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Econ. 7250
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NOTES ON FISHERIES ECONOMICS

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Section 1. Optimal Control Theory

The core of the material in this handout is based on *Mathematical Bioeconomics: The Optimal Management of Renewable Resources* by Colin W. Clark (Second Edition, 1990).

Consider the problem of how to find

$$J(x_0) = \max_u \int_0^T f(x, u, t)\, dt \tag{1}$$

such that

$$\dot{x}_t = g(x, u, t) \tag{2}$$

with x_0 given, $u_t \in U$ for all t, and raised dots denote derivatives with respect to time. Here x is the "state variable" and u is the "control variable." For simplicity, consider them scalars instead of vectors. Often we will use a final time $T = \infty$. Form the "Hamiltonian":

$$\mathscr{H} = f + \lambda g$$

where λ is a function of time called the "adjoint variable" or the "costate variable."[1] The "Maximum Principle" gives two necessary conditions for optimality (using asterisks to denote optimal values):

$$\boxed{\max_{u_t \in U} \mathscr{H}_t \ \forall\, t} \tag{3}$$

$$\boxed{\dot{\lambda}_t^* = -\frac{\partial \mathscr{H}_t^*}{\partial x}\ \forall\, t\,.} \tag{4}$$

The Maximum Principle also states that λ_t^* is continuous and is piecewise continuously differentiable (where "continuously differentiable" means that the derivative exists and the derivative is itself a continuous function, though the derivative may not be differentiable).

There are infinitely many maximizations implied by (3), one for every t, but each one is a problem in standard calculus. Usually (3) implies

$$\partial \mathscr{H}^* / \partial u = 0\,, \tag{5}$$

but if there are constraints on u then the standard Kuhn-Tucker complementary slackness conditions would hold. If $U = [\underline{u}\,, \overline{u}]$, the best way to write

[1] In cases of no economic importance, it is mathematically possible that the correct Hamiltonian is actually $0f + \lambda g = \lambda g$.

these is

$$\left.\begin{array}{ll}\partial\mathcal{H}^*/\partial u \le 0 & \text{if } u^* = \underline{u}\\ \partial\mathcal{H}^*/\partial u = 0 & \text{if } u^* \in (\underline{u}\,, \overline{u})\\ \partial\mathcal{H}^*/\partial u \ge 0 & \text{if } u^* = \overline{u}.\end{array}\right\} \tag{6}$$

When we discuss fisheries below, $\underline{u}$ will be zero, and we will not mention an upper bound, which implicitly means taking $\overline{u} = \infty$.

A solution u_t^* is said to be "interior" if it is an element of the interior of U. For example, if $U = [\underline{u}\,, \overline{u}]$, then u_t^* would be "interior" if it was an element of $(\underline{u}\,, \overline{u})$.

An interesting special case occurs when $\mathcal{H}$ is linear in u: say, $\mathcal{H} = \sigma u + z$ or, more explicitly,

$$\mathcal{H} = \sigma(x, \lambda, t)\, u + z(x, \lambda, t) \tag{7}$$

where σ and z are some functions which do not depend on u. (A mathematician would object to this use of the word "linear" because of the presence of the "$+\,z(x, \lambda, t)$" term; a mathematician would call such a function "affine" not "linear." In these notes, when we say "linear" we technically usually mean "affine" instead.) In economics, this often occurs when firms are competitive and have constant returns to scale.[2] In this case, $\partial\mathcal{H}/\partial u = \sigma$, so (6) implies

$$\left.\begin{array}{ll}\sigma \le 0 & \text{if } u^* = \underline{u}\\ \sigma = 0 & \text{if } u^* \in (\underline{u}\,, \overline{u})\\ \sigma \ge 0 & \text{if } u^* = \overline{u}\end{array}\right\} \tag{8}$$

or, seen from another perspective,

$$\left. u^* \begin{cases} = \underline{u} & \text{if } \sigma < 0\\ \in [\underline{u}\,, \overline{u}] & \text{if } \sigma = 0\\ = \overline{u} & \text{if } \sigma > 0.\end{cases}\right\} \tag{9}$$

The $\sigma \ne 0$ solutions are called "bang-bang" solutions because in many problems, as time goes on, the optimal control changes discontinuously from $\underline{u}$ to $\overline{u}$ or vice versa as the "switching function" σ changes sign. The $\sigma = 0$ solutions are called "singular" solutions. Since (7) implies that $\partial\mathcal{H}/\partial u = \sigma$, and since $\sigma = 0$ on a singular solution:

[2] A firm's profit is total revenue (price times output) minus total cost (average cost times output). Let "output" be the control, u. If the firm is competitive then price "p" is not a function of u. If the firm has constant returns to scale then average cost "AC" is not a function of u. Thus profit would be $pu - AC\,u$, which would be linear in u since neither p nor AC would depend on u. Then $\mathcal{H} = (p - AC)\, u + \lambda\, g$, whose the first term is linear in u. If g is linear in u as well, then $\mathcal{H}$ would be linear in u, as in (7).

> (5) characterizes optimality both for interior solutions to nonlinear problems and for singular solutions to linear problems.

We could almost say:

> (5) characterizes optimality for interior solutions to all problems.

The only error in this formulation is that it ignores the fact that singular solutions to linear problems could have u^* being equal to $\underline{u}$ or $\overline{u}$. In the fisheries problems we study in these Notes, the second formulation is good enough.

Defining $J_0 = J(x_0)$ as in (1), it can be shown that

$$\frac{\partial J_0}{\partial x_0} = \lambda_0 \tag{10}$$

as long as J_0 is differentiable. (In economics, the left-hand side is how much the present discounted value J_0 of the initial resource stock x_0 would increase if the initial stock rose by one unit; so λ_0 is the "shadow value" or "shadow price" of the resource.) Although (2) does not allow x_t to make discontinuous jumps, discontinuous jumps would be permitted if (2) only held "almost everywhere" ("a.e.," synonymous with "virtually everywhere," "v.e.," which all mean "except on a set of measure zero"). If we were to allow x_t to make a discontinuous jump at time $\tau > 0$, and define J_τ as $\max_u \int_\tau^T f(x, u, t)\, dt$ subject to (2) and $u_t \in U \; \forall\, t > \tau$ and the new x_τ, then

$$\frac{\partial J_\tau}{\partial x_\tau} = \lambda_\tau \tag{11}$$

(see p. 323 of Sydsæter, Hammond, Seierstad, and Strøm, *Further Mathematics for Economic Analysis*, 2005). A result which can be helpful in signing the costate variable is due to Caputo (*Foundations of Dynamic Economic Analysis*, 2005, (13) p. 57): there is a function $\beta_t > 0 \, \forall\, t$ (whose definition you do not need to know) such that

$$\lambda_t^* = \frac{1}{\beta_t} \int_t^T \beta_\tau f_x'(x_\tau^*, u_\tau^*, \tau)\, d\tau\,. \tag{12}$$

This can be helpful because often one knows the sign which f_x' will have in the future.

If the final time T in (1) is finite, then (since we put no conditions on x_T except nonnegativity, which I don't go into here), the following "transversality condition" is necessary for an extremum (that is, a maximum or a minimum):

$$\lambda_T^* = 0\,. \tag{13}$$

However, if $T = \infty$, the situation requires special treatment. If neither f nor g depend explicitly on t, the optimal control problem is called "autonomous." We will not study any autonomous problems. However, if f takes the special form $\hat{f}(x,u)\, e^{-\delta t}$ and g does not depend explicitly on t, we say that the problem is "autonomous except for geometric discounting." Some of the problems we study are such problems (namely, the monopolist's problem and the social planner's problem—but *not* the problem of the competitive firm). In such problems, if $T = \infty$ the following "transversality condition" is necessary for an extremum:

$$\lim_{t\to\infty} \mathscr{H}^* = 0 \tag{14}$$

(see Caputo, op. cit., Theorem 14.9).

Conditions (3) and (4), together with the appropriate transversality condition, are necessary conditions for optimality. Sometimes it is useful to know sufficient conditions for optimality. For a maximum, the sufficient conditions are the necessary conditions plus either:

Mangasarian Sufficient Condition: $\mathscr{H}(x, u, \lambda^*, t)$ is concave in (x, u) for all $t \in [0, T]$ and for all admissible (x, u). If $\mathscr{H}$ is strictly concave, the optimal solution is unique.

Arrow Sufficient Condition: $\mathscr{H}(x, u^*, \lambda^*, t)$ is concave in x at x^* for all $t \in [0, T]$. If $\mathscr{H}$ is strictly concave, the optimal path of x is unique but the optimal path of u is not necessarily unique.

(See for example Caputo, op. cit., pp. 53 and 60–61.) For a minimum, change "concave" to "convex." For a refresher on how to do the concavity check called for in the Mangasarian sufficient condition (concavity of a function of more than one variable), see the Econ. 7005 mathematical prerequisites notes.

Section 2. Elementary Mathematical Ecology

Let x denote the number of fish in a population (or let it denote their biomass), and let $F(x)$ denote their excess of births over natural deaths. In the absence of human interference, $\dot{x} = F(x)$.

In the Examples and Exercises of these Notes we will consider the important special case where F is the "logistic" growth function

$$F(x) = rx\left(1 - \frac{x}{K}\right), \tag{15}$$

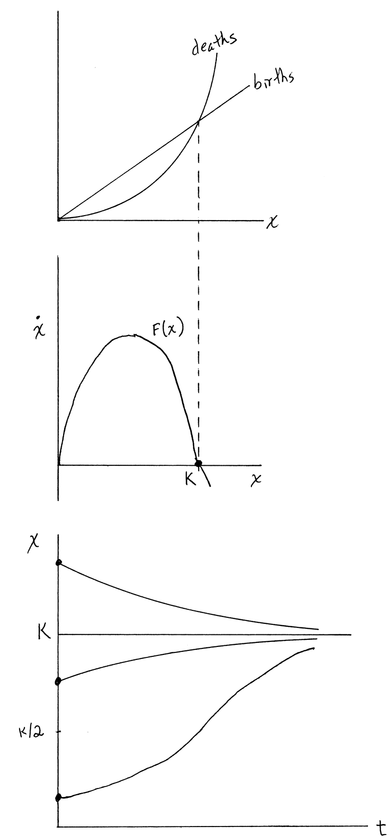

Figure 1. Logistic growth. The bottom graph is in the time dimension; its lowest function shows the "S" shape for which logistic growth is well known.

where r is called the "intrinsic growth rate" and K is the "carrying capacity." While we call $\dot{x}$ the "growth" of the population, we call $\dot{x}/x$ its "growth rate." With logistic growth and no human interference, $\dot{x}_t = F(x_t) = rx - (r/K)x^2$. Figure 1 shows how the logistic growth function's $F(x)$ could be derived from birth and death curves, and what the solution of the nonlinear differential equation $\dot{x} = F(x) = rx\,(1-x/K)$ can look like for three example values of x_0. An analytical expression for the solution of the nonlinear differential equation $\dot{x} = F(x)$ is given in (81).

Under logistic growth, the species' growth rate is

$$\frac{\dot{x}}{x} = r - \frac{r}{K}x \leq r\,, \quad \text{and}$$

$$\lim_{x \to 0} \left(\frac{\dot{x}}{x} \right) = r \,.$$

Hence the intrinsic growth rate r is the maximum growth rate, and the population grows at rate r in the limit as x goes to zero. For logistic growth,

$$F'(x) = r - \frac{2r}{K} x \le r \quad \text{and}$$
$$F'(0) = r \,,$$

which provides another interpretation of r: it is the maximum value of F', and that maximum value occurs at $x = 0$. (The two interpretations are different because $\dot{x}/x \neq F'$, as can be seen from their expression above.)

Cases, such as logistic growth, in which $F'' < 0$ are called cases of "pure compensation" as shown in Figure 2. Another case shown in that figure is the case of "depensation," in which F is convex for small x, then becomes concave, while $F > 0 \, \forall x \in (0, K)$. The final case shown in the figure is of "critical depensation" which is when $F < 0$ for some small x, $F > 0$ for medium values of x, and $F < 0$ for large values of x. In critical depensation, there are two positive values of x for which $F(x) = 0$, labeled K_0 and K in Figure 2 (c), and if x is ever less than the lower of these two values (K_0), then $F(x)$ is negative, the population falls, then $F(x)$ is still negative, so the population falls further, and this repeats until the population becomes extinct. Large mammals, such as elephants and whales, tend to be characterized by critical depensation: if the population falls below a critical level, it is doomed. Small organisms, such as bacteria, tend to be characterized by pure compensation.

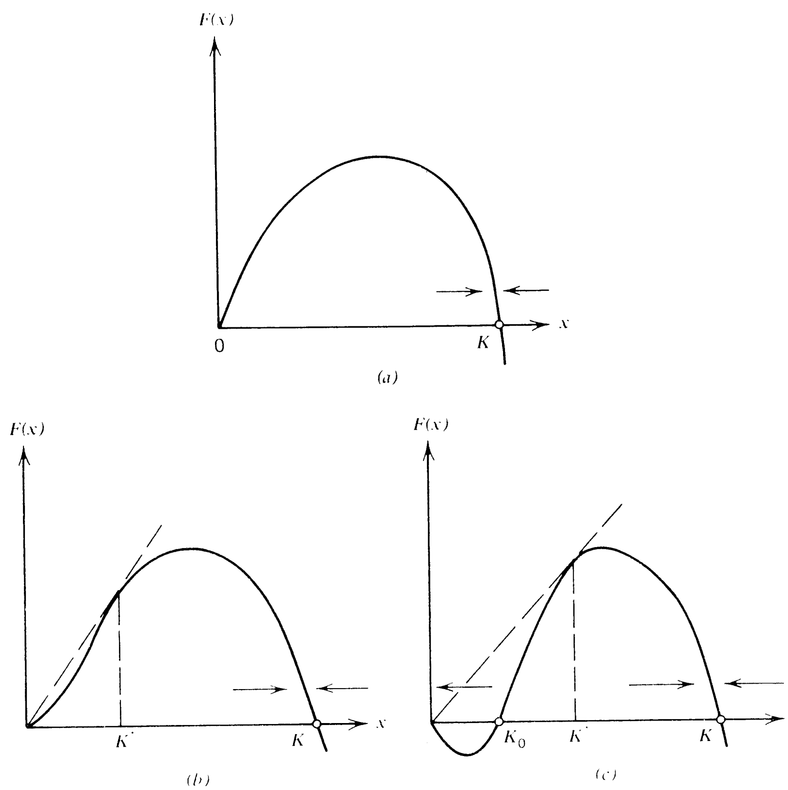

Figure 2. Clark’s Figure 1.5, types of biological growth functions. Compenstion is (a), depensation is (b), and critical depensation is (c).

Section 3. Private-Property Fishery: The General Formula

The problem of each firm is to

$$\max_{\langle h_t \rangle} \int_0^\infty \pi(x_t, h_t, t)\, e^{-\delta t}\, dt \tag{16}$$

subject to

$$\dot{x}_t = F(x_t) - h_t \tag{17}$$

and

$$h_t \geq 0\, \forall\, t \tag{18}$$

where x and F are as in Section 2, h is the amount of fish harvested, π is profit, and δ is the discount rate.

To start, form

$$\mathscr{H} = e^{-\delta t} \pi(x_t, h_t, t) + \lambda_t [F(x_t) - h_t]\,. \tag{19}$$

Let $M\Pi$ denote marginal profit $\partial\pi/\partial h$ and let $F'(x)$ mean $\partial F/\partial x$ not $\partial F/\partial t$.

In general, π is nonlinear in h. However, if the production function has constant returns to scale (and there are no fixed costs), and the firm is competitive, then π is linear in h (see footnote 2). In that case, (19) implies that $\mathscr{H}$ is linear in h, which is the case described by (7).

In this paragraph, assume either that h_t^* is interior or, if π is linear in h, then assume the solution is singular. Then as explained in Section 1, (5) characterizes the optimal solution. We have:

Proposition 1. *Assume h_t^* is interior or, if π is linear in h, assume the solution is singular. Then the solution to (16) subject to (17) and (18) is*

$$\boxed{\delta M\Pi_t = M\Pi_t\, F_t' + \dot{M\Pi}_t + \tfrac{\partial \pi_t}{\partial x}} \tag{20}$$

which can be rewritten as

$$\boxed{\delta = F'(x_t) + \frac{\dot{M\Pi}_t}{M\Pi_t} + \frac{\partial\pi(x_t, h_t, t)/\partial x}{M\Pi_t} \quad \text{if } M\Pi \neq 0.} \tag{21}$$

This implicitly defines h_t^* as a function of δ, x_t, and t.

Proof. This proof rests on using (5). To begin, (5) and (19) imply

$$0 = \frac{\partial \mathscr{H}^*}{\partial h} = e^{-\delta t}\,\frac{\partial \pi_t}{\partial h} - \lambda_t = e^{-\delta t} M\Pi_t - \lambda_t\,,$$

so

$$\lambda_t = e^{-\delta t} M\Pi_t\,. \tag{22}$$

On the other hand, (4) implies

$$\dot{\lambda}_t = -e^{-\delta t}(\partial \pi_t/\partial x) - \lambda_t F_t'(x)\,. \tag{23}$$

Using (22), this is equal to $-e^{-\delta t}(\partial \pi_t/\partial x) - e^{-\delta t} M\Pi_t F'(x_t)$, or

$$\dot{\lambda}_t = -e^{-\delta t}\left[\tfrac{\partial \pi_t}{\partial x} + M\Pi_t F_t'\right]\,. \tag{24}$$

But differentiating (22) with respect to time gives $\dot{\lambda}_t = -\delta e^{-\delta t} M\Pi_t + e^{-\delta t} \dot{M\Pi}_t = e^{-\delta t}(\dot{M\Pi}_t - \delta M\Pi_t)$. Equating this with $\dot{\lambda}_t$ from (24) gives:

$$\begin{aligned} -e^{-\delta t}\left[\tfrac{\partial \pi_t}{\partial x} + M\Pi_t\, F_t'\right] &= e^{-\delta t}(\dot{M\Pi}_t - \delta M\Pi_t) \\ -\tfrac{\partial \pi_t}{\partial x} - M\Pi_t\, F_t' &= \dot{M\Pi}_t - \delta M\Pi_t\,. \end{aligned}$$

This leads directly to (20). ∎

Proposition 1 assumes interior solutions; we should now determine the circumstances under which solutions actually are interior. The proof uses "$\triangleq$" to mean "is defined to be."

Lemma 1. *Suppose that there are no fixed costs, i.e., that if $h = 0$ then $\pi = 0$ (formally: $\pi(x_t, 0, t) = 0$ for all t and all $x \geq 0$). Then $h_t^* > 0$ if and only if*

$$\pi(h, x_t, t)/h > \lambda_t\, e^{\delta t} \quad \text{for some } h > 0. \tag{25}$$

Proof. Maximizing $\mathscr{H}$ over h would be equivalent to maximizing $\bar{\mathscr{H}} \triangleq \mathscr{H} - \lambda_t\, F(x_t)$ since the last term does not involve h. We have $\bar{\mathscr{H}} = e^{-\delta t}\pi(x_t, h_t, t) - \lambda_t\, h_t$. Since there are no fixed costs, if $h_t = 0$ then $\bar{\mathscr{H}} = 0$.

If (25) holds, then $e^{-\delta t}\pi > \lambda h$ for some $h > 0$, so $\bar{\mathscr{H}} = e^{-\delta t}\pi - \lambda h > 0$ for some $h > 0$. Hence if (25) holds, setting $h_t^* = 0$ would not maximize $\bar{\mathscr{H}}_t$, nor maximize $\mathscr{H}_t$; and not maximizing $\mathscr{H}_t$ violates the Maximum Principle's (3). ∎

Lemma 1's condition is not easy to check. In particular, it implies that even if there exists some $\bar{h} > 0$ such that $\pi(\bar{h}, x_t, t) > 0$, there is no guarantee that $h_t^* > 0$, because the hurdle to be overcome is not $\pi > 0$, it is $\pi/h > \lambda e^{\delta t}$.

Lemma 1 said roughly that if (potential) profit at a particular date is large enough, then harvest at that date will be strictly positive. The next result is a partial converse of that result: it says that if harvest is strictly positive, profit at that date is (weakly) positive.

Lemma 2. *Suppose that there are no fixed costs. Then if $h_t^* > 0$ and $\lambda_t^* \geq 0$, one has $\pi_t^* > 0$.*

Proof. Suppose not; then $h_t^* > 0$ and $\lambda_t^* \geq 0$ but $\pi_t^* \leq 0$. From this and (19),

$$\mathcal{H}_t = e^{-\delta t}\pi(x_t, h_t, t) + \lambda_t F(x_t) - \lambda_t h_t < \lambda_t F(x_t)\,. \qquad (26)\,.$$

However, if instead one set $h_t = 0$, then the Hamiltonian would be $\mathcal{H} = e^{-\delta t}\pi(x_t, 0, t) + \lambda_t[F(x_t) - 0] = \lambda_t\, F(x_t)$, which is larger than the Hamiltonian in (26). Hence setting $h_t^* > 0$ does not maximize the Hamiltonian and is not actually optimal. This is a contradiction. ▮

Lemma 2 involves the sign of λ^*; the next result discusses that sign. In all of the fisheries we study later, $\partial\pi_t/\partial x_t \geq 0$ will hold, so this lemma will ensure that $\lambda^* \geq 0$.

Lemma 3. *We have $\lambda_t^* > 0$ (strictly) if and only if $\partial\pi_\tau/\partial x_\tau > 0$ (strictly) over an interval of time of positive measure where $\tau \geq t$. Also, $\lambda_t^* = 0$ if and only if $\partial\pi_\tau/\partial x_\tau = 0$ for all $\tau \geq t$ (except possibly on a set of measure zero).*

Proof. Using (16), the "f" of (1) is $\pi_t\, e^{-\delta t}$. It follows that (12) implies

$$\lambda_t^* = \frac{1}{\beta_t}\int_t^T \beta_\tau\, e^{-\delta\tau}(\partial\pi_\tau/\partial x_\tau)\, d\tau\,. \qquad (27)$$

Since $\beta_\tau > 0$ from the discussion concerning (12), and since $e^{-\delta\tau} > 0$, one can conclude from (27) that as long as $\partial\pi_\tau/\partial x_\tau$ is strictly positive over some interval of time after t, it will be true that $\lambda_t^* \geq 0$, and if $\partial\pi_\tau/\partial x_\tau$ is zero for all dates after t, then $\lambda_t^* = 0$. ▮

For dates when the solution is not interior (which happens for problems linear in the control for dates when the solution is not singular), $h_t^* = 0$. On those dates the Kuhn-Tucker conditions (6) imply that instead of $\partial\mathcal{H}/\partial h =$

$e^{-\delta t} M\Pi_t - \lambda_t$ being equal to zero, it would only have to be less than or equal to zero on those dates; so on those dates, (22) is replaced by

$$\lambda_t \geq e^{-\delta t} M\Pi_t \tag{28}$$

and (21) does not hold.

We next find conditions sufficient to ensure that h_t^* will not take jumps in the interior of the (x, h) plane.

Lemma 4. *If marginal profit is continuous in h for all $h > 0$, and if h_t^* jumps at time τ, then either the value of harvest just before τ, $h_{\tau-}^*$, or the value of harvest just after τ, $h_{\tau+}^*$, must be zero.*

Proof. Suppose to the contrary that h_t^* jumps at time τ but that both $h_{\tau-}^*$ and $h_{\tau+}^*$ are strictly greater than zero. Then $\lambda_t = e^{-\delta t} M\Pi_t$ (which is (22)) rather than $\lambda_t \geq e^{-\delta t} M\Pi_t$ (which is (28)) holds for both $\tau-$ and $\tau+$:

$$\begin{aligned} \lambda_{\tau-}\, e^{\delta \tau-} &= M\Pi_{\tau-} \\ \lambda_{\tau+}\, e^{\delta \tau+} &= M\Pi_{\tau+}\,. \end{aligned} \tag{29}$$

Both λ_t^* and $e^{-\delta t}$ are continuous at all dates, the former because continuity of λ is a basic property asserted by the Maximum Principle. Hence from (29), $M\Pi^*$ must be continuous at τ. Since marginal profit is assumed to be continuous in h for all $h > 0$, and since by assumption $h > 0$ both before and after τ, the continuity of $M\Pi$ at τ in turn implies that h^* is continuous at τ. However, that contradicts the supposition that h_t^* jumps at time τ. This contradiction establishes the proof. ▮

Since the transversality condition (14) is only applicable to problems that are autonomous (except possibly for geometric discounting), it only applies when $\pi(x_t, h_t, t)$ in (19) does not depend explicitly on t. Hence *it does not apply when studying competitive firms*, because their profit depends on the exogenous time trend of prices (exogenous to them). When it does apply, it requires

$$\lim_{t \to \infty} [e^{-\delta t} \pi + \lambda\, (F - h)] = 0\,. \tag{30}$$

If the solution is interior (or singular), (22) holds; substituting it into (30) yields

$$\lim_{t \to \infty} e^{-\delta t} [\pi + M\Pi\ (F - h)] = 0\,. \tag{31}$$

This will hold as long as the term in brackets, $[\pi + M\Pi (F - h)]$, grows with time more slowly than $e^{\delta t}$. This will be the case if the system approaches a steady state or a limit cycle, where a "steady state" is defined to be a situation where all time derivatives are zero, and a "limit cycle" is an "isolated closed trajectory." (Mathematicians sometimes call a steady state an "equilibrium," but we will not do that, reserving the term "equilibrium" to mean "quantity supplied equals quantity demanded," which may happen outside of a steady state.)

This completes listing the necessary conditions for solving (16) subject to (17) and (18), both for the nonlinear and the linear case. Traditionally, however, the linear case has been analyzed in a different way, and I explain that traditional way for the rest of this paragraph. When π is linear in h (a mathematician would say "when π is affine in h" as explained after (7)), it can be written as

$$\pi(x_t, h_t, t) = \pi_1(x_t, t)\, h_t + \pi_2(x_t, t)$$

for some functions π_1 and π_2. To rule out fixed costs, which give rise to nonconvexities which could imperil existence of an optimal solution, it is necessary to require that $\pi(x_t, 0, t)$ be identically zero; this means $\pi_2 \equiv 0$. That means that π_1 is equal to average profit and is equal to marginal profit; accordingly, I will rename π_1 to "$AM\Pi$." The Hamiltonian then becomes

$$\begin{aligned} \mathscr{H} &= e^{-\delta t} AM\Pi(x_t, t)\, h_t + \lambda_t [F(x_t) - h_t] \\ &= [e^{-\delta t} AM\Pi(x_t, t) - \lambda_t]\, h_t + \lambda_t\, F(x_t) \\ &\triangleq \sigma(x_t, \lambda_t, t)\, h_t + \lambda_t\, F(x_t) \end{aligned} \tag{32}$$

defining the "switching function" σ as $e^{-\delta t} AM\Pi - \lambda$. This has the form of (7), so from (9) the optimal solution is:

$$h_t^* = \begin{cases} 0 & \text{if } e^{-\delta t} AM\Pi_t - \lambda_t < 0 \\ \in [0, \infty) & \text{if } e^{-\delta t} AM\Pi_t - \lambda_t = 0 \\ +\infty & \text{if } e^{-\delta t} AM\Pi_t - \lambda_t > 0. \end{cases} \tag{33}$$

In the singular solution, $e^{-\delta t} AM\Pi_t - \lambda_t = 0$, which means that $e^{-\delta t} M\Pi - \lambda_t = 0$, which is the same as (22). This is to be expected because (22) came from (5), which is valid both for interior solutions to nonlinear problems and for singular solutions to linear problems, as discussed just after (9). The nonsingular solution $h_t^* = 0$ has $e^{-\delta t} AM\Pi_t - \lambda_t \le 0$, which means that $\lambda_t \ge e^{-\delta t} M\Pi_t$, just as concluded in (28). Note that if $\lambda_t > 0$ and $AM\Pi_t = p_t - c(x_t) < 0$ then the first line of (33) implies that $h_t^* = 0$; this loosely resembles a converse of Lemma 2.

Section 4. Private-Property Dynamic Competitive Equilibrium

In most of these Notes we will be concerned with the dynamic competitive equilibrium. In such an equilibrium, firms take the future time path of prices, p_t for $t \in [0, \infty)$, given. On that basis, each of the N firms (we suppose them to be identical) solve an optimal control problem to obtain their supply in future periods, Nh_t^* for $t \in [0, \infty)$. Meanwhile, consumers also take the future time path of prices as given, and their market demand curve determines quantity of fish demanded at each date. There is a dynamic competitive equilibrium if quantity supplied by firms is equal to quantity demanded by consumers at every date, now and in the future.

Let p_t be the market price. The total revenue of a competitive firm at date t is $p_t\, h_t$, where the firm takes p_t as given, that is to say, the firm perceives absolutely no connection between its h_t and the prevailing p_t.

Suppose the market inverse demand curve is denoted by $\phi_t(h)$. The basic assumptions we will make throughout these Notes are that $\phi(h)$ is decreasing in h (a downward-sloping demand curve); that $\phi(h)$ is continuous in h except possibly at $h = 0$; and that $\phi'(h)$ is continuous in h except possibly at $h = 0$.

If supply equals demand at date t then $p_t = \phi_t(N_t\, h_t)$. This is the basic condition for market equilibrium. I will always suppose that neither N nor the demand curve are time-varying, so the condition for market equilibrium at date t can be simplified to $p_t = \phi(Nh_t)$. For notational simplicity, I will always assume $N = 1$. (The firm does not know that $N = 1$ so it still thinks it is in a competitive industry, not that it is a monopolist.) This makes the market equilibrium condition

$$p_t = \phi(h_t)\,. \tag{34}$$

Another basic assumption we will make throughout these Notes is that the total cost function $TC(x, h)$ is (weakly) convex in h. If this were to fail, existence of a competitive equilibrium would be questionable. We will also assume that $TC(x, h)$ is twice differentiable in h for all values of $h \geq 0$. Therefore $\partial TC/\partial h \geq 0$, and this derivative is itself differentiable.[3] Call these assumptions together "the basic assumptions on costs."

We can show that the only jumps which h_t^* may take are down to zero or up from zero.

Lemma 5. *Make the basic assumptions on costs. Then in dynamic competitive equilibrium, h_t^* will not take jumps in the interior of the (x, h) plane.*

[3] All we actually need below is for this derivative to be continuous.

Proof. From Lemma 4, it will suffice to show that marginal profit is continuous in h for all $h > 0$. Substituting the market equilibrium condition (34) into $\pi = ph - TC(x, h)$, total profit is $\phi(h)\,h - TC(x, h)$, and

$$M\Pi = \phi'(h_t)\,h_t + \phi(h_t) - \partial TC/\partial h\,.$$

We always assume in these Notes that $\phi(h)$ and $\phi'(h)$ are continuous for all $h > 0$. We have also assumed that $\partial TC/\partial h$ is continuous in h. This confirms that marginal profit is continuous in h for all $h > 0$. ▮

Furthermore, we can show that the only jumps which h_t^* may take are up from zero.

Proposition 2. *Make the basic assumptions on costs. Then in dynamic competitive equilibrium, h_t^* is continuous.*

Proof. Because of Lemma 5, the only kind of jumps which can be taken are either from or to $h = 0$. So all we need to prove is that h_t^* never jumps up from or down to zero.

Suppose by way of contradiction that at time τ, h_t^* did jump up from or down to zero. If it jumped down to zero, then for some small $\epsilon > 0$, $h(\tau-\epsilon) > 0$ and $h(\tau+\epsilon) = 0$. If it jumped up from zero, then for some small $\epsilon < 0$ (note the sign), $h(\tau+\epsilon) = 0$ and $h(\tau-\epsilon) > 0$. In both cases, therefore, $h(\tau-\epsilon) > 0$ and $h(\tau+\epsilon) = 0$; all that distinguishes the two cases is the sign of ϵ. Note this implies that $h(\tau-\epsilon) > h(\tau+\epsilon) = 0$.

Since $h(\tau-\epsilon) > 0$ and $h(\tau+\epsilon) = 0$, from (22) and (28),

$$\begin{aligned} \lambda(\tau-\epsilon)\,e^{\delta\tau-} &= M\Pi(\tau-\epsilon) \\ \lambda(\tau+\epsilon)\,e^{\delta\tau+} &\geq M\Pi(\tau+\epsilon)\,. \end{aligned}$$

Since both λ and $e^{-\delta t}$ are continuous, and since the proof of Lemma 5 showed that marginal profit is continuous in h for all $h > 0$, to get a discontinuous jump in h we need a discontinuous jump in $M\Pi$: $M\Pi(\tau-\epsilon) > M\Pi(\tau+\epsilon)$. This implies (using MC for "marginal cost"): $p(\tau-\epsilon) - MC(\tau-\epsilon) > p(\tau+\epsilon) - MC(\tau+\epsilon)$, and hence that

$$p(\tau-\epsilon) - p(\tau+\epsilon) > MC(h(\tau-\epsilon)) - MC(h(\tau+\epsilon)) = MC(h(\tau-\epsilon)) - MC(0)\,.$$

By the (weak) convexity of total cost in h, $MC(h(\tau-\epsilon)) - MC(0) \geq 0$. Then $p(\tau-\epsilon) > p(\tau+\epsilon)$. In competitive equilibrium, $p_t = \phi(h_t)$ (that is (34)), so the implication is that $\phi(h(\tau-\epsilon)) > \phi(h(\tau+\epsilon))$. Since ϕ is strictly decreasing in h, this implies that $h(\tau-\epsilon) < h(\tau+\epsilon)$, which contradicts the previous paragraph's conclusion that $h(\tau-\epsilon) > h(\tau+\epsilon) = 0$. ▮

As we noted in Section 3, the transversality conditions at infinity (30) and (31) are inapplicable to the problem of competitive firms, whose problem is non-autonomous because it involves the exogenously-varying function p_t ("exogenous" to the firm, that is). On page 97 of the third (2010) edition of Clark's book, he suggests "first looking at the case of a finite time horizon T, and then letting $T \to \infty$." At the end of Section 5 we will follow Clark's hint, but in order to do so, we need to know the finite-time transversality condition:

Lemma 6. *If $T < \infty$ and if $h_T > 0$ then $M\Pi_T = p_T - MC_T = 0$, or, imposing market equilibrium, $\phi(h_T) = MC_T$.*

Proof. If $T < \infty$, (13) has to hold. From (22), $\lambda_t = e^{-\delta t} M\Pi_t$ when $h_t > 0$. ▮

From (11), $\partial J/\partial x_T = \lambda_T = 0$, so the fish stock is valueless at the margin at T—which makes intuitive sense because if it were not, then it would not be optimal to stop fishing at T. This is merely "intuitive" because it is not the firm's choice whether or not to stop fishing at T: it is an external constraint imposed on the firm.

Section 5. Private-Property Competition: Search Fisheries & Constant Returns to Scale

In general, total cost is a function of both stock size and harvest, and it could be an explicit function of time as well (if, for example, input prices are exogenously changing with time): $TC(x_t, h_t, t)$. In these Note, we will assume that there is no exogenous time change in total cost: $TC(x_h, h_t)$. Mathematically, both TC/x and TC/h could be referred to as "average cost, but in the rest of economics, "average cost" only means total cost divided by output, TC/h, so we follow that convention in these notes. Similarly, mathematically, both $\partial TC/\partial x$ and $\partial TC/\partial h$ could be referred to as "marginal cost, but in the rest of economics, "marginal cost" only means the derivative of total cost with respect to output, $\partial TC/\partial h$, so we follow that convention in these notes. In the rest of economics, "constant returns to scale," which we assume in this section (see the section title), implies that average cost is a constant, which is to say that TC/h is not a function of h. In the context of fisheries, it could still be a function of x; as long as it is not a function of h, the technology will be characterized has having constant returns to scale. It follows that $TC = c(x) \cdot h$ would be an appropriate specification for fisheries costs under constant returns to scale, because it implies that average cost $TC/h = c(x)$, which is not a function of h.

Given these remarks, profit under constant returns to scale and perfect competition can be expressed for the static and dynamic cases as

$$\pi(x, h) = [\, p - c(x)]\, h \quad \text{and} \tag{35s}$$

$$\pi(x_t, h_t, t) = [\, p_t - c(x_t)]\, h_t \,. \tag{35d}$$

Because the function c is total cost divided by h, the function c is the average cost function. Because the function c is the derivative of total cost with respect to h, the function c is also the marginal cost function. The fact that average cost is equal to marginal cost comes from the assumption of constant returns to scale.

We will always assume that $c(x)$ is a continuous function of x.

As pointed out above, in both (35s) and (35d), price is not a function of h, in order to describe competitive behavior. (35s) is the special case of (35d) in which all the variables are constant. Therefore, if (35d) ever reaches a steady state, the steady state will be the solution to (35s).

To summarize this section so far: in general, the total cost function is written as $TC(x_t, h_t, t)$, and average cost TC/h would in general also be a function of x_t, h_t, and t. If, as in (35s) and (35d), the average cost function c does not depend on h, and the firm is competitive, then

- marginal cost is equal to average cost (where "marginal" means the derivative with respect to output h, not with respect to x, and "average" means divided by output not x);
- we say that there is "constant average cost," even though $c(x)$, which is both average and marginal cost, may vary with x, because "constant average cost" in the rest of economics means that average cost *does not vary with output* and we want to use the same terminology in these Notes;
- the production function has constant returns to scale;
- π is linear in h;
- average profit is equal to marginal profit; and finally, as mentioned in Section 3 in the paragraph after (19),
- $\mathscr{H}$ is linear in h.

In order for harvest to be neither zero nor infinity, (33) requires that the solution be singular. Therefore we will use (21) for studying both (35s) and (35d).

If c actually does not depend on x, the fishery is said to be a "schooling" fishery. By contrast if, as intended in (35s) and (35d), c does depend on x, the fishery is said to be a "search" fishery (see Philip A. Neher, *Natural Resource Economics: Conservation and Exploitation*, Cambridge University Press 1990, p. 177, p. 195). In a search fishery, $c'(x) < 0$: as the stock declines, average costs go up, the "stock effect." In this section, we consider a search fishery (as reflected in the section title); Section 6 will consider schooling fisheries.

Throughout this section, we will make the additional assumption that

$$c''(x) > 0\,. \tag{36}$$

The first reason to make this assumption is that it is difficult to make the opposite assumption, $c'' < 0$, and draw a $c(x)$ function with $c' < 0$ that still obeys $c(x) > 0$ for all x. On the other hand, "difficult" does not mean impossible, and one could in addition argue that c'' could be negative for small and medium x and positive for x's so large as to be economically irrelevant. Another reason to assume (36) is that it is plausible that the stock effect is largest for very small x, where it becomes difficult to find any fish at all.

We need to establish two short technical results before continuing.

Lemma 7. *With a private-property search fishery having constant returns to scale, $\lambda_t^* > 0$ if and only if $h_\tau^* > 0$ (strictly) over an interval of time of positive measure where $\tau \geq t$. Also, and $\lambda_t^* = 0$ if and only if $h_\tau^* = 0$ for all $\tau \geq t$ (except possibly on a set of measure zero).*

Proof. From (35d), $\partial\pi_\tau/\partial x_\tau = -c'(x_\tau)\,h_\tau$. Under the "search fishery" assumption of this section, $c' < 0$, so $\partial\pi_\tau/\partial x_\tau > 0$ if $h_\tau > 0$ and $\partial\pi_\tau/\partial x_\tau = 0$ if $h_\tau = 0$. The conclusion follows from Lemma 3. ▌

Corollary. *With a private-property search fishery having constant returns to scale, if $h_t^* > 0$ then $\pi_t^* > 0$.*

Proof. Combine Lemmas 2 and 7. ▌

Subsection a. The Steady State Case

First consider (35s). It implies that $\partial\pi/\partial x = -c'(x)\,h$ and $AM\Pi = p - c(x)$.

We will want to use (20) from Proposition 1, but that requires h_t^* to be interior or singular. The following result explains how to rule out situations when fishing never occurs. Recall from Section 2 that K is the carrying capacity. (If growth is not logistic, define K as the largest strictly positive stock size x at which $F(x) = 0$.)

Lemma 8. *Fishing never occurs if*

$$\phi(0) < c(K)\,. \tag{37}$$

Fishing never occurs only if $\phi(0) \le c(K)$.

Proof. Sufficiency ("if"):

From (35s), $AM\Pi = p - c(x)$.

Because the inverse demand curve is downward-sloping, the largest possible equilibrium value of price is $\phi(0)$, at $h = 0$.

Because $c'(x) < 0$, the smallest possible steady-state value of $c(x)$ is $c(K)$. Smaller values of x would give rise to larger values of c, and larger values of x could never be steady states because they would always have $\dot{x} = F(x) - h < F(x) < 0$.

It follows that the largest possible value of $AM\Pi$ is $\phi(0) - c(K)$; in other words, $AM\Pi \le \phi(0) - c(K)$.

From Lemma 7, $\lambda^* \ge 0$.

If $\phi(0) - c(K) < 0$, then from the steps so far, we have

$$AM\Pi \le \phi(0) - c(K) < 0 \le \lambda\,.$$

From (33), if $e^{-\delta t} AM\Pi_t < \lambda_t$ then $h_t^* = 0$. Evaluated at $t = 0$, the premise is $AM\Pi < \lambda$. The preceding paragraph shows that if $\phi(0) - c(K) < 0$ then this premise $AM\Pi < \lambda$ is met, and therefore that $h^* = 0$. This proves sufficiency.

Necessity ("only if"):

If $h = 0$, then price is $\phi(0)$, average and marginal cost is $c(K)$, meaning that $AM\Pi = p - c(x) = \phi(0) - c(K)$.

If $h = 0$ then $\partial\pi/\partial x = -c'(x)\,h = 0$, implying from Lemma 3 that $\lambda = 0$.

From (33), if $h = 0$ then $e^{-\delta t} AM\Pi_t - \lambda_t \le 0$. Evaluated at $t = 0$, this is $AM\Pi - \lambda \le 0$. Substituting from the previous two paragraphs, if $h = 0$ then $\phi(0) - c(K) - 0 \le 0$, so $\phi(0) \le c(K)$. This proves necessity. ▮

Sometimes we will follow other authors and call $\phi(0)$, which is the intersection of the demand curve with the price axis, the "choke price," denoted "p_c," since this is the price that "chokes off" demand. As noted in the sufficiency part of this proof, the choke price is the largest possible equilibrium value of price.

Accordingly, for the rest of this section, assume that $\phi(0) > c(K)$. This ensures that some fishing will occur. Then h_t^* is interior or singular, meaning that we can use (20) from Proposition 1. That equation leads to the following:

$$\begin{aligned} \delta\,[p - c(x)] &= [p - c(x)]\,F'(x) + 0 - c'(x)\,h \\ [\delta - F'(x)]\,[p - c(x)] &= -c'(x)\,h\,. \end{aligned} \tag{38}$$

In a search fishery the right-hand side is not zero (assuming that h is not zero, for which see Lemma 8). Therefore, $\delta - F'(x) \neq 0$ and

$$\begin{aligned} p - c &= \frac{-hc'}{\delta - F'} \\ p &= c(x) - \frac{hc'}{\delta - F'}\,. \end{aligned} \tag{39}$$

(Time subscripts have been omitted because (35s) pertains to the steady state.) In a sense, (39) is a steady-state supply curve for fish—it can be rewritten as

$$\begin{aligned} h &= -\frac{\delta - F'(x)}{c'(x)}\,p + \frac{[c(x)]\,[\delta - F'(x)]}{c'(x)} \\ &= \frac{\delta - F'(x)}{c'(x)}\,[c(x) - p]\,, \end{aligned} \tag{40}$$

which explicitly shows h depending on p—but (40) contains x, which is tied to the value of h in the steady state in a way not captured by (40).[4] To get a complete "supply relationship" (I would not strictly call it a "supply curve"), substitute the steady-state condition $\dot{x} = 0$ into (17), obtaining

$$\left\{ \begin{aligned} & h = F(x)\,, \quad \text{then use (39) to obtain} && (41) \\ & p = c(x) - \frac{c'(x)\,F(x)}{\delta - F'(x)}\,. && (42) \end{aligned} \right.$$

(42) and (41), which appear as equations (5.8) and (5.9) on page 135 of the second edition of Clark's book, work together to yield a complete supply relationship between price and the steady-state value of quantity h: each value of x will give a value for price p from (42), and it will give a value for h from (41), so running through values of x will give rise to price–quantity combinations; the graph of these combinations is the steady-state supply relationship of this firm. For details, see the Example 1 below or see Figure 5.12, p. 136 of Clark, reproduced as Figure 3 here. In that figure, Quadrant IV shows (41) and Quadrant II shows (42) in a special case which

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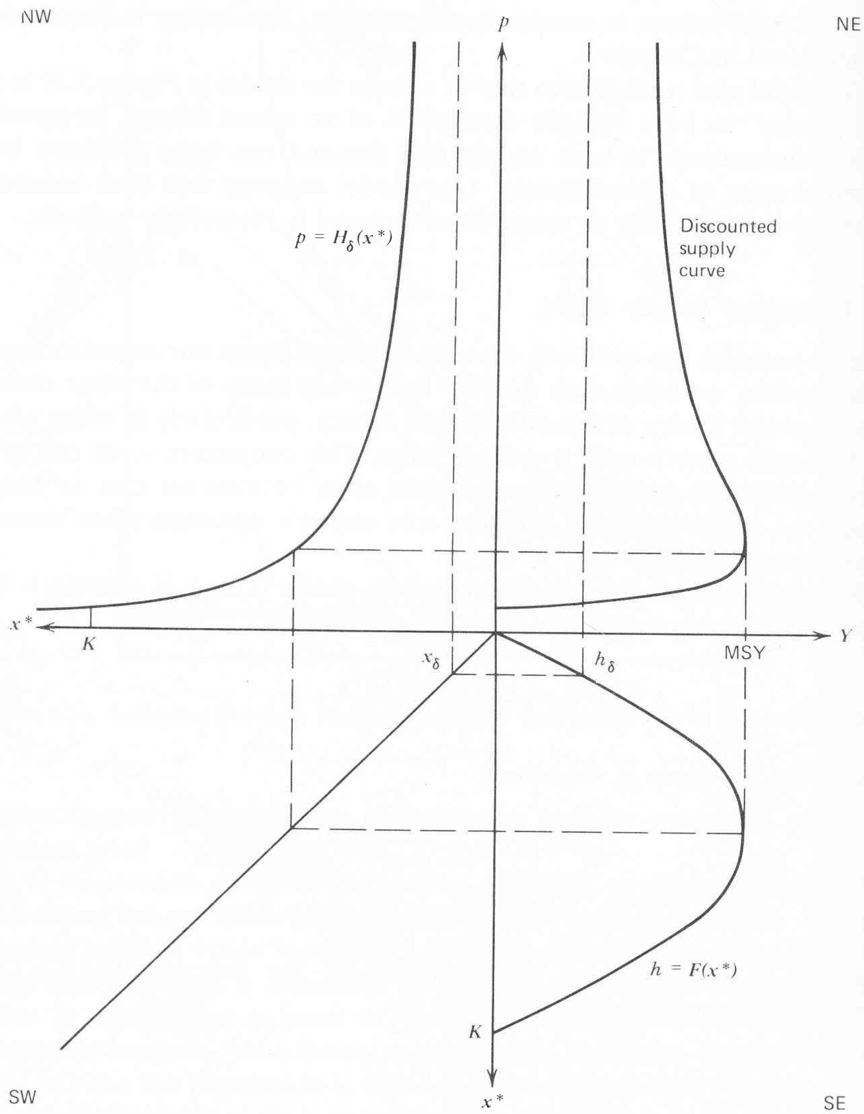

Figure 5.12. Discounted supply curve (Schaefer model).

Figure 3. Clark's Figure 5.12. The axis he labels "*Y*" is our "*h*."

is analyzed in the paragraph after next; Clark denotes the right-hand side of (42) by "$H_\delta(x^*)$."

Before analyzing Quadrant II of Figure 3, we explain what that figure's "x_δ" means.

Lemma 9. *Implicitly define x_δ by*

$$\begin{aligned} F'(x_\delta) &= \delta \quad \text{if such an } x_\delta \geq 0 \text{ exists, and} \\ x_\delta &= 0 \quad \text{otherwise.} \end{aligned} \tag{43}$$

If $h_t^* > 0$, then $x_t^* \neq x_\delta$, $F'(x_t^*) < \delta$, and $AM\Pi_t^* > 0$.

Proof. Suppose to the contrary that $x^* = x_\delta$. There are two cases.

In the first case, $x_\delta = 0$, implying that $x^* = 0$, which contradicts the assumption that $h^* > 0$.

In the second case, $x_\delta > 0$, implying that $\delta = F'(x_\delta) = F'(x^*)$. Then the left-hand side of (38) in the steady state is zero. However, the right-hand side of (38) cannot be zero, because $h^* > 0$ was assumed, and because in a search fishery, $c'(x) \neq 0$. This contradiction establishes the proof that $x \neq x_\delta$.

To show that $F'(x^*) < \delta$, suppose on the contrary that $F'(x^*) \geq \delta$. Then $\delta - F'(x) \leq 0$. Also, $p - c(x) \geq 0$ because from Lemma 2, for harvest to strictly positive (which we assumed in the lemma), profit must be nonnegative. It follows that the left-hand side of (38) is less than or equal to zero. However, the right-hand side of (38) is strictly positive, because $h^* > 0$ was assumed, and because in a search fishery, $-c'(x) > 0$. This contradiction establishes the proof that $F'(x^*) < \delta$.

The $AM\Pi_t > 0$ claim follows from the Corollary to Lemma 7. ∎

To say more requires putting more structure on $F(x)$.

Lemma 10. *Assuming logistic growth,*

$$x_\delta = \begin{cases} \dfrac{K}{2} \cdot \dfrac{r - \delta}{r} & \text{if } r \geq \delta \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

Also, assuming logistic growth,

$$h_t^* > 0 \Longrightarrow F'(x_t^*) < \delta,\, x_t^* \geq x_\delta \text{ and } AM\Pi_t^* > 0\,; \tag{45}$$

$$x_t^* < x_\delta \Longrightarrow F'(x_t^*) > \delta \text{ and } h_t^* = 0\,. \tag{46}$$

[4]From (38), another not-particularly-useful expression is $[\delta - F'(x)]\,[\phi(F(x)) - c(x)] = -c'(x)\,F(x)$ where ϕ is the inverse demand curve, as defined just before Proposition 3.

