_t F(x_t)\,.$$

(This has the form of (28).) When h_t^* is interior, the term in brackets must be equal to zero (as in (29)), so

$$p_t = c(x_t) + \lambda_t e^{\delta t}\,. \qquad (68)$$

Looking at the right-hand side of (68), x is continuous at τ regardless of whether h is continuous at τ or not; so $c(x)$ is continuous at τ. Also $\lambda_t e^{\delta t}$ is continuous at τ since continuity of λ is a basic property asserted by the Maximum Principle. It follows that the entire righthand side of (68) is continuous at τ. So p_t has to be continuous at τ; but in equilibrium, $p_t = \phi(h_t)$, so continuity of p at τ (and continuity of $\phi(h)$, which we certainly assume for $h > 0$ and sometimes assume for $h = 0$ as well) implies continuity of h at τ.

In this paragraph we show that $\lambda_t^* > 0$ (strictly) if and only if $h_\tau > 0$ (strictly) over an interval of positive measure where $\tau > t$. Using (30d), the "f" of (1) is $\pi_t\, e^{-\delta t} = e^{-\delta t}[p_t - c(x_t)]\, h_t$. It follows that setting $t = 0$ in (12) implies

$$\lambda_0^* = \frac{1}{\beta_t} \int_0^T \beta_\tau\, e^{-\delta \tau}(-c'(x_\tau^*))\, h_\tau^*\, d\tau\,. \tag{69}$$

Since $\beta_\tau > 0$ from the discussion concerning (12), and since $e^{-\delta\tau} > 0$, and since $-c'(x_\tau^*) \geq 0$, and since $h_\tau^* \geq 0$, one can conclude from (69) that as long as h^* is strictly positive over some interval of time—that is, as long as some fishing is going to occur sometime—it will be true that $\lambda_0^* > 0$, and from (10), $\partial J/\partial x_0 > 0$.

While (65) and (16) follow from the necessary conditions for solving (15) and (16) given (30d), the question of sufficiency arises. The Mangasarian sufficiency condition of Section 1 requires

$$\nabla^2 \mathscr{H} = \begin{bmatrix} \mathscr{H}''_{hh} & \mathscr{H}''_{hx} \\ \mathscr{H}''_{xh} & \mathscr{H}''_{xx} \end{bmatrix}$$

to be negative semidefinite. However, here (using $\triangleq$ to mean "is defined to be," as in (28))

$$\begin{aligned} \mathscr{H} &= e^{-\delta t}\big[p_t - c(x_t)\big] h_t + \lambda_t \big[F(x_t) - h_t\big] \\ &= \big\{e^{-\delta t}[p_t - c(x_t)] - \lambda_t\big\} h_t + \lambda_t\, F(x_t) \\ &\triangleq \sigma(x_t, \lambda_t, t)\, h_t + \lambda_t\, F(x_t)\,, \end{aligned} \tag{70}$$

meaning that $\mathscr{H}''_{hh} = 0$, hence that $|\nabla^2\mathscr{H}| = -(\mathscr{H}''_{hx})^2 = -(\partial\sigma/\partial x)^2 = -[-e^{-\delta t}c'(x_t)]^2$, which means $\mathscr{H}$ fails the test for strict concavity, and fails the test for concavity whenever $\mathscr{H}''_{hx} \neq 0$ (in other words, whenever $c' \neq 0$), as is always the case for search fisheries. To check the Arrow sufficiency condition, first note that either we follow a singular solution, in which case $\sigma = 0$, or $h^* = 0$ (or $h^* = \infty$, which is not interesting); in either case, $\mathscr{H}^* = \lambda_t\, F(x_t)$. Along a singular solution, λ is given by (21) as $e^{-\delta t} M\Pi$, and here $M\Pi = p - c$, so

$$\mathscr{H}^* = e^{-\delta t}[p_t - c(x_t^*)] F(x_t^*)\,.$$

Then $\mathscr{H}_x^{*\prime} = e^{-\delta t}[-c'F + (p - c)F']$ and $\mathscr{H}_{xx}^{*\prime\prime} = e^{-\delta t}[-c''F - c'F' - c'F' + (p - c)F'']$; collecting terms,

$$\mathscr{H}_{xx}^{*\prime\prime} = e^{-\delta t}[(p - c)F'' - 2c'F' - c''F]\,. \tag{71}$$

In many situations (though not all) we can rule out F being negative; for example, with logistic growth, as long as x is never greater than the carrying capacity, F will not be negative. Typically, $p-c > 0$ (from Lemma 2), $F'' < 0$ (from logistic growth), $c' < 0$ (which characterizes a search fishery), and $c'' > 0$ (which is (31)). F' can have either sign, but if $F'(x_t^*) < 0\,\forall\, t$ together with the other common conditions, (71) would be negative and the Arrow sufficient condition for a maximum would hold. For the case of logistic growth, a sufficient condition for $F'(x_t^*) < 0\,\forall\, t$ is that $x_t^* > K/2\,\forall\, t$.

We will further illustrate the qualitative theory of dynamic systems such as (65) and (16) by going through "Example 1, dynamics" below. This paragraph briefly discusses the quantitative theory (see *Natural Resource Economics: Notes and Problems* by Jon M. Conrad and Colin W. Clark, 1987, pages 45 and 52). General mathematical notation for such systems is

$$\dot{x}_t = F(x_t, y_t)$$
$$\dot{y}_t = G(x_t, y_t)$$

where this F is unrelated to fisheries. Let (x^*, y^*) be a steady-state point, i.e., a point making both F and G equal to zero. A first-order Taylor Series approximation to F can be written

$$F(x, y) \approx F(x^*, y^*) + F'_x(x^*, y^*)(x - x^*) + F'_y(x^*, y^*)(y - y^*)\,.$$

A similar approximation holds for G. Indeed, if we take gradient vectors, e.g. $\nabla F^* = [F'_x \;\; F'_y]$, to be row vectors, we could write

$$\begin{bmatrix} F \\ G \end{bmatrix} \approx \begin{bmatrix} F^* \\ G^* \end{bmatrix} + \begin{bmatrix} \nabla F^* \\ \nabla G^* \end{bmatrix} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}\,.$$

Since (x^*, y^*) is defined to be a steady-state point, both F^* and G^* are zero. Taking this into account and writing the gradients explicitly, to first order,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} F'^*_x & F'^*_y \\ G'^*_x & G'^*_y \end{bmatrix} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}\,.$$

Note that the first matrix on the right-hand side consists entirely of numbers (constants), not variables. If we define $\xi = x - x^*$ and $\eta = y - y^*$, then the (ξ, η) plane has its origin precisely at the point (x^*, y^*). Clearly $\dot{\xi} = \dot{x}$ and $\dot{\eta} = \dot{y}$. So

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} F'^*_x & F'^*_y \\ G'^*_x & G'^*_y \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}\,,$$

a system of *linear* differential equations whose solution is

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = a\,\mathbf{v}_1\,e^{R_1 t} + b\,\mathbf{v}_2\,e^{R_2 t}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors, and R_1 and R_2 are the corresponding eigenvalues, of the matrix of the differential equation system. It can be shown that if the eigenvalues are both positive, the steady-state point (the "node") is unstable; if both are negative, the node is stable; if one is positive and one is negative, it is a saddle point; if the eigenvalues are complex then they will occur in a "conjugate pair" $\alpha \pm \beta i$ where $i = \sqrt{-1}$, and if the real part ("α") is positive, the node is an unstable spiral, whereas if the real part is negative, the node is a stable spiral (if the real part is zero, the node is a center). If, as often happens in economics, both eigenvalues are real and one is positive and one is negative—pick $R_2 < 0 < R_1$ since it is arbitrary which one is positive and which one is negative—then in order for an arbitrary (ξ_0, η_0) to go to $(\xi_t^*, \eta_t^*) = (0, 0)$ as $t \to \infty$, we need $a = 0$. If we therefore impose $a = 0$, then the ratio of ξ to η as $t \to \infty$ is given by $\mathbf{v}_2$. For example, if $\mathbf{v}_2 = \begin{bmatrix} 1 \\ m \end{bmatrix}$, then the slope of the convergent separatrix is m (in the ξ-η plane, but it has the same slope in the x-y plane).

Example 1, dynamics. Assume (30d) with (just as "Example 1, steady state"): $c(x) = \frac{\gamma}{qx}$, $q = 1$, $\gamma = 50$; $F(x) = rx(1 - \frac{x}{K})$ with $r = 0.1$, $K = 100$; and $\delta = 0.2$. Unlike in "Example 1, steady state," it is necessary to also assume a particular demand function $\phi(h)$. We will assume linear demand curves which pivot as in Figure 3, not as in Figure 4. Since these demand curves are all linear and all pivot around the point $h = 2.8$, they all have the form $p = (-b/2.8)h + b$ where b is their p-intercept. We will use five such demand curves:

$$p = (-15/2.8)h + 15 \tag{72}$$

$$p = (-12.2/2.8)h + 12.2 \tag{73}$$

$$p = (-10/2.8)h + 10 \tag{74}$$

$$p = (-9/2.8)h + 9 \tag{75}$$

$$p = (-5/2.8)h + 5. \tag{76}$$

Two of these demand curves ((73) and (75)) are the dashed lines in Figure 3; the other three lie in the three sectors into which the dashed lines divide Figure 3.

Assuming that the *Mathematica* code in the example above has been read in, the procedure to draw isoclines with *Mathematica* follows.

```
(* For Isoclines of the Phase Diagram *)

(* Definitions: *)

xdot[x_, h_] := F[x] - h
hdot[x_, h_] :=
  ((delta - D[F[x], x])*(phi[h] - c[x]) + D[c[x], x]*F[x]
  )/D[phi[h], h]

xdotisocline =
```

```
ContourPlot[Evaluate[xdot[x,h]],{x,1,125},{h,0,3.5},
     Contours->{0},ContourShading->False,PlotPoints->50];
(* this got the xdot=0 isocline *)

(* First demand curve *)
intercept = 15;
phi[h_] := (-intercept*h)/2.8 + intercept
hdotisocline=
ContourPlot[Evaluate[hdot[x,h]],{x,1,125},{h,0,3.5},
     Contours->{0},ContourShading->False,PlotPoints->50];
Show[hdotisocline,xdotisocline];
(* for a finite final date *)
terminalsurface=
  Plot[h/.Solve[phi[h]==c[x], h][[1]][[1]],
  {x,1,125},PlotRange->{0,3.5}];
Show[hdotisocline,xdotisocline,terminalsurface,
     PlotRange->{0,3.5}];


(* Second Demand Curve*)
intercept = 12.2;
phi[h_] := (-intercept*h)/2.8 + intercept
hdotisocline=
ContourPlot[Evaluate[hdot[x,h]],{x,1,125},{h,0,3.5},Contours->{0},
         ContourShading->False,PlotPoints->50];
Show[hdotisocline,xdotisocline];
terminalsurface=
  Plot[h/.Solve[phi[h]==c[x], h][[1]][[1]],
  {x,1,125},PlotRange->{0,3.5}];
Show[hdotisocline,xdotisocline,terminalsurface,
     PlotRange->{0,3.5}];

(* etc. *)
```

Using these techniques, demand curve (72) yields Fig. 5A; similarly, (73) yields Fig. 6A; (74) yields Fig. 7A; (75) yields Fig. 8A; and (76) yields Fig. 9A. See also Figure 6.12, p. 187 of Clark, reproduced as Figure 11 here. The “B” versions of each graph, and Figure 10, are explained after the Exercises below.

In this example, growth is logistic and $\delta = 0.2 > 0.1 = r \geq F'(x)$, so $\delta > F'(x)$ for all x. This means from (67) that $\partial \dot{h}/\partial h > 0$ (the area of the (x, h) plane which lies above the $\dot{h} = 0$ isocline has $\dot{h} > 0$). It means, from the notation of (43), that $x_\delta = 0$.

In this example, the given functional forms (without needing to replace p with $\phi(h)$) imply that (71) is

$$\mathcal{H}^{*}_{xx}{}'' = \frac{-2pr}{K}\,.$$

This is strictly negative and shows according to the Arrow sufficiency result that any path in this example which satisfies the necessary conditions is optimal, and that there is only one optimal path for x.

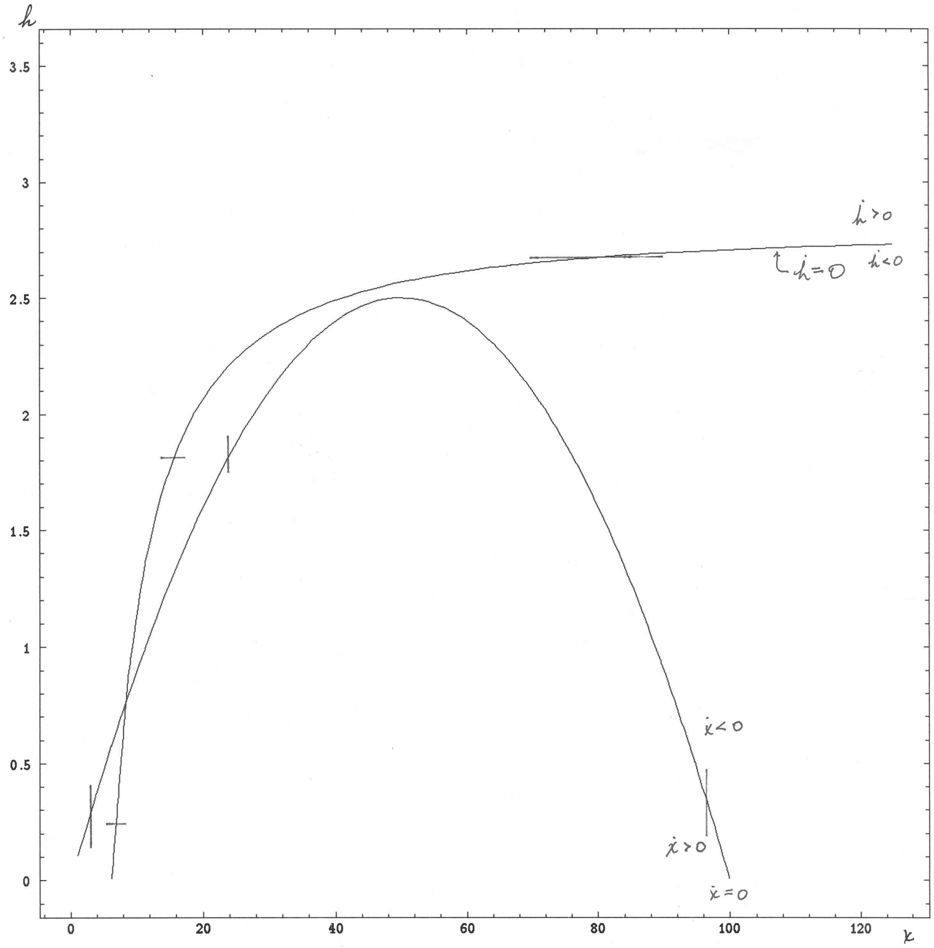

Figure 5A. Demand curve is $p = -\frac{15}{2.8}h + 15$ (equation (72), corresponding to an intersection in the upper-most "1 intersection" region of Figure 3).

