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This result can be applied whenever a fraction is raised to a negative power. For example,

$$\left(\frac{5}{4}\right)^{-3} = \frac{4^3}{5^3} = \frac{64}{125}$$

Example A.1

If $ab^2 = 2$, compute the following:

$$\text{(a) } a^2b^4 \qquad \text{(b) } a^{-4}b^{-8} \qquad \text{(c) } a^3b^6 + a^{-1}b^{-2}$$

Solution

(a) $a^2b^4 = (ab^2)^2 = 2^2 = 4$

(b) $a^{-4}b^{-8} = (ab^2)^{-4} = 2^{-4} = 1/2^4 = 1/16$

(c) $a^3b^6 + a^{-1}b^{-2} = (ab^2)^3 + (ab^2)^{-1} = 2^3 + 2^{-1} = 8 + 1/2 = 17/2$

Note: An important motivation for introducing definitions [A.2] and [A.3] is that we would like the properties in [A.4] to be valid for all exponents. For example, consider the consequences of requiring [A.4](a) to be valid for $a^5 \cdot a^0$. We obtain $a^{5+0} = a^5$, so that $a^5 \cdot a^0 = a^5$, and hence we must choose $a^0 = 1$. If [A.4](a) is to be valid for $m = -n$, we must have $a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1$. Because $a^n \cdot (1/a^n) = 1$, we *must* define a^{-n} by [A.3].

Compound Interest

Powers are used in practically every branch of applied mathematics, including economics. To illustrate their use, consider how they are needed to calculate compound interest.

Suppose you deposit \$1000 in a bank at 8% interest per year.[1] After one year you will have earned $\$1000 \cdot 0.08 = \$80$ in interest, so the amount in your bank account at the end of the year will be \$1080. This can be rewritten as

$$1000 + \frac{1000 \cdot 8}{100} = 1000\left(1 + \frac{8}{100}\right) = 1000 \cdot 1.08$$

If this new amount of $\$1000 \cdot 1.08$ is left in the bank for another year at an interest

[1] Remember that 1% means one in a hundred, or 0.01. To calculate, say, 23% of \$4000, we write

$$\frac{4\,000 \cdot 23}{100} = 920 \qquad \text{or} \qquad 4000 \cdot 0.23 = 920$$

rate of 8%, after a second year, the amount will have grown to a total of

$$1000 \cdot 1.08 + \frac{(1000 \cdot 1.08) \cdot 8}{100} = 1000 \cdot 1.08 \left(1 + \frac{8}{100}\right) = 1000 \cdot (1.08)^2$$

Each year the amount will increase by the factor 1.08, and we see that at the end of t years, it will have grown to $\$1000 \cdot (1.08)^t$. If the original amount is $\$K$ and the interest rate is $p\%$ per year, by the end of the first year, the amount will be $K + K \cdot p/100 = K(1 + p/100)$ dollars. The growth factor per year is thus $1 + p/100$. In general, after t (whole) years, the original investment of $\$K$ will have grown to an amount

$$K\left(1 + \frac{p}{100}\right)^t$$

when the interest rate is $p\%$ per year (and interest is added to the capital every year—that is, compound interest).

If you see an expression like $(1.08)^t$, you should immediately be able to recognize it as the amount to which \$1 has grown after t years when the interest rate is 8% per year. What would be the interpretation of $(1.08)^0$? You deposit \$1 at 8% per year, and leave the amount for 0 years. Then you still have only \$1, because there has been no time to accumulate any interest, so that $(1.08)^0$ *must* equal 1.

Are Negative Exponents Useful?

How much money should you have deposited in the bank 5 years ago in order to have \$1000 today, given that the interest rate has been 8% per year over this period? If we call this amount x, the requirement is that $x \cdot (1.08)^5$ must equal \$1000, or that

$$x \cdot (1.08)^5 = 1000$$

The solution for x is

$$x = \frac{1000}{(1.08)^5} = 1000 \cdot (1.08)^{-5}$$

(which is approximately \$681). It turns out that $\$(1.08)^{-5}$ is what you should have deposited 5 years ago in order to have \$1 today, given the constant interest rate of 8%.

In general, $\$P\,(1 + p/100)^{-t}$ is what you should have deposited t years ago in order to have $\$P$ today if the interest rate has been $p\%$ every year.