Proof. For logistic growth, (43) means $\delta = F'(x_\delta) = r - 2rx_\delta/K$. Equation (44) follows.

Lemma 9 showed that if $h^* > 0$ then $F'(x^*) < \delta$. For logistic growth, this means that $\delta > r - 2rx/K$, so $x > K(r-\delta)/(2r)$. This implies from (44) that $x > x_\delta$. This and Lemma 9 prove (45).

It remains to prove (46). If $x_\delta = 0$, the premise of (46) cannot hold. So assume $x_\delta > 0$, that is, $F'(x_\delta) = \delta$. The first paragraph in this proof gives $F'(x)$; inspection shows that it is decreasing in x. So if $x < x_\delta$, $F'(x) > F'(x_\delta) = \delta$. This means the first term on the left-hand side of (38) is negative. If h_t^* were positive, (38) would imply that $p - c(x) < 0$, but that violates the Corollary to Lemma 7, so h_t^* cannot be positive. ▮

In Figure 3, Clark uses x_δ as defined in (43), and defines

$$h_\delta = F(x_\delta)\,. \tag{47}$$

Having explained what x_δ means, we now turn attention to Quadrant II of Figure 3. First note that in general, one cannot show that $p = H_\delta(x)$ (the right-hand side of (42)) is monotonic in x. This is because we will not be able to sign

$$\frac{dp}{dx} = c'(x) - \frac{c''(x)F(x)}{\delta - F'(x)} - \frac{c'(x)F'(x)}{\delta - F'(x)} - \frac{F''(x)}{(\delta - F'(x))^2}c'(x)F(x)\,.$$

We can sign $c'(x) < 0$, $\delta - F'(x) > 0$ from (45), $F(x) > 0$ (because $F(x) < 0$ cannot be a steady state, as noted in the sufficiency part of the proof of Lemma 8), and $c''(x) > 0$ from (36), but $F'(x)$ and $F''(x)$ are ambiguous (although in the special case of (15), $F'' < 0$ and F' is positive or negative as x is less than or greater than $K/2$). This leads to

$$\begin{aligned}\frac{dp}{dx} &= (-) - \frac{(+)(+)}{(+)} - \frac{(-)F'(x)}{(+)} - \frac{(-)\text{if logistic}}{(+)}(-)(+)\ ;\text{ if logistic,}\\ &= (-) - (+) + \{(+)\text{ for small } x,\ (-)\text{ for large } x\} + (-)\,.\end{aligned}$$

For the logistic case (Quadrant IV shows this case), for large x this can be signed, and it is negative, as drawn in Quadrant II of Figure 3, but for small x it is ambiguous even in the logistic case. (We are not concerned with values of x smaller than x_δ because they could not be steady states with fishing.) If we impose not only (15) but also

$$c(x) = \gamma/x \tag{48}$$

for some constant γ then we have what Clark (p. 45) calls "the Schaefer model." Figure 4 uses *Mathematica* to analyze it. The basic conclusions are as follows. (1) Although in Quadrant II of Clark's figure the relationship between p and x is monotonic, it does not seem possible to guarantee monotonicity, because dp/dx in '`Out[7]`' is difficult to sign. (2) '`Out[6]`' confirms Clark's figure's property that $\lim_{x\to\infty} p(x) = 0$. (3) The results after '`In[12]`' are that $\lim_{p\to\infty} x(p)$ is zero if $\delta > r$ and it is $K\,(r-\delta)/(2r)$ if $\delta \leq r$. From (44) this means that

$$\lim_{p\to\infty} x(p) = x_\delta \tag{49}$$

as shown in Clark's graph, meaning that our definition of x_δ is the same as his, and that (49) is connected to requiring that profit be positive.

This completes our mathematical analysis of Figure 3, which comes from (41) and (42) and which characterizes the steady-state supply curve in this section. For the intuition behind that figure's Quadrant I, note that at low prices, the steady state supply curve is upward-sloping, and it is generated from steady state values of x which are rather large, between $K/2$ and K, where $F'(x) < 0$. As price rises along this upward-sloping supply curve, x falls towards $K/2$ and h rises towards maximum sustainable yield "*MSY*." But once price rises sufficiently, x will fall below $K/2$, making h fall. These low values of x form the backward-bending part of the steady state supply curve. When price is high, the fishing industry can make x small, but it cannot make h large. Indeed, by responding to high prices by pushing stock size down, the industry causes harvest to fall. This is in part because we have imposed a sustainability *requirement* on this analysis: steady states are by definition sustainable. In the "real world," with no sustainability requirement and with firms that are not far-sighted, high prices might not cause low harvests, they might cause high harvests, harvests that are unsustainable.

Clearly in this graph, in light of (45), the interesting situation is where $x > x_\delta$ and $\pi > 0$. But why is $\pi > 0$—under constant returns to scale, we are used to thinking, from other microeconomics courses, that $\pi = 0$. The reason is that the fish here are a scarce resource and are privately owned, so the earn an economic rent, just like land does. Profit in this fishery is rent.[5]

Next, I slightly extend an interesting result of Clark (pp. 60–61) to show that if δ is sufficiently large and if, unlike in (48), $c(0)$ is sufficiently small,

[5] Some authors call either marginal profit or $\lambda_t = e^{-\delta t} M\Pi_t$ (from (22)) rent, but that term is incorrect. Marginal profit or λ could be called marginal rent.

```
In[1]:= (* Schaefer model, p. 45 of Clark 1976 *)
        F[x_] := r x (1 - x / K)
        c[x_] := gamma / x

In[3]:= (* Clark p. 157 H_delta *)
        c[x] - D[c[x], x] F[x] / (delta - D[F[x], x])
```

Out[3]= $\frac{\text{gamma}}{x} + \frac{\text{gamma}\ r \left(1 - \frac{x}{K}\right)}{x \left(\text{delta} + \frac{r\,x}{K} - r \left(1 - \frac{x}{K}\right)\right)}$

```
In[4]:= Simplify[%]
```

Out[4]= $\frac{\text{gamma}\ (\text{delta}\ K + r\ x)}{x\ (\text{delta}\ K - K\ r + 2\ r\ x)}$

```
In[5]:= p[x_] = Simplify[%%]
```

Out[5]= $\frac{\text{gamma}\ (\text{delta}\ K + r\ x)}{x\ (\text{delta}\ K - K\ r + 2\ r\ x)}$

```
In[6]:= Limit[p[x], x → Infinity]
```

Out[6]= 0

```
In[7]:= D[p[x], x] // Simplify
```

Out[7]= $-\frac{\text{gamma}\ \left(\text{delta}^2\ K^2 - \text{delta}\ K\ r\ (K - 4\ x) + 2\ r^2\ x^2\right)}{x^2\ (\text{delta}\ K - K\ r + 2\ r\ x)^2}$

```
In[8]:= Solve[p == p[x], x]
```

Out[8]= $\left\{\left\{x \to \frac{-\text{delta}\ K\ p + \text{gamma}\ r + K\ p\ r - \sqrt{8\ \text{delta}\ \text{gamma}\ K\ p\ r + (\text{delta}\ K\ p - \text{gamma}\ r - K\ p\ r)^2}}{4\ p\ r}\right\},\right.$
$\left.\left\{x \to \frac{-\text{delta}\ K\ p + \text{gamma}\ r + K\ p\ r + \sqrt{8\ \text{delta}\ \text{gamma}\ K\ p\ r + (\text{delta}\ K\ p - \text{gamma}\ r - K\ p\ r)^2}}{4\ p\ r}\right\}\right\}$

```
In[9]:= x[p_] := x /. Part[Solve[p == p[x], x] , 2]

In[10]:= x[p]
```

Out[10]= $\frac{-\text{delta}\ K\ p + \text{gamma}\ r + K\ p\ r + \sqrt{8\ \text{delta}\ \text{gamma}\ K\ p\ r + (\text{delta}\ K\ p - \text{gamma}\ r - K\ p\ r)^2}}{4\ p\ r}$

```
In[11]:= Limit[x[p], p → Infinity]
```

Out[11]= $\frac{-\text{delta}\ K + \sqrt{K^2\ (\text{delta} - r)^2} + K\ r}{4\ r}$

```
In[12]:= FullSimplify[Limit[x[p], p → Infinity], Assumptions → {K > 0}] /. Sqrt[x_^2] → Abs[x]

         Simplify[%, Assumptions → {delta - r < 0}]
         Simplify[%%, Assumptions → {delta - r >= 0}]
```

Out[12]= $\frac{K\ (-\text{delta} + r + \text{Abs}[\text{delta} - r])}{4\ r}$

Out[13]= $\frac{K\ (-\text{delta} + r)}{2\ r}$

Out[14]= 0

Figure 4. A *Mathematica* analysis of Quadrant II of Figure 3 and of (42). Lines ‘`In[13]`’ and ‘`In[14]`’ are not marked in this printout, but they are the lines which immediately follow ‘`In[12]`.”

the stock may be driven to extinction. This would be "socially optimal extinction," assuming the First Welfare Theorem holds (optimality of competitive equilibrium)—but only if no human, in the present or in the future, assigns any value to the existence of this population. ("Population" rather than "species" because this might be one isolated "population" of species, which would have other populations as well.) If some human, now or in the future, would value this fish beyond just wanting to eat it, then externalities come into play, and the model becomes quite different. For example, there would be a value placed on x; our model has no value directly placed on x, only value placed on h, and x then only obtains value via imputation, like agricultural land, being valued not in and of itself, but only as an instrument to produce something else which is of value.

Proposition 3. *Let* $\max_x F(x) \triangleq MSY$ *for "maximum sustainable yield." Suppose that (41) holds, that* $x > 0$, *and that:*

$$c''(x) \geq 0 \text{ (this is (36))},$$
$$c(0) < \phi(MSY)\,, \tag{50}$$
$$\delta > 2F'(0)\,, \tag{51}$$
$$F''(x) < 0 \,\forall\, x > 0, \text{ and}$$
$$F(0) = 0.$$

Then (42) cannot hold.

Proof. Rewrite (42) as

$$\frac{-c'(x)\,F(x)}{p - c(x)} = \delta - F'(x)\,.$$

In market equilibrium, this means steady-state x and h obey

$$\frac{-c'(x)\,F(x)}{\phi(h) - c(x)} = \delta - F'(x)\,. \tag{52}$$

Suppose ξ is in $(0, x)$. I will show below that

$$\frac{-c'(x)\,F(x)}{\phi(h) - c(x)} \leq \frac{-c'(x)\,F(x)}{\phi(MSY) - c(x)} \tag{53}$$
$$< \frac{-c'(x)\,F(x)}{c(0) - c(x)} \tag{54}$$
$$= \frac{c'(x)}{c'(\xi)}\,F'(\xi) \tag{55}$$

$$
\begin{aligned}
&\leq F'(\xi) && (56)\\
&< F'(0) && (57)\\
&< F'(0) + [F'(0) - F'(x)] && (58)\\
&= 2F'(0) - F'(x) && (59)\\
&< \delta - F'(x)\,. && (60)
\end{aligned}
$$

Since the left-hand side ("LHS") of (53) is the LHS of (52), and the right-hand side ("RHS") of (60) is the RHS of (52), this will prove it is impossible for (52) to hold.

To prove (53): First note that $-c' \geq 0$. By definition, $F(x) \leq MSY$. By (41), this means $h \leq MSY$. Since ϕ is downward-sloping, $\phi(h) \geq \phi(MSY)$.

To prove (54): use the second assumption of the proposition.

To prove (55): By the Generalized Mean Value Theorem (sometimes known as the Cauchy Mean Value Theorem),[6]

$$\frac{F(0) - F(x)}{c(0) - c(x)} = \frac{F'(\xi)}{c'(\xi)} \quad \text{for some } \xi \in (0, x). \tag{61}$$

The last assumption of the proposition gives $F(0) = 0$; hence

$$\frac{-F(x)}{c(0) - c(x)} = \frac{F(0) - F(x)}{c(0) - c(x)} = \frac{F'(\xi)}{c'(\xi)} \quad \text{for some } \xi \in (0, x).$$

To prove (56): $\xi < x$, and by the proposition's first assumption, $c'' \geq 0$, so $c'(\xi) \leq c'(x)$. Dividing by $c'(\xi)$ and recalling that $c'(x) < 0$ for all x, we get $1 \geq c'(x)/c'(\xi) > 0$.

To prove (57): $0 < \xi$, and by the proposition's fourth assumption, $F'' < 0$, so $F'(0) > F'(\xi)$.

To prove (58): The proposition assumes that $0 < x$. Then as in the proof of (57), $F'(0) > F'(x)$; so the term in brackets in (58) is positive.

(59) is trivial, and (60) follows from the proposition's third assumption.
∎

[6]The Mean Value Theorem itself states, for example, that $c(0) - c(x) = c'(\xi)(0 - x)$ for some $\xi \in (0, x)$. In words: there will exist at least one point ξ between zero and x at which the tangent line to $c(x)$ will be exactly parallel to the line joining $c(0)$ with $c(x)$. Bartle (*Elements of Real Analysis*, Second Edition, p. 197) writes, "In fact the Mean Value Theorem is a wolf in sheep's clothing and is *the* Fundamental Theorem of the Differential Calculus." One could divide $F'(\hat{\xi}) = (F(0)-F(x))/(0-x)$ for $\hat{\xi} \in (0, x)$ by $c'(\xi) = (c(0)-c(x))/(0-x)$ to obtain the left-hand side of (61), but the other side would be $F'(\hat{\xi})/c'(\xi)$ instead of the right-hand side of (61), so the Generalized Mean Value Theorem is not a trivial consequence of the Mean Value Theorem.

Corollary ("Extinction"). *Under the conditions of Proposition 3, the only possible steady state would have $x = 0$.*

Remark 1. Since $h \leq MSY$, in the right-hand side (50), $\phi(MSY) \leq \phi(h)$, that is, $\phi(MSY)$ is the smallest market equilibrium price possible. Since c is a decreasing function of x, $c(x) < c(0)$, that is, $c(0)$ is the largest marginal cost possible. The equation (50) thus guarantees that marginal cost is less than price for all possible equilibrium x and h. (This is rather strong.)

Remark 2. The conditions for Proposition 3, which result in a great deal of fishing, are in some sense the opposite of the conditions for Lemma 8, which result in no fishing. Proposition 3's equation (50) postulates that the lowest possible steady state price, $\phi(MSY)$, is higher than the highest possible average cost, $c(0)$. In contrast, Lemma 8's equation (37) postulates that the highest possible steady state price, $\phi(0)$, is lower than the lowest possible steady state average cost, $c(K)$.

The assumption $F(0) = 0$ is true in any fishery. The assumption $F'' < 0$, pure compensation, usually means that $F(x)$ is larger for small x than in the cases of depensation, and always means that $F(x)$ is larger for small x than in cases of critical depensation (those ecological terms were defined in Section 2). Typically, extinction is more likely under depensation, and even more under critical depensation, than under pure compensation. The most important assumptions on the economic side are (50) and (51). Equation (50) states that the cost of driving the stock to extinction is not too high, and that the demand for fish is not too low. Equation (51) requires δ to be high. If growth is logistic, we know from Section 2 that $F'(0) = r$ and that r is the highest possible growth rate $\dot{x}/x$, so(51) requires δ to be larger than twice the highest possible growth rate.[7] As stated in the corollary, under the conditions of the proposition the only possible steady-state outcome would be extinction. Exercise 5 below suggests a graphical method by which one might be able to show that what happens is indeed a steady state with extinction. However the proposition does not rule out non-steady-state outcomes (such as limit cycles), so if a graphical analysis such as that suggested by Exercise 5 does not show that the outcome is a steady state with extinction, then a dynamic analysis would be needed to determine whether or not extinction is the inevitable outcome of the situation in the proposition.

[7]The proposition goes through even if either the third or the fourth strict inequality in its assumptions is turned into a weak inequality.

(Clark p. 61 by contrast says that '(41) holds but (42) does not hold' "clearly implies that extinction is optimal." This seems hasty.[8])

We next return to Figure 3 and show how to calculate its Quadrant I precisely, using *Mathematica*. To do this requires giving each parameter a numerical value. The example uses the Schaefer model.

Example 1, steady state: (35s) with: $c(x) = \frac{\gamma}{qx}$, $q = 1$, $\gamma = 50$; $F(x) = rx(1 - \frac{x}{K})$ with $r = 0.1$, $K = 100$; and $\delta = 0.2$. (The function $c(x)$ has two constants instead of one by tradition.) Since $r < \delta$, $x_\delta = 0$ by (44), so x is unrestricted. Since $c(0) = \infty$, Proposition 3's condition (50) fails, which suggests that extinction is unlikely. To ensure that, on the other hand, some fishing does occur, Lemma 8's condition (37) involves calculating $c(K)$, which is $50/(1 \cdot 100) = 0.5$; so if $\phi(0) > 1/2$, h^* will not be zero, but if $\phi(0) < 1/2$, h^* will be zero.

In this example (42) can be proven to lead to

$$p = \frac{\gamma}{qx} \frac{K\delta + rx}{K(\delta - r) + 2rx} . \tag{62}$$

The proof begins by noting that $c'(x) = -\gamma/(qx^2)$ and $F'(x) = r - (2rx/K)$. Substituting these into (42) yields

$$\begin{aligned} p &= \frac{\gamma}{qx} - \frac{\frac{-\gamma}{qx^2} rx(1 - \frac{x}{K})}{\delta - r + \frac{2rx}{K}} \\ &= \frac{\gamma}{qx} - \frac{\frac{-\gamma}{qx} r(K - x)}{K(\delta - r) + 2rx} \\ &= \frac{\gamma}{qx} + \frac{\gamma r(K - x)}{qK(\delta - r) + 2qrx} \\ &= \frac{\gamma}{qx} \left[1 + \frac{r(K - x)}{K(\delta - r) + 2rx} \right] \\ &= \frac{\gamma}{qx} \frac{K\delta - Kr + 2rx + rK - rx}{K(\delta - r) + 2rx} , \end{aligned}$$

which simplifies to (62).

Figures 5 and 6 show the (identical) Quadrant I steady-state supply curve as derived using *Mathematica* in the following way. The expression for `p[x_]` is from (42).

[8] Clark p. 61 also says "We show that in this case Eq. (2.42) has no solution $x \geq 0$," but his proof only goes through if it is assumed that $x > 0$.

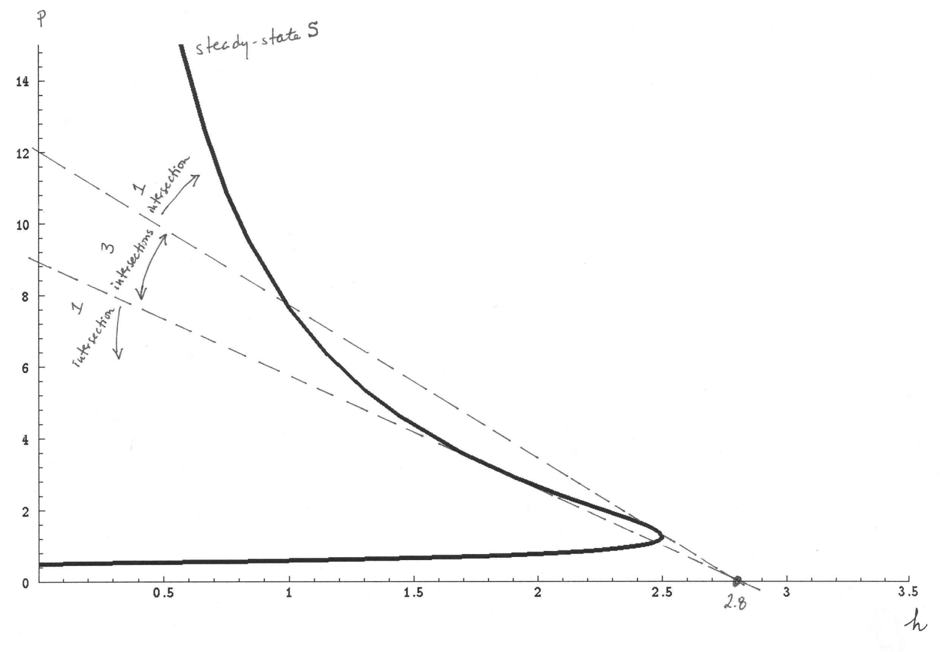

Figure 5. The steady state supply curve for Example 1. Superimposed on it are two hypothetical linear demand curves whose *h* intercept is 2.8. Dynamic analysis of the upper-most “1 intersection” region is given in Figure 12A based on (63); of the middle “3 intersections” region is given in Figure 14A based on (65); and of the lower-most “1 intersection” region is given in Figure 16A based on (67).

```
(* For Steady-State Supply Curve *)

(* Definitions:

        x: stock size (of fish)
     c[x]: average & marginal cost (decreases in x, constant in h)
     F[x]: excess of natural births over deaths
    delta: interest rate
        r: intrinsic growth rate
        K: carrying capacity
        h: harvest size
   phi[h]: inverse demand curve, not used below
xdot[x,h]: derivative of x with respect to time
hdot[x,h]: derivative of h with respect to time
*)


p[x_] := c[x] - (D[c[x], x]*F[x])/(delta - D[F[x], x])
   (* steady-state price as a function of x; see above *)
h[x_] := F[x]
   (* steady-state yield as a function of x; see above. *)

(* One example *)

c[x_] := gamma/(q*x)
F[x_] := r*x*(1 - x/K) (* logistic growth *)

gamma = 50
q = 1
r = 0.1  (* intrinsic growth rate *)
K = 100  (* carrying capacity *)
delta = 0.2 (* interest rate *)

(* Steady-State Supply Curve *)
ParametricPlot[Evaluate[{h[x],p[x]}], {x,1,100}];
Show[%, PlotRange->{{0,3.5},{0,15}}];
```

Intersections of Figure 5 and 6's the steady-state supply curve with the market demand curve will determine the steady-state equilibrium values of h and p. However, if the demand curve's choke price $p_c = \phi(0)$ is less than $1/2$, then as noted at the beginning of this Example, no fishing will occur, even if the demand curve intersects Figure 5's supply curve.[9]

[9] Preliminary numerical work suggests that the steady-state supply curve intersects the price axis at $1/2$. If that is the case, then if the demand curve's choke price $p_c = \phi(0)$ is less than $1/2$, it cannot intersect Figure 5's supply curve.

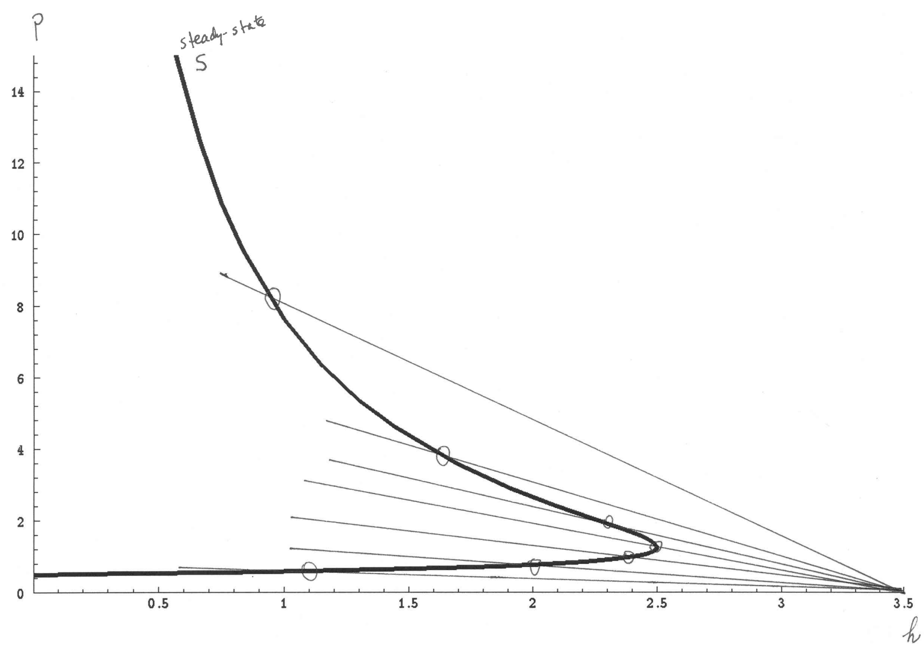

Figure 6. The steady state supply curve for Example 1. Superimposed on it are some hypothetical linear demand curves whose h intercept is 3.5.

For concreteness, suppose the demand curve is linear and pivots around the point $h = 2.8$, as sketched in Figure 5. As the slope gets progressively steeper, demand intersects supply: once; twice (but only for a single slope); three times; twice (but only for a single slope); and once. So, depending on the slope of the demand curve, there may be one, two, or three steady-state equilibria.

On the other hand, if the demand curve is linear and pivots around the point $h = 3.5$, as sketched in Figure 6, then demand always intersects supply exactly once.

