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Econ. 7250
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NOTES ON TIMBER ECONOMICS

Assume the volume of wood in a tree expressed as a function of tree age t (for "time") is $V(t)$ and the total cost of cutting down a tree is C. Suppose the price of a unit volume of wood (such as 1 liter of wood) is fixed at one dollar. (This ignores issues of demand and supply at various dates, and so does not address questions of competitive equilibrium.) Let T be the age at which the tree is cut down, noting that

$$\frac{d}{dT} = \frac{d}{dt}\frac{dt}{dT} = \frac{d}{dt} \cdot 1 = \frac{d}{dt}\,. \tag{1}$$

The firm wishes to

$$\max_T e^{-\delta T}[V(T) - C]\,. \tag{2}$$

Proposition 1. *The value of T which solves (2) is implicitly defined by*

$$\delta = \frac{V'(T)}{V(T) - C}\,, \tag{3}$$

which can be rewritten as

$$\delta = \frac{d}{dT}\ln[V(T) - C]\,. \tag{4}$$

Proof. The first-order condition for (2) is

$$0 = -\delta e^{-\delta T}[V(T) - C] + e^{-\delta T}V'(T)\,, \text{implying}$$
$$V'(T) = \delta\,[V(T) - C]\,,$$

which leads to (3). This and Lemma 1 leads to (4). ∎

Lemma 1.

$$\frac{V'(T)}{V(T) - C} = \frac{d}{dT}\ln[V(T) - C]\,.$$

Proof. We have

$$\frac{V'(T)}{V(T) - C} = \frac{\frac{d}{dT}[V(T) - C]}{V(T) - C} = \frac{d}{dT}\ln[V(T) - C]\,.$$

∎

The right-hand side of (4) is the semi-elasticity of $V(T) - C$ with respect to T (because the derivative in (4) is taken with respect to T instead of with respect to $\ln T$). In the graph of $\ln[V(T) - C]$ shown in the bottom part of Figure 1, the condition (4) is satisfied at the T (there labeled "I_1^*" for reasons explained later) at which the slope of $\ln[V(T) - C]$ is equal to δ.

Next, suppose the tree is replanted after being harvested. Subsume the planting costs into C, since replanting occurs just after cutting down (we model them as occurring at the same date). If T_i denotes the time at which the i^{th} generation of trees is cut down, then the present value of the plot of land is

$$e^{-\delta T_1}[V(T_1)-C] + e^{-\delta T_2}[V(T_2-T_1)-C] + e^{-\delta T_3}[V(T_3-T_2)-C] + \cdots .$$

Given that neither costs C nor price (which is defined to be equal to one) depends on time, each generation of trees will be cut down at the same age. Call this age "I" for "rotation interval" (or "I_m" where the m stands for multiple cropping): $I = T_1 = T_2 - T_1 = T_3 - T_2 = \cdots$. Then the present value is

$$\begin{aligned} & e^{-\delta I}[V(I)-C] + e^{-\delta \cdot 2I}[V(I)-C] + e^{-\delta \cdot 3I}[V(I)-C] + \cdots \\ & \quad = [V(I)-C] \sum_{k=1}^{\infty} e^{-k\delta I} = \frac{V(I) - C}{e^{\delta I} - 1} \end{aligned} \tag{5}$$

where the last equality comes from setting $r = e^{-\delta I}$ in the formula for the sum of an infinite geometric series

$$\sum_{k=1}^{\infty} r^k = \frac{r}{1-r} = \frac{1}{1/r - 1} . \tag{6}$$

Proposition 2. *The value of I which maximizes (5), called I_m^*, is implicitly defined by*

$$\frac{\delta}{1 - e^{-\delta I_m^*}} = \frac{V'(I_m^*)}{V(I_m^*) - C} , \tag{7}$$

which can be rewritten as

$$\frac{\delta}{1 - e^{-\delta I_m^*}} = \frac{d}{dI} \ln[V(I) - C] \bigg|_{I_m^*} \tag{8m}$$

(where the "m" in the equation number is for the multiple-cropping case).