Problems

1. Compute the following:

a. 6^3 **b.** $\left(\frac{2}{3}\right)^2$ **c.** $(-1)^5$ **d.** $(0.3)^2$

e. $(4.5 - 2.5)^4$ **f.** $2^2 \cdot 2^4$ **g.** $2^2 \cdot 3^2 \cdot 4^2$ **h.** $(2^2 \cdot 3^2)^3$

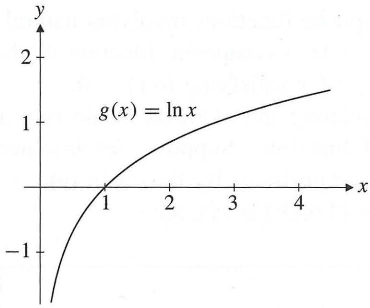

FIGURE 8.5 The graph of the natural logarithmic function $g(x) = \ln x$.

In Fig. 8.5 we have drawn the graph of $g(x) = \ln x$. The shape of this graph ought to be remembered. According to Example 8.3, we have $g(1/e) = -1$, $g(1) = 0$, and $g(e) = 1$. Observe that this corresponds well with the graph.

Differentiation of Logarithmic Functions

If we assume that $g(x) = \ln x$ has a derivative for all $x > 0$, then this derivative can be easily found. Differentiate implicitly the equation

$$e^{g(x)} = x \qquad [*]$$

with respect to x, using the result in [8.4]. This gives

$$e^{g(x)}g'(x) = 1$$

Because $e^{g(x)} = x$, so $xg'(x) = 1$. Hence:

$$\boxed{g(x) = \ln x \quad \Longrightarrow \quad g'(x) = \frac{1}{x}} \qquad [8.9]$$

Thus, the derivative of $\ln x$ at point x is simply the number $1/x$. For $x > 0$, we have $g'(x) > 0$, so that $g(x)$ is *strictly* increasing. Note moreover that $g''(x) = -1/x^2$, which is less than 0 for all $x > 0$, so that $g'(x)$ is *strictly* decreasing. This confirms the shape of the graph in Fig. 8.5. In fact, the growth of $\ln x$ is quite slow. For example, $\ln x$ first attains the value 10 when $x > 22,026$, because $\ln x = 10$ gives $x = e^{10} \approx 22,026.5$.

Note: We derived [8.9] *assuming* that $g(x) = \ln x$ was differentiable. In fact, by Theorem 7.9 in Section 7.6, the logarithmic function g *is* differentiable. Because the derivative of $f(x) = e^x$ is e^x, applying (7.24) to $y_0 = e^{x_0}$ tells us that $g'(y_0) = 1/e^{x_0} = 1/y_0$. This is the same as [8.9], except that the symbol y_0 has replaced x.

A Characterization of the Number e

In Section 8.2, we showed by implicit differentiation that if $g(x) = \ln x$ is differentiable, then $g'(x) = 1/x$. More specifically, $g'(1) = 1$. If we use the *definition* of $g'(1)$ and [8.7](c), together with the fact that $\ln 1 = 0$, we obtain

$$1 = g'(1) = \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0} \frac{1}{h} \ln(1+h) = \lim_{h\to 0} \ln(1+h)^{1/h}$$

Because $\ln(1+h)^{1/h}$ tends to 1 as h tends to 0, it follows that $(1+h)^{1/h}$ itself must tend to e, and so

$$e = \lim_{h\to 0} (1+h)^{1/h} \tag{8.16}$$

TABLE 8.1 *Values of* $(1+h)^{1/h}$

h	1	1/2	1/10	1/1000	1/100000	1/1000000
$(1+h)^{1/h}$	2.00	2.25	2.5937...	2.7169...	2.71825...	2.718281828...

Table 8.1 has been computed using a scientific calculator. The results seem to confirm that the decimal expansion we gave for e is correct. From the table, we can see that a closer and closer approximation to e is obtained by choosing h smaller and smaller. If we let $h = 1/n$, where the natural number n becomes larger and larger, we obtain the following:

$$\boxed{e = \lim_{n\to\infty} (1 + 1/n)^n} \tag{8.17}$$

Another Important Limit

If a is an arbitrary number greater than 1, then $a^x \to \infty$ as $x \to \infty$. For example, $(1.0001)^x \to \infty$ as $x \to \infty$. Furthermore, if p is an arbitrary positive number, then $x^p \to \infty$ as $x \to \infty$. If we compare $(1.0001)^x$ and x^{1000}, it is clear that the former increases quite slowly at first, whereas the latter increases very quickly. Nevertheless, $(1.0001)^x$ eventually "overcomes" x^{1000}. In general, we claim the following:

$$\boxed{\lim_{x\to\infty} \frac{x^p}{a^x} = 0 \qquad (a > 1,\ p \text{ is a fixed number})} \tag{8.18}$$

8.5 Compound Interest and Present Discounted Values

Equation [8.21], $f'(t) = rf(t)$ for all t, has a particularly important application to economics. After t years, a deposit of $\$K$ earning interest at the rate $p\%$ per year will increase to

$$K(1+r)^t \quad (\text{where} \quad r = p/100) \tag{1}$$

(see Section A.1, Appendix A). Each year the principal increases by the factor $1+r$.

