Qualifying Exam in Resource Economics

September 1996

The following pages are an excerpt from a paper you read. On the last page is a modern version of the Hotelling Rule, together with its proof. However, there are eight "blanks" in the proposition and its proof. On a separate sheet of paper, write down what should go in these eight blanks. Adding some explanation will probably be appropriate.

Hints: Some of the blanks are easier to fill in than others, and most do not depend on getting the right answers to the previous blanks. Some of the blanks are easier to fill in if you read ahead. The answer to one blank is an equation number; the answer to the other blanks are mathematical expressions, possibly including operators ($+$, $\div$, etc.) and relations ($=$, $\leq$, etc.).

Good luck!

1. The Perfect-Foresight Competitive Model

Assume that there are N resource-owning firms, identical in all respects (to avoid order-of-exploitation problems), each owning $S > 0$ units of an exhaustible natural resource with certainty. These firms have perfect foresight of all future prices and take these prices parametrically. They act to maximize the net present value of their resource extraction with a rate of time preference (or time discount) denoted r_t for each time t. Define $\delta_t = e^{\int_0^t r_s\, ds}$; if $r_t \equiv r$ then $\delta_t = e^{rt}$. If q_t^i denotes the extraction rate of a firm i at time t, assume that for each t, extraction costs $C_t(q_t^i)$ are twice differentiable in q with $C_t(0) = 0$, $C_t'(q) > 0$, and $C_t''(q) > 0$. For each q, $C_t(q)$ is assumed to be continuous in t. Also assume $C_t''(q)$ is bounded away from zero for large q and for all t.

The problem of firm i is to

$$\max_{q_t^i} \int_0^\infty \frac{p_t\, q_t^i - C_t(q_t^i)}{\delta_t}\, dt \tag{1}$$

subject to $\dot{x}^i(t) = -q^i(t)$, $q^i(t) \geq 0$, $x^i(0) = S^i$, and $\lim_{t\to\infty} x^i(t) \geq 0$, where p_t is the resource price at time t, $x^i(t)$ is the stock of the resource remaining in the ground at time t for firm i, and where a raised dot denotes differentiation with respect to time. These constraints collectively ensure that $x^i(t)$ is nonincreasing and nonnegative for every $t \in [0, \infty)$. Restrict attention to controls q_t which are piecewise continuous and are everywhere continuous from the left. The symbol "$\triangleq$" will mean "is defined to be."

For the i^{th} competitive firm, the Hamiltonian is $[p_t\, q_t^i - C_t(q_t^i)]/\delta_t - \lambda_t^i\, q_t^i$. Optimal control theory implies $\dot{\lambda}_t^i = 0$ (so $\lambda_t^i \equiv \lambda^i$) along with the following decision rule: if

$C_t'(q) > p_t - \lambda^i \delta_t$ for every $q > 0$, or equivalently

$$\text{if} \quad C_t'(0) \geq p_t - \lambda^i \delta_t, \quad \text{then extract } q_t^i = 0; \tag{2}$$

otherwise, extract the $q_t^{i*} > 0$ implicitly given by

$$C_t'(q_t^{i*}) = p_t - \lambda^i \delta_t \tag{3}$$

(the asterisk denotes equilibrium and is omitted when the meaning is clear). Because $C_t''(q)$ is bounded away from zero for large q and for all t, if (2) does not occur then it is always possible to satisfy (3). One has $\lambda \geq 0$. This completes specification of the supply side of the market.

Assume that the market demand curve $D_t(p_t)$ is decreasing and differentiable in p except possibly as it approaches the p or q axes, and that it is continuous in t for all p. Letting the "choke price" $\min\{\, p : D_t(p) = 0 \,\}$ be $p_c(t)$, define an inverse demand function ϕ as: $\phi_t(Q) = D_t^{-1}(Q)$ for $Q > 0$ and $\phi_t(Q) = p_c(t)$ for $Q = 0$. This completes specification of the demand side of the market.

Finally, equate supply and demand, since this paper is exclusively concerned with equilibrium paths:

$$p_t \geq \phi_t(Q_t) \text{ for all } t, \text{ and } p_t = \phi_t(Q_t) \text{ if } Q_t > 0. \tag{4}$$

To ensure a nontrivial equilibrium, assume that $p_c(t) > C_t'(0)$ for some t.

Since (1) is a strictly concave problem, identical firms act identically; hence $\lambda^i \equiv \lambda$, $q_t^i \equiv q_t$, and so forth. From the supply side, decision rule (2)–(3) implies that $Q_t = 0$ if and only if (henceforth, "iff") $C_t'(0) \geq p_t - \lambda \delta_t$; hence $Q_t > 0$ iff $p_t > C_t'(0) + \lambda \delta_t$. From the demand side, $Q_t > 0$ iff $p_c(t) > p_t$. Putting these together,

Proposition 1. *$Q_t > 0$ iff $p_c(t) > C_t'(0) + \lambda\delta_t$.*

Thus extraction occurs whenever demand is large and costs (extraction costs plus "user cost") are small.

One can then show:

Proposition 2. *Let $\mathcal{T}$ be the set of all dates at which* ___blank 1___. *The only possible competitive equilibrium quantity and price paths are:*

(i) *$\phi_t(Nq_t) - C_t'(q_t) = \lambda\delta_t$, $Q_t = Nq_t > 0$, and $p_t = \phi_t(Nq_t)$ for all $t \in \mathcal{T}$; and*

(ii) *$Q_t = q_t = 0$ and $p_t \in [\,p_c(t),$ ___blank 2___$]$ for all other t.*

The q_t implicitly defined in (i) exists, is unique, and is positive.

Proof. Case (i): If $p_c(t) > C_t'(0) + \lambda\delta_t$ then $Q_t > 0$ from Proposition 1, so $q_t^i > 0$ for all i. Hence ___blank 3___ from (3), implying $\lambda\delta_t = \phi_t(Q_t) - C_t'(q_t)$ from (4).

Case (ii): If $p_c(t) \leq$ ___blank 4___ then $Q_t = q_t = 0$ from Proposition 1. Hence $p_t \leq$ ___blank 5___ from (2). It only remains to show that $p_t \geq p_c(t)$. Substitute $Q_t = 0$ into (___blank 6___)'s inequality and use the definition of p_c.

To prove that the q_t implicitly defined in (i) exists, is unique, and is positive, begin by supposing for simplicity that both C and ϕ are finite on $(0, \infty)$ (this can be relaxed). Fix the time index $t \in \mathcal{T}$. Clearly $\phi_t(0)$___blank 7___ > 0 because otherwise $p_c(t) \triangleq \phi_t(0)$___blank 8___, contradicting $t \in \mathcal{T}$. Also, $\lim_{q\to\infty} \phi_t(Nq) < \infty$, $\lim_{q\to\infty} C_t'(q) = \infty$, so for sufficiently large q, $\phi_t(Nq) - C_t'(q) - \lambda\delta_t < 0$. Finally, $\phi_t(Nq) - C_t'(q)$ is continuous on $(0, \infty)$ under the maintained assumptions. Apply the Intermediate Value Theorem, with uniqueness due to the monotonicity of ϕ_t and C_t' in q. ∎