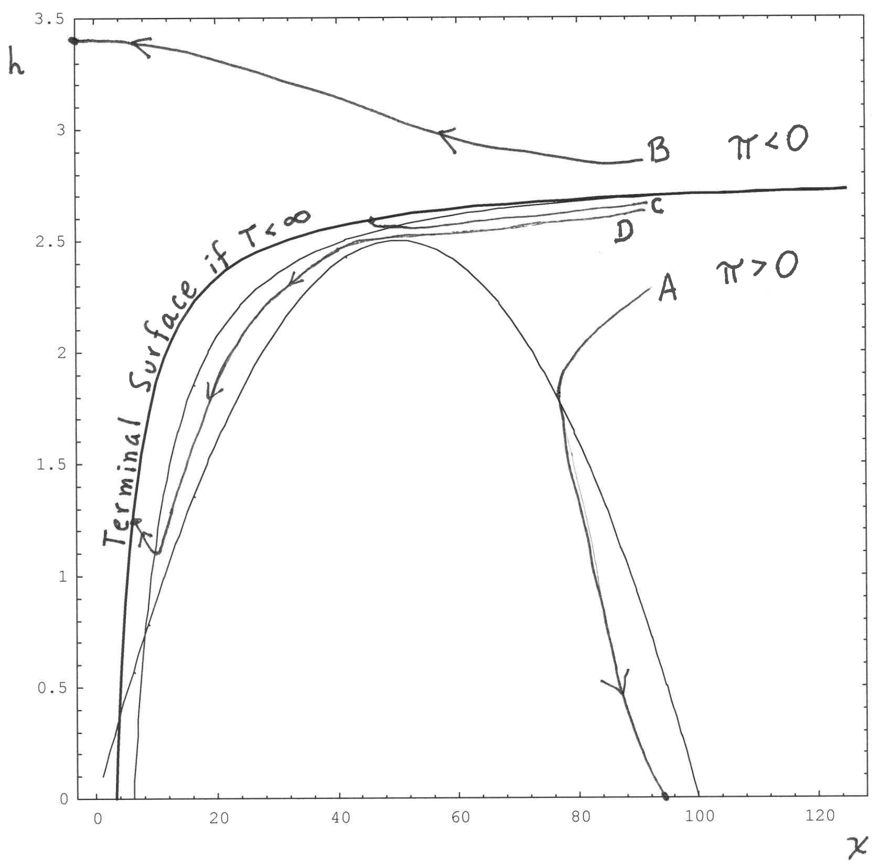

Figure 5B. Figure 5A combined with the finite-time terminal surface.

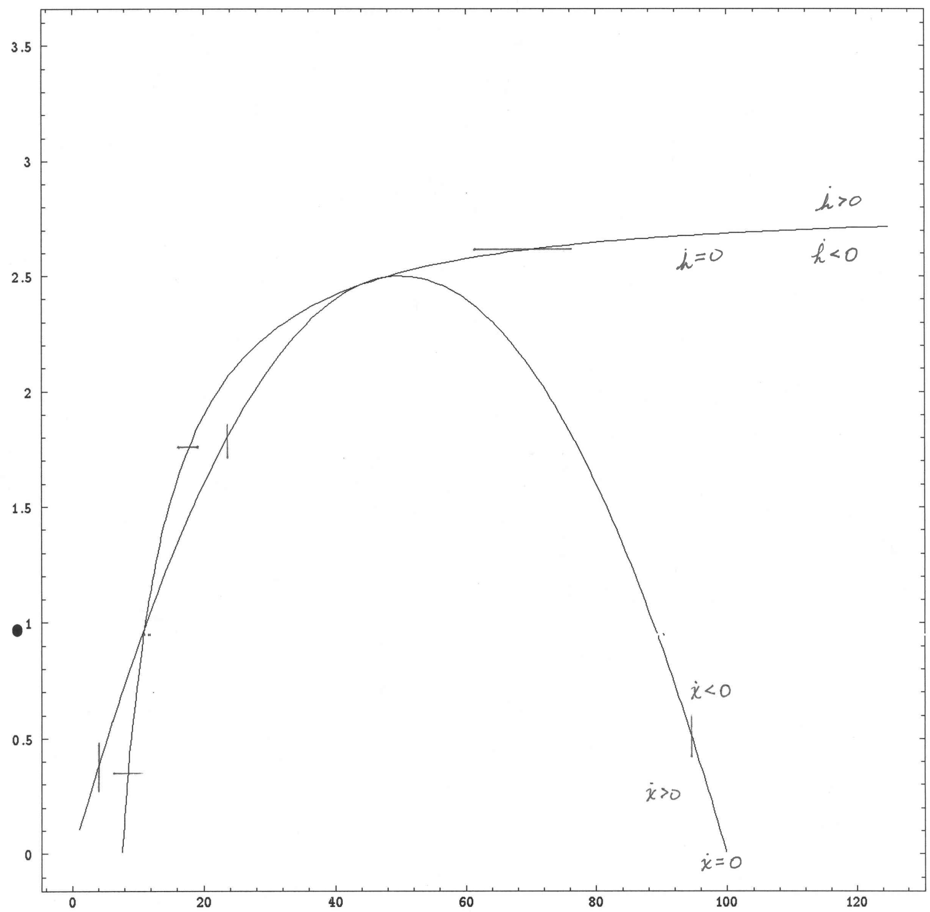

Figure 6A. Demand curve is $p = -\frac{12.2}{2.8}h + 12.2$ (equation (73), corresponding to a demand curve on the knife edge between the middle, "3 intersections" region of Figure 3 and that figure's upper-most "1 intersection" region).).

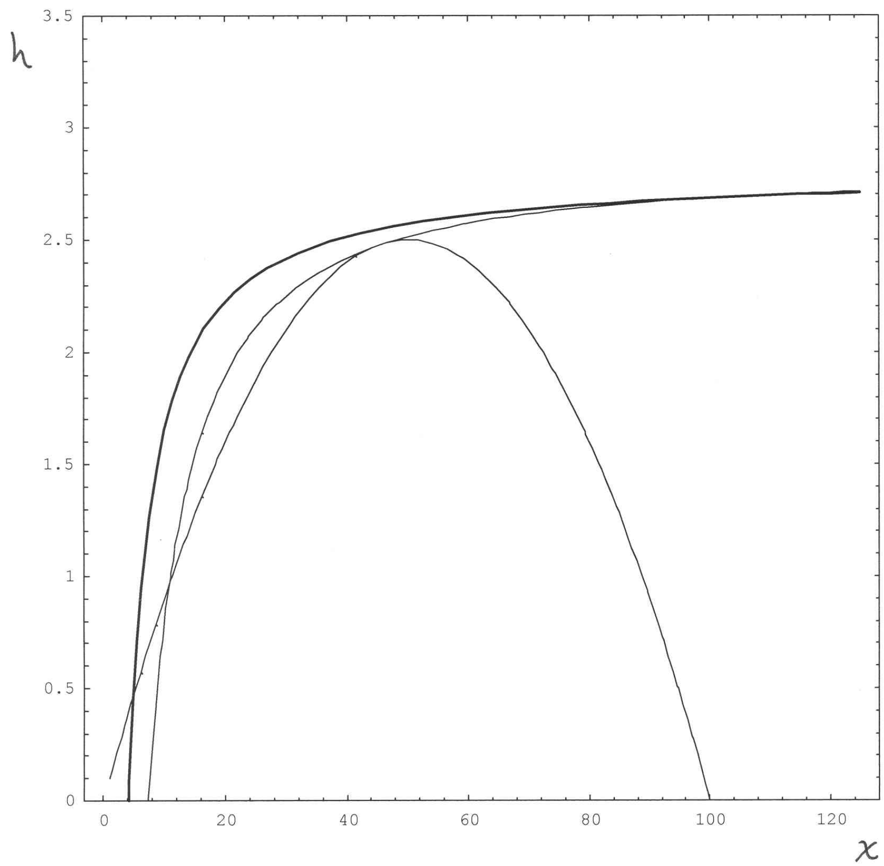

Figure 6B. Figure 6A combined with the finite-time terminal surface.

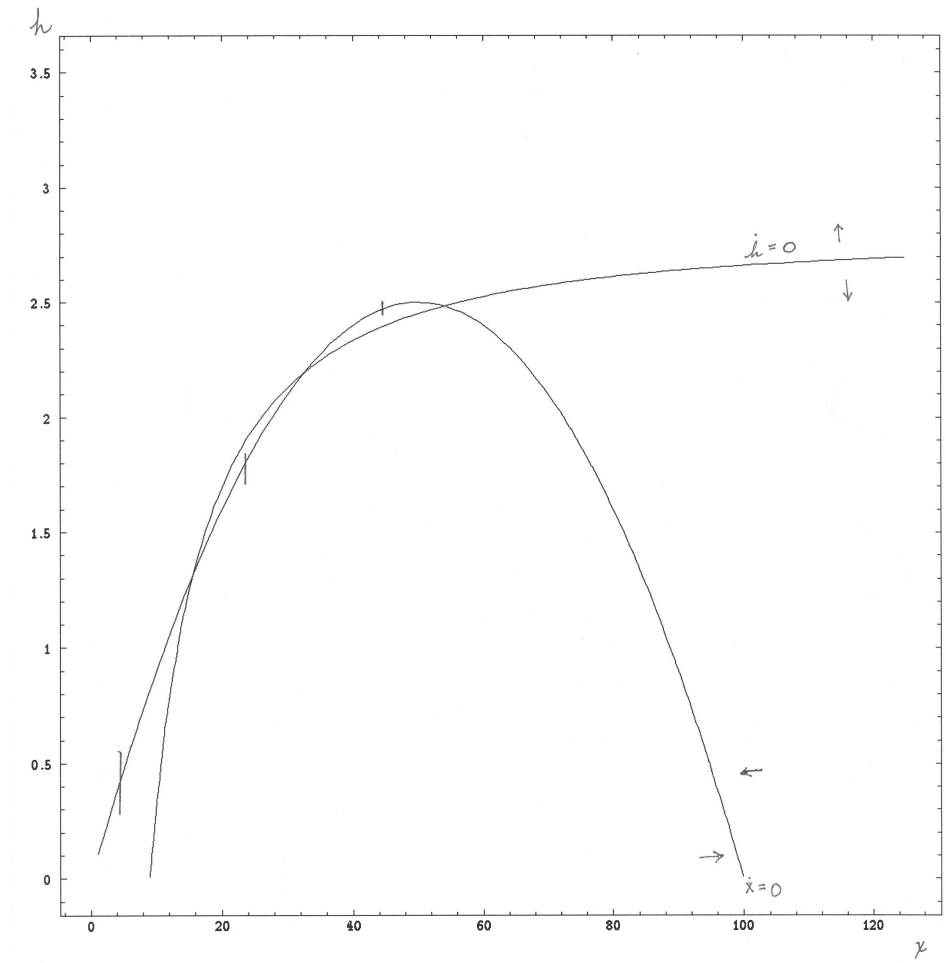

Figure 7A. Demand curve is $p = -\frac{10}{2.8}h + 10$ (equation (74), corresponding to an intersection in the middle, "3 intersections" region of Figure 3).

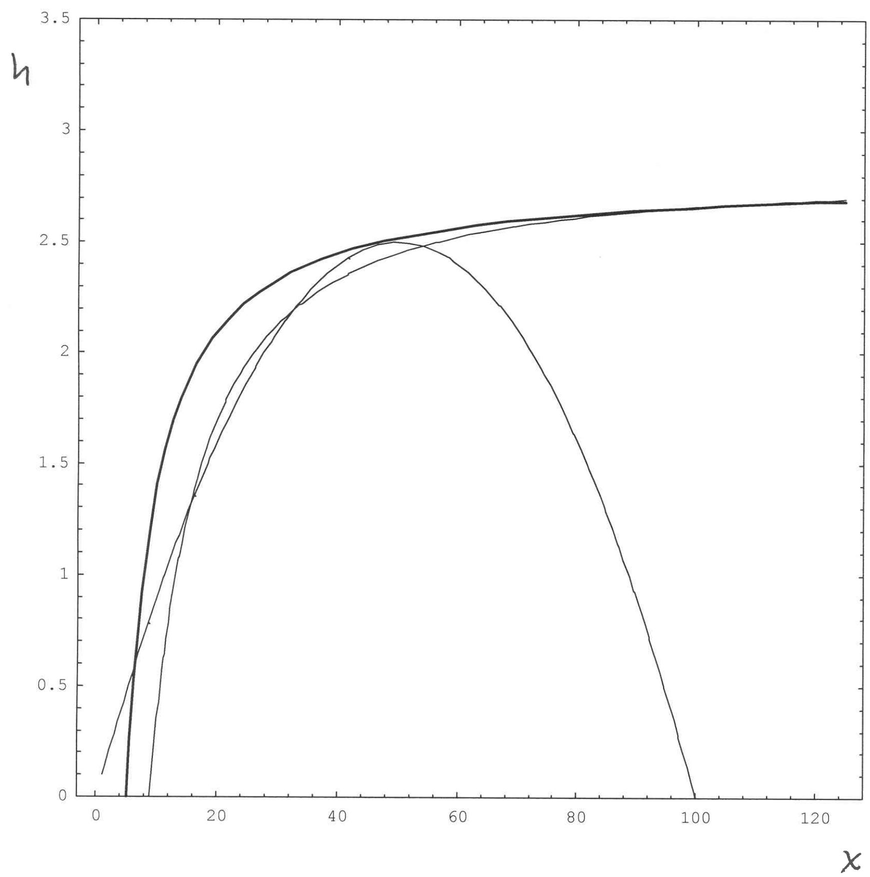

Figure 7B. Figure 7A combined with the finite-time terminal surface.