Formula [1] assumes that the interest is added to the principal at the end of each year. Suppose instead that payment of interest is offered each half year, but at an interest rate $p/2$. Then the principal after 1/2 year will have increased to

$$K + K\frac{p/2}{100} = K\left(1+\frac{r}{2}\right)$$

Therefore, the principal increases by the factor $1+r/2$ each half year. After 1 year, the principal will have increased up to $K(1+r/2)^2$, and after t years it will be

$$K\left(1+\frac{r}{2}\right)^{2t} \tag{2}$$

It is clear that a biannual interest payment at the rate $\frac{1}{2}p\%$ is better for a lender than an annual interest payment at the rate $p\%$. This is easily seen also from the fact that $(1+r/2)^2 = 1+r+r^2/4 > 1+r$.

More generally, suppose that interest at the rate $p/n\%$ is added to the principal at n different times distributed evenly over the year. Then the principal will be multiplied by a factor $(1+r/n)^n$ each year. After t years, the principal is

$$K\left(1+\frac{r}{n}\right)^{nt} \tag{3}$$

The greater is n, the more profitable is the investment for the lender. See Problem 3.

In practice, there is a limit to how frequently interest can be added to savings accounts. However, let us examine what happens to the expression in [3] as the annual frequency n tends to infinity. We put $r/n = 1/m$. Then $n = mr$ and so

$$K\left(1+\frac{r}{n}\right)^{nt} = K\left(1+\frac{1}{m}\right)^{mrt} = K\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt} \tag{4}$$

As $n \to \infty$ (with r fixed), so $m = n/r \to \infty$, and according to [8.17], we have $(1+1/m)^m \to e$. Hence, the expression in [4] approaches Ke^{rt} as n tends to infinity. When we let n approach infinity, the accumulation of interest happens more and more frequently. In the limit, we talk about **continuous compounding** of interest. After t years, an initial amount $\$K$ will have increased to

$$K(t) = Ke^{rt} \qquad \textbf{(continuous compounding)} \qquad [8.26]$$

The number r is often referred to as the **rate of interest**. By differentiating [8.26], we have the following important fact.

> With continuous compounding of interest at rate r, the principal increases at the constant relative rate r, so that $K'(t)/K(t) = r$.

From [8.26], we infer that $K(1) = Ke^r$, so that the principal increases by the factor e^r during the first year. In general, $K(t+1) = Ke^{r(t+1)} = Ke^{rt}e^r = K(t)e^r$, so that with continuous compounding of interest, the principal increases *each* year by the fixed factor e^r.

Comparing Different Forms of Interest

At an interest rate of $p\%$ $(= 100r)$ per year, continuous compounding of interest is best for the lender. (See Problem 3.) For comparatively low interest rates, however, the difference between annual and continuous compounding of interest is quite small.

Example 8.9

Find the amount by which \$1 increases in the course of a year when the interest rate is 8% per year and interest is added:

(a) only at the end of the year
(b) at the end of each half year
(c) continuously

Solution In this case, $r = 8/100 = 0.08$, so we obtain the following:

(a) $K = (1 + 0.08) = 1.08$
(b) $K = (1 + 0.08/2)^2 = 1.0816$
(c) $K = e^{0.08} \approx 1.08329$

If we increase the interest rate or increase the number of years over which interest accumulates, then the difference between yearly and continuous compounding of interest increases.

Note: A consumer who wants to take out a loan may be faced with several offers from financial institutions. It is therefore of considerable importance to compare the various offers. The concept of **effective interest rate** is often used in making such comparisons. Imagine an offer that implies a yearly interest rate $p\%$ with interest p/n added n times during the year. A principal amount of K will then have increased after 1 year to $K(1+r/n)^n$, where $r = p/100$. Define the *effective interest rate* P as the annual percentage interest rate that, when compounding is continuous, gives the same total interest over the year. If $R = P/100$, then after 1 year, the initial amount K increases to Ke^R. Hence, R is defined by the equation

$$Ke^R = K(1+r/n)^n$$

Canceling K and then taking ln of both sides gives

$$R = n\ln(1+r/n) \tag{8.27}$$

If $r = 0.08$ and $n = 1$, for example, then $R = \ln(1+0.08) \approx 0.077$. Thus, a yearly interest rate of 8% corresponds to an effective interest rate (with continuous compounding) of about 7.7%.