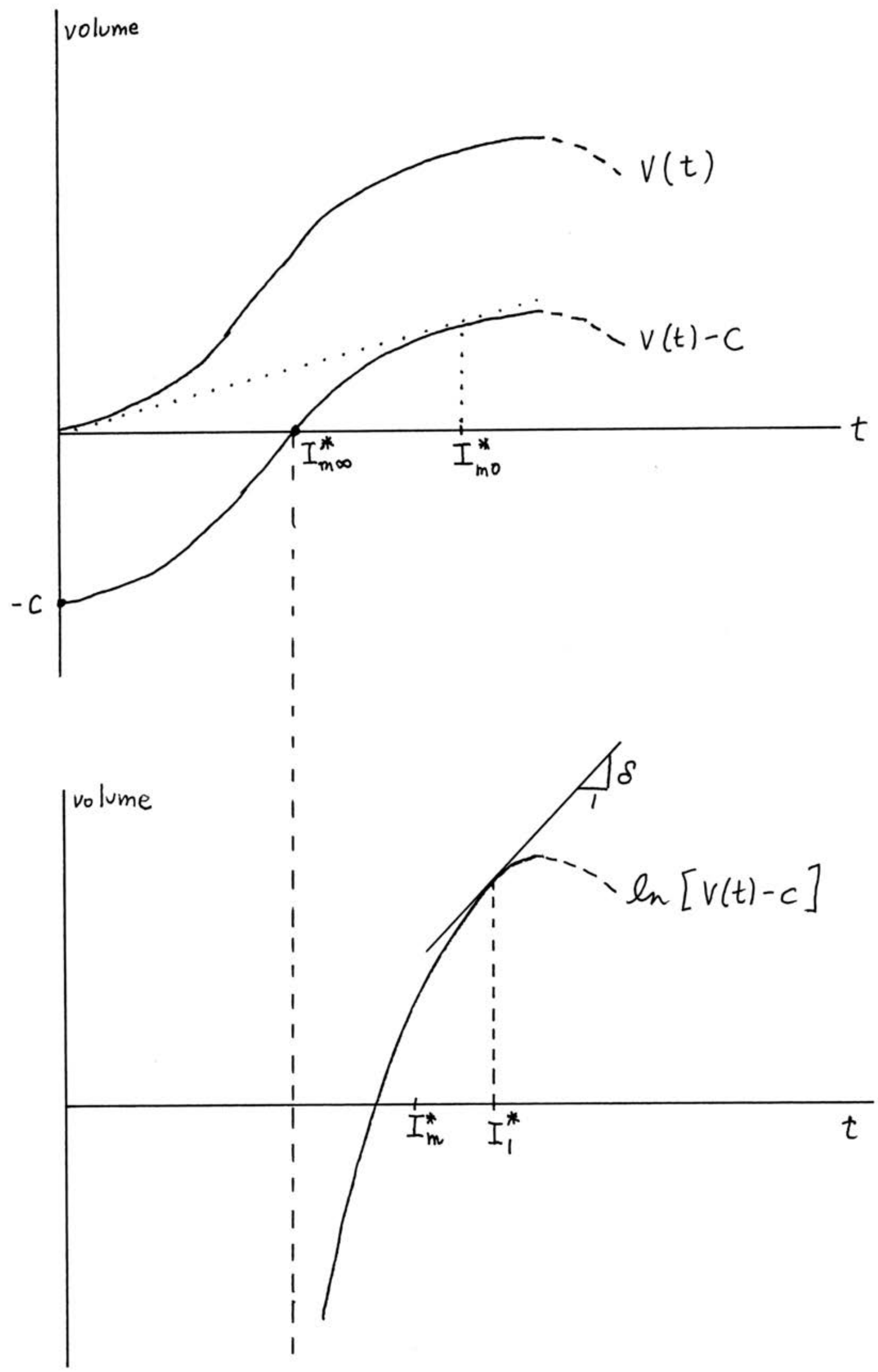

Figure 1. Top diagram: volume of wood as a function of age (with the dotted curve representing average shrinking of volume as very old trees die); and volume of wood minus cutting costs. Bottom diagram: the logarithm of the latter.

Proof. Maximizing (5) with respect to I yields the first-order condition

$$0 = \frac{V'(I)}{e^{\delta I} - 1} - \frac{V(I) - C}{(e^{\delta I} - 1)^2}\,\delta e^{\delta I}$$

$$\frac{V(I) - C}{e^{\delta I} - 1}\,\delta e^{\delta I} = V'(I)$$

$$\frac{V(I) - C}{1 - e^{-\delta I}}\,\delta = V'(I)$$

from which (7) follows. Use Lemma 1 to obtain (8m). ▮

Let $T = I_1^*$ be the optimal age to cut down the tree in the single-crop case. Then

Proposition 3. *We have $I_m^* < I_1^*$.*

Proof. Note that, using (1),

$$\frac{d}{dI} = \frac{d}{dt}\frac{dt}{dI} = \frac{d}{dt} \cdot 1 = \frac{d}{dt} = \frac{d}{dT}\,. \tag{9}$$