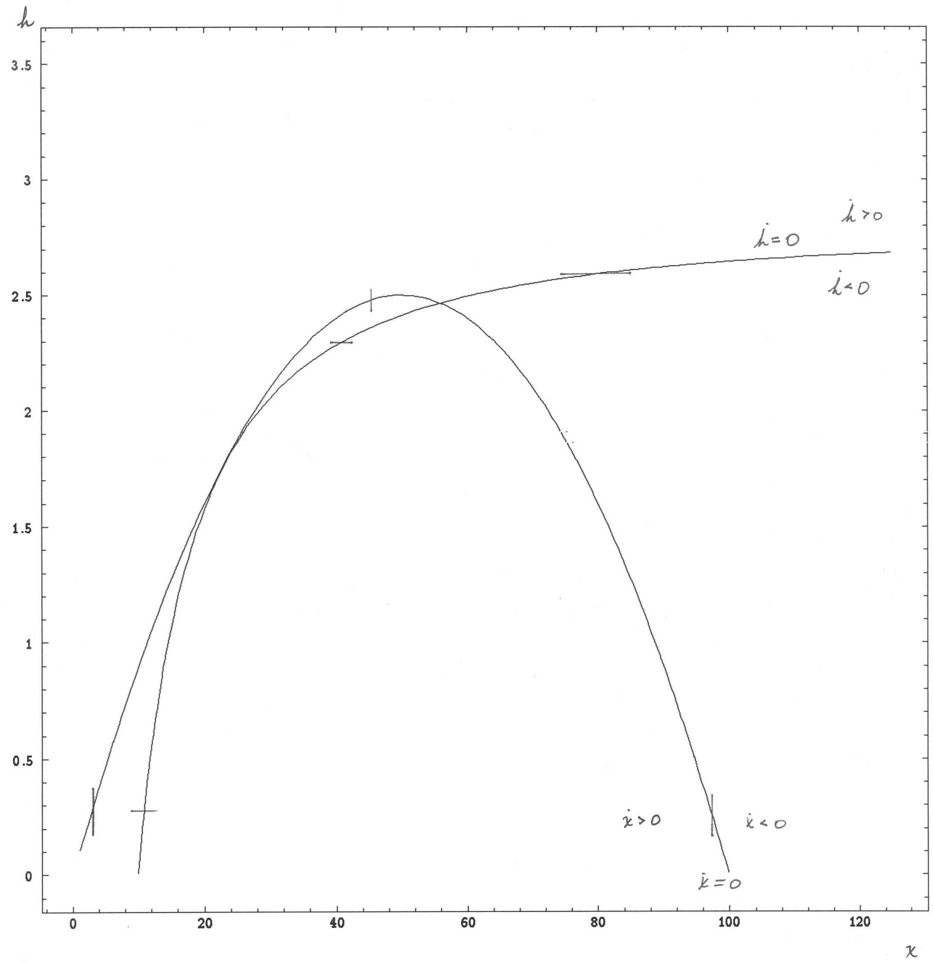

Figure 8A. Demand curve is $p = -\frac{9}{2.8}h + 9$ (equation (75), corresponding to a demand curve on the knife edge between the middle, “3 intersections” region of Figure 3 and that figure’s lower-most “1 intersection” region).

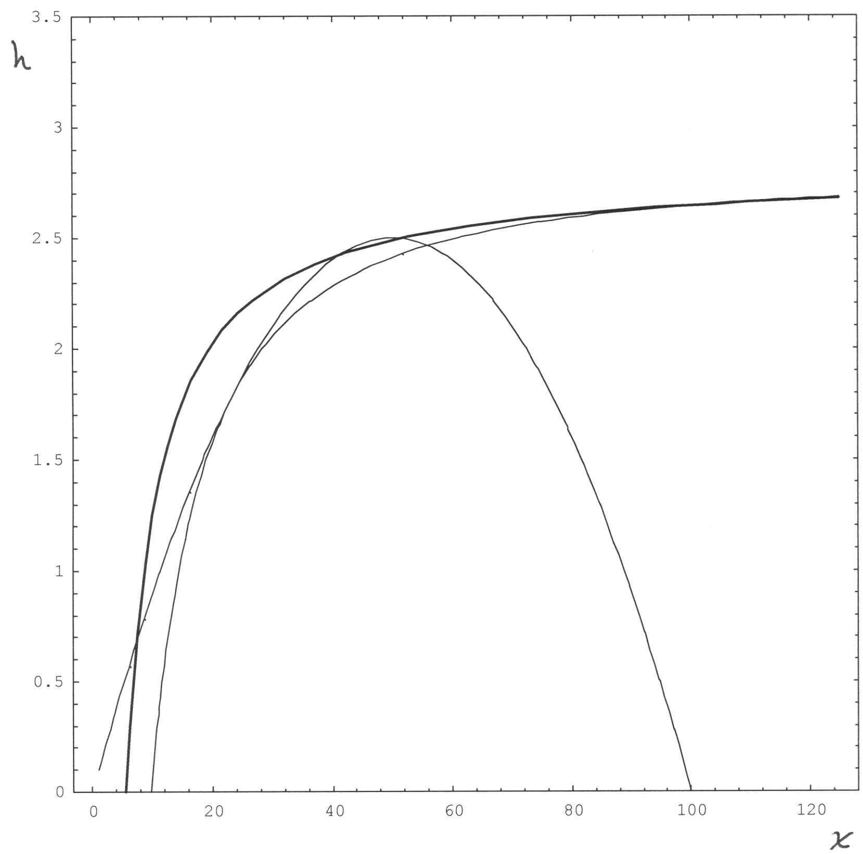

Figure 8B. Figure 8A combined with the finite-time terminal surface.

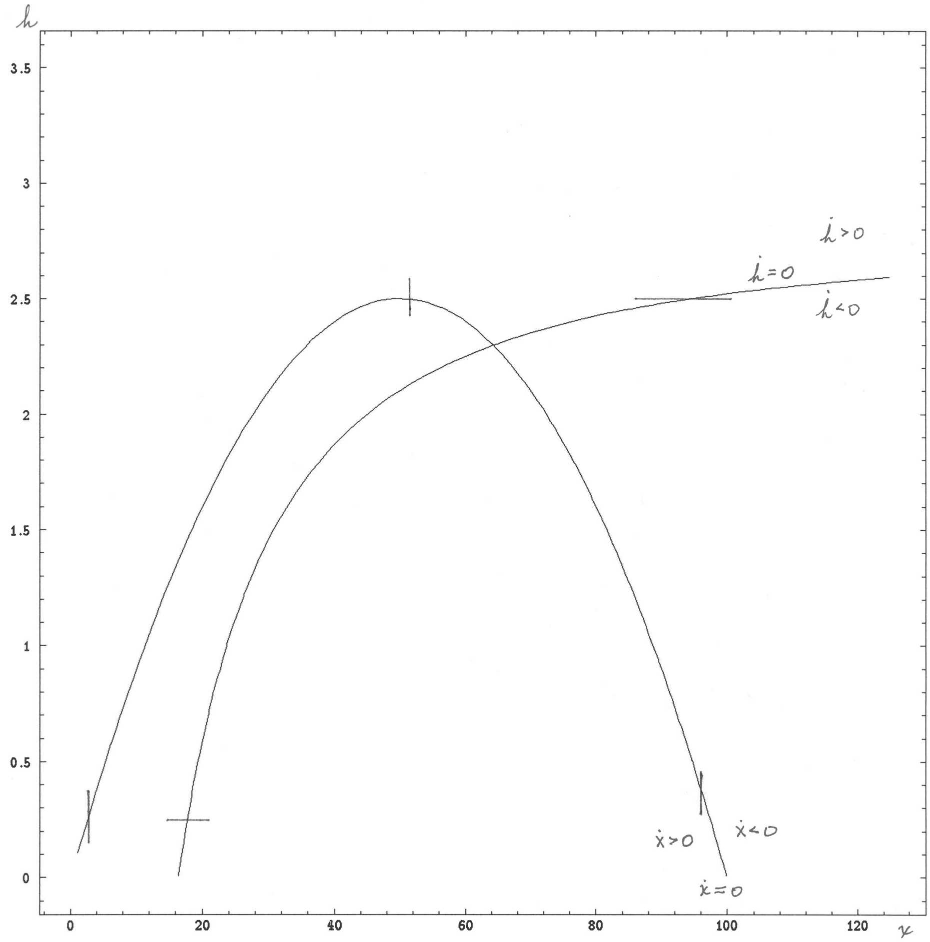

Figure 9A. Demand curve is $p = -\frac{7}{2.8}h + 7$ (equation (76), corresponding to an intersection in the lower-most "1 intersection" region of Figure 3).

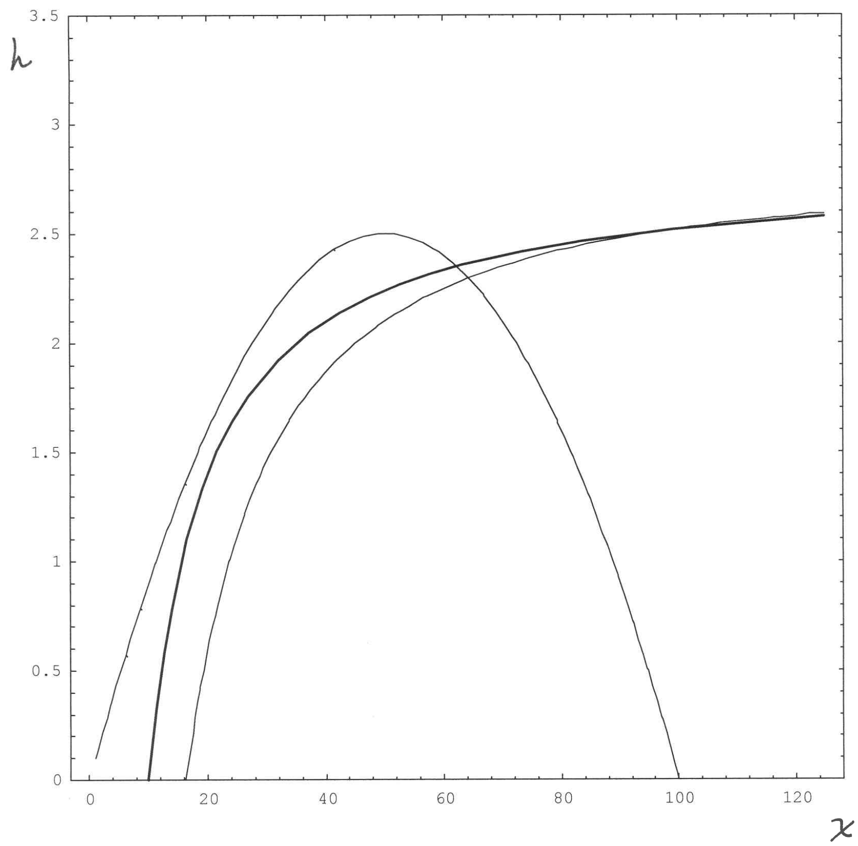

Figure 9B. Figure 9A combined with the finite-time terminal surface.

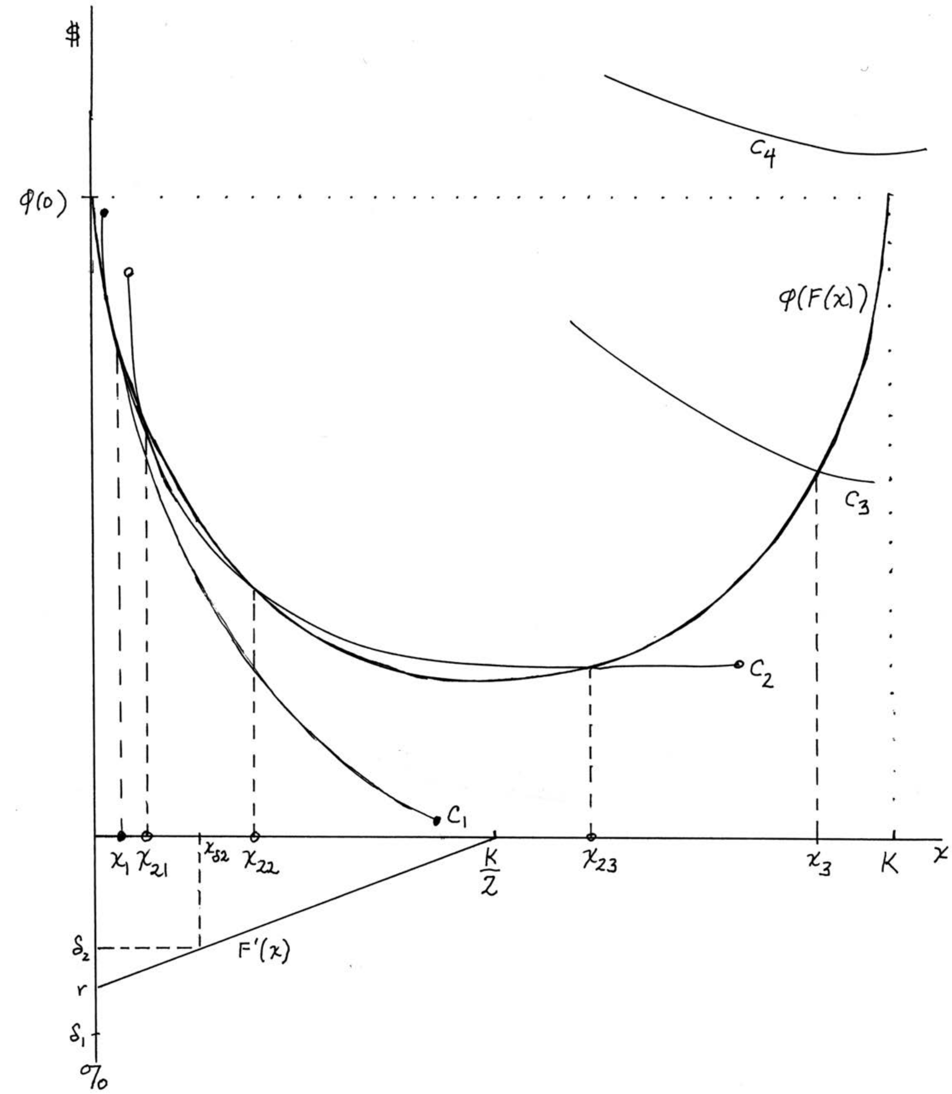

Figure 10. One way of summarizing some of the conclusions of Figure 5A–Figure 9B.

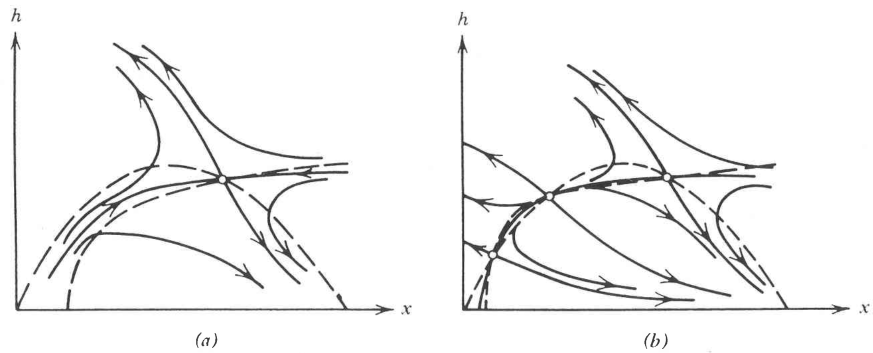

Figure 6.12. Bifurcation of supply and demand equilibria in the nonlinear fishery model.

Figure 11. Clark's Figure 6.12.

Exercise 1. Finish drawing the phase plane diagrams in Figs. 5–9. Locate the trajectory to the steady state in each diagram assuming $x_0 = K$ (recall that $K = 100$). (At the end of this section, we will show that these trajectories are optimal.)

Exercise 2. Using the results of Exercise 1, as the demand curve pivots clockwise in Fig. 3, locate the equilibrium points. Note the bifurcation when, as demand increases, suddenly quantity jumps down and price jumps up.

Exercise 3. Re-solve "Example 1, dynamics" with a value of δ which is less than $r = 0.1$.

Exercise 4. Re-solve "Example 1, dynamics" with demand curves like those in Fig. 4 instead of those in Fig. 3. The phase plane diagram will look as in Figs. 5 and 9. (Why?) As demand pivots clockwise in Fig. 4, there will not be a bifurcation. (Why?)

Exercise 5. Re-solve "Example 1, steady-state" with a growth function $F(x)$ that exhibits critical depensation and show that this results in Clark's Figure 5.18a, p. 144, reproduced as Panel a of Figure 12 here. May extinction result from this model? What is the dynamic behavior like?

Exercise 6. Re-solve "Example 1, steady-state" with a cost function $c(x)$ that has $c(0) < \infty$ and show that this results in Panel b of Figure 12. May extinction result from this model? What is the dynamic behavior like?

Exercise 7. Verify the dependence of the steady-state supply curve on δ which is illustrated in Fig. 5.13, p. 137 of Clark, reproduced as Figure 13 here.

In phase diagrams such as the ones in the Exercises, it is unfortunately not trivial to prove that the convergent separatrix is optimal (that is, that it is optimal to approach the steady state). The transversality conditions (26) and (27) do not help because they are inapplicable to the problem of competitive firms, whose problem is non-autonomous.

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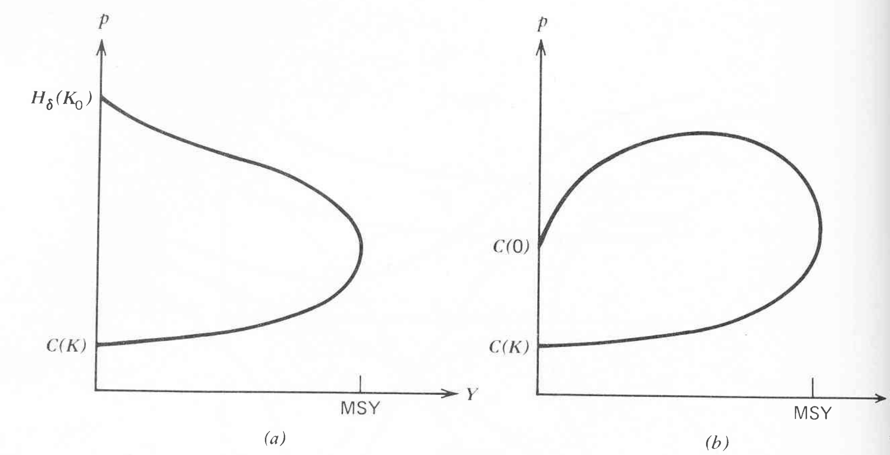

Figure 5.18. Discounted supply curves: (*a*) critical depensation model, with $F'(K_0) < \delta$; (*b*) finite extinction cost, with $F'(0) < \delta$.

Figure 12. Clark's Figure 5.18.

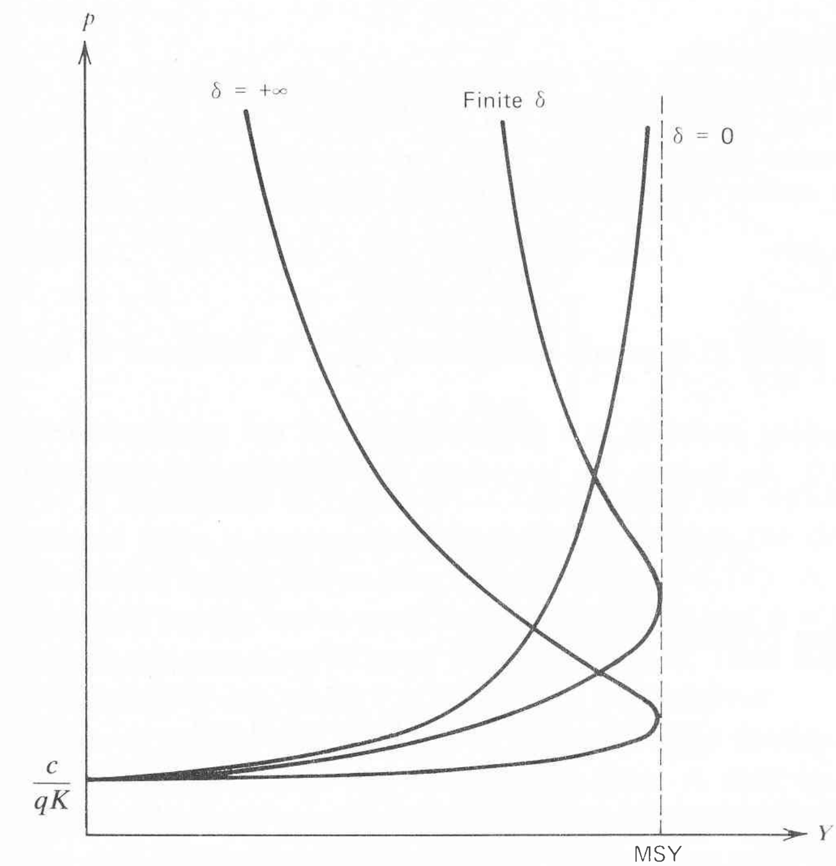

Figure 5.13. Discounted supply curve and its limiting positions.

Figure 13. Clark's Figure 5.13.

On page 97 of the third (2010) edition of Clark's book, he writes: "(Is there some way to prove that the optimal solution must approach [the steady state] (x^*, h^*) as $t \to \infty$? This can be done by first looking at the case of a finite time horizon T, and then letting $T \to \infty$. We skip the details.)" Following Clark's hint, recall that if $T < \infty$, (13) has to hold—intuitively, from (11),

$$\frac{\partial J}{\partial x_T} = \lambda_T = 0$$

so the fish stock is valueless at the margin at T—so from (21), $M\Pi_t = p_T - c_T = 0$, or $\phi(h_T) = c(x_T)$ imposing market equilibrium. In the *Mathematica* code, the `terminalsurface` lines find the (x, h) points which satisfy this $\phi(h_T) = c(x_T)$ finite-time transversality condition. Fig. 5B superimposes that set of points (that curve) onto Fig. 5A (and similarly for the "B" versions of the other figures). Call this curve the "finite-time terminal surface." The size of T determines which point on this finite-time terminal surface is the right one; if T is small, the path from the initial point to the terminal surface will be short (such as "Path C" of Figure 5B), but if T is 5 billion years, it will be long (such as "Path D" of Figure 5B). The greater the value of T, the closer the path has to get to the steady state, where motion slows because the path is so close to the nullclines. This is known as the "turnpike" property (the analogy being that one does not go from F to G in Figure 14 via a straight line, but rather by detouring near—not literally on—the JS "turnpike").

Furthermore, since the finite-time terminal surface is where $M\Pi = p - c = 0$, from (30d), it also where $\pi = 0$. Hence it describes a zero-profit surface.[7] Starting from a point on it and increasing h while keeping x constant, $p - c = \phi(h) - c(x)$ will fall from zero to a negative value; so it divides the phase plane into areas of negative profit above it and positive profit below it.

Figure 14 enables a complete analysis. It is a redrawing of Figure 5A, based on (72). I first note that, like Path B in Figure 5B, Paths GHI and RTU in this figure are correctly drawn as reaching $x = 0$ in finite time.[8] Suppose

[7]There is another zero-profit surface, at $h \equiv 0$, but it is not a finite-time terminal surface. Proof: on that surface, $\pi = AM\Pi \cdot h$ is zero but $AM\Pi$ is not zero, so, from (21), $\lambda_t = e^{-\delta t} M\Pi_t$ is not zero; but on a finite-time terminal surface $\lambda_T = 0$ from (13), so $h \equiv 0$ is not a finite-time terminal surface.

[8]In other words, if (x_0, h_0) begins above the $\dot{h} = 0$ isocline, the path reaches $x = 0$ in finite time. To prove this, begin by noting that from such a starting point, $h_t > h_0 \, \forall \, t > 0$. By the definition of Maximum Sustainable Yield, $F(x_t) \leq MSY$ for all x and therefore for all t. If for the (x_0, h_0) of interest, $h_0 < MSY$, then study a different initial point with the same x_0 but an h_0 raised until $h_0 > MSY$. If a path from this new starting point reaches $x = 0$ in finite time, as we will prove, then since phase paths (of autonomous systems, a class to which our

$T < \infty$. If $x_0 = \tilde{x}$, then the paths from A, K, and L cannot be optimal because they cannot reach the finite-time terminal surface. Similarly, if $x_0 = \hat{x}$, then the paths from M, T, and H cannot be optimal because they cannot reach the finite-time terminal surface. From $\tilde{x}$, a small T would result in an optimal path being like BC; for a larger T, an optimal path would be like DE; and for an even larger T, it would be like FG. From $\hat{x}$, a small T would result in an optimal path being like VW; for a larger T, an optimal path would be like QR.

One alternative at this point is to forget about the $T = \infty$ case and instead note that for very large T, the optimal path is extremely close to the convergent separatrix until dates so far into the future that they have no economic importance.

Eschewing that alternative, optimal paths for $T = \infty$ must resemble optimal paths for large finite T. From $\hat{x}$ the optimal path will thus either look like QRTU or like PS. However, QRTU cannot be optimal because once it reaches T, its continuation TU has $\pi_t < 0 \, \forall \, t$ (until $\pi = 0$), whereas there is a feasible alternative, PS, which has $\pi_t > 0 \, \forall \, t$. So from $\hat{x}$, PS is optimal. From $\tilde{x}$ the optimal path will either look like FGHI or like JS. However, FGHI cannot be optimal because once it reaches H, its continuation HI is nonoptimal, for the same reason that TU was nonoptimal. So from $\tilde{x}$, JS is optimal.[9]

Path JS is rather close to the zero-profit surface; one may wonder why it is optimal, suspecting that there are other feasible points which generate more profit. Figure 15 shows that the highest steady-state profit occurs on the $F(x)$ function at approximately $x = 80$, $h = 1.3$. If for example $x_0 = 80$ then it would be feasible to stay at that point forever, earning, at each date, more profit than at any date on Path JS. However, Figure 15

---

problem belongs) cannot cross, the path from the original starting point will also reach $x = 0$ in finite time.

Define a new variable "χ" which has $\chi_0 = x_0$, but instead of $\dot{x} = F(x) - h$, define $\dot{\chi} = MSY - h_0$. Clearly $\dot{\chi} \geq F(x_t) - h_0 > F(x_t) - h_t = \dot{x}$. Also, $\dot{\chi} = MSY - h_0$ is negative as explained in the previous paragraph, so $\dot{x}$ is more negative than $\dot{\chi}$. Since $\chi_0 = x_0$, this means $\chi_t > x_t \, \forall \, t > 0$. The differential equation defining χ is trivial to solve: $\chi_t = (MSY - h_0)\, t + x_0$ (ensuring that $\chi_0 = x_0$). Clearly χ_t reaches zero in finite time (in fact, at time $-x_0/(MSY - h_0) > 0$). Since x_t is less than χ_t, x_t will reach zero before χ_t does. This completes the proof that x_t reaches zero in finite time.

[9] The nonoptimality of QRTU can also be shown by appealing to a result of R. Hartl (*Journal of Economic Theory* 1987), namely that in a one-state-variable infinite-horizon autonomous (or autonomous except for geometric discounting) problem, the optimal path of the state variable must be monotonic. However, the nonoptimality of FGHI cannot be shown in that way.

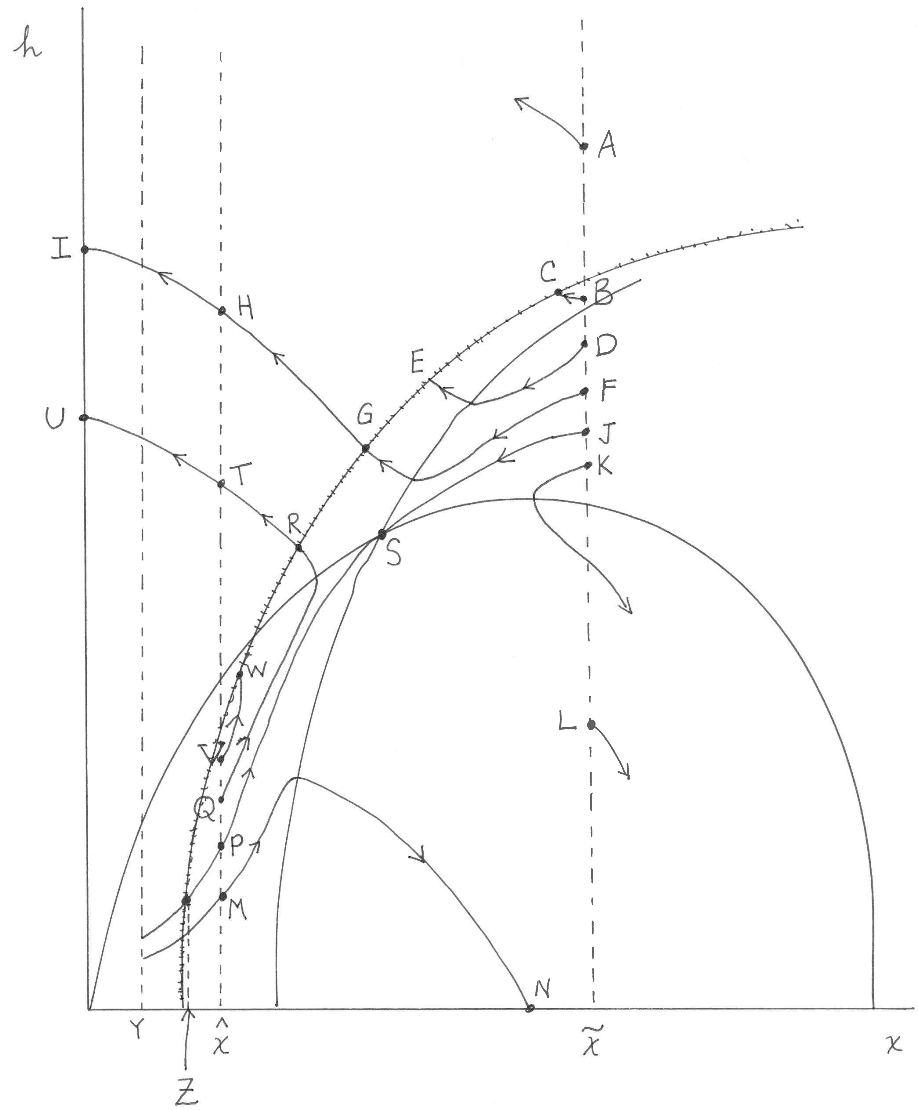

Figure 14. Based on Figure 5A, equation (72). The finite-time terminal surface is the solid line with small hatch marks.

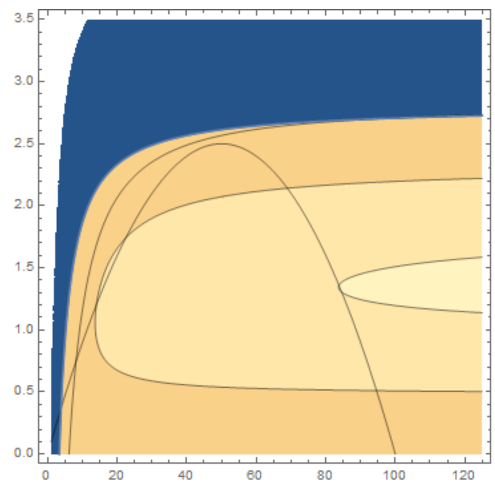

Figure 15. Isoprofit lines for Figure 14. Blue is negative profit, and profit increases with lighter colors.

shows *equilibrium* profit, that is, taking the demand curve into account, but the competitive firm does not know where the demand curve is. The $(x, h) \approx (80, 1.3)$ point may maximize profit for a monopolist (whether it does or not is not relevant here), but the competitive firm has to maximize profit given the exogenous prices it faces, and Path JS is the correct answer to that problem.

This proves that PS or JS are optimal, except for the case of x_0 being very small, e.g. at Point Y. In this case, instead of making negative profits by, for example, taking the path to Point P, I conjecture the optimal path sets $h^* = 0$ and lets the fish stock recover to Point Z; then, h jumps up to Point P and continues on to Point S.

The next three paragraphs prove that this behavior—namely h^* being zero at the beginning before jumping to the convergent separatrix—is consistent with the mathematics of Optimal Control Theory. Our phase-plane paths were constructed from (16) and (65); the latter came from (61), which came from (20), which came from (5), which came from assuming an interior (or, in the case of $\mathscr{H}$ linear in h as we have, a "singular") solution. Therefore there is an alternative to following a phase-plane path, namely by *not* assuming an interior (technically, a "singular") solution. That means setting the control to be at its minimum or maximum allowed value (which is called taking a "most rapid approach path," abbreviated "MRAP"); in our case, it means setting $h_t^* = 0$. This confirms that it is allowed to have $h_t^* = 0$ for some time.

Next we need to show that it is allowed to have $h_t^* = 0$ for an *initial* period of time, since that is what I claimed was possible. Accordingly, assume throughout this paragraph that we are discussing an initial time period in which $h_t^* = 0$. Whenever $h_t^* = 0$, (25) gives $e^{-\delta t} M\Pi_t \leq \lambda_t$. In market equilibrium this is

$$e^{-\delta t}[\phi(0) - c(x_t)] \leq \lambda_t \,. \tag{77}$$

Also, $h = 0$ implies $\dot{\lambda}_t = -\partial\mathcal{H}/\partial x = -\partial[e^{-\delta t}(p - c)h + \lambda(F - h)]/\partial x = -[-e^{-\delta t}c'h + \lambda F'] = -\lambda_t F'(x_t)$. From (69), $\lambda_0 > 0$ as long as fishing will occur at some point; and since for small x, $F' > 0$ because we have assumed logistic growth, we have $\dot{\lambda}_0 = -\lambda_0 F_0' < 0$, so the right-hand side of (77) starts by falling. When $h = 0$, the left-hand side of (77) rises: the proof begins with $\partial e^{-\delta t}[\phi(0) - c(x_t)]/\partial t = -\delta e^{-\delta t}[\phi(0) - c] - e^{-\delta t}c'\dot{x} = -\delta e^{-\delta t}[\phi(0) - c] - e^{-\delta t}c'F = -e^{-\delta t}\{\delta[\phi(0) - c] + c'F\}$, then notes that $c'F < 0$ and $\phi(0) - c < 0$ for points such as Y that lie to the left of the finite-time terminal surface. Hence the left-hand side and the right-hand side are getting closer, and at some date, in order not to have h being equal to zero forever, (77) will be met with equality, and h will cease being zero. That is called the "switching point."

You can ignore the following paragraph, which derives the precise time paths followed by λ and x during the previous paragraph's MRAP from $(x_0, 0)$ to Point P's x coordinate. While $h = 0$, (16) and (38) give $\dot{x} = rx(1 - (x/K))$. (In this exercise, $K = 100$ and $r = 0.1$.) The solution of this differential equation is $x_t = K/(1 + c_1 e^{-\delta t})$ where constant $c_1 = (K - x_0)/x_0$ is known. When $h = 0$ we saw above that $\dot{\lambda} = -\lambda F'$; dividing by λ and substituting $F' = r - \frac{2r}{K}x$ and substituting for x_t from the previous sentence gives

$$\frac{d\lambda/dt}{\lambda} = \frac{2r}{K}x - r = \frac{2r}{K}\frac{K}{1 + c_1 e^{-\delta t}} - r \,. \tag{78}$$

Moving dt to the other side and integrating both sides turns out to give

$$\ln\lambda = 2r\left[t + \frac{\ln(1 + c_1 e^{-\delta t})}{r}\right] - rt + c_2 \tag{79}$$

for some constant c_2. (I did the difficult integral with *Mathematica*, but it is easy to verify its correctness by differentiating.) Taking exp of both sides gives us λ as a function of t, although the constant c_2 is still undetermined. The full procedure, then, is to find the value of Z using computer numerical methods. Once Z is found, the switch time t_s can be calculated because we know the rate at which x moves from x_0 towards Z:

$$\mathrm{Z} = \frac{K}{1 + c_1 e^{-\delta t_s}} .$$

Having calculated t_s from this equation, the next step is to note that at t_s, $e^{-\delta t_s} M\Pi_{t_s} = \lambda_s$. Substituting previous results into $e^{-\delta t_s} M\Pi_{t_s} = \lambda_s$:

$$e^{-\delta t_s} [\phi(0) - c(\mathrm{Z})] = \exp\left\{2r\left[t_s + \frac{\ln(1 + c_1 e^{-\delta t_s})}{r}\right] - rt_s + c_2\right\} .$$

The only unknown remaining in this equation is c_2, so this equation can be used to calculate c_2. Once c_2 has been calculated, the time path of λ before the switch time is fully determined; and since we already determined the pre-t_s time paths of x and, trivially, of h, this fully completes the pre-t_s specification.[10]

This concludes our proof of the optimality of the convergent separatrix, as well as showing that if x_0 is small then the convergent separatrix is not optimal at the beginning of the program.

Figure 10 is a way of summarizing the conclusions we have come to for Figures 5A–9B; we omit the rather simple demonstration that for logistic growth, $\phi(F(x))$ appears as it does in the graph. In working out the Exercise which gave rise to Figures 5A–9B you will have observed that the basic governing equation for the steady state can be written as

$$[\delta - F'(x_{ss})][\phi(F(x_{ss})) - c(x_{ss})] = -c'(x_{ss})\, F(x_{ss}) . \qquad (80)$$

At $x_{ss} = 0$ or K one has $F = 0$ and hence the right-hand side of (80) is zero, but for all $x_{ss} \in (0, K)$ the right-hand side of (80) is strictly positive, so between 0 and K we need the left-hand side of (80) to be strictly positive as well. Because of Lemma 2, the second term on the left-hand side of (80) is strictly positive; thus we need the first term to be positive as well. If $\delta > r$, as δ_1 is in Figure 10, then the first term of (80) is always positive. If $\delta < r$, as δ_2 is in the graph, then the first term of (80) is only positive to the right of $x_{\delta 2}$, so we will require $x_{ss} > x_{\delta 2}$.

[10] The reason we were able to analytically solve the pre-t_s period was that we (with *Mathematica*) were able to solve two nonlinear differential equations. This was only possible because we got lucky: $h = 0$ resulted in easy equations. The post-t_s specification cannot be analytically solved, even using *Mathematica 6.0.*

For a $c(x)$ function like c_4 in Figure 10, it is impossible for the second term on the LHS of (80) to be positive, so no x_{ss} exists. For the parameters we chose, this case is not represented by any of Figures 5A–9B.

For the cost function c_3, x_{ss} will lie between x_3 and K. We can use a small table and the Intermediate Value Theorem to prove that there exists at least one $x_{ss} \in (x_3, K)$ which satisfies (80) (that is, for which the left-hand side of (80) is equal to the right-hand side of (80)):

x	LHS of (80)	RHS of (80)	LHS – RHS
x_2	0	+	−
K	+	0	+

This example, with as few as one x_{ss} that is close to K, seems to correspond to Figure 9, although it had a high demand curve, which does not seem to match with c_3's high costs.

For the cost function c_2, I claim that $x_{ss} \in \{(x_{21}, x_{22}) \cup (x_{23}, K)\}$. In fact just as in the c_3 case, we can show that there exists at least one x_{ss} in (x_{23}, K) which satisfies (80), but there might be one (or more than one) x in (x_{21}, x_{22}) which also satisfies it. If $\delta < r$ one needs to recall the additional restriction that $x_{ss} > x_{\delta 2}$. This case seems to correspond to Figure 7.

For the cost function c_1, using similar reasoning to the c_3 case there is an x_{ss} in (x_1, K) satisfying (80) in the $\delta > r$ case. In the $\delta < r$ case, the additional restriction that $x_{ss} > x_{\delta 2}$ makes it impossible to prove in principal that a suitable x_{ss} exists, though it is likely to. This case seems to correspond to Figure 5, with x_{ss} close to zero.

I would like to close this section with a note about how other authors approach this constant-returns-to-scale, private-property fishery. Most other analyses (e.g., Caputo op. cit. p. 137) assume that price p_t is fixed in time. This implicitly means either that one just wishes to derive the steady-state supply curve and leave determination of equilibrium to further work (note that there is no analogous procedure for the dynamic analysis), or that the actual market demand curve for this fish is horizontal, which would violate consumers' budget constraints. To analyze this case, start with assuming an interior (or, technically, a singular) solution; then (5) characterizes the optimum, and it leads to (61). When p is fixed in time, the $\dot{p}$ term in (61) drops out, and (61) becomes

$$\delta = F'(\bar{x}) + \frac{0 - c'(\bar{x})\, F(\bar{x})}{p - c(\bar{x})} \tag{81}$$

for a constant x called "$\bar{x}$" which implicitly defined by (81). If $x_0 \neq \bar{x}$, the system has to get from x_0 to $\bar{x}$ in the beginning. If $\bar{x} \neq x_T$, the system has to get from $\bar{x}$ to x_T at the end. In such cases, the system has to spend some time off of the interior (technically, the singular) path which is given by (81). The only alternative to (61) is following a noninterior (technically, a nonsingular) path, i.e., setting harvest to be one of its extreme values, either $h = 0$ or $h = \max h$ (which we have taken to be infinite, but which in reality would be finite). Hence the solution is to follow a MRAP from x_0 to $\bar{x}$, then follow the interior (singular) path $x_t = \bar{x}$ as long as possible, then follow a MRAP from $\bar{x}$ to x_T.

Section 4. Private-Property Competition: Other Examples (Schooling Fisheries)

The next pair of examples is

$$\pi(x, h) = [\, p - c\,]\, h \quad \text{and} \tag{82s}$$

$$\pi(x_t, h_t, t) = [\, p_t - c\,]\, h_t. \tag{82d}$$

As noted when discussing (30), this average cost function $c(x_t, h_t) = c$ is appropriate for "schooling" fisheries; the average cost function for (30), which was $c(x_t, h_t) = c(x_t)$, is appropriate for "search" fisheries. In working out case (82) you should start with (19), because it is unknown whether the additional assumptions used to derive (20) (namely $M\Pi \neq 0$) and (36) (namely $\delta - F'(x) \neq 0$) hold in this section. You should find that when, among other conditions, $\delta > r$, it is possible for no steady state to exist. That may imply extinction.[11] Figures 16–21 can help you work through these cases. The following notation is used in those graphs:

x_δ as in (43);
$h_\delta = F(x_\delta)$;
$\hat{h}$ such that $\phi(\hat{h}) = c$.

Figure 22 is a summary of some aspects of Figures 16–19 and Figure 23 is a summary of the similar aspects of Figures 20–21. The governing equation for the steady state is

$$[\delta - F'(x_{ss})]\,[\phi(F(x_{ss})) - c] = 0\,. \tag{83}$$

[11] Extinction can occur in cases (30) but not with the particular functional form we chose for $c(x)$, namely $c(x) = \gamma/(qx)$ with the traditional constant "q" being set equal to one, because with that functional form the marginal cost of driving x to zero is infinite (see Exercises 4 and 5 above).

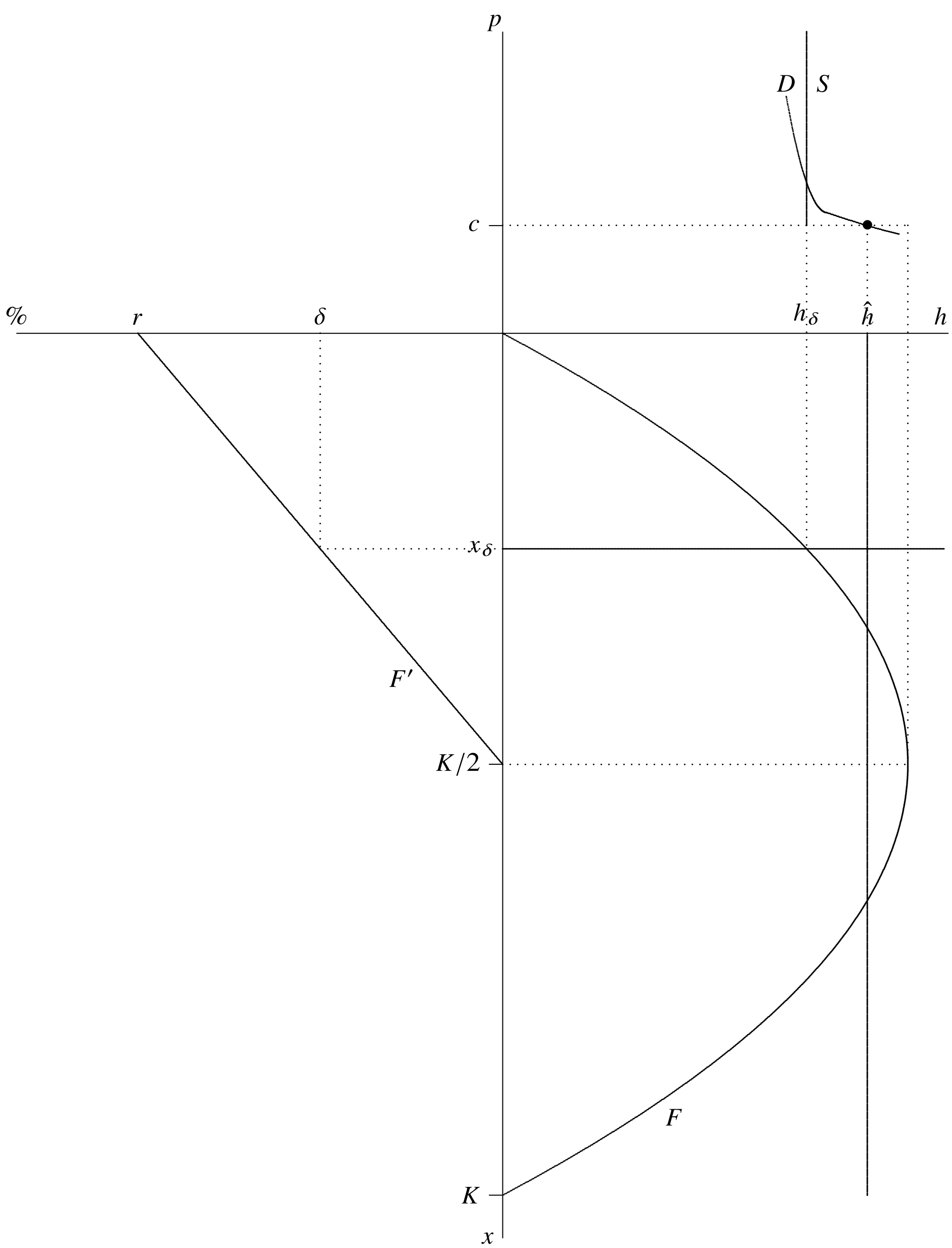

Figure 16. The supply "curve" is the solid vertical line and the bullet.

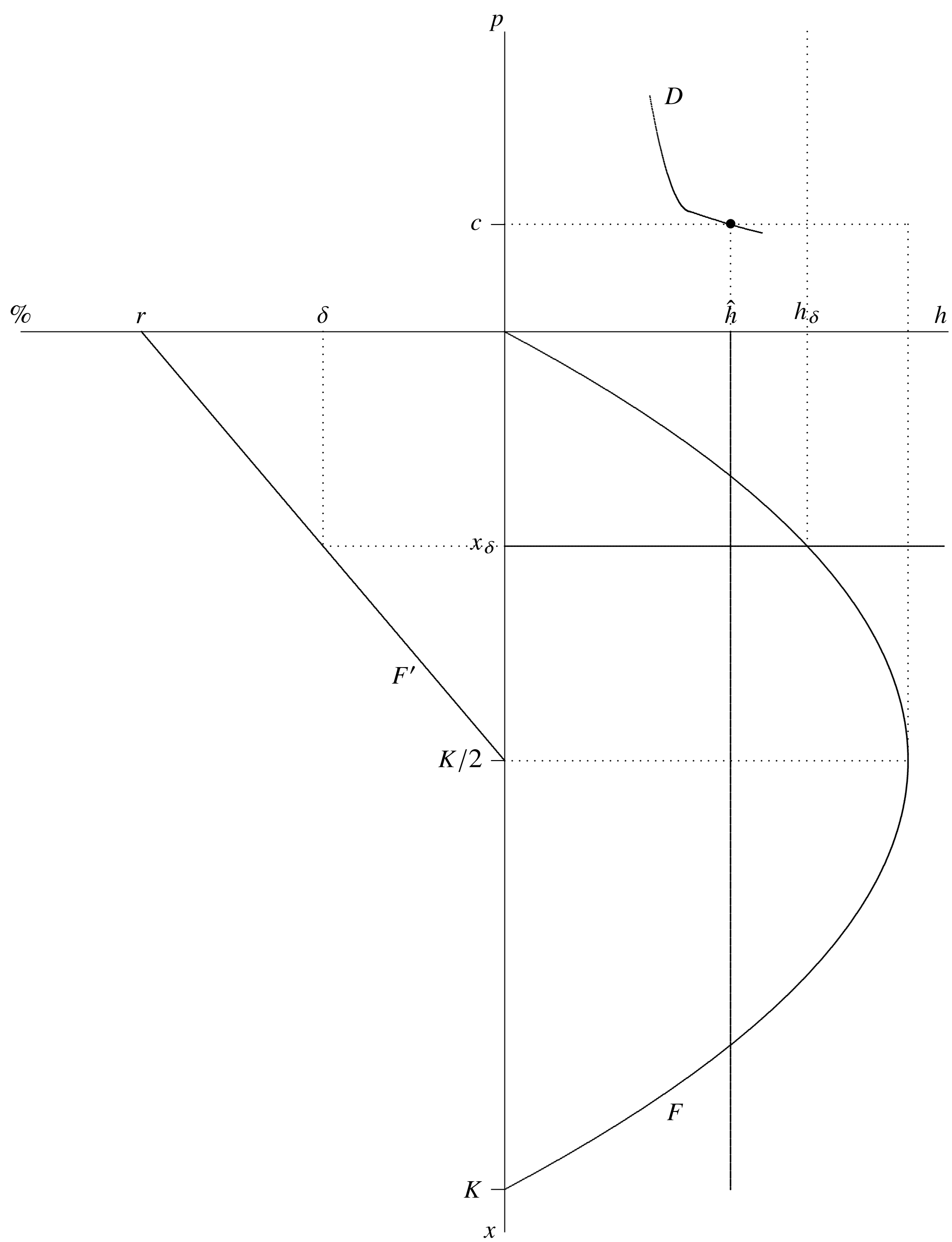

Figure 17. At h_δ, profit is negative. The supply “curve” is only the bullet.

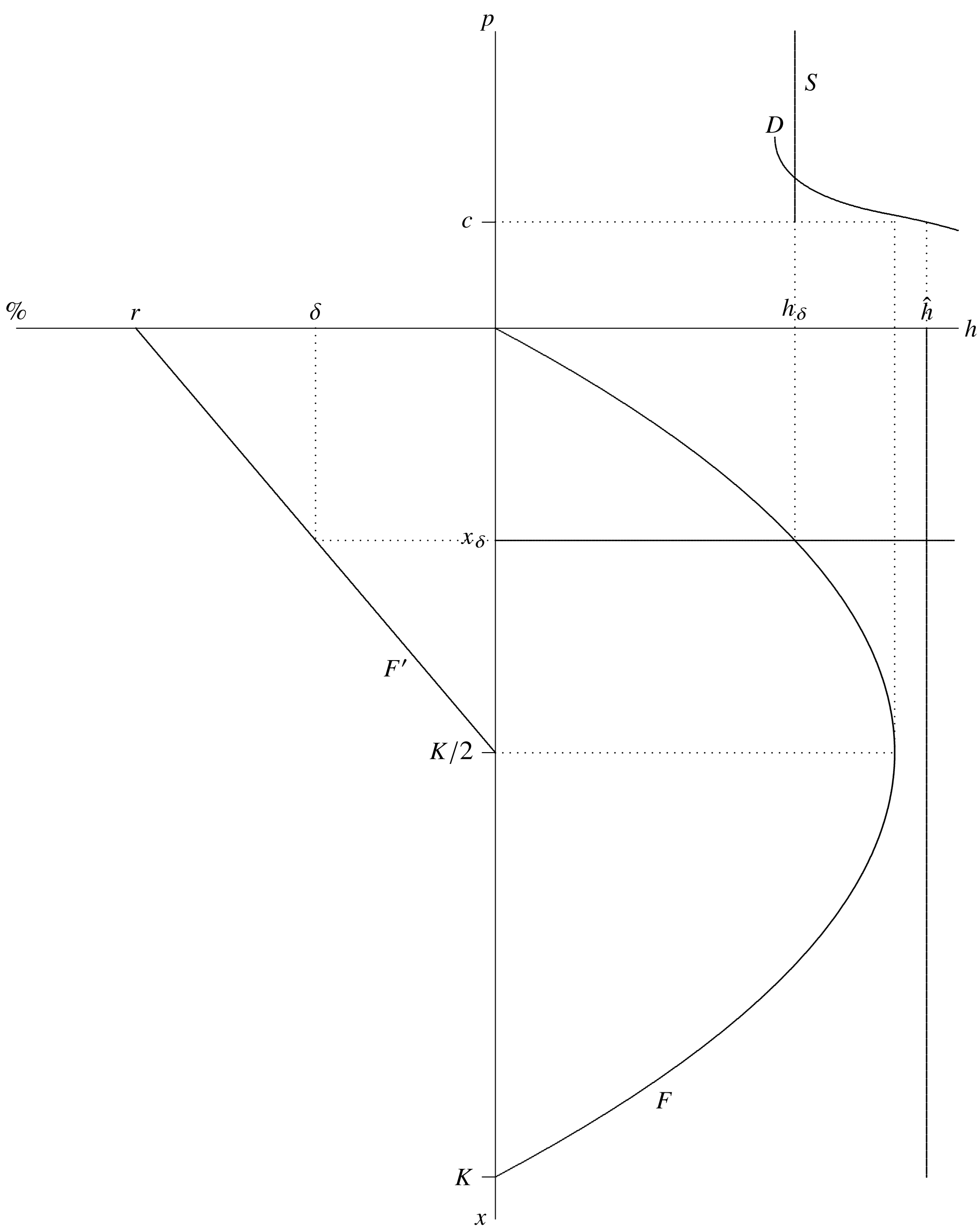

Figure 18.

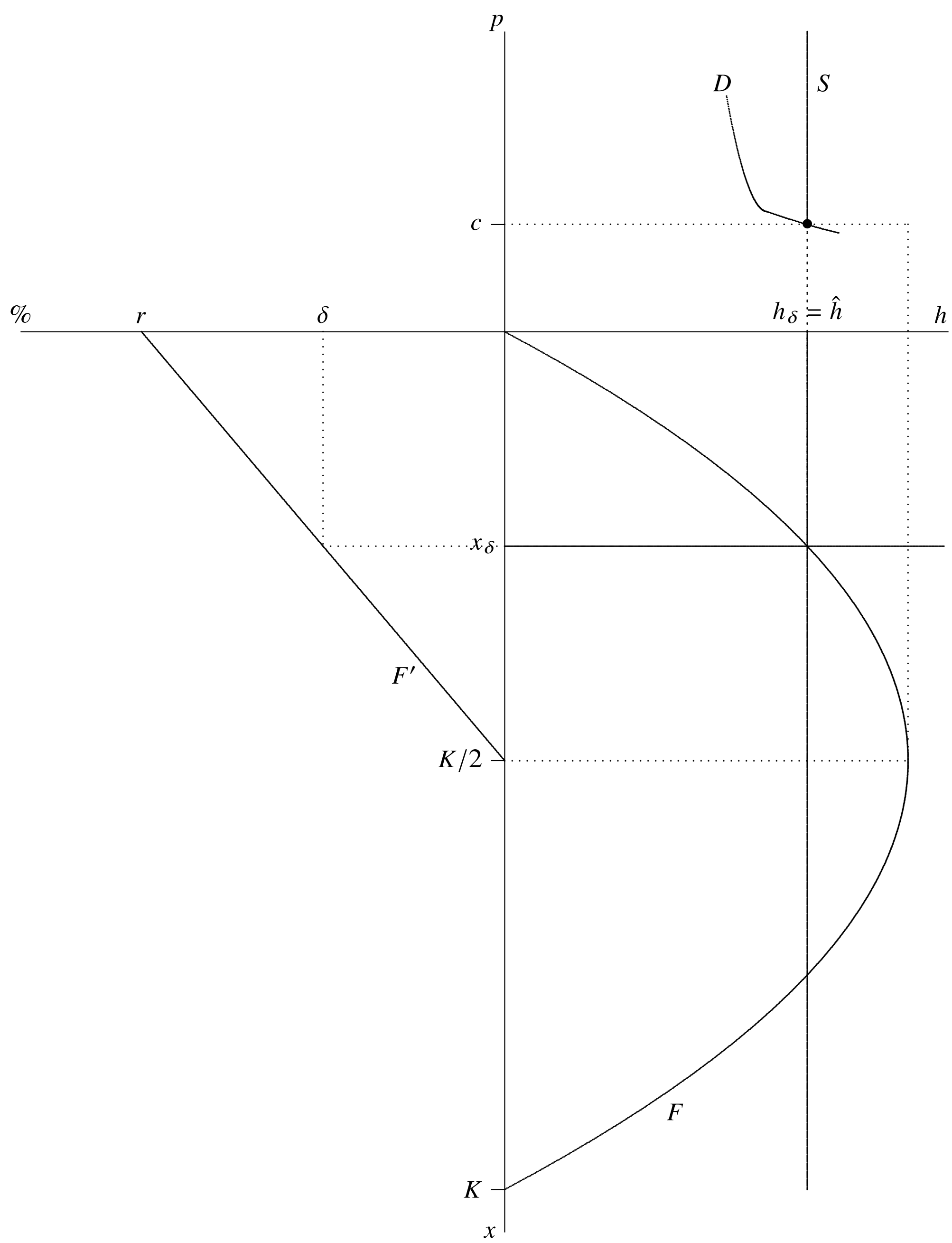

Figure 19. The supply “curve” is the solid vertical line and the bullet.

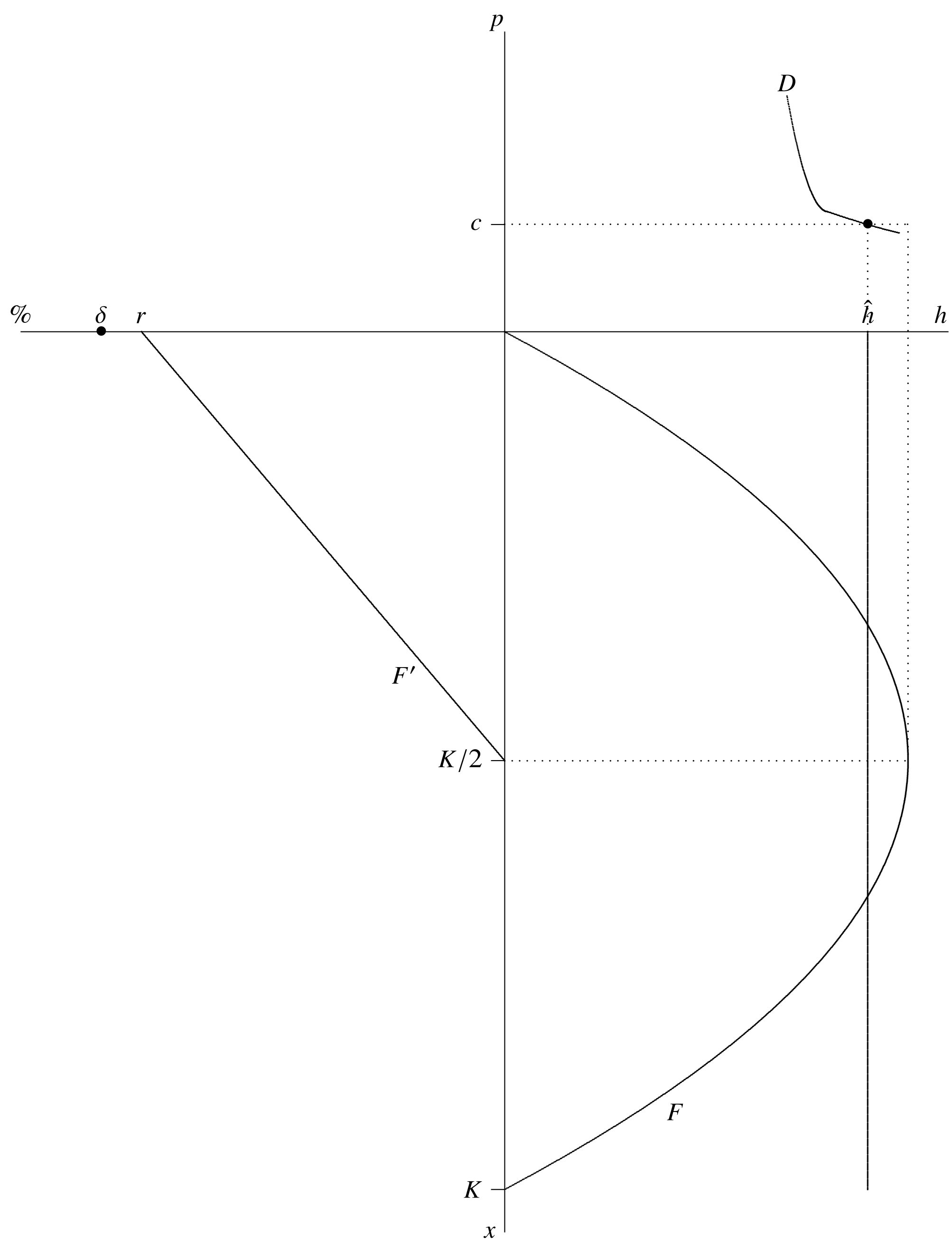

Figure 20. In Quadrant I, the supply “curve” is only the bullet.

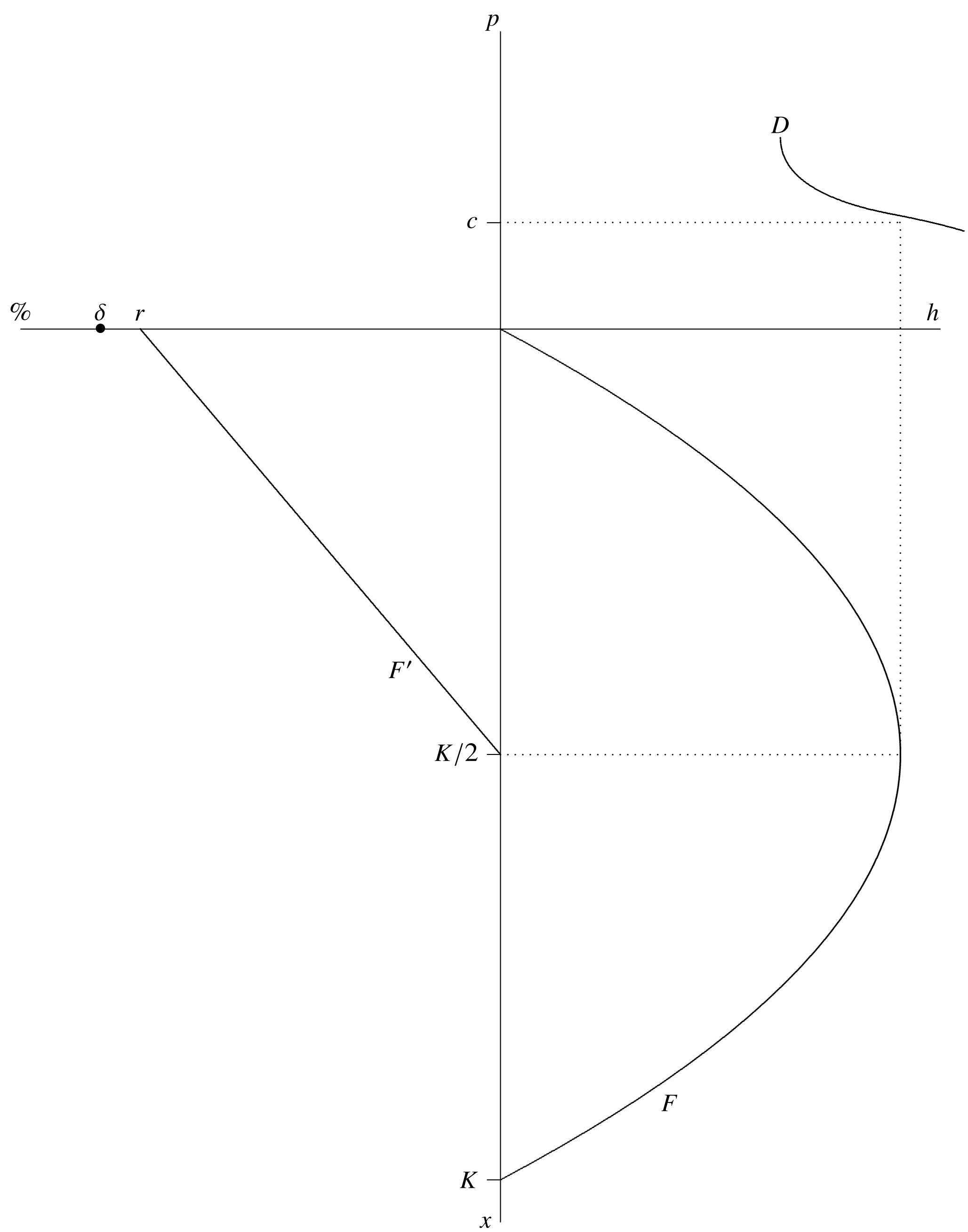

Figure 21.

This is similar to (80) for a search fishery (the left-hand sides of the two equations are the same and the right-hand sides are different), but (83) is easier to analyze because it is satisfied wherever $\phi(F(x_{ss}))$ is exactly equal to c, and because c is a constant here, instead of being a declining function as it was for search fisheries.

Another pair of examples is

$$\pi(x, h) = [\, p - c(x, h)]\, h \quad \text{and} \tag{84s}$$

$$\pi(x_t, h_t, t) = [\, p_t - c(x_t, h_t, t)]\, h_t. \tag{84d}$$

This is the most general form of the competitive firm's problem.

Section 5. Monopoly

In the case of monopoly,

$$\pi(x_t, h_t, t) = [\, \phi_t(h_t) - c(x_t, h_t, t)]\, h_t \tag{85}$$

is the most general form of the problem, where $\phi(h)$ is the market demand curve. (Compare with (84). Also note that before (65), imposing $p_t = \phi(h_t)$ was the last step, whereas here it's the first step.) Just as in the elementary case of a producible good, the monopolist has no supply curve; there is no relationship mapping p to h because the monopolist never takes p as given. Therefore, there is no steady-state analysis for the monopolist analogous to that giving rise to steady-state supply curves for the competitive industry.[12]

Substituting (85) into (20) gives the general solution for the monopolist. Instead of exhibiting this, consider the following special case of (85):

$$\pi(x_t, h_t) = [\, \phi(h_t) - c(x_t)]\, h_t\, . \tag{86}$$

This uses the same cost function as (30). Let *TR* be total revenue $\phi(h_t)\, h_t$ and *MR* be marginal revenue $d\,TR/dh$. One has $M\Pi = MR(h) - c(x)$, $\dot{M\Pi} = \dot{MR} - \dot{c}$, $\partial\pi/\partial x = c'(x)\, h$, and so substituting (86) and these results into (20) yields

$$\delta = F'(x_t) + \frac{\dot{MR} - \dot{c}(x_t) - c'(x_t)\, h_t}{MR - c(x_t)}\, .$$

[12]There is a relationship mapping $\phi(h)$ to h for the monopolist, but this is not a supply curve (which maps $\mathbf{R}^1$ to $\mathbf{R}^1$) but a supply functional (mapping a function space into $\mathbf{R}^1$). Any given demand curve $\phi_1(h)$ does have a point (call it (p_1, h_1)) which the monopolist would choose, but this is not a point on a supply curve because one could draw another demand curve $\phi_2(h)$ through the same (p_1, h_1) and with ϕ_2 the monopolist might not choose to produce at (p_1, h_1). All this is the same as in the case of a static producible resource.

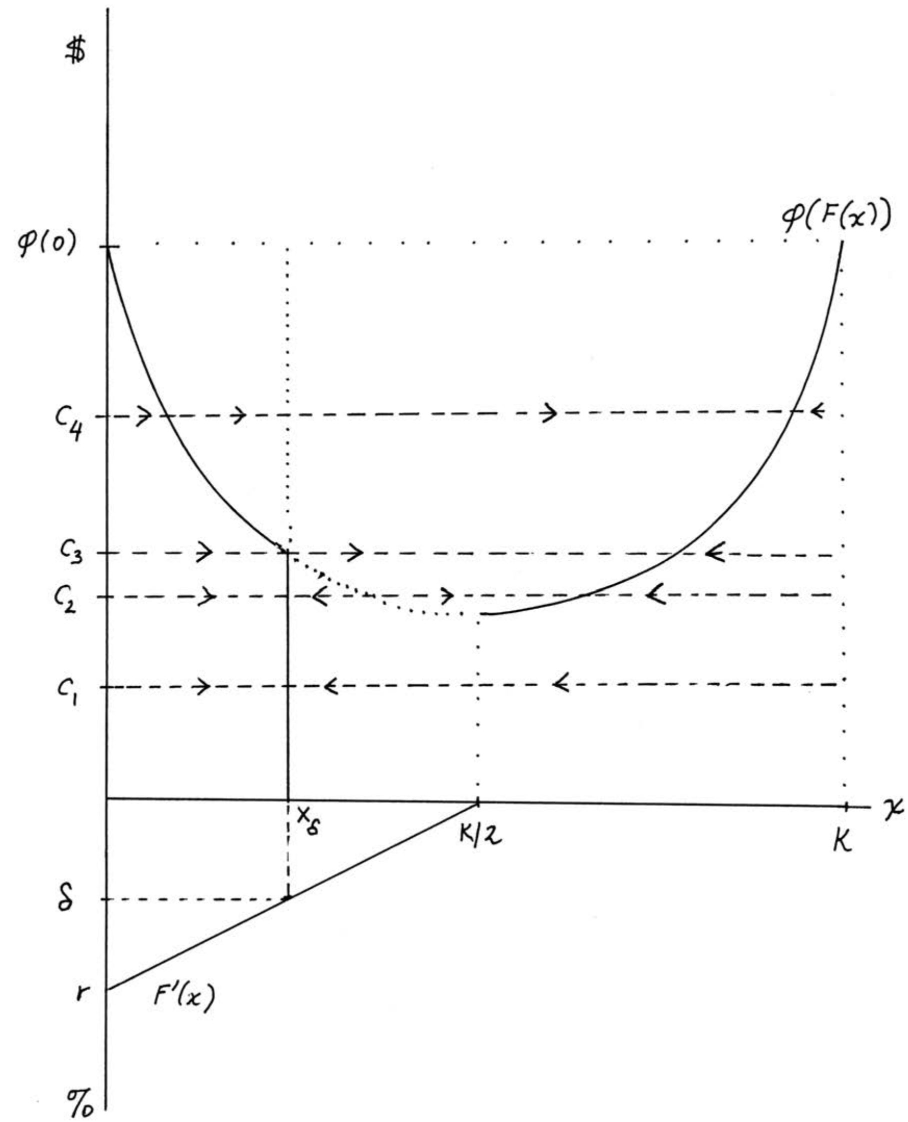

Figure 22. A summary of Figures 16–19. The dotted portion of the vertical line above x_δ has negative profit. The dotted portion of $\phi(F(x))$ is unstable.