Figure 5's behavior is more complicated than Figure 6's, and repays a closer look. Figure 7 superimposes onto Figure 5 a set of five alternative demand curves which are all linear and all pivot around the point $h = 2.8$. They all have the form $p = (-b/2.8)h + b$ where b is their p-intercept. These demand curves are:

$$p = (-15/2.8)\,h + 15 \quad \text{(greatest demand)} \tag{63}$$

$$p = (-12.2/2.8)\,h + 12.2 \tag{64}$$

$$p = (-10/2.8)\,h + 10 \tag{65}$$

$$p = (-9/2.8)\,h + 9 \tag{66}$$

$$p = (-5/2.8)\,h + 5 \quad \text{(least demand).} \tag{67}$$

Two of these demand curves ((64) and (66)) are the dashed lines in Figure 5; the other three lie in the three sectors into which the dashed lines divide Figure 5.

One way to derive the arrows in Figure 7 would be to apply a conventional, dubious "stability" analysis, in which price is assumed to fall if quantity demanded is less than quantity supplied, and price is assumed to rise if the opposite occurs. This is dubious because the analysis assumes competitive firms and consumers, so no agents think they can change the price. The correct way to derive the arrows in Figure 7 is to do a fully dynamic analysis, which is done in the next subsection. The result turns out to be the same in this situation (see Exercise 1, below).

So far, we have analyzed the pair (41) and (42) by using only one $c(x)$ and considering different demand curves. However, one could also analyze it by using only one demand curve and considering different forms for $c(x)$. To do that, use the four-quadrant diagram of Figure 8 to derive $\phi(F(x))$ for *one given* inverse demand curve $\phi(h)$. Next, transfer the derived $\phi(F(x))$ to Figure 9 and consider different $c(x)$ curves.

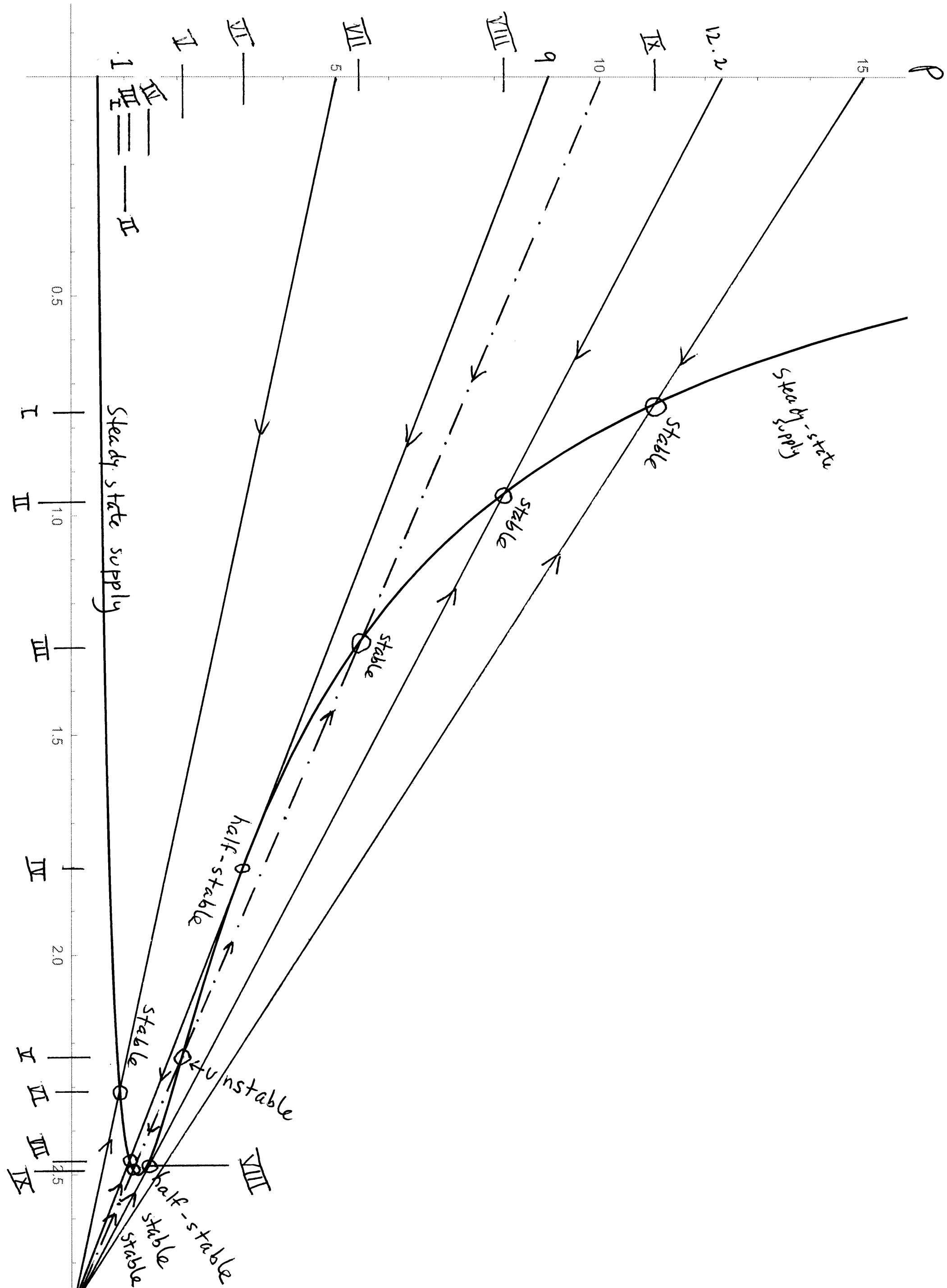

Figure 7. Superimposing demand curves (63)–(67) onto Figure 5. Steady-state market equilibrium occurs at the intersection(s) of demand and supply. Bifurcations occur at choke prices of 9 and 12.2. The demand curve with p_c = 10 is “dash dot” marked simply to aid in legibility. The roman numerals are used to help create Figure 18 from this figure. The arrows and characterizations of steady-state points by type come from the subsection on dynamics.

Lemma 11. *For the situation graphed in Figure 9, if cost is c_4, no x_{ss} exists.*

If cost is c_3, $x_{ss} \in (x_3, K)$.

If cost is c_2, $x_{ss} \in \{(x_{21}, x_{22}) \cup (x_{23}, K)\}$; if $\delta < r$, there is an additional restriction that $x_{ss} > x_{\delta 2}$.

If cost is c_1, $x_{ss} \in (x_1, K)$; if $\delta < r$, there is an additional restriction that $x_{ss} > x_{\delta 2}$, which may mean that no x_{ss} exists.

Proof. Equation (42) was derived from (38), and it is more convenient to use (38) than (42) to analyze Figure 9. Imposing market equilibrium ($p = \phi(h)$) onto (38) yields

$$[\delta - F'(x_{ss})][\phi(F(x_{ss})) - c(x_{ss})] = -c'(x_{ss})\, F(x_{ss})\,. \tag{68}$$

At $x_{ss} = 0$ or K one has $F = 0$ and hence the right-hand side of (68) is zero, but for all interesting $x_{ss} \in (0, K)$ the right-hand side of (68) is strictly positive, so for x_{ss} between 0 and K we need the left-hand side of (68) to be strictly positive as well. Because of Lemmas 2 and 7, assuming $h^* > 0$, the second term on the left-hand side of (68) is strictly positive; thus we need the first term to be positive as well. If $\delta > r$, as δ_1 is in Figure 9, then the first term of (68) is always positive. If $\delta < r$, as δ_2 is in the graph, then the first term of (68) is only positive to the right of $x_{\delta 2}$, so we will require $x_{ss} > x_{\delta 2}$.

For a $c(x)$ function like c_4 in Figure 9, it is impossible for the second term on the LHS of (68) to be positive, so no x_{ss} exists. This is to be expected from Lemma 8, because $c_4(K) > \phi(0)$.

As noted above, if $h^* > 0$ then the second term on the left-hand side of of (68) is strictly positive: $\phi(F(x)) > c(x)$. For the cost function c_3, x_{ss} thus has to lie between x_3 and K. We can use a small table and the Intermediate Value Theorem to prove that there exists at least one $x_{ss} \in (x_3, K)$ which satisfies (68) (that is, for which the left-hand side of (68) is equal to the right-hand side of (68)):

x	LHS of (68)	RHS of (68)	LHS − RHS
x_3	0	+	−
K	+	0	+

For the cost function c_2, to prove the claim given in the lemma, note that just as in the c_3 case, we can show that there exists at least one x_{ss} in (x_{23}, K) which satisfies (68), but there might be one (or more than one) x in (x_{21}, x_{22}) which also satisfies it. If $\delta < r$ one needs to recall the additional restriction that $x_{ss} > x_{\delta 2}$.

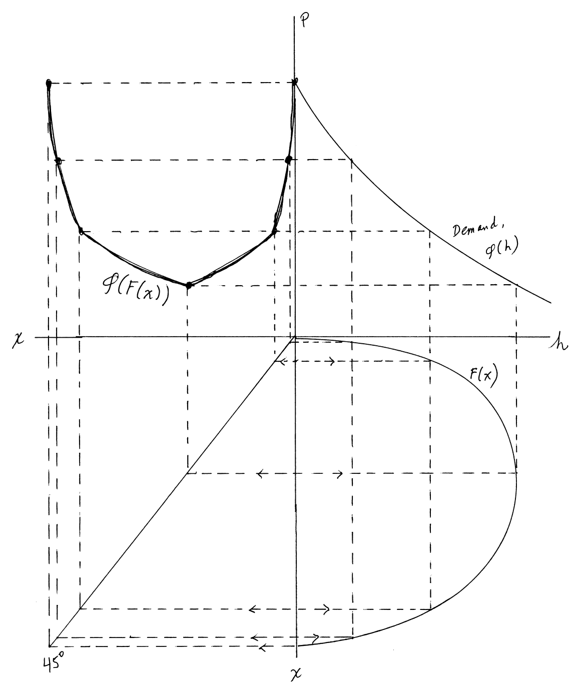

Figure 8. The derivation of $\phi(F(x))$ for a given demand curve ϕ.

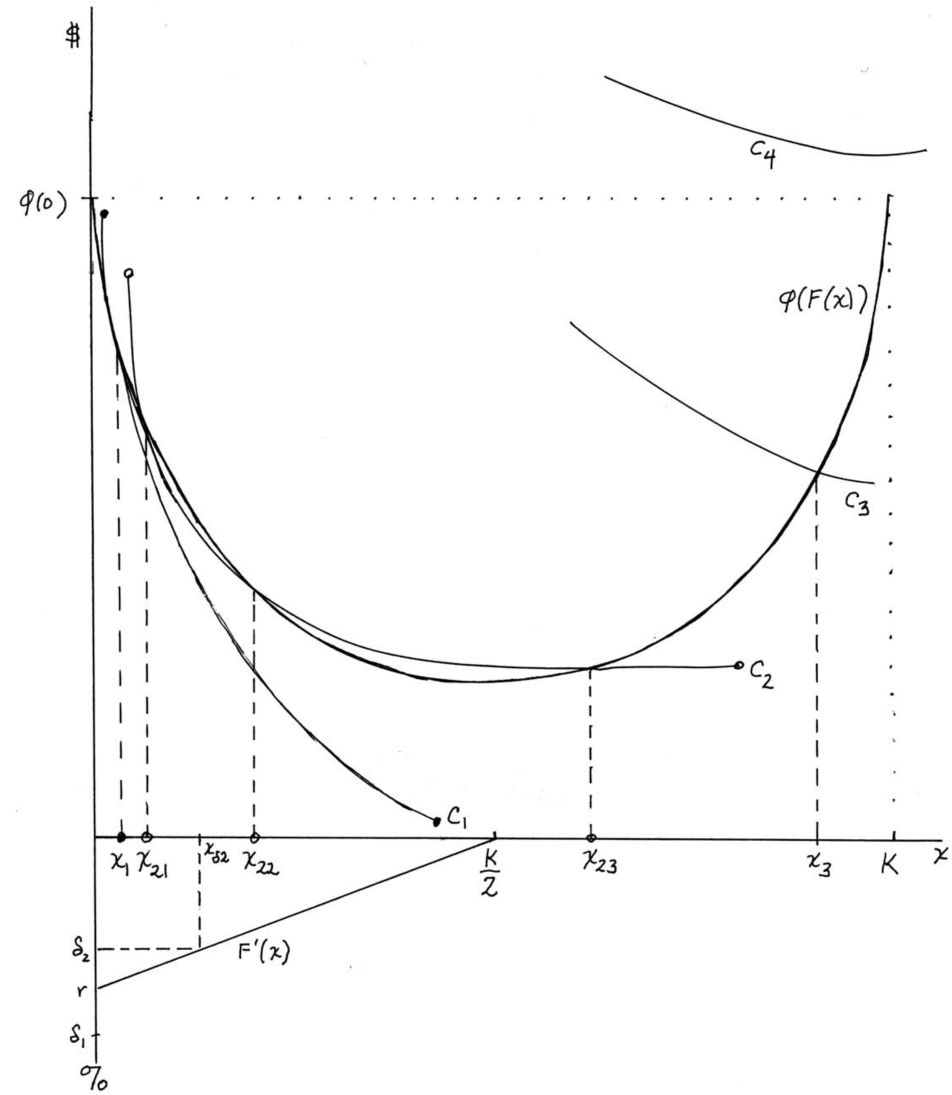

Figure 9. Superimposing some potential search-fishery average cost functions $c(x)$ onto the $\phi(F(x))$ function of Figure 8. Because in this section we assume constant returns to scale, $c(x)$ is not a function of harvest.

For the cost function c_1, using similar reasoning to the c_3 case there is an x_{ss} in (x_1, K) satisfying (68) in the $\delta > r$ case. In the $\delta < r$ case, the additional restriction that $x_{ss} > x_{\delta 2}$ makes it impossible to prove in principal that a suitable x_{ss} exists, though it is likely to. ▮

Subsection b. The Dynamic Case

Now consider (35d). As before (though now with time subscripts), $M\Pi_t = p_t - c(x_t)$ and $\partial\pi/\partial x = -c'(x_t)\, h_t$. We also have $\dot{M\Pi}_t = \dot{p}_t - c'(x_t)\,\dot{x}_t$. If $h_t > 0$, (20) holds, and substituting (35d) into (20) therefore gives

$$\begin{aligned} \delta\,(p-c) &= (p-c)F' + \dot{p} - c'\dot{x} - c'h \\ (\delta - F')(p-c) &= \dot{p} - c'\dot{x} - c'h \qquad (69) \\ &= \dot{p} - c'F \qquad (70) \end{aligned}$$

where (69) becomes (70) due to (17).

One could solve (69) for h to give

$$h = \frac{[-p + c(x)]\,[\delta - F'(x)]}{c'(x)} + \frac{\dot{p} - c'(x)\,\dot{x}}{c'(x)}\,. \qquad (71)$$

If all these time derivatives are zero, (71) is the same as (40). However, (71) is too complicated to be particularly enlightening. Alternatively, one could solve (70) for p to get

$$p = c(x) + \frac{\dot{p} - c'(x)\,F(x)}{\delta - F'(x)} \qquad (72)$$

then combine it with (from (17))

$$h = F(x_t) - \dot{x}_t \qquad (73)$$

to get a dynamic version of the pair (42) and (41). However, the presence of time derivatives makes it impossible to proceed as before to sketch a (now dynamic) supply curve.

At this point, we abandon any attempt to further describe the firm's supply response, and instead proceed to combine: (i) what we know about the firm's supply response, with (ii) a market demand curve. This will enable us to derive the market equilibrium dynamic paths. Substituting the market equilibrium condition $p_t = \phi(h_t)$ (namely (34)) into (72) gives:

$$\begin{aligned} \phi(h) &= c(x) + \frac{\phi'(h)\,\dot{h} - c'(x)\,F(x)}{\delta - F'(x)} \\ (\delta - F')(\phi - c) &= \phi'\dot{h} - c'\,F\,. \end{aligned}$$

This leads to

$$\dot{h}_t = \frac{[\delta - F'(x_t)][\phi(h_t) - c(x_t)] + c'(x_t)\,F(x_t)}{\phi'(h_t)}, \tag{74}$$

which along with

$$\dot{x}_t = F(x_t) - h_t \tag{17}$$

forms a dynamical system of the form

$$\begin{aligned} \dot{x}_t &= f_1(x_t, h_t) \\ \dot{h}_t &= f_2(x_t, h_t) \end{aligned} \tag{75}$$

where f_1 is the right-hand side of (17) and f_2 is the right-hand side of (74). To derive the phase-plane diagrams of the dynamics one finds the isoclines (more properly, the "nullclines"), which are the set of all (x, h) which make $\dot{x}$ or $\dot{h}$ equal to zero. (Notice that if one sets the time derivatives of the system (74)/(17) equal to zero, one obtains the basic equations we used for the steady state, (41) and (42).)

Because neither f_1 nor f_2 in (75) depends explicitly on t, the dynamic system (75) is autonomous. It follows that paths in its phase space cannot cross, because every (x, h) generates a unique $(\dot{x}_t, \dot{h}_t)$, that is, a unique direction of motion in phase space.

The $\dot{x} = 0$ and $\dot{h} = 0$ isoclines divide the (x, h) phase plane into "isosectors"; inside one isosector, neither $\dot{x}$ nor $\dot{h}$ change signs.

Lemma 12. *The area of the (x, h) plane which lies above the $\dot{x} = 0$ isocline has $\dot{x} < 0$; similarly, the area of the (x, h) plane which lies below the $\dot{x} = 0$ isocline has $\dot{x} > 0$.*

If $\delta > F'(x)$, the area of the (x, h) plane which lies above the $\dot{h} = 0$ isocline has $\dot{h} > 0$, and the area of the (x, h) plane which lies below the $\dot{h} = 0$ isocline has $\dot{h} < 0$.

If $\delta < F'(x)$, the area of the (x, h) plane which lies above the $\dot{h} = 0$ isocline has $\dot{h} < 0$, and the area of the (x, h) plane which lies below the $\dot{h} = 0$ isocline has $\dot{h} > 0$.

Proof. If one started on a point (x, h) which made $\dot{x} = 0$, then kept x the same but increased h, (17) implies that $\dot{x}$ would change from being zero to being negative. This proves the first paragraph.

If one started on a point (x, h) which made $\dot{h} = 0$, then kept x the same but increased h, (74) implies that $\dot{h}$ would change from being zero. To see how it would change, abbreviate (74)'s numerator so that

$$\dot h = \frac{\alpha(x,h)}{\phi'(h)} .$$

Then on the $\dot h = 0$ curve,

$$\begin{aligned}
\frac{\partial \dot h}{\partial h} &= \frac{\partial\alpha/\partial h}{\phi'} - \frac{\phi''\alpha}{(\phi')^2} = \frac{\partial\alpha/\partial h}{\phi'} - \frac{\phi''}{\phi'}\dot h \\
&= \frac{\partial\alpha/\partial h}{\phi'} - \frac{\phi''}{\phi'}\cdot 0 = \frac{\partial\alpha/\partial h}{\phi'} \\
&= \frac{(\delta - F')\phi'}{\phi'} = \delta - F' . \qquad (76)
\end{aligned}$$

Hence if $\delta > F'(x)$, deviating from the $\dot h = 0$ isocline by raising h will raise $\dot h$ from zero to something positive—implying that the area of the (x,h) plane which lies above the $\dot h = 0$ isocline has $\dot h > 0$, and the area of the (x,h) plane which lies below the $\dot h = 0$ isocline has $\dot h < 0$. Conversely, if $\delta < F'(x)$, then deviating from the $\dot h = 0$ isocline by raising h will lower $\dot h$ from zero to something negative ∎

In order to ensure that Proposition 2 governs harvest in this case, we need to verify that the basic assumptions on costs of Section 4 hold in this section. Those assumptions were that total cost $TC(x,h)$ was (weakly) convex in h and that $\partial TC/\partial h$ was nonnegative and differentiable. In this section, from (35), total cost is $c(x)\,h$, which is linear in h and so is convex in h, and $\partial TC/\partial h = c(x)$, which is nonnegative and differentiable. Hence Proposition 2 does govern harvest in this case.

While (74) and (17) follow from the necessary conditions for solving (16) and (17) given (35d), the question of sufficiency arises. The Mangasarian sufficiency condition of Section 1 requires

$$\nabla^2\mathcal{H} = \begin{bmatrix} \mathcal{H}''_{hh} & \mathcal{H}''_{hx} \\ \mathcal{H}''_{xh} & \mathcal{H}''_{xx} \end{bmatrix}$$

to be negative semidefinite. However, here (using $\triangleq$ to mean "is defined to be," as in (32))

$$\begin{aligned}
\mathcal{H} &= e^{-\delta t}\big[p_t - c(x_t)\big]h_t + \lambda_t\big[F(x_t) - h_t\big] \\
&= \big\{e^{-\delta t}[p_t - c(x_t)] - \lambda_t\big\}h_t + \lambda_t\,F(x_t) \\
&\triangleq \sigma(x_t,\lambda_t,t)\,h_t + \lambda_t\,F(x_t)\,, \qquad (77)
\end{aligned}$$

meaning that $\mathcal{H}''_{hh} = 0$, hence that $|\nabla^2\mathcal{H}| = -(\mathcal{H}''_{hx})^2 = -(\partial\sigma/\partial x)^2 = -[-e^{-\delta t}c'(x_t)]^2$, which means $\mathcal{H}$ fails the test for strict concavity, and fails the test for concavity whenever $\mathcal{H}''_{hx} \neq 0$ (in other words, whenever $c' \neq 0$), as is always the case for search fisheries. To check the Arrow sufficiency condition, first note that either we follow a singular solution, in which case $\sigma = 0$, or $h^* = 0$ (or $h^* = \infty$, which is not interesting); in either case, $\mathcal{H}^* = \lambda_t F(x_t)$. Along a singular solution, λ is given by (22) as $e^{-\delta t}M\Pi$, and here $M\Pi = p - c$, so

$$\mathcal{H}^* = e^{-\delta t}[p_t - c(x_t^*)]F(x_t^*) .$$

Then $\mathcal{H}^{*\prime}_x = e^{-\delta t}[-c'F + (p-c)F']$ and $\mathcal{H}^{*\prime\prime}_{xx} = e^{-\delta t}[-c''F - c'F' - c'F' + (p-c)F'']$; collecting terms,

$$\mathcal{H}^{*\prime\prime}_{xx} = e^{-\delta t}[(p-c)F'' - 2c'F' - c''F] . \tag{78}$$

In many situations (though not all) we can rule out F being negative; for example, with logistic growth, as long as x is never greater than the carrying capacity, F will not be negative. Typically, $p - c > 0$ (from Lemma 2), $F'' < 0$ (from logistic growth), $c' < 0$ (which characterizes a search fishery), and $c'' > 0$ (which is (36)). F' can have either sign, but if $F'(x_t^*) < 0 \,\forall\, t$ together with the other common conditions, (78) would be negative and the Arrow sufficient condition for a maximum would hold. For the case of logistic growth, a sufficient condition for $F'(x_t^*) < 0 \,\forall\, t$ is that $x_t^* > K/2 \,\forall\, t$.

We will further illustrate the qualitative theory of dynamic systems such as (74) and (17) by going through “Example 1, dynamics” below. This paragraph briefly discusses the quantitative theory (see *Natural Resource Economics: Notes and Problems* by Jon M. Conrad and Colin W. Clark, 1987, pages 45 and 52). General mathematical notation for such systems is

$$\begin{aligned} \dot{x}_t &= F(x_t, y_t) \\ \dot{y}_t &= G(x_t, y_t) \end{aligned}$$

where this F is unrelated to fisheries—a notation clash between mathematics and ecology. Let (x^*, y^*) be a steady-state point, i.e., a point making both F and G equal to zero. (This is a notation clash between mathematics and economics: economists use asterisks to denote optima, not steady states.) A first-order Taylor Series approximation to F can be written

$$F(x,y) \approx F(x^*,y^*) + F'_x(x^*,y^*)(x-x^*) + F'_y(x^*,y^*)(y-y^*) .$$

A similar approximation holds for G. Indeed, if we take gradient vectors, e.g. $\nabla F^* = [F'_x \;\; F'_y]$, to be row vectors, we could write

$$\begin{bmatrix} F \\ G \end{bmatrix} \approx \begin{bmatrix} F^* \\ G^* \end{bmatrix} + \begin{bmatrix} \nabla F^* \\ \nabla G^* \end{bmatrix} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix} .$$

Since (x^*, y^*) is defined to be a steady-state point, both F^* and G^* are zero. Taking this into account and writing the gradients explicitly, to a first order approximation,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} F'^*_x & F'^*_y \\ G'^*_x & G'^*_y \end{bmatrix} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix} . \tag{79}$$

Note that the first matrix on the right-hand side consists entirely of numbers (constants), not variables. (It is a Jacobian matrix, a matrix of first derivatives, where each row is the gradient of a function.) If we define $\xi = x - x^*$ and $\eta = y - y^*$, then the (ξ, η) plane has its origin precisely at the point (x^*, y^*). Clearly $\dot{\xi} = \dot{x}$ and $\dot{\eta} = \dot{y}$. So

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} F'^*_x & F'^*_y \\ G'^*_x & G'^*_y \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} ,$$

a system of *linear* differential equations whose solution is

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = a\, \mathbf{v}_1\, e^{R_1 t} + b\, \mathbf{v}_2\, e^{R_2 t}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors, and R_1 and R_2 are the corresponding eigenvalues, of the matrix of the differential equation system.

It can be shown that if the eigenvalues are both real and positive, the steady-state point is unstable. Such steady-state points are called "unstable nodes" or "repellers." See Figure 10 for what different types of steady-state points look like.[10]

If the eigenvalues are both real and negative, the steady-state point is stable. Such steady-state points are called "stable nodes" or "attractors."

If the eigenvalues are both real and one is positive and the other is negative, the steady-state point is called a "saddle point" or just a "saddle." Saddle points are common in economic models. Mathematicians call saddle points "unstable" but economists do not, because in economics, a firm or consumer or social planner often can control the system so that it stays on the saddle point's convergent separatrix, which is defined to be the only

[10] The source of Figure 10 is `http://www.scholarpedia.org/article/Equilibrium`.

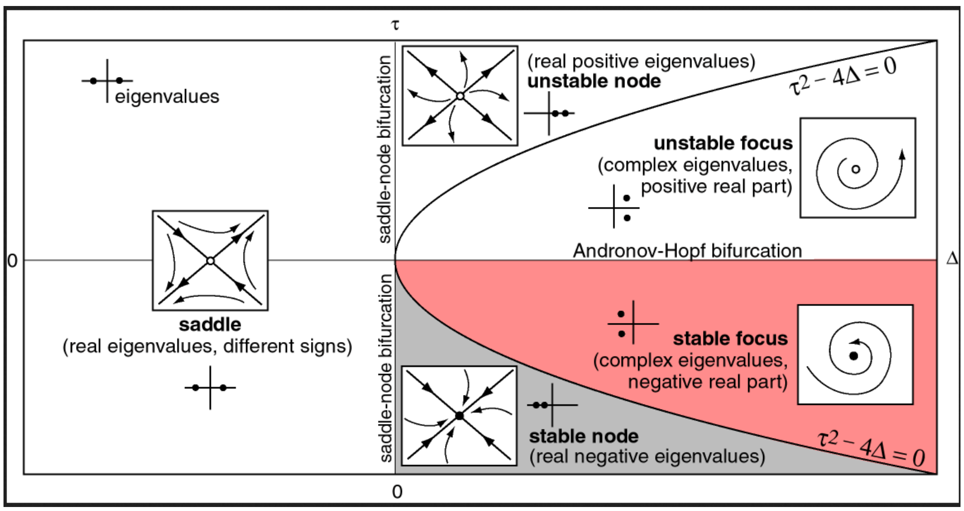

Figure 10. Types of steady-state points. The vertical axis τ denotes the trace of the Jacobian matrix of (79) and the horizontal axis Δ denotes the determinant of that Jacobian matrix, although I prefer to analyze behavior using that matrix's eigenvalues rather than using τ and Δ. In each of the six parts of this diagram, there is a small graph with two dots; the dots represent the eigenvalues, and the horizontal and vertical axes represent, respectively, the real and the imaginary component of each eigenvalue.

path leading to the saddle point. (The divergent separatrix is the only path leading away from the saddle point.) It can be useful to know how to calculate the position of the convergent separatrix. To do this, pick $R_2 < 0 < R_1$ since it is arbitrary which eigenvalue is positive and which one is negative. Then in order for an arbitrary (ξ_0, η_0) to go to $(\xi_t^*, \eta_t^*) = (0,0)$ as $t \to \infty$, we need $a = 0$. If we therefore impose $a = 0$, then the ratio of η to ξ as $t \to \infty$ is given by $\mathbf{v}_2$. For example, if $\mathbf{v}_2 = \begin{bmatrix} 1 \\ m \end{bmatrix}$, then the slope of the convergent separatrix is m (in the ξ-η plane, but it has the same slope in the x-y plane).

If the eigenvalues are complex then they will occur in a "conjugate pair" $\alpha \pm \beta i$ where $i = \sqrt{-1}$. It can be shown[11] that in this case, the solution can be rewritten for some constants $\mathbf{a}_1$ and $\mathbf{b}_1$ as

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \mathbf{a}_1\, e^{\alpha t} \cos(bt) + \mathbf{b}_1\, e^{\alpha t} \sin(bt)\,.$$

If the real part ("α") is positive, the steady-state point is an unstable spiral, whereas if the real part is negative, the steady-state point is a stable spiral (and if the real part is exactly zero, the steady-state point is a center).

We will encounter one type of steady-state point which is not depicted in Figure 10, called a "saddle node." Saddle nodes look like a combination of a saddle and a (stable or unstable) node; Figure 11 depicts a saddle node which is a combination of a saddle and a stable node.[12] A saddle node's Jacobian matrix in (79) has one eigenvalue which is real and positive and the other which is real and exactly zero. In Figure 10, this occurs at the boundary between its Quadrants I (two positive eigenvalues) and II (one positive and one negative eigenvalue), and at the boundary between Quadrants IV (two negative eigenvalues) and III. These two boundaries are labeled "Saddle-Node Bifurcation" in the figure. (Mathematicians could have called each boundary by its own name, a "Saddle-Repeller Bifurcation" or a "Saddle-Attractor Bifurcation," but they did not.)

[11] See for example https://math.libretexts.org/Bookshelves/Analysis/Supplemental_Modules_(Analysis)/Ordinary_Differential_Equations/3%3A_Second_Order_Linear_Differential_Equations/3.4%3A_Complex_Roots_of_the_Characteristic_Equation. That reference assumes you know Euler's Formula, for which see https://www.mathsisfun.com/algebra/eulers-formula.html. That reference also assumes you know that problems like ours, which is a system of two first-order differential equations, can be rewritten as a single second-order differential equation, as shown for example in https://mjo.osborne.economics.utoronto.ca/index.php/tutorial/index/1/sim/t. This mathematics is also covered in Clark's book.

[12] The source of Figure 11 is https://www.larserikpersson.se/webcourse/ix-introduction-to-the-theory-of-dynamical-systems-chaos-stability-and-bifurcations/7-classification-of-critical-points/.

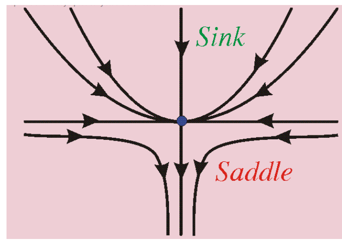

Figure 11. Phase-plane paths near a "saddle node." This particular saddle node has characteristics of a saddle point in the lower part of the diagram, and of an attractor in the upper part of the diagram; this author refers to attractors as "sinks." Therefore, this saddle node corresponds to the boundary between Quadrants III and IV in Figure 10. There is another type of saddle node which has characteristics of a saddle point and a repeller, which corresponds to the boundary between Quadrants I and II in Figure 10.

Example 1, dynamics. Assume (35d) with, just as "Example 1, steady state": $c(x) = \frac{\gamma}{qx}$, $q = 1$, $\gamma = 50$; $F(x) = rx(1 - \frac{x}{K})$ with $r = 0.1$, $K = 100$; and $\delta = 0.2$. Unlike in "Example 1, steady state," it is necessary to also assume a particular demand function $\phi(h)$. We will assume linear demand curves which pivot as in Figure 5, not as in Figure 6, and in particular, we will study each of the demand curves given by (63)–(67). They are the ones graphed in Figure 7.

Assuming that the *Mathematica* code in "Example 1, steady state" above has been read in, the procedure to draw isoclines with *Mathematica* follows. The first non-comment line of code is (17) and the second is (74).

```
(* For Isoclines of the Phase Diagram *)

(* Definitions: *)

xdot[x_, h_] := F[x] - h
hdot[x_, h_] :=
  ((delta - D[F[x], x])*(phi[h] - c[x]) + D[c[x], x]*F[x]
  )/D[phi[h], h]

xdotisocline =
ContourPlot[Evaluate[xdot[x,h]],{x,1,125},{h,0,3.5},
     Contours->{0},ContourShading->False,PlotPoints->50];
(* this got the xdot=0 isocline *)

(* First demand curve *)
intercept = 15;
phi[h_] := (-intercept*h)/2.8 + intercept
hdotisocline=
ContourPlot[Evaluate[hdot[x,h]],{x,1,125},{h,0,3.5},
```

```
     Contours->{0},ContourShading->False,PlotPoints->50];
Show[hdotisocline,xdotisocline];
(* for a finite final date *)
terminalsurface=
  Plot[h/.Solve[phi[h]==c[x], h][[1]][[1]],
  {x,1,125},PlotRange->{0,3.5}];
Show[hdotisocline,xdotisocline,terminalsurface,
     PlotRange->{0,3.5}];


(* Second Demand Curve*)
intercept = 12.2;
phi[h_] := (-intercept*h)/2.8 + intercept
hdotisocline=
ContourPlot[Evaluate[hdot[x,h]],{x,1,125},{h,0,3.5},Contours->{0},
         ContourShading->False,PlotPoints->50];
Show[hdotisocline,xdotisocline];
terminalsurface=
  Plot[h/.Solve[phi[h]==c[x], h][[1]][[1]],
  {x,1,125},PlotRange->{0,3.5}];
Show[hdotisocline,xdotisocline,terminalsurface,
     PlotRange->{0,3.5}];

(* etc. *)
```

Using these techniques, demand curve (63) yields Fig. 12A; similarly, (64) yields Fig. 13A; (65) yields Fig. 14A; (66) yields Fig. 15A; and (67) yields Fig. 16A. See also Figure 6.12, p. 187 of Clark, reproduced as Figure 17 here. The "B" versions of each graph, and Figure 9, are explained after the Exercises below.

In this example, growth is logistic and $\delta = 0.2 > 0.1 = r \geq F'(x)$, so $\delta > F'(x)$ for all x. This means from (76) that $\partial \dot{h}/\partial h > 0$ (the area of the (x, h) plane which lies above the $\dot{h} = 0$ isocline has $\dot{h} > 0$). It means, from the notation of (44), that $x_\delta = 0$.

In this example, the given functional forms (without needing to replace p with $\phi(h)$) imply that (78) is

$$\mathcal{H}^{*}_{xx}{}'' = \frac{-2pr}{K} .$$

This is strictly negative and shows according to the Arrow sufficiency result that any path in this example which satisfies the necessary conditions is optimal, and that there is only one optimal path for x.

Exercise 1. Finish drawing the phase plane diagrams in Figs. 12–16. Locate the trajectory to the steady state in each diagram assuming $x_0 = K$ (recall that $K = 100$). Just after these exercises, we will show, using the "B" versions of the graphs, that these trajectories are optimal. Locate the optimal trajectories from other values of x_0, and confirm the arrows shown in Figure 7.

Exercise 2. Using the results of Exercise 1, as the demand curve pivots clockwise in Fig. 5, locate the equilibrium points. Note the bifurcation when, as demand increases, suddenly quantity jumps down and price jumps up. One way to summarize the results of Exercise 1 and Figure 7's analysis is to see how steady-state price and harvest varied with the choke price p_c. Explain how Figure 7 leads to Figure 18. Bifurcations,

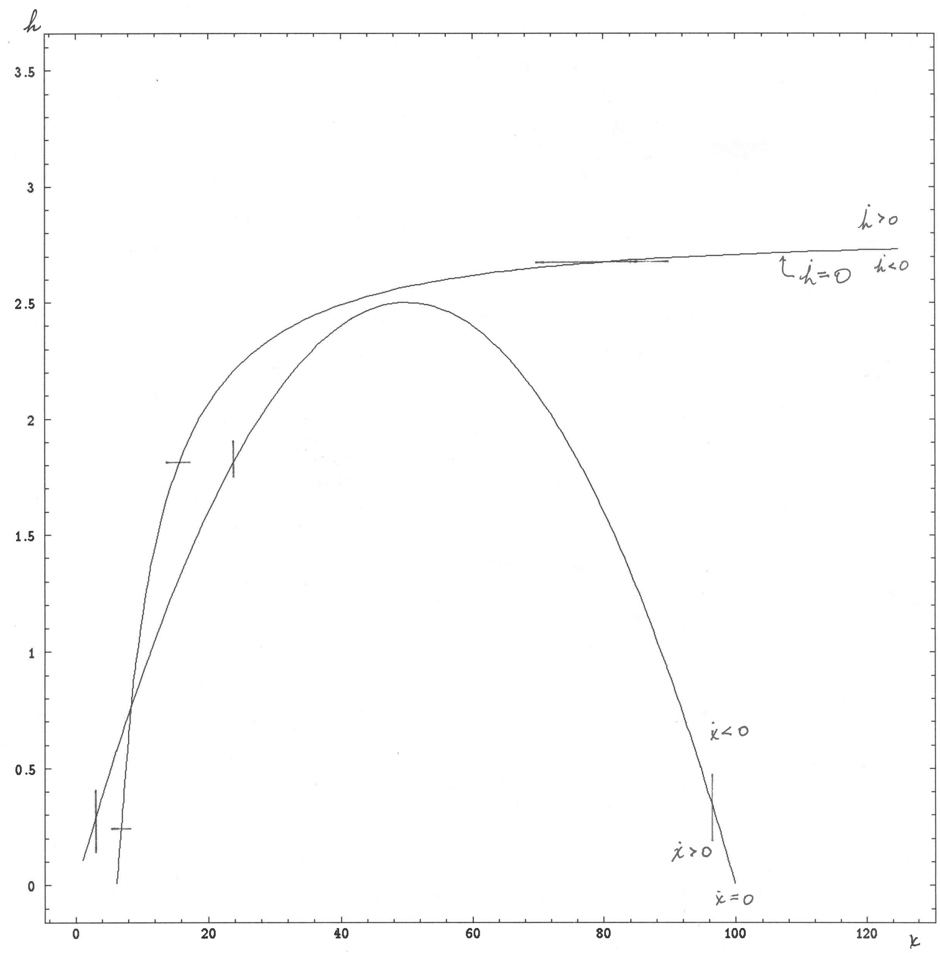

Figure 12A. Demand curve is $p = -\frac{15}{2.8}h + 15$ (equation (63)), corresponding to a choke price of 15 and an intersection in the upper-most "1 intersection" region of Figure 5. Drawing the phase plan paths is left to the reader, who should conclude that the steady-state point is a saddle point whose convergent separatrixes approach from the northeast and southwest. Note: the plural form of separatrix is sometimes spelled "separatrices."

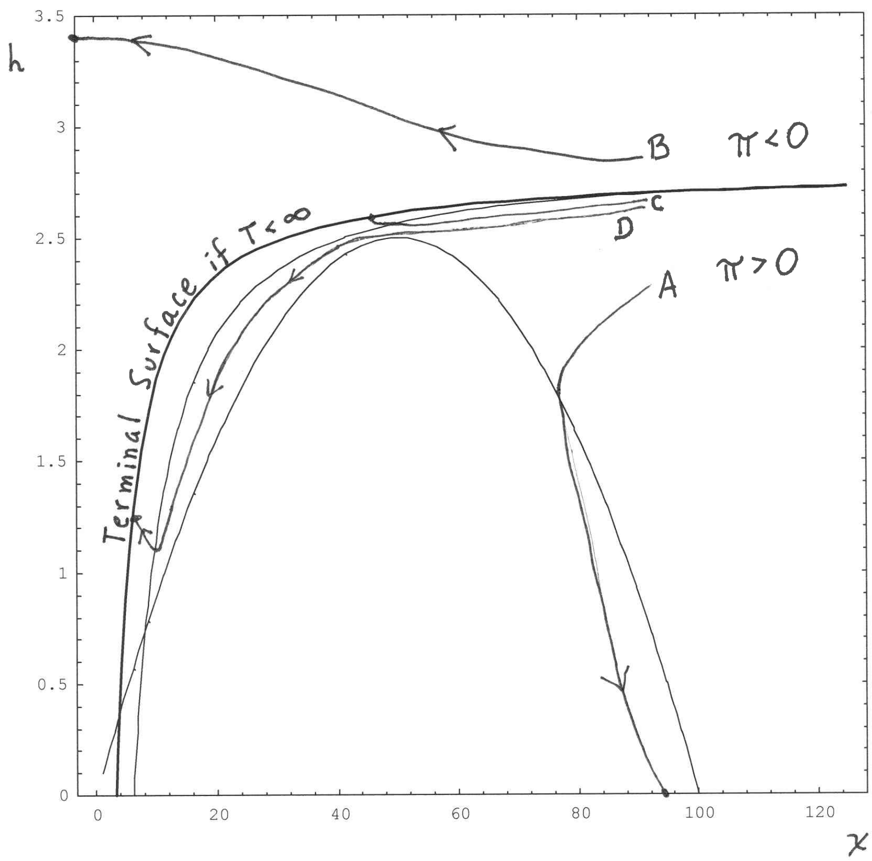

Figure 12B. Figure 12A combined with the finite-time terminal surface. The infinite-horizon behavior implied by studying the finite-horizon behavior in this figure is reflected in Figures 7 and 18.

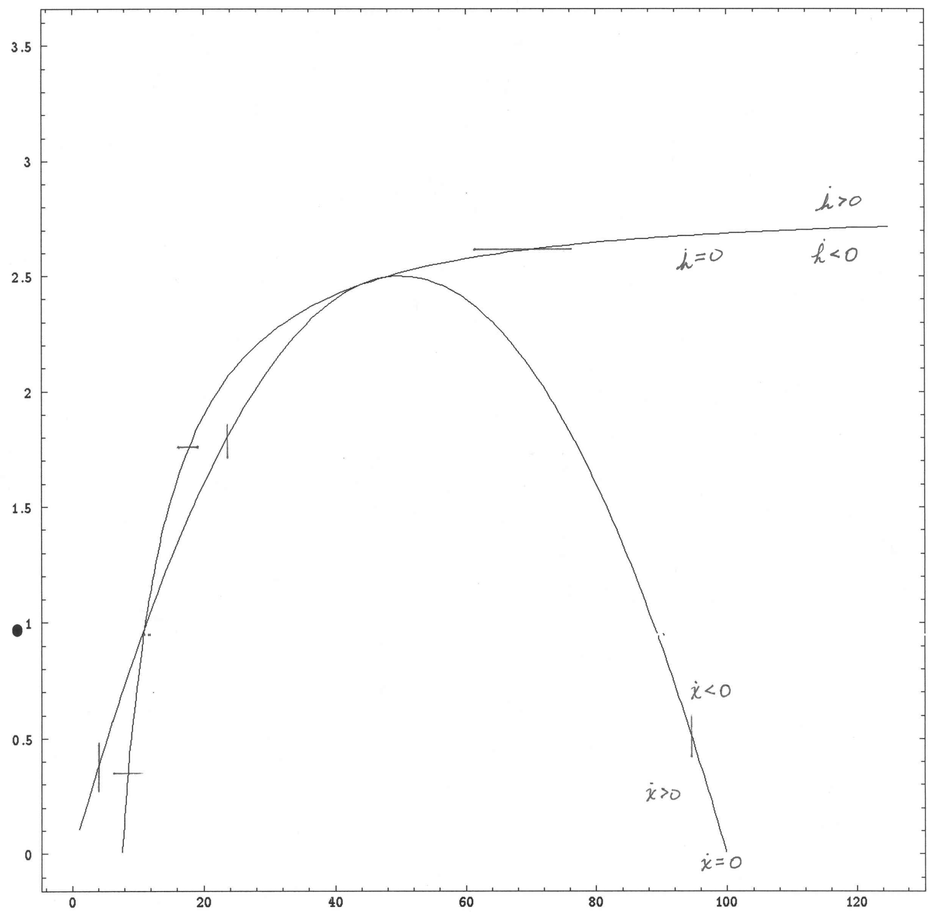

Figure 13A. Demand curve is $p = -\frac{12.2}{2.8}h + 12.2$ (equation (64)), corresponding to a choke price of 12.2 and a demand curve on the knife edge between the middle, "3 intersections" region of Figure 5 and that figure's upper-most "1 intersection" region. Drawing the phase plan paths is left to the reader, who should conclude that the left steady-state point is a saddle point whose convergent separatrixes approach from the northeast and southwest, and that the right-most steady-state point is a saddle node (repeller on the left, saddle on the right). The steady-state points are joined by a path, because the path which converges to the left-hand steady-state point from the right cannot have, in its past, crossed either the $\dot{x} = 0$ isocline or the $\dot{h} = 0$ isocline, because paths crossing those isoclines leave the isosector and thus cannot converge to the steady-state point. Paths that join two steady-state points are called "heteroclinic orbits." If $x_0 = K$, the equilibrium path gets stuck at the saddle node.

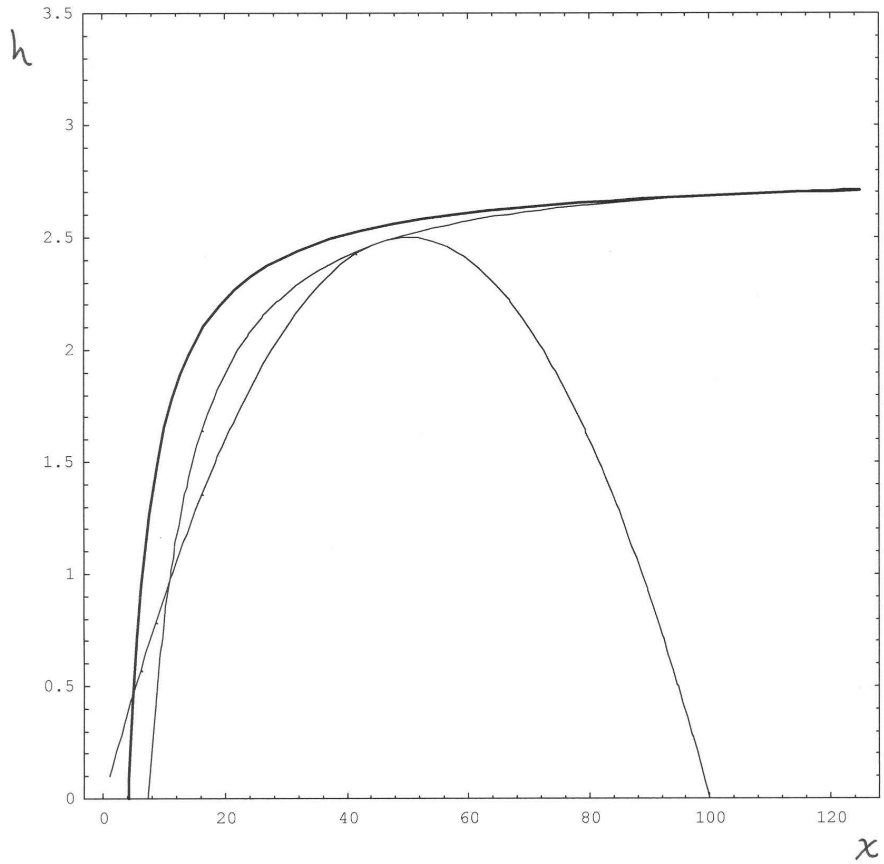

Figure 13B. Figure 13A combined with the finite-time terminal surface. The infinite-horizon behavior implied by studying the finite-horizon behavior in this figure is reflected in Figures 7 and 18.

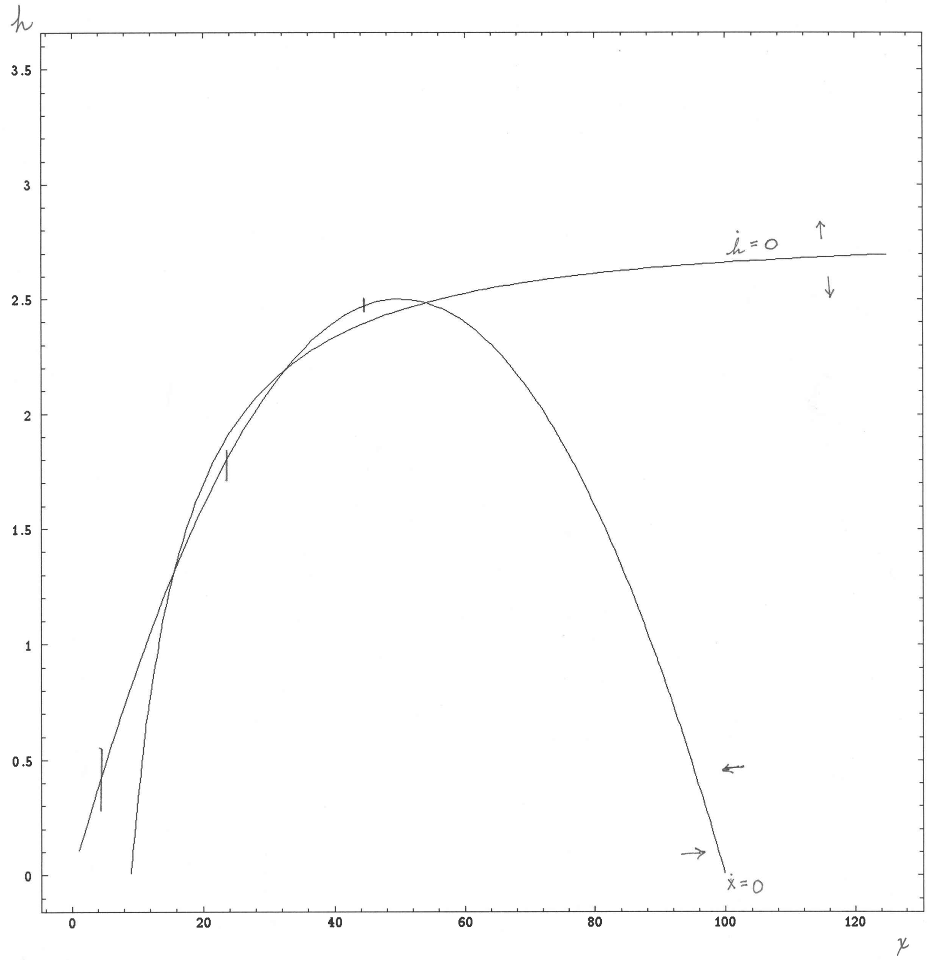

Figure 14A. Demand curve is $p = -\frac{10}{2.8}h + 10$ (equation (65)), corresponding to a choke price of 10 and an intersection in the middle, "3 intersections" region of Figure 5. Drawing the phase plan paths is left to the reader, who should conclude that the left and right steady-state points are saddle points whose convergent separatrixes approach from the northeast and southwest, and that the middle steady-state point is a repeller. Each saddle point is joined to the middle steady-state point (repeller) by a path (a "heteroclinic orbit") which converges to that saddle point, for geometrical reasons like those given at the end of the caption of Figure 13A.

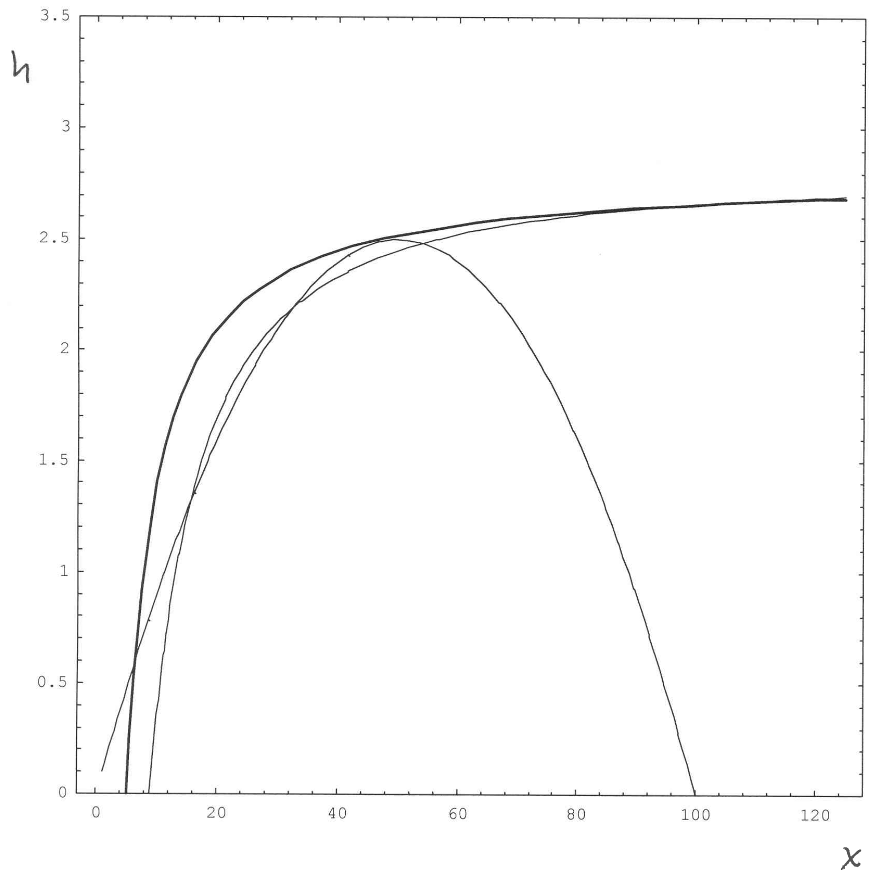

Figure 14B. Figure 14A combined with the finite-time terminal surface. The infinite-horizon behavior implied by studying the finite-horizon behavior in this figure is reflected in Figures 7 and 18.

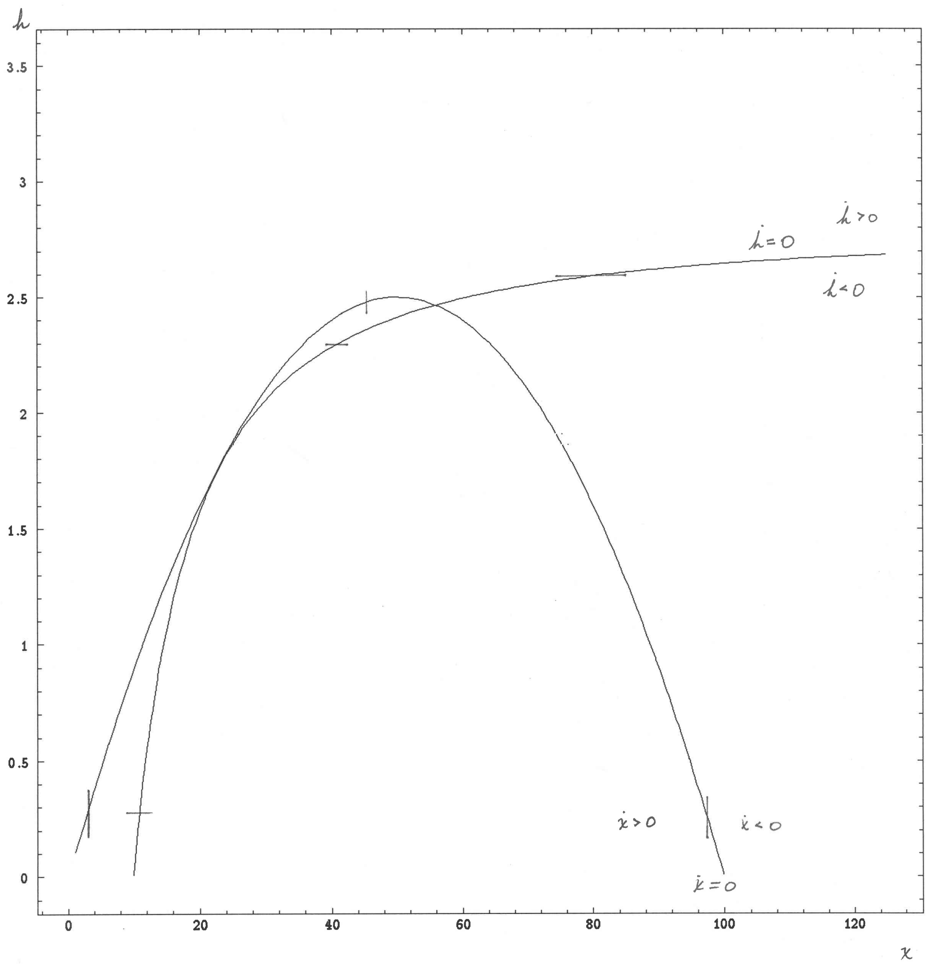

Figure 15A. Demand curve is $p = -\frac{9}{2.8}h + 9$ (equation (66)), corresponding to a choke price of 9 and a demand curve on the knife edge between the middle, "3 intersections" region of Figure 5 and that figure's lower-most "1 intersection" region. Drawing the phase plan paths is left to the reader, who should conclude that the left steady-state point is a saddle node (saddle on the left, repeller on the right) and the right-most steady-state point is a saddle point whose convergent separatrixes approach from the northeast and southwest. The steady-state points are joined by a path (a "heteroclinic orbit") for geometrical reasons like those given at the end of the caption of Figure 13A.

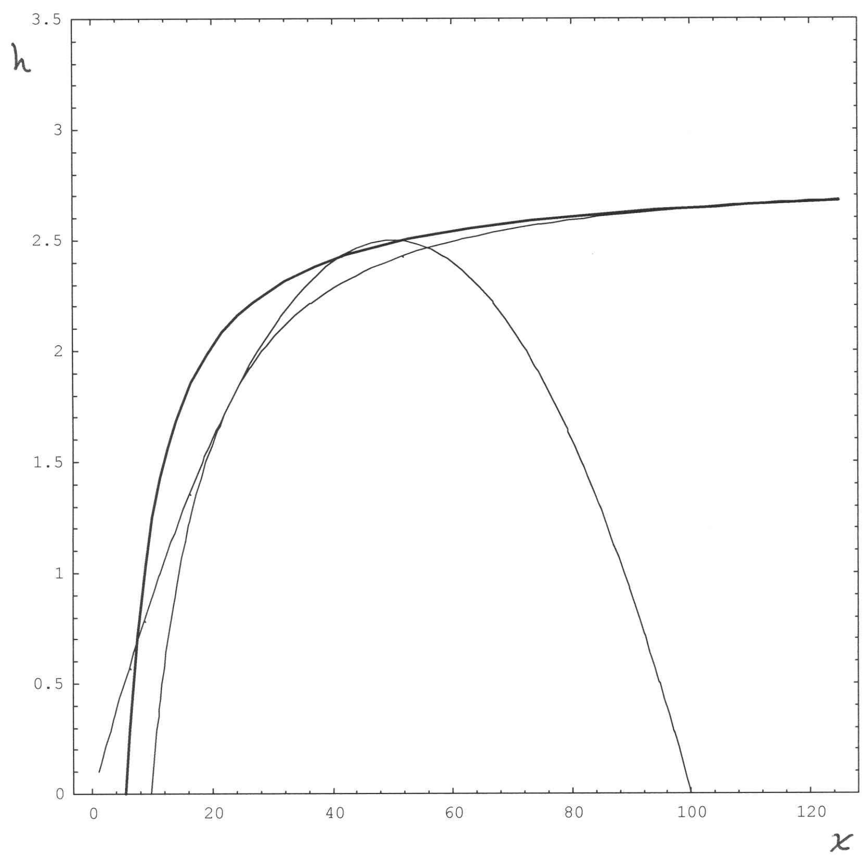

Figure 15B. Figure 15A combined with the finite-time terminal surface. The infinite-horizon behavior implied by studying the finite-horizon behavior in this figure is reflected in Figures 7 and 18.

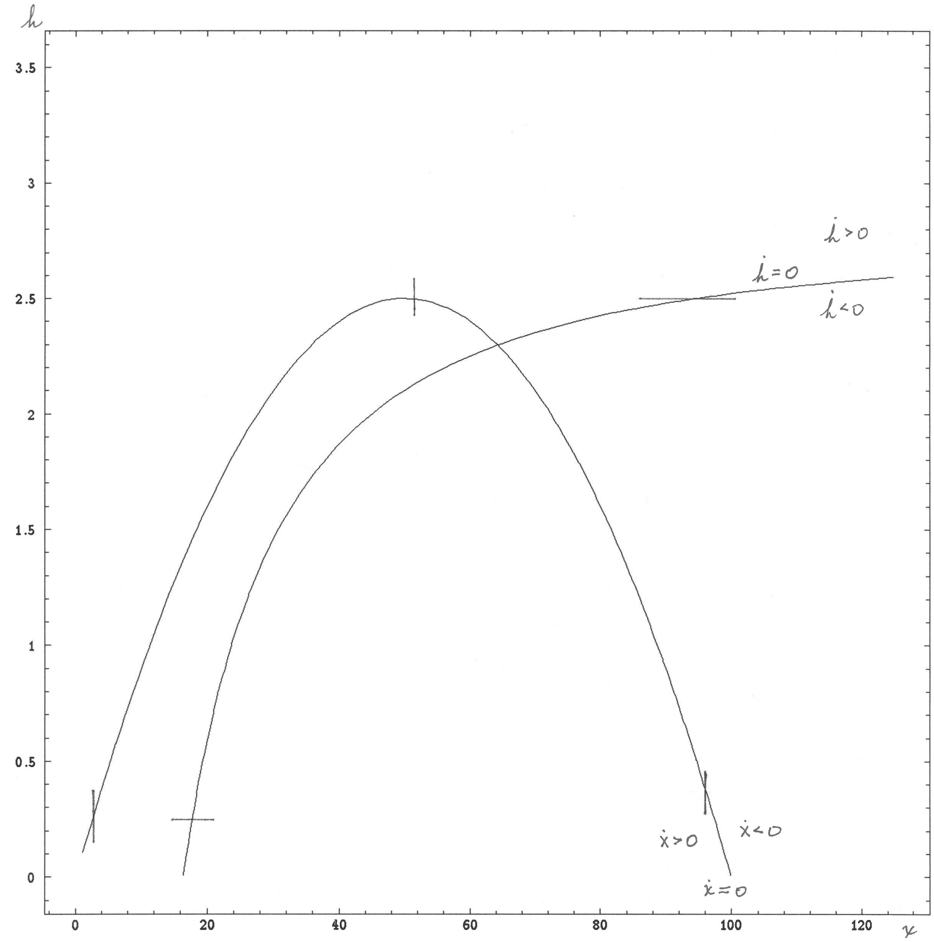

Figure 16A. Demand curve is $p = -\frac{7}{2.8}h + 7$ (equation (67)), corresponding to a choke price of 7 and an intersection in the lower-most "1 intersection" region of Figure 5. Drawing the phase plan paths is left to the reader, who should conclude that the steady-state point is a saddle point whose convergent separatrixes approach from the northeast and southwest. This case represents a lower demand curve than Figure 14A, yet is has a higher h^*_{ss} (and higher x^*_{ss} and lower p), as anticipated in Figure 5, since Figure 14A is along the backward-bending part of the steady state supply curve.

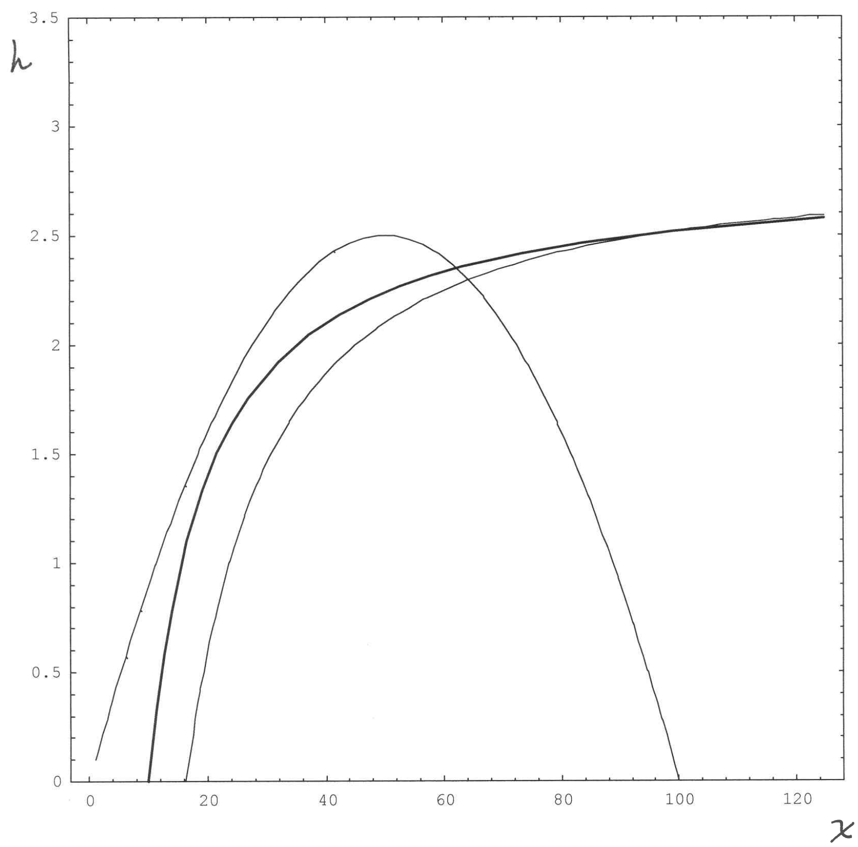

Figure 16B. Figure 16A combined with the finite-time terminal surface. The infinite-horizon behavior implied by studying the finite-horizon behavior in this figure is reflected in Figures 7 and 18.

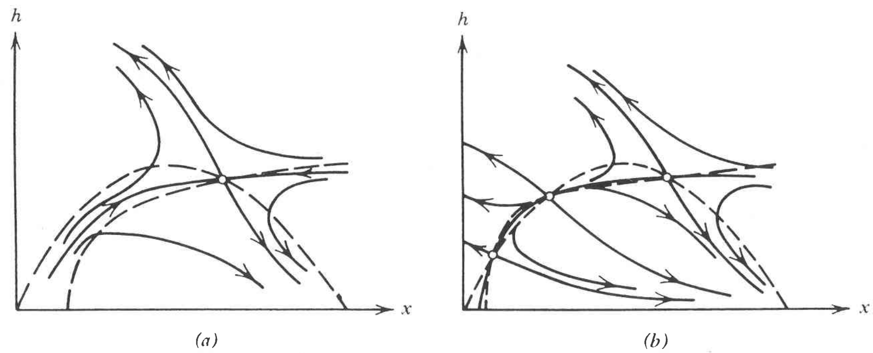

Figure 6.12. Bifurcation of supply and demand equilibria in the nonlinear fishery model.

Figure 17. Clark's Figure 6.12.

which are large qualitative changes resulting from infinitesimally small parameter changes, occur around the choke prices 12.2 and 9. The bifurcations in Figure 18 are "saddle-node bifurcations" (sometimes called "fold bifurcations"). Use Figure 18 and Figs. 12–16 to explain why "saddle-node bifurcations" is an appropriate name ("saddle-repeller bifurcations" would have been an even more appropriate name, but mathematicians do not use it). Also, each of the two bifurcations in Figure 18, the one at $p_c = 9$ and the one at $p_c = 12.2$, correspond to moving from one quadrant of Figure 10 to another; which two quadrants of Figure 10 are they?

Exercise 3. Re-solve "Example 1, dynamics" with a value of δ which is less than $r = 0.1$. Note from Lemma 12 that $\partial\dot{h}/\partial h$ has a more complicated behavior than when $\delta > r$.

Exercise 4. Re-solve "Example 1, dynamics" with demand curves like those in Fig. 6 instead of those in Fig. 5. The phase plane diagram will look as in Figs. 12 and 16. (Why?) As demand pivots clockwise in Fig. 6, there will not be a bifurcation. (Why?)

Exercise 5. Re-solve "Example 1, steady-state" with a growth function $F(x)$ that exhibits critical depensation and show that this results in Clark's Figure 5.18a, p. 144, reproduced as Panel a of Figure 19 here. May extinction result from this model? What is the dynamic behavior like?

Exercise 6. Re-solve "Example 1, steady-state" with a cost function $c(x)$ that has $c(0) < \infty$ and show that this results in Panel b of Figure 19. May extinction result from this model? What is the dynamic behavior like?

Exercise 7. Verify the dependence of the steady-state supply curve on δ which is illustrated in Fig. 5.13, p. 137 of Clark, reproduced as Figure 20 here.

In phase diagrams such as the ones in the Exercises, it is unfortunately not trivial to prove that the convergent separatrix is optimal (that is, that it is optimal to approach the steady state). In Section 3, we pointed out that the transversality conditions (30) and (31) do not help because they are

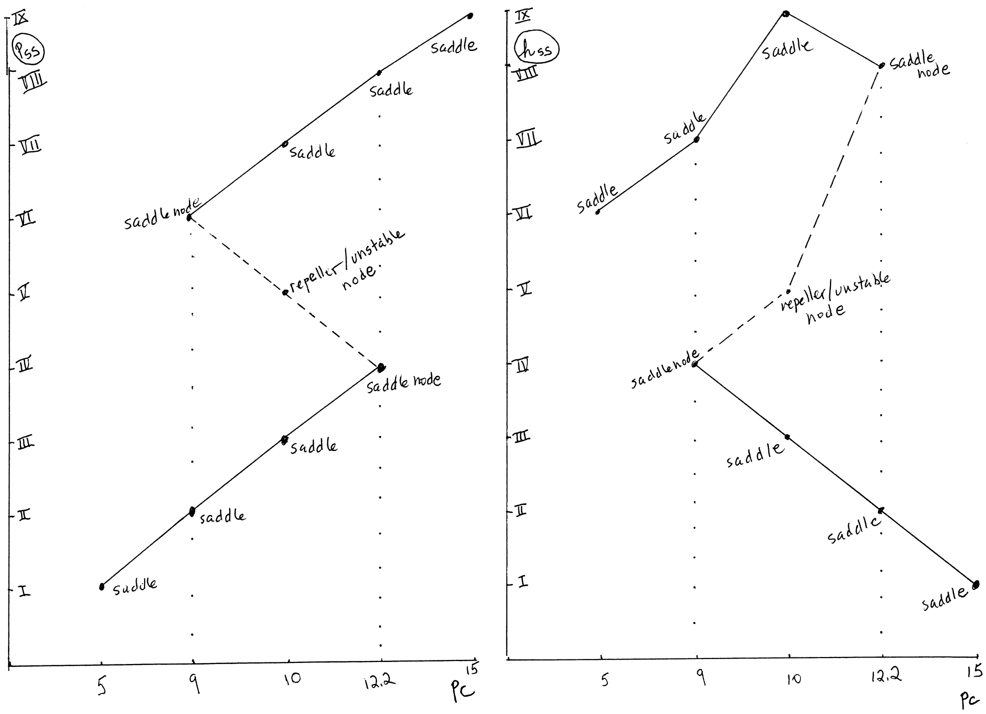

Figure 18. A not-to-scale summary of bifurcations from Figure 7. The horizontal axis of both graphs is the demand curve's choke price, p_c. Bifurcations occur at $p_c = 9$ and $p_c = 12.2$, and are called "saddle-node bifurcations." The dashed lines trace unstable steady state points, as conjectured by a naive stability analysis and as confirmed by the dynamic analysis subsection. The axes are distorted, using evenly-spaced roman numerals instead of actual coordinates, to make the figure more legible.

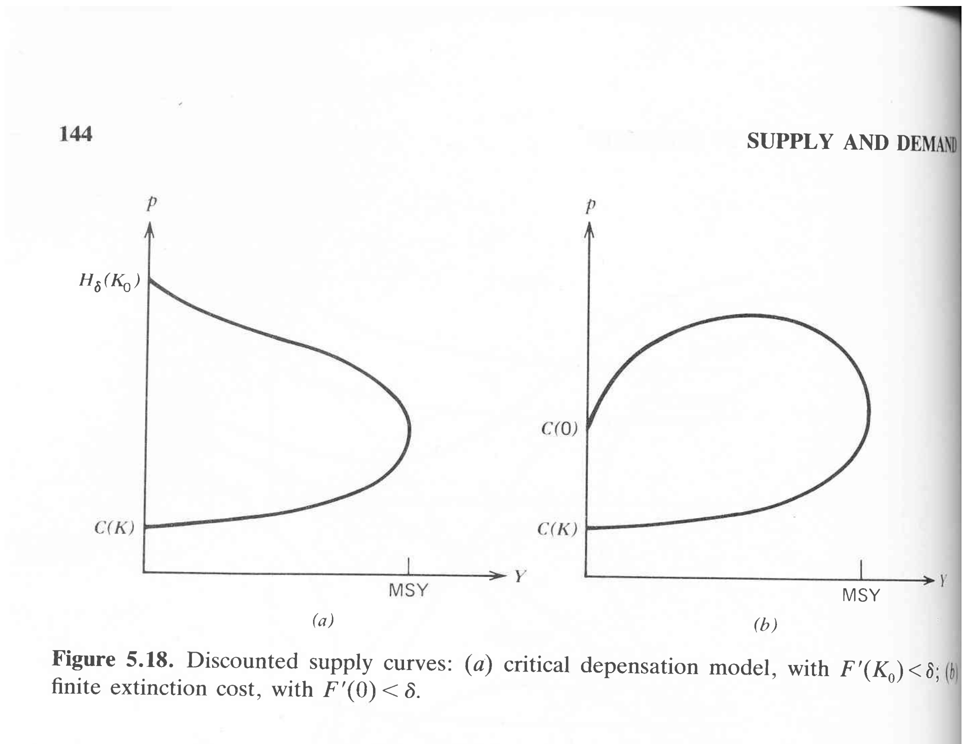

Figure 19. Clark's Figure 5.18.

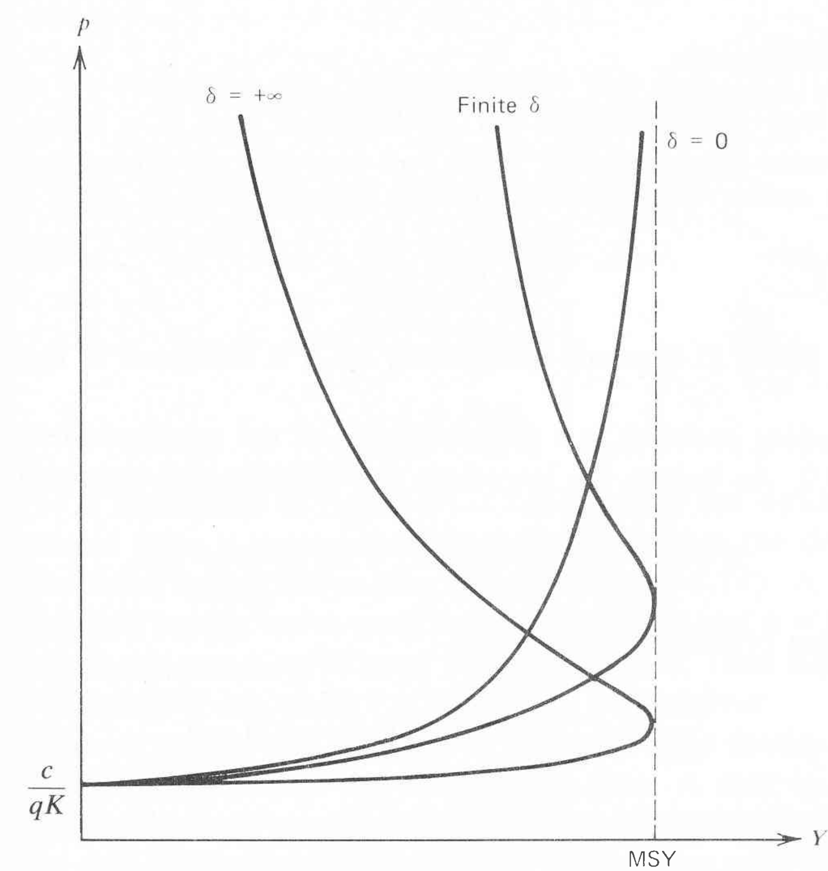

Figure 5.13. Discounted supply curve and its limiting positions.

Figure 20. Clark's Figure 5.13.

inapplicable to the problem of competitive firms, whose problem is non-autonomous. In Section 4, we mentioned Colin Clark's approach to this problem. His precise words were:

> (Is there some way to prove that the optimal solution must approach [the steady state] (x^*, h^*) as $t \to \infty$? This can be done by first looking at the case of a finite time horizon T, and then letting $T \to \infty$. We skip the details.)
>
> —p. 97 of the third (2010) edition

At the end of Section 4 (which assumed perfect competition), we showed that in Lemma 6 that if the final date were not infinity but instead $T < \infty$, then the transversality condition would be that at T, price should equal marginal cost. Using the constant returns to scale assumption of the current section, that means that at T, price should equal average cost, and so profit should equal zero. Imposing equilibrium, this condition is $\phi(h_T) = c(x_T)$. The (x, h) points satisfying this equation form a curve we can call the "finite time terminal surface." With our constant returns to scale assumption, this is also a zero-profit surface. We can plot this curve using *Mathematica*. In the *Mathematica* code given above in "Example 1, dynamics," this is done in the `terminalsurface` lines. Fig. 12B superimposes that set of points (that curve) onto Fig. 12A, and similarly for the "B" versions of the other figures. The size of T determines which point on this finite-time terminal surface is the right one; if T is small, the path from the initial point to the terminal surface will be short (such as "Path C" of Figure 12B), but if T is 5 billion years, it will be long (such as "Path D" of Figure 12B). The greater the value of T, the closer the path has to get to the steady state, where motion slows because the path is so close to the nullclines. This is known as the "turnpike" property (the analogy being that one does not go from H to U in Figure 21 via a straight line, but rather by detouring near—not literally on—the JS "turnpike").[13]

As noted in the preceding paragraph, the finite-time terminal surface in this section is also a zero-profit surface.[14] Starting from a point on it and increasing h while keeping x constant, $p - c = \phi(h) - c(x)$ will fall from

[13]Here, "turnpike" refers to high-speed highway which is the fastest route between two cities even though it is not the shortest route between two cities.

[14]There is another zero-profit surface, at $h \equiv 0$, but it is not a finite-time terminal surface. Proof: on that surface, $\pi = AM\Pi \cdot h$ is zero but $AM\Pi$ is not zero, so, from (22), $\lambda_t = e^{-\delta t} M\Pi_t$ is not zero; but on a finite-time terminal surface $\lambda_T = 0$ from (13), so $h \equiv 0$ is not a finite-time terminal surface.

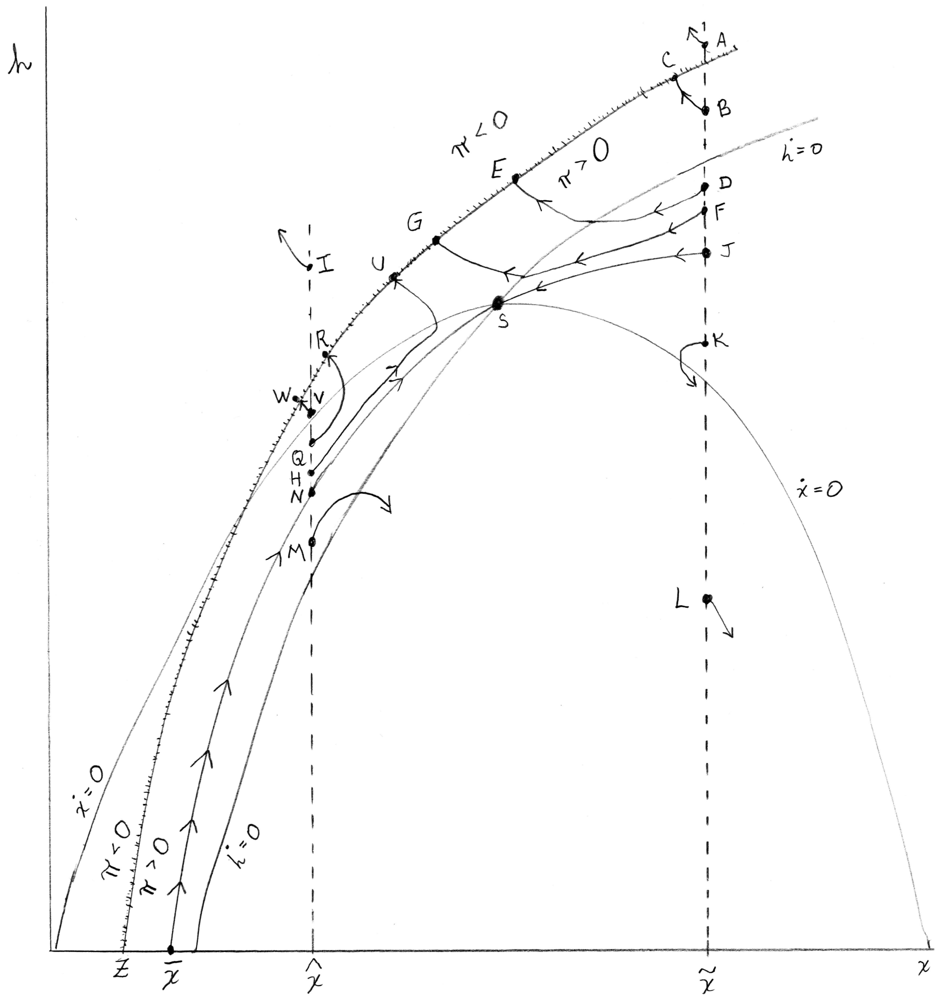

Figure 21. A generic phase-plane diagram loosely based on Figure 12B, equation (63). The finite-time terminal surface is the solid line with small hatch marks dividing the plane into negative-profit and positive-profit areas.

zero to a negative value; so it divides the phase plane into areas of negative profit above it and positive profit below it.

Figure 21 enables a complete analysis. It is a redrawing of Figure 12A, based on (63), distorted in the interest of legibility. Suppose $T < \infty$. If $x_0 = \tilde{x}$, then the paths from A, K, and L cannot be optimal because they cannot reach the finite-time terminal surface. Similarly, if $x_0 = \hat{x}$, then the paths from M and I cannot be optimal because they cannot reach the finite-time terminal surface. From $\tilde{x}$, a small T would result in an optimal path being like BC; for a larger T, an optimal path would be like DE; and for an even larger T, it would be like FG. From $\hat{x}$, a small T would result in an optimal path being like VW; for a larger T, an optimal path would be like QR; and for an even longer T, an optimal path would be like HU.

For very large T, evidently the optimal path is will be extremely close to the convergent separatrix until dates far into the future. We could then either conclude, somewhat heuristically, that for $T = \infty$, the convergent separatrix is optimal; or we could conclude that for very large T, the optimal path is so close to the convergent separatrix for so long that any eventual differences between them have no economic importance and can be ignored. So from now on, we will take the convergent separatrix as being the optimal path when T is infinite, which is our usual case.

This proves that NS or JS are optimal, except for the case of x_0 being very small.

Lemma 13. *Let $\bar{x}$ denote the point at which the convergent separatrix intersects the x axis. If $x_0 < \bar{x}$, the optimal strategy is to set $h^* = 0$ for an initial period of time. This path follows the x axis going to the right. Once x reaches $\bar{x}$, h^* follows the convergent separatrix.*

Proof. We first show that a plan having an initial period with $h^* = 0$ is consistent with the mathematics of Optimal Control Theory. Recall that our phase-plane paths were constructed from (17) and (74); the latter came from (70), which came from (21), which came from (5), which came from assuming an interior (or, in the case of $\mathscr{H}$ linear in h as we have, a "singular") solution. Therefore there is indeed an alternative to following a phase-plane path, namely by *not* assuming an interior (technically, a "singular") solution. That means setting the control to be at its minimum or maximum allowed value (which is called taking a "most rapid approach path," abbreviated "MRAP"); in our case, it means setting $h_t^* = 0$. This confirms that the equations of Optimal Control Theory do allow one to set $h_t^* = 0$ for some time.

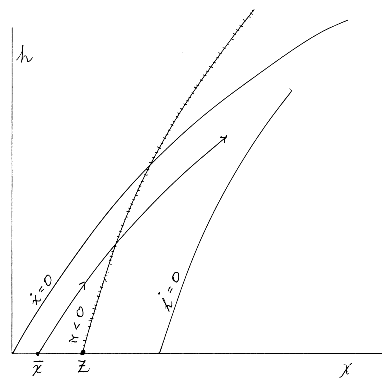

Figure 22. This situation with $\bar{x}$ < Z cannot occur.

To determine when the initial no-harvest period will end, first note that during it, x rises because there is no harvesting.

Proposition 2 requires that h_t be continuous. If h were kept at zero while x grew beyond $\bar{x}$, the only way at any future date to get to the convergent separatrix would be for h to jump up, which is not allowed. If h were kept at zero for such a short time that x had not yet reached $\bar{x}$, there would be no way to adjust h so as to be on the convergent separatrix, because the convergent separatrix does not exist at such small levels of x. So h should be kept at zero just until x reaches $\bar{x}$. ▮

We also need to rule out a geometry which is different from Figure 21's.

Lemma 14. *Let Z be the point at which the finite-time terminal surface intersects the x axis. Then $Z < \bar{x}$.*

Proof. Suppose not; then Figure 22 holds. Recall (as shown in Figure 12B) that the region above and to the left of the finite-time terminal surface is a region of negative profit. This region is so marked ("$\pi < 0$") in Figure 22.

As the proof of Lemma 13 pointed out, the path of h_t^* has to be continuous, and the only continuous phase-plane path which sets $h = 0$ over an initial interval and then joins the convergent separatrix has to join the convergent separatrix at $\bar{x}$. Once that has happened, in Figure 22, following the convergent separatrix to the right of $\bar{x}$ leads through the $\pi^* < 0$ region. However, according to Lemma 2, $h^* > 0$ is incompatible with $\pi^* < 0$ (recalling from Lemma 7 that we know that $\lambda_t^* \geq 0$). This contradiction establishes the proof. ▮

Presumably, if one used a computer to calculate the location of $\bar{x}$, one would find that it satisfies this lemma.

Rather remarkably, despite the nonlinearity of the system, we can find *analytical* expressions for the time paths of all the variables (harvest, price, stock size, and the adjoint variable) during any initial period of no harvesting. (This presumes one first knows the coordinate of $\bar{x}$, which requires using a computer.) This result is not very important, however, because Lemma 13 already described the qualitative features of this path.

Lemma 15. *If $x_0 < \bar{x}$, the optimal plan is to set $h_t^* = 0$ until the switching time t_s implicitly defined by*

$$\bar{x} = \frac{K}{1 + \frac{K - x_0}{x_0} e^{-rt_s}} . \tag{80}$$

For $t < t_s$, price is (an arbitrary path which remains greater than or equal to) $\phi(0)$. For $t < t_s$,

$$x_t^* = \frac{K}{1 + \frac{K - x_0}{x_0} e^{-rt}} . \tag{81}$$

Using the value of c_2 implicitly defined by

$$e^{-\delta t_s}[\phi(0) - c(\bar{x})] = \exp\left\{2r\left[t_s + \frac{\ln(1 + \frac{K - x_0}{x_0} e^{-rt_s})}{r}\right] - rt_s + c_2\right\} , \tag{82}$$

for $t < t_s$ the time path of λ_t^* is

$$\lambda_t^* = \exp\left\{2r\left[t + \frac{\ln(1 + \frac{K - x_0}{x_0} e^{-rt})}{r}\right] - rt + c_2\right\} . \tag{83}$$

Proof. During the period of time when harvest is kept at zero, it is sufficient to describe the time path of price by saying that it will be constant at $\phi(0)$. Technically, however, price could follow a completely arbitrary path greater than or equal to $\phi(0)$; all that matters is that price remains always at or above the choke price, so that quantity demanded is zero.

Next, note that whenever $h_t^* = 0$, (28) gives $e^{-\delta t} M\Pi_t \le \lambda_t$. In market equilibrium this is

$$e^{-\delta t}[\phi(0) - c(x_t)] \le \lambda_t . \tag{84}$$

Also, $h = 0$ implies $\dot{\lambda}_t = -\partial\mathcal{H}/\partial x = -\partial[e^{-\delta t}(p - c)h + \lambda(F - h)]/\partial x = -[-e^{-\delta t} c' h + \lambda F'] = -\lambda_t F'(x_t)$. From Lemma 7), $\lambda_0 > 0$ as long as fishing will occur at some point; and since for small x, $F' > 0$ because we have assumed logistic growth, we have $\dot{\lambda}_0 = -\lambda_0 F'_0 < 0$, so the right-hand side of (84) starts by falling. When $h = 0$, the left-hand side of (84) rises: the proof begins with $\partial e^{-\delta t}[\phi(0) - c(x_t)]/\partial t = -\delta e^{-\delta t}[\phi(0) - c] - e^{-\delta t}c'\dot{x} = -\delta e^{-\delta t}[\phi(0) - c] - e^{-\delta t}c'F = -e^{-\delta t}\{\delta[\phi(0) - c] + c'F\}$, then notes that $c'F < 0$ and $\phi(0) - c < 0$ for points such as Y that lie to the left of the finite-time terminal surface. Hence the left-hand side and the right-hand side are getting closer, and at some date, in order not to have h being equal to zero forever, (84) will be met with equality, and h will cease being zero. That time is called the "switching time," which we denote by t_s. To calculate t_s, note that while $h = 0$, (17) and (15) give $\dot{x} = rx(1 - (x/K))$. (In this exercise, $K = 100$ and $r = 0.1$.) The solution of this nonlinear differential equation is (81). To find the switch time t_s, set x_t in (81) equal to $\bar{x}$ and solve for the corresponding t. That will be t_s, and proves (80). The equation (80) is simple enough that t_s could be expressed analytically, an exercise left to the reader.

Meanwhile, when $h = 0$ we saw above that $\dot{\lambda} = -\lambda F'$; dividing by λ and substituting $F' = r - \frac{2r}{K}x$ and substituting for x_t from (81) ago gives

$$\frac{d\lambda/dt}{\lambda} = \frac{2r}{K}x - r = \frac{2r}{K}\frac{K}{1 + \frac{K-x_0}{x_0}e^{-rt}} - r\,. \tag{85}$$

The solution to this nonlinear differential equation is obtained by moving dt to the other side and integrating both sides, which turns out to give

$$\ln\lambda = 2r\left[t + \frac{\ln(1 + \frac{K-x_0}{x_0}e^{-rt})}{r}\right] - rt + c_2 \tag{86}$$

for some constant c_2. (I did the difficult integral with *Mathematica*, but it is easy to verify its correctness by differentiating.) So the right-hand side of (84) follows (83). Evaluating (83) at t_s must yield, by (84), $e^{-\delta t_s}[\phi(0) - c(\bar{x}]$. That yields (82), which is one equation in the one unknown, c_2, and is simple enough that c_2 could be determined analytically, an exercise left to the reader. ∎

The reason we were able to analytically solve the pre-t_s period was that we (with *Mathematica*) were able to solve two nonlinear differential equations. This was only possible because we got lucky: $h = 0$ resulted in easy equations. The post-t_s specification cannot be analytically solved, even using *Mathematica 6.0*. Moreover, we need to use the post-t_s specification to find the coordinate of $\bar{x}$, which is used in this lemma.

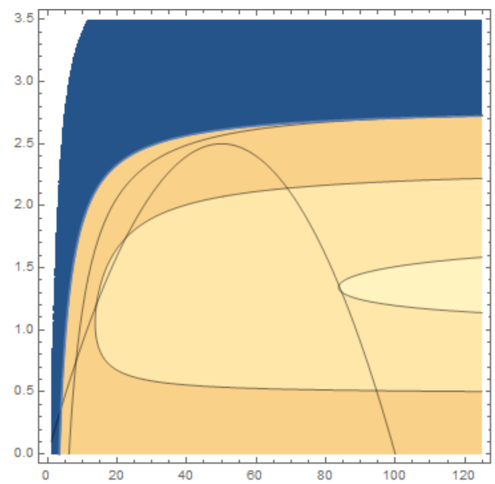

Figure 23. Isoprofit lines for Figure 21. Blue is negative profit, and profit increases with lighter colors.

Path JS is rather close to the zero-profit surface; one may wonder why it is optimal, suspecting that there are other feasible points which generate more profit. Figure 23 shows that the highest steady-state profit occurs on the $F(x)$ function at approximately $x = 80$, $h = 1.3$. If for example $x_0 = 80$ then it would be feasible to stay at that point forever, earning, at each date, more profit than at any date on Path JS. However, Figure 23 shows profit taking the demand curve into account, but the competitive firm has no idea where the demand curve is. The $(x, h) \approx (80, 1.3)$ point may maximize profit for a monopolist (whether it does or not is not relevant here), but the competitive firm has to maximize profit given the exogenous prices it faces, and Path JS is the correct answer to that problem.

We have extensively discussed Example 1, based on one cost function and various demand curves, originating from Figure 5, but we have not said anything in this subsection about a dynamic analysis for Figure 9, based on one demand curve and different cost functions. The main reason is that it was easy to parameterize the difference in the demand curves of Figure 5, since they only differ in one parameter, their choke prices, but I drew the different cost functions in Figure 9 freehand, without any analytical expressions, let alone expressions which differed in just one parameter. It would be interesting to investigate how different cost functions affect the dynamic behavior of the system, but search fisheries have complicated cost functions (even if one assumes constant returns to scale), so I have not done this. In the next section, on schooling fisheries, the cost functions are simpler, and there I do analyze how different costs affect the dynamic behavior of the system.

It is appropriate to close this section with a note about how other authors approach this constant-returns-to-scale, private-property fishery. Most other analyses (e.g., Caputo op. cit. p. 137) assume that price p_t is fixed in time. This implicitly means either that one just wishes to derive the steady-state supply curve and leave determination of equilibrium to further work (note that there is no analogous procedure for the dynamic analysis), or that the actual market demand curve for this fish is horizontal, which would violate consumers' budget constraints. To analyze this case, start with assuming an interior (or, technically, a singular) solution; then (5) characterizes the optimum, and it leads to (70). When p is fixed in time, the $\dot{p}$ term in (70) drops out, and (70) becomes the following, for a constant value of x called $x^\dagger$ which implicitly defined by (87):

$$\delta = F'(x^\dagger) + \frac{0 - c'(x^\dagger)\,F(x^\dagger)}{p - c(x^\dagger)}\,. \tag{87}$$

If $x_0 \neq x^\dagger$, the system has to get from x_0 to $x^\dagger$ in the beginning. If $x^\dagger \neq x_T$, the system has to get from $x^\dagger$ to x_T at the end. In such cases, the system has to spend some time off of the interior (technically, the singular) path which is given by (87). The only alternative to (70) is following a noninterior (technically, a nonsingular) path, i.e., setting harvest to be one of its extreme values, either $h = 0$ or $h = \max h$ (which we have taken to be infinite, but which in reality would be finite). Hence the solution is to follow a MRAP from x_0 to $x^\dagger$, then follow the interior (singular) path $x_t = x^\dagger$ as long as possible, then follow a MRAP from $x^\dagger$ to x_T.

Section 6. Private-Property Competition: Schooling Fisheries and Constant Returns to Scale

The next pair of examples is

$$\pi(x, h) = [\,p - c\,]\,h \quad \text{and} \tag{88s}$$

$$\pi(x_t, h_t, t) = [\,p_t - c\,]\,h_t\,. \tag{88d}$$

As noted when discussing (35), this average cost function $c(x_t, h_t) = c$ is appropriate for "schooling" fisheries; the average cost function for (35), which was $c(x_t, h_t) = c(x_t)$, is appropriate for "search" fisheries. In working out case (88) you should start with (20), because it is unknown whether the additional assumptions used to derive (21) (namely $M\Pi \neq 0$) and (42) (namely $\delta - F'(x) \neq 0$) hold in this section. You should find that

$$\dot{h} = \frac{[\delta - F'(x)]\,[\phi(h) - c]}{\phi'(h)}, \tag{89}$$

which together with (17) governs the schooling fishery phase-plane diagrams. Figures 26–29 can help you work through these cases. The following notation is used in those graphs, where $D(p)$ denotes the market demand curve:

x_δ as in (44);
$h_\delta = F(x_\delta)$ as in (47);
$\hat{h}$ such that $\phi(\hat{h}) = c$ or $\hat{h} = D(c)$.

You should find that when, among other conditions, $\delta > r$, it is possible for no steady state to exist. That may imply extinction.[15]

Figure 30 is a summary of some aspects of Figures 26–25 and Figure 31 is a summary of the similar aspects of Figures 28–29. The governing equation for the steady state is

$$[\delta - F'(x_{ss})]\,[\phi(F(x_{ss})) - c] = 0\,. \tag{90}$$

This is similar to (68) for a search fishery (the left-hand sides of the two equations are the same and the right-hand sides are different), but (90) is easier to analyze because it is satisfied wherever $\phi(F(x_{ss}))$ is exactly equal to c, and because c is a constant here, instead of being a declining function as it was for search fisheries. In search fisheries, there was no one single diagram that showed all the different possible types of behavior: Figure 18 only applied to demand curves which were all linear and all had an h-coordinate of 2.8 and was analyzed with identical costs, and Figure 9, which had one demand curve and different costs, could not even be connected to a bifurcation diagram because the difference in costs could not be summed up in a single parameter. By contrast, in schooling fisheries, Figures 30 and 31 make clear that only one parameter is needed to summarize the different situations, and that parameter, $\hat{h} = D(c)$, combines both the demand side through D and the supply side through c. The details of what the rest of the

[15] Extinction can occur in cases (35) but not with the particular functional form we chose for $c(x)$ in Example 1, namely $c(x) = \gamma/(qx)$ with the traditional constant "q" being set equal to one, because with that functional form the marginal cost of driving x to zero is infinite (see Exercises 4 and 5 above).

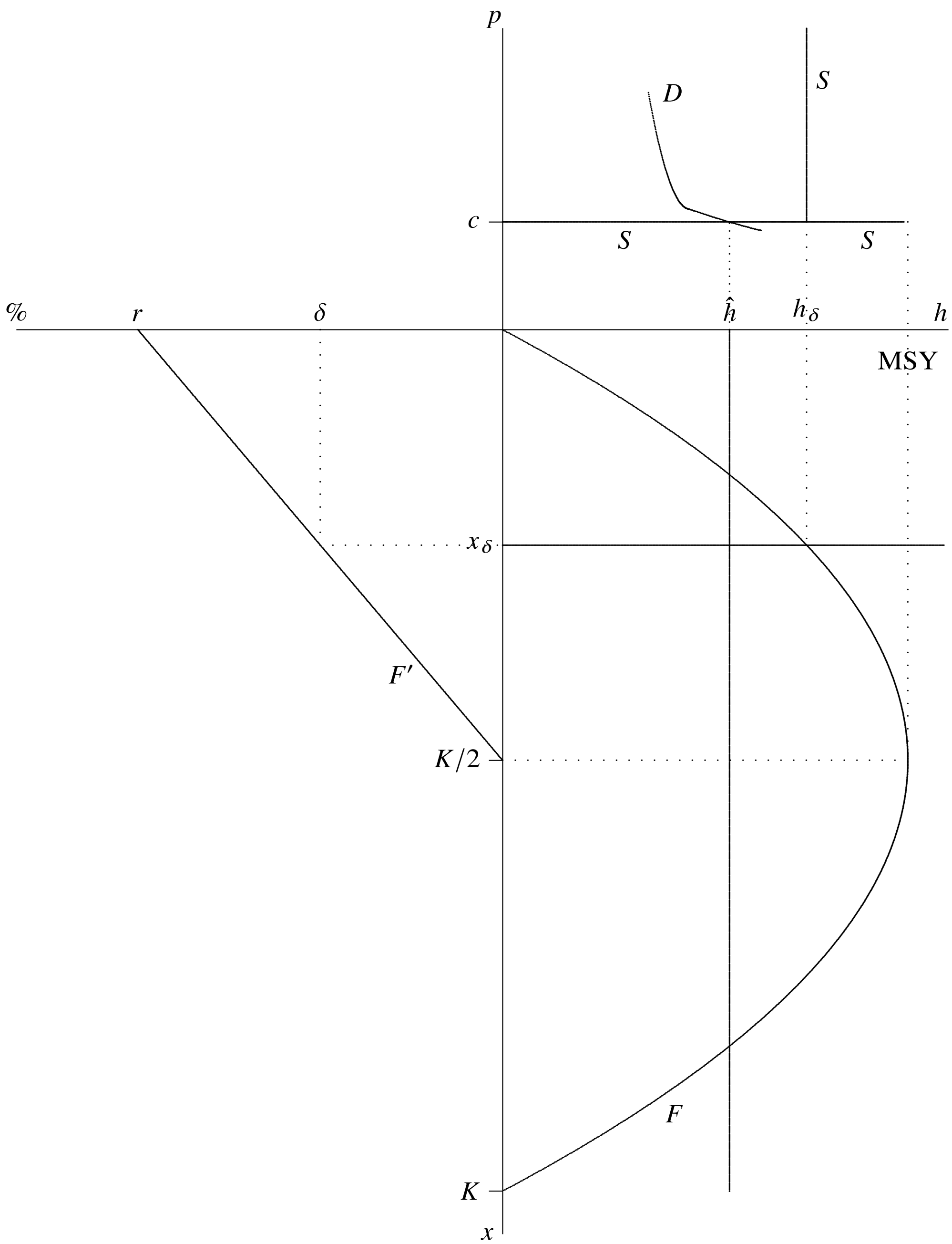

Figure 24. The steady-state supply curve is denoted by "S." At h_δ, profit is negative. Here $\delta < r$ and $D(c) = 28 = \hat{h} < h_\delta$.

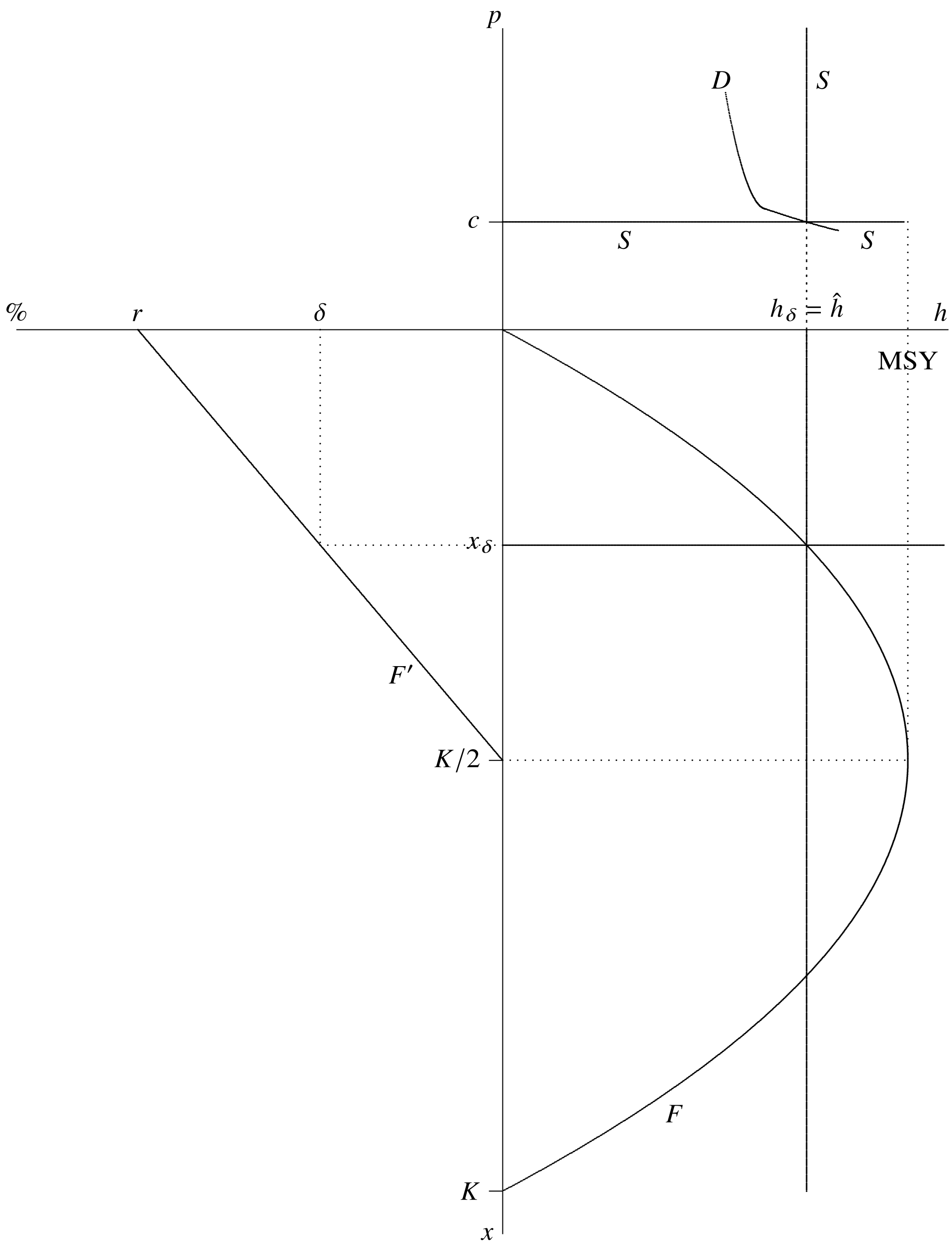

Figure 25. The steady-state supply curve is denoted by "S." Here $\delta < r$ and $D(c) = 37.5 = \hat{h} = h_\delta$.

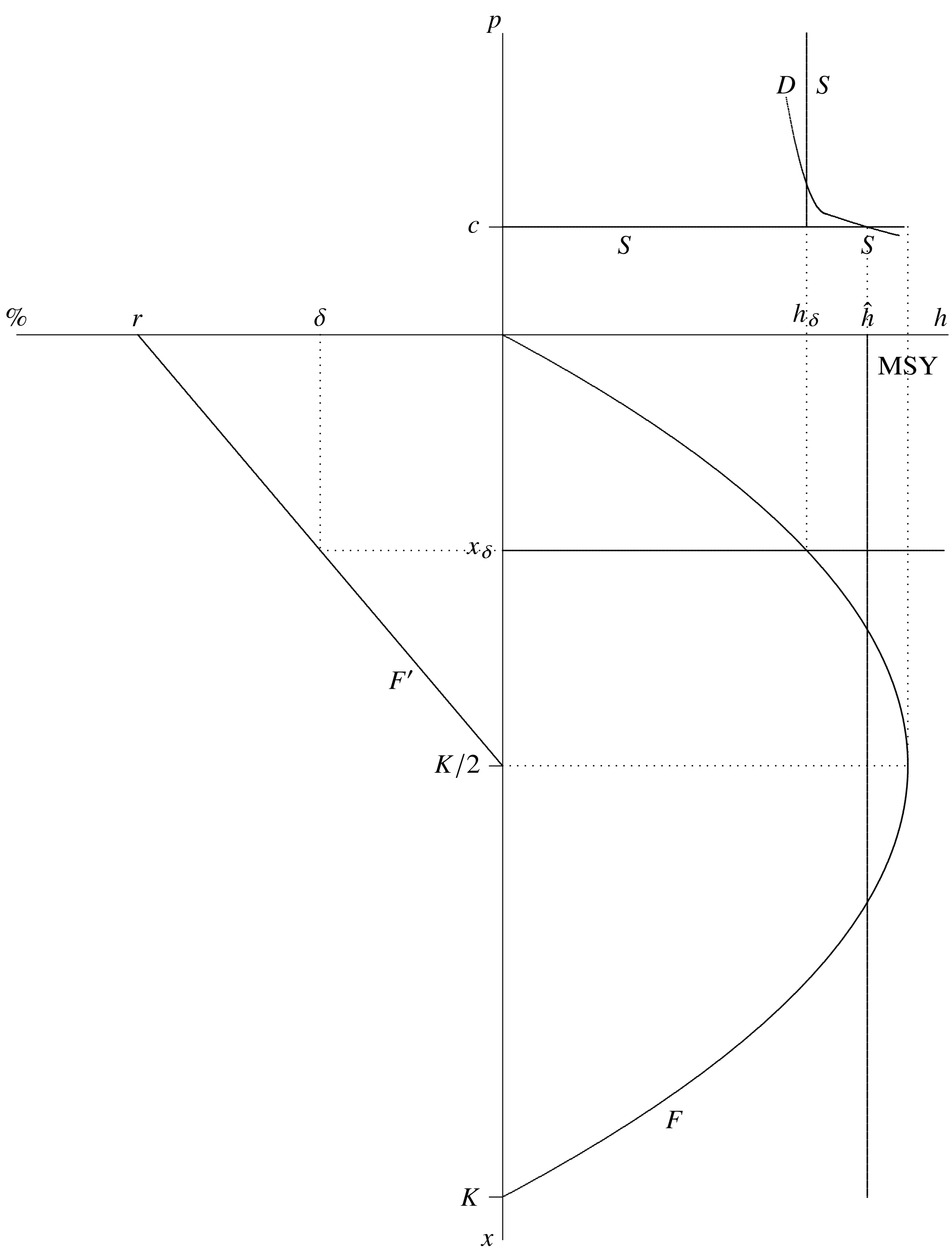

Figure 26. The steady-state supply curve is denoted by "S." Here $\delta < r$ and $D(c) = 45 = \hat{h} > h_\delta$ but $\hat{h} <$ MSY.

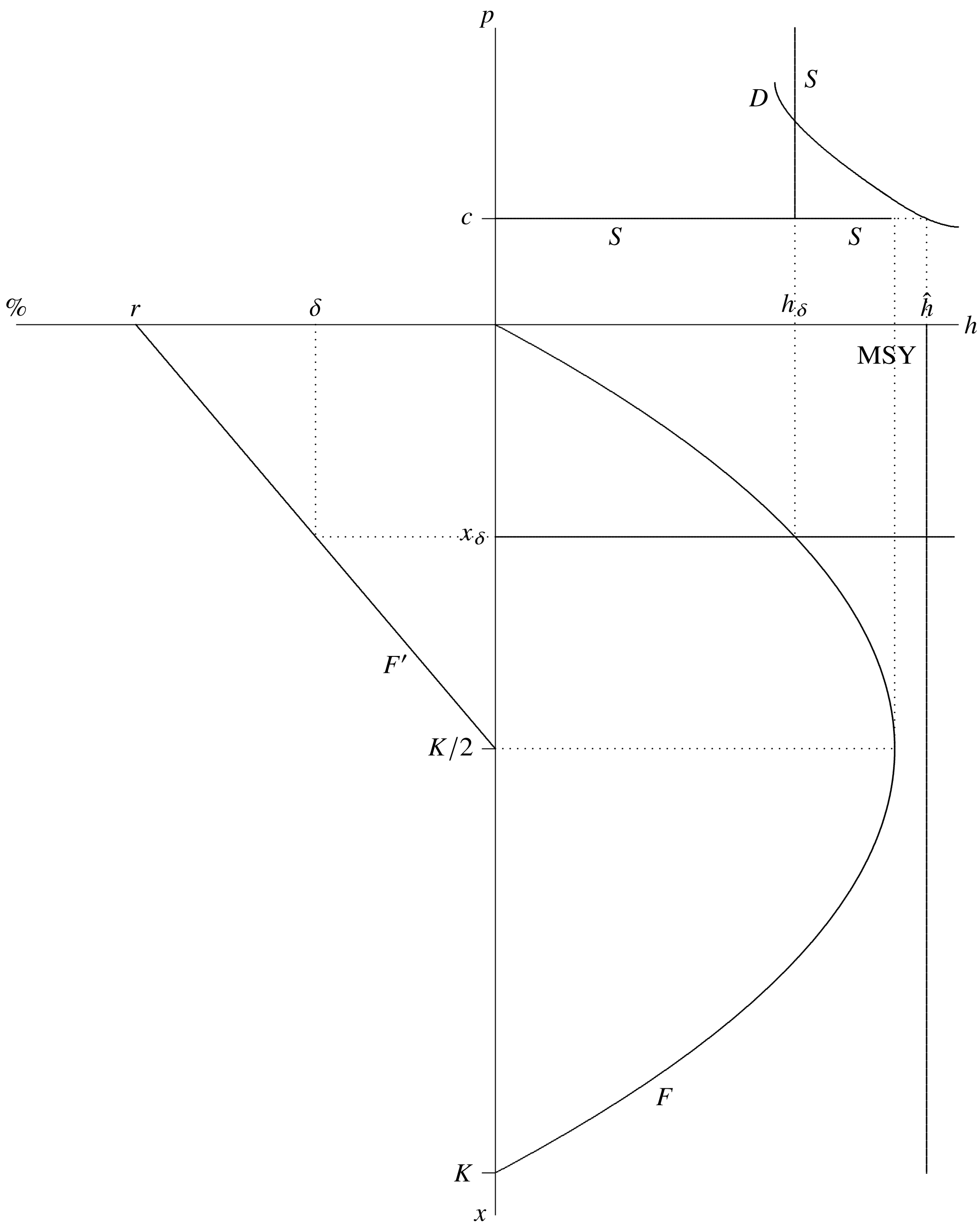

Figure 27. The steady-state supply curve is denoted by "S." Here $\delta < r$ and $D(c) = 54 = \hat{h} > h_\delta$ and $\hat{h}$ is bigger than this figure's MSY, which is 50.

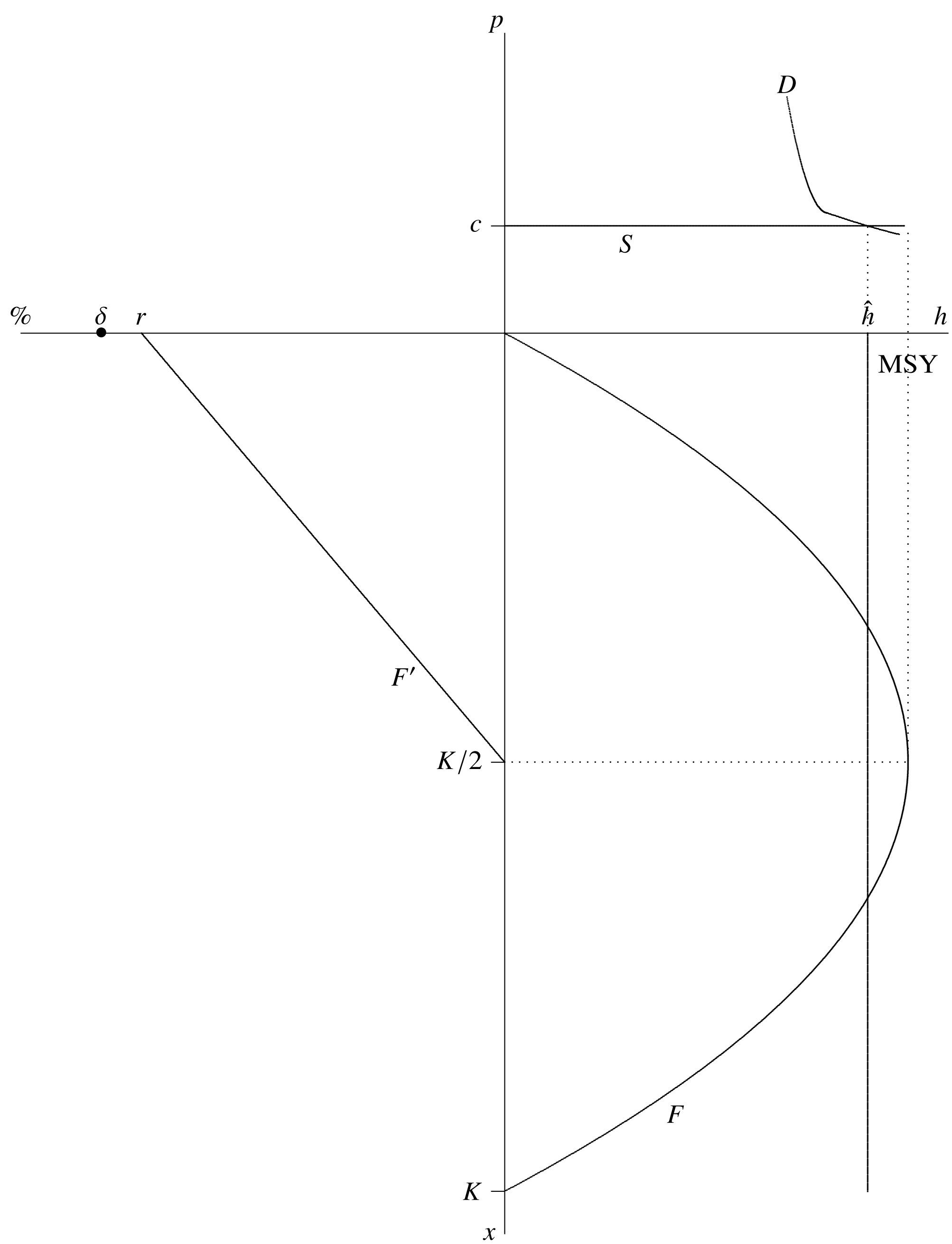

Figure 28. The steady-state supply curve is denoted by "S." Here $\delta > r$ and $D(c) = 45 = \hat{h} <$ MSY.

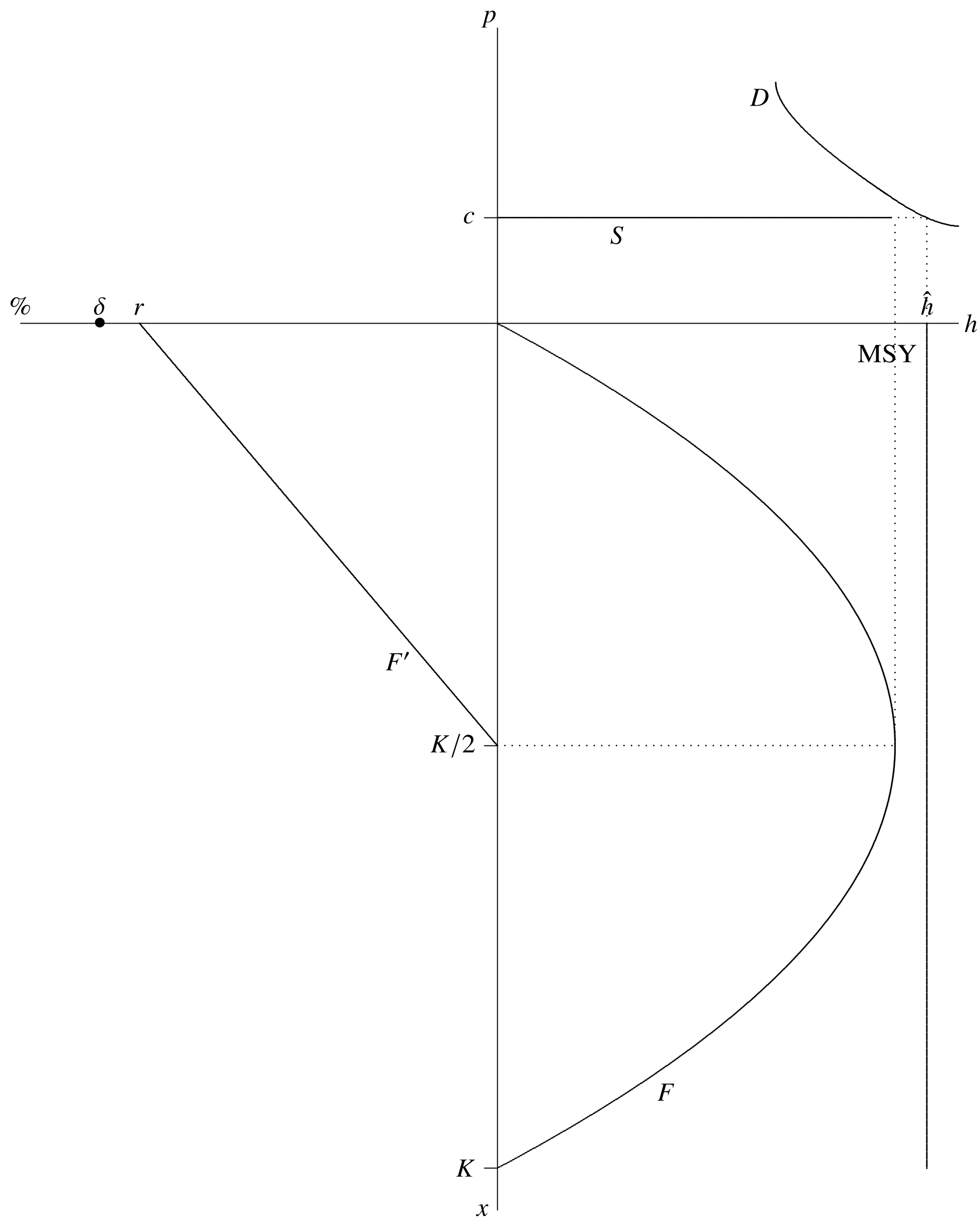

Figure 29. The steady-state supply curve is denoted by "S." Here $\delta > r$ and $D(c) = 54 = \hat{h}$, which is bigger than this figure's MSY, which is 50.

demand curve looks like are irrelevant to Figures 30 and 31. Those figures of course do require the very simple cost structure of schooling fisheries, however.

Another pair of examples is

$$\pi(x, h) = [\, p - c(x, h)]\, h \quad \text{and} \tag{91s}$$

$$\pi(x_t, h_t, t) = [\, p_t - c(x_t, h_t, t)]\, h_t. \tag{91d}$$

This is the most general form of the competitive firm's problem.

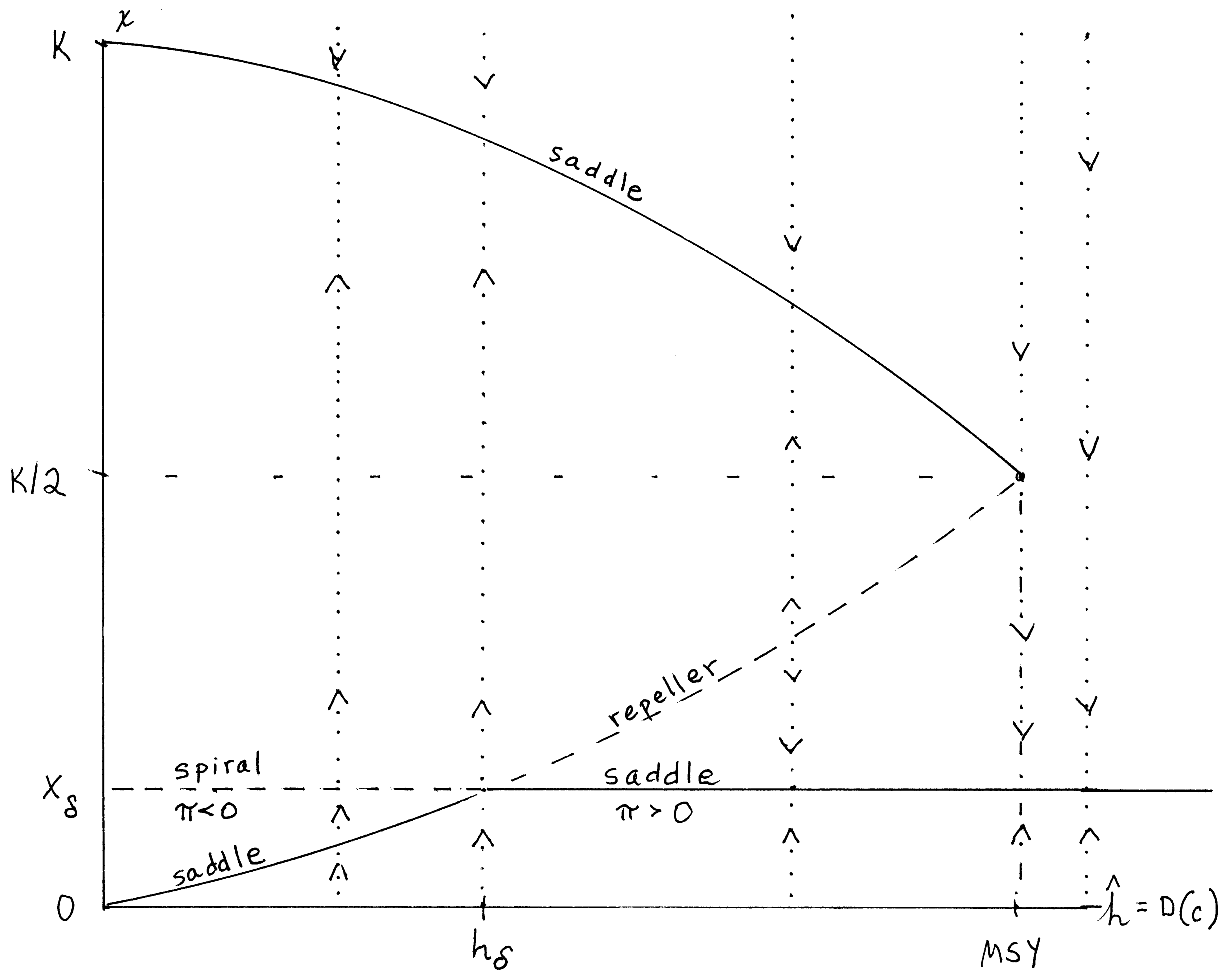

Figure 30. A summary of how steady-state points change with $\hat{h} = D(c)$ in Figures 26–25, which cover the case of $\delta < r$. The part to the left of h_δ is from Figure 24; the part at h_δ is from Figure 25; the part between h_δ and MSY is from Figure 26; and the part to the right of MSY is from Figure 27. The arrows are not phase paths, but do show how x, beginning at different values of x_0 would change with time, depending on where $\hat{h}$ lies.