Therefore, (4) is

$$\delta = \frac{d}{dI}\ln[V(I) - C]\bigg|_{I_1^*} \tag{8s}$$

where the "s" in the equation number is for the single-cropping case. Since the left-hand side of (8m) is larger than the left-hand side of (8s), the right-hand side of (8m) has to be larger than the right-hand side of (8s). The right-hand side of these equations is the slope of $\ln[V(I) - C]$. Therefore, the slope of $\ln[V(I) - C]$ has to be larger in the multi-cropping case than it is in the single-crop case. As can be seen from our graph of $\ln[V(I) - C]$, this means that the multi-cropping case's I must be smaller. ▮

Proposition 4. *Another way of comparing the single- and multiple-cropping cases is*

$$V'(I_m^*) = \delta\,[V(I_m^*) - C] + \delta \cdot \underbrace{[V(I_m^*) - C]\sum_{k=1}^{\infty} e^{-k\delta I_m^*}}_{\text{“site value”}} \tag{10}$$

compared with

$$V'(I_1^*) = \delta\,[V(I_1^*) - C]\,. \tag{11}$$

Proof. (11) follows from (3) and the definition of I_1. To obtain (10), rewrite (7) as

$$\frac{V'(I_m^*)}{V(I_m^*) - C} = \delta\,\frac{1 - e^{-\delta I_m^*} + e^{-\delta I_m^*}}{1 - e^{-\delta I_m^*}} = \delta\left[1 + \frac{e^{-\delta I_m^*}}{1 - e^{-\delta I_m^*}}\right] = \delta\left[1 + \frac{1}{e^{\delta I_m^*} - 1}\right].$$

Then

$$V'(I_m^*) = \delta\,[V(I_m^*) - C] + \delta\frac{V(I_m^*) - C}{e^{\delta I_m^*} - 1}, \tag{12}$$

from which (10) follows using (6). ∎

Clark (first edition, section 8.1 p. 259) writes (reading I_m^* for T)[1]:

> "[(7)] for the optimal rotation period T is called the *Faustmann formula;* it was derived in 1849 by M. Faustmann, a German forester. . . . The third term [in (12)] reflects the rotation aspect of the problem. The expression
>
> $$\frac{V(T) - C}{e^{\delta T} - 1}$$
>
> is the present value of this stream of future revenues; in the forestry literature this expression is called the *site value.* The condition of Eq. [(12)] is that the forest be cut at age T, when the marginal increment to the value of the trees equals the sum of the opportunity costs of investment tied up in the standing trees and in the site."

Proposition 5. *In the limit as $\delta \to 0$, the marginal value of $V(I_m) - C$ is equal to its average value.*

Proof. The left-hand side of (7) has the property that

$$\lim_{\delta \to 0} \frac{\delta}{1 - e^{-\delta I}} = \text{``}\,\frac{0}{0}\,\text{''} = \lim_{\delta \to 0} \frac{1}{I e^{-\delta I}} = \frac{1}{I}$$

using L'Hôpital's Rule. Denoting I_m^* in the limit as $\delta \to 0$ by I_{m0}^*, it follows that in the limit as $\delta \to 0$, (7) implies

$$\frac{1}{I_{m0}} = \frac{V'(I_{m0}^*)}{V(I_{m0}^*) - C}$$
$$\frac{V(I_{m0}^*) - C}{I_{m0}} = V'(I_{m0}^*)\,.$$

∎

[1] See also `https://en.wikipedia.org/wiki/Faustmann%27s_formula`.

This is illustrated by the position of I^*_{m0} in the upper graph of Figure 1.

Proposition 6. *In the limit as $\delta \to \infty$, one has $V(I_m) - C = 0$.*

Proof. The left-hand side of (7) has the property that

$$\lim_{\delta \to \infty} \frac{\delta}{1 - e^{-\delta I}} = \infty\,.$$

Denoting I^*_m in the limit as $\delta \to \infty$ by $I^*_{m\infty}$, it follows that in the limit as $\delta \to \infty$, (7) implies

$$\infty = \frac{V'(I^*_{m\infty})}{V(I^*_{m\infty}) - C} \quad \Rightarrow \quad V(I^*_{m\infty}) = C$$

since there is no way in general to make $V'(I)$ equal to infinity. ▮

This is illustrated by the position of $I^*_{m\infty}$ in the upper graph of Figure 1. It follows that one can put bounds $(I^*_{m\infty}, I^*_{m0})$ on the location of I^*_m in the upper graph of Figure 1 (as in Clark 1e Fig. 8.3).

Corollary. *One has $I^*_{m\infty} < I^*_{m0} < I^*_1$.*

Proof. First inequality: Propositions 5 and 6, and Figure 1. Second inequality: Proposition 3. ▮