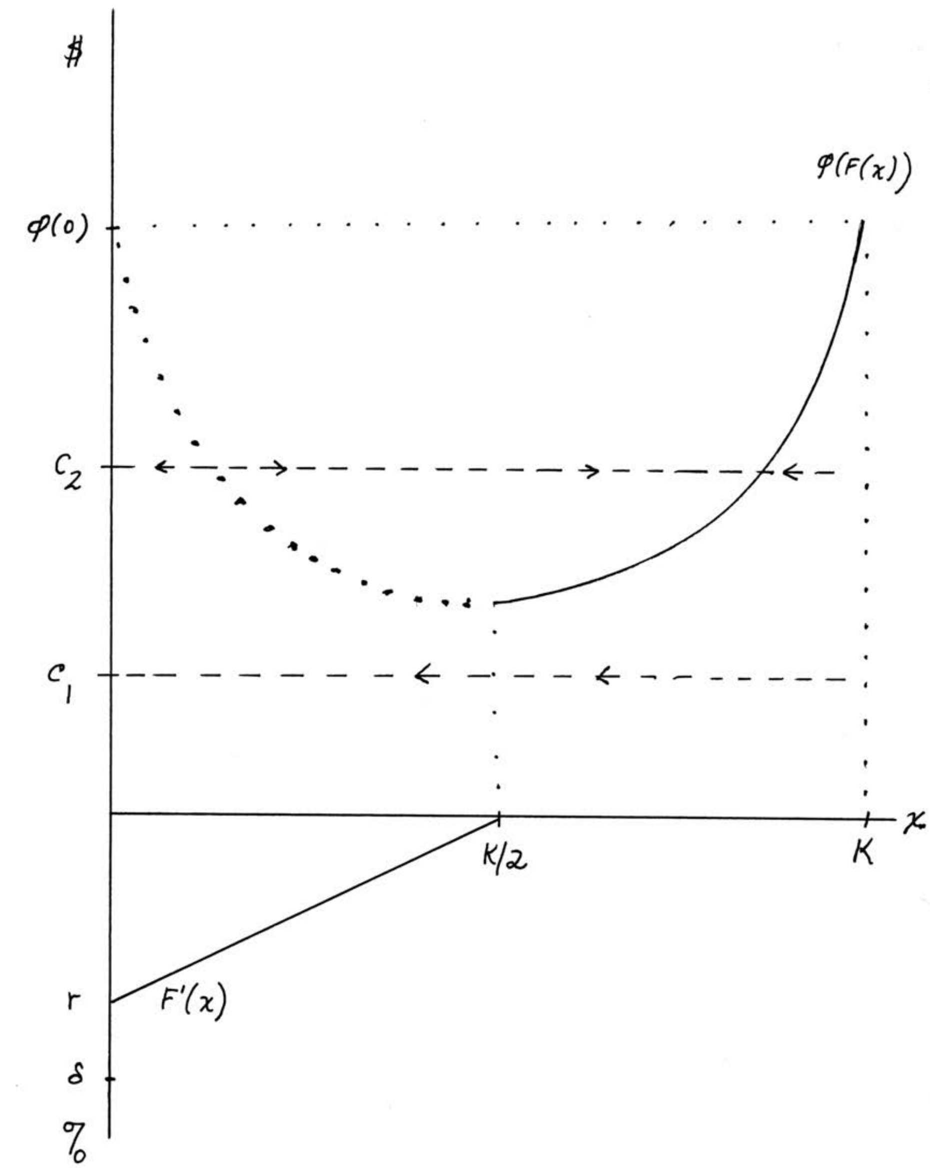

Figure 23. A summary of Figures 20–21. The dotted portion of $\phi(F(x))$ is unstable.

But

$$
\begin{aligned}
MR &= \tfrac{d}{dh}[\phi(h)\cdot h] = \phi'(h)\cdot h + \phi \;\Rightarrow \\
\dot{MR} &= \phi''\dot{h}h + \phi'\dot{h} + \phi'\dot{h} \\
&= \phi''\dot{h}h + 2\phi'\dot{h}
\end{aligned}
$$

and

$$\dot{c} = \tfrac{d}{dt}c(x) = c'(x)\cdot\dot{x}\,,$$

so

$$\delta = F' + \frac{\phi''\dot{h}h + 2\phi'\dot{h} - c'\dot{x} - c'h}{\phi' h + \phi - c}\;;$$

gathering the $\dot{h}$ terms and noting that $-c'\dot{x} - c'h = -c'F$ from (16) yields

$$\delta = F' + \frac{(\phi'' h + 2\phi')\dot{h} - c'F}{\phi' h + \phi - c}\,.$$

Solving this last equation for $\dot{h}$ gives the following dynamic system:

$$\dot{h}_t = \frac{[\delta - F'(x_t)][\phi'(h_t)h_t + \phi(h_t) - c(x_t)] + c'(x_t)\,F(x_t)}{\phi''(h_t)h_t + 2\phi'(h_t)} \tag{87}$$

$$\dot{x}_t = F(x_t) - h_t\,. \tag{16}$$

(Compare this with the system formed by (65) and (16) in Section 3.)

Exercise. Re-work "Example 1, dynamics" for the case of a monopolist. Use the same functional forms $c(x)$ and $F(x)$ as in "Example 1, dynamics," together with the same parameters q, γ, r, K, and δ. Also use the same demand curves, (72)–(76). Once you have found the steady-state price-quantity combination, sketch a graph showing each demand curve and marking the steady-state price-quantity combination on it (obviously there is no steady-state monopoly supply curve). Compare this graph with Figure 3 and especially with the graph from Exercise 2 of Section 3.

Exercise. In (65), show that the $\dot{h} = 0$ isocline is characterized by $dh/dx > 0$ and $d^2h/dx^2 < 0$. Is this true for the $\dot{h} = 0$ isocline under (87)?

Section 6. Competitive, Open Access Fishery: Steady States

In open access steady-state equilibrium, total profit is zero: $0 = \pi(x,h) = ph - c(x,h)\,h$ (so $p = c(x,h)$). Also, $0 = \dot{x} = F(x) - h$. Combining these equations yields

$$
\begin{aligned}
p &= c(x, F(x)) \\
h &= F(x).
\end{aligned} \tag{88}
$$

If, as a special case, we have constant average cost (hence constant returns to scale and average cost equals marginal cost), then $c(x, h) = c(x)$ and (88) becomes

$$\begin{aligned} p &= c(x) \\ h &= F(x). \end{aligned} \tag{89}$$

(88) and especially (89) should be compared with the system formed by (36) and (35). As in the previous system, each value of x will give a value for price p and for quantity h from (88) or (89). Running through values of x will thus give rise to price–quantity combinations; the graph of these combinations is the steady-state supply curve. Alternatively, the steady-state supply curve can be derived graphically, using the same technique illustrated in Figure 1. (Note that in the limit as $\delta \to \infty$, the system formed by (36) and (35) approaches (89).)

The traditional way of deriving this supply curve is completely different. In this traditional approach, harvest h (also called yield Y) is a function of x and of "fishing effort" E: $Y = h(E, x)$. In the steady state, $\dot{x} = 0$ so $Y = F(x)$ from (16). Hence, given E_{ss} (whose subscript denotes steady state), x_{ss} solves the equation $h(E_{ss}, x_{ss}) - F(x_{ss}) = 0$. Write the solution of this equation as $x_{ss}(E_{ss})$. Then the "yield-effort curve" is defined by $Y(E_{ss}) = F(x_{ss}(E_{ss}))$. Total cost incurred by the firm is some function $C(E)$. Total revenue is $pY(E_{ss})$. In open-access equilibrium, total revenue equals total cost, so $C(E_{ss}) = pY(E_{ss})$. This yields E_{ss} as an implicit function of p; call this $E_{ss}(p)$. Then steady-state harvest is $Y(E_{ss}(p))$, which is the supply curve.

Exercise. Find the open-access steady-state supply curve for: (a) logistic growth; (b) depensation; (c) critical depensation. Use $c(x) = \gamma/(qx)$.

Note that between the cases of private property and open access there are intermediate cases where enforcement of property rights exists but is imperfect.

Section 7. Competitive, Open Access Fishery: Dynamics

Along with the biological equation (16), we require an equation describing the dynamic behavior of firms. Open-access firms are usually not modeled as solving an explicit intertemporal maximization problem because there is no benefit to them of leaving fish in the ocean (because someone else will fish them out today). For open-access dynamics, the most common *ad hoc* assumption is that, if E is fishing effort, then $\dot{E}$ is proportional to profit:

$$\dot{E} = k\pi = k[\,ph(E, x) - C(E)]. \tag{90}$$

(The constant of proportionality k is not to be confused with the usual notation for carrying capacity K; nor is the total cost function C to be confused with the average cost function c.) In order to compare this with our previous results, we would like to change from (90) to an equation giving $\dot{h}$. To do this, note that since it is natural to assume $\partial h(E,x)/\partial E \neq 0$ (and in fact that $\partial h(E,x)/\partial E > 0$ since x is held constant in this derivative), the Implicit Function Theorem assures us that it is possible for the equation $h(E,x)$ to be inverted to give effort as a function of h and x: $E(x,h)$. Then (90) becomes

$$\frac{d}{dt}E(x,h) = k[\,ph - C(E(x,h))] \quad \text{or}$$

$$\frac{\partial E}{\partial h}\dot{h} + \frac{\partial E}{\partial x}\dot{x} = k[\,ph - C(E(x,h))]$$

so

$$\dot{h}_t = \frac{k[\,p_t h_t - C(E(x_t,h_t))] - \frac{\partial E}{\partial x}[F(x_t) - h_t]}{\partial E/\partial h}\,. \tag{91}$$

Combined with (16), (91) gives a dynamical system in (x,h) as in previous sections above. The *final* step would be to impose equilibrium, namely $p_t = \phi(h_t)$. (Setting $p_t = \phi(h_t)$ was also the final step in Section 3, because it also dealt with competition, and it was the first step in Section 5, because that was about monopoly.) The final dynamical system would be

$$\begin{aligned}\dot{h}_t &= \frac{k[\,\phi(h_t)h_t - C(E(x_t,h_t))] - \frac{\partial E}{\partial x}[F(x_t) - h_t]}{\partial E/\partial h}\\ \dot{x}_t &= F(x_t) - h_t\end{aligned} \tag{92}$$

Note that if in (92) $\dot{x} = 0$ together with $\dot{h} = 0$, then we do get (88).

In deriving the explicit dynamic equation for private-property competition, (65), and for monopoly, (87), I assumed that the total cost incurred by the firm had the special form $c(x_t)\,h_t$ (which is constant average cost). It is possible to make special assumptions on $C(E)$ here which also imply that total cost is $c(x_t)\,h_t$:

Proposition 2. *Suppose that $C(E)$ is proportional to E (so that $C(E)$ can be written as αE for some $\alpha > 0$). Suppose that $E(x,h)$ is linear in h and is separable in x and h (so that $E(x,h)$ can be written as $E_1(x)\cdot h$ for some function $E_1(x)$). Then if a new function $c(x_t)$ is defined as $\alpha E_1(x_t)$ and a new constant j is defined as $k\cdot\alpha$, total cost $C(E)$ can be written $c(x)\cdot h$, and in addition (92) becomes*

$$\begin{aligned}\dot{h}_t &= \frac{j[\,\phi(h_t) - c(x_t)] - c'(x_t)[F(x_t) - h_t]}{c(x_t)}\,h_t\\ \dot{x}_t &= F(x_t) - h_t\,.\end{aligned} \tag{93}$$

Proof. To prove that total cost can be written $c(x)\,h$:

$$\begin{aligned} C(E(x,h)) &= \alpha\, E(x,h) \quad \text{since } C(E) = \alpha E \\ &= \alpha\, E_1(x)\, h \quad \text{since } E(x,h) = E_1(x)\, h \\ &= c(x)\, h \quad \text{by the definition of } c(x). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\partial E}{\partial h} &= E_1(x) = c(x)/\alpha \\ \frac{\partial E}{\partial x} &= h\frac{\partial E_1(x)}{\partial x} = h\frac{\partial[c(x)/\alpha]}{\partial x} = \frac{h}{\alpha}c'(x)\,. \end{aligned}$$

Making these substitutions into (92), along with $j = k\alpha$, results in (93). ▮

Example. Suppose $C(E) = \gamma \cdot E$ and $h = qEx$ (so that $E(x,h) = h/(qx)$). Then the conditions of the above proposition are satisfied with $\alpha = \gamma$ and $E_1(x) = 1/(qx)$, so $c(x) = \gamma/(qx)$ and $j = k \cdot \gamma$. From (93),

$$\dot{h}_t = \frac{k \cdot \gamma[\,\phi(h_t) - \gamma/(qx)] + (\gamma/(qx^2))[F(x_t) - h_t]}{\gamma/(qx)}\, h_t\,.$$

Simplifying yields the dynamic system

$$\begin{aligned} \dot{h} &= \left[kq\,\phi(h_t)\,x_t - k\,\gamma + \frac{F(x_t) - h_t}{x_t}\right] h_t \\ \dot{x}_t &= F(x_t) - h_t\,. \end{aligned} \tag{94}$$

Exercise. (See the Exercise in Section 5.) Re-work "Example 1, dynamics" for the case of open access. Use the same functional forms $c(x)$ and $F(x)$ as in "Example 1, dynamics" (which, because of the above comments on costs, means (93) holds), together with the same parameters q, γ, r, K, and δ. Also use the same demand curves, (72)–(76). Once you have found the steady-state price-quantity combination, sketch a graph showing each demand curve and marking the steady-state price-quantity combination on it. Compare this graph with Figure 3 and especially with the graph from Exercise 2 of Section 3.

Summary

Throughout this section assume that total costs have the form $c(x) \cdot h$ (constant returns to scale).

The steady-state supply curve for a private-property competitive fishery is

$$p = c(x) - \frac{c'(x)\,F(x)}{\delta - F'(x)}\,. \tag{36}$$

$$h = F(x) \tag{35}$$

whereas the steady-state supply curve for an open-access fishery is

$$\begin{aligned} p &= c(x) \\ h &= F(x). \end{aligned} \tag{89}$$

There is no steady-state supply curve for a monopolist.

The dynamic system for a private-property competitive fishery is

$$\dot{h}_t = \frac{[\delta - F'(x_t)][\phi(h_t) - c(x_t)] + c'(x_t)\,F(x_t)}{\phi'(h_t)}\,. \tag{65}$$

$$\dot{x}_t = F(x_t) - h_t\,. \tag{16}$$

The dynamic system for a monopolist is

$$\dot{h}_t = \frac{[\delta - F'(x_t)][\phi'(h_t)h_t + \phi(h_t) - c(x_t)] + c'(x_t)\,F(x_t)}{\phi''(h_t)h_t + 2\phi'(h_t)} \tag{87}$$

$$\dot{x}_t = F(x_t) - h_t\,. \tag{16}$$

The dynamic system for an open-access fishery is

$$\begin{aligned} \dot{h}_t &= \frac{j[\,\phi(h_t) - c(x_t)] - c'(x_t)[F(x_t) - h_t]}{c(x_t)}\; h_t \\ \dot{x}_t &= F(x_t) - h_t\,. \end{aligned} \tag{93}$$

By the First Theorem of Welfare Economics, competitive equilibria are socially optimal. Therefore it would have been possible to find the competitive equilibrium by first solving the social planner's problem and then appealing to the First Theorem of Welfare Economics. That would have been easier than what we did above, which was to find the competitive equilibrium path of prices; but it is good to know how to find competitive equilibria.