The Present Value of a Future Claim

Suppose that an amount K is due for payment t years after the present date. What is the *present value* of this amount when the interest rate is $p\%$ per year? Equivalently, how much must be deposited today earning $p\%$ annual interest in order to have the amount K after t years?

If interest is paid annually, the amount A will have increased to $A(1+p/100)^t$ after t years, so that we need $A(1+p/100)^t = K$. Thus, $A = K(1+p/100)^{-t} = K(1+r)^{-t}$, where $r = p/100$. If interest is compounded continuously, however, then the amount A will have increased to Ae^{rt} after t years. Hence, $Ae^{rt} = K$, or $A = Ke^{-rt}$. Altogether, we have the following:

> If the interest rate is $p\%$ per year and $r = p/100$, an amount K that is payable in t years has the present value:
>
> $K(1+r)^{-t}$, with yearly interest payments
>
> Ke^{-rt}, with continuous compounding of interest

[8.28]

Problems

1. An amount \$1000 earns interest at 5% per year. What will this amount have grown to after (a) 10 years, and (b) 50 years, when interest is compounded (i) yearly, (ii) monthly, (iii) continuously?

http://www.mas.ncl.ac.uk/~nzal/MAS267.dir/contvarint.pdf
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The point of these remaining pages is to derive the last formula on the last page of this handout (present value for continuously varying discount rates).

7. Continuously Varying Interest Rates

7.1 The Continuous Varying Interest Rate Formula.

Suppose that interest is continuously compounded with a rate which is changing in time. Let the present time be time 0, and let $r(s)$ denote the interest rate per unit at time s units, $s \geq 0$.

The quantity $r(s)$ is called the **spot** or the **instantaneous** interest rate at time s.

Let $D(t)$ be the amount that you will have in your account at time t if you deposit $\mathbf{P}$ at time 0. In order to determine $D(t)$ in terms of the interest rates $r(s), 0 \leq s \leq t$, note that by the Simple Interest Formula 1.8, for small h, we have

$$D(s+h) \approx D(s)(1 + r(s) \cdot h),$$

($\approx$ means "is approximately equal to")

$$D(s+h) \approx D(s) + D(s) \cdot r(s) \cdot h,$$

$$D(s+h) - D(s) \approx D(s) \cdot r(s) \cdot h,$$

$$\frac{D(s+h) - D(s)}{h} \approx D(s) \cdot r(s).$$

By the definition of the derivative 4.7, taking the limit as $h \to 0$, we have

$$D'(s) = D(s) \cdot r(s)$$

or

$$\frac{D'(s)}{D(s)} = r(s).$$

To solve this differential equation, integrate both sides:

$$\int_0^t \frac{D'(s)}{D(s)} ds = \int_0^t r(s) ds.$$

Thus

$$[\ln D(s)]_0^t = \int_0^t r(s) ds$$

or

$$\ln D(t) - \ln D(0) = \int_0^t r(s)ds.$$

Since $D(0) = \mathbf{P}$, we obtain from the preceding equation that

$$\ln D(t) - \ln \mathbf{P} = \int_0^t r(s)ds$$

$$\exp\left(\ln D(t) - \ln \mathbf{P}\right) = \exp\left(\int_0^t r(s)ds\right)$$

$$\exp\left(\ln D(t)\right) \cdot \exp\left(\ln \mathbf{P}^{-1}\right) = \exp\left(\int_0^t r(s)ds\right)$$

$$D(t) \cdot \mathbf{P}^{-1} = \exp\left(\int_0^t r(s)ds\right)$$

and

$$D(t) = \mathbf{P} \cdot \exp\left(\int_0^t r(s)ds\right).$$

7.4 The Present Value. The Continuous Varying Interest Rate Formula 7.1 states that a principal $\mathbf{P}$ earning a continuously compounded interest $r(s)$ per unit at time s units will be worth

$$D(t) = \mathbf{P} \cdot \exp\left(\int_0^t r(s)ds\right)$$

at time t. Therefore

$$\mathbf{P} = D(t) \cdot \left[\exp\left(\int_0^t r(s)ds\right)\right]^{-1}$$

$$= D(t) \cdot \exp\left[-\left(\int_0^t r(s)ds\right)\right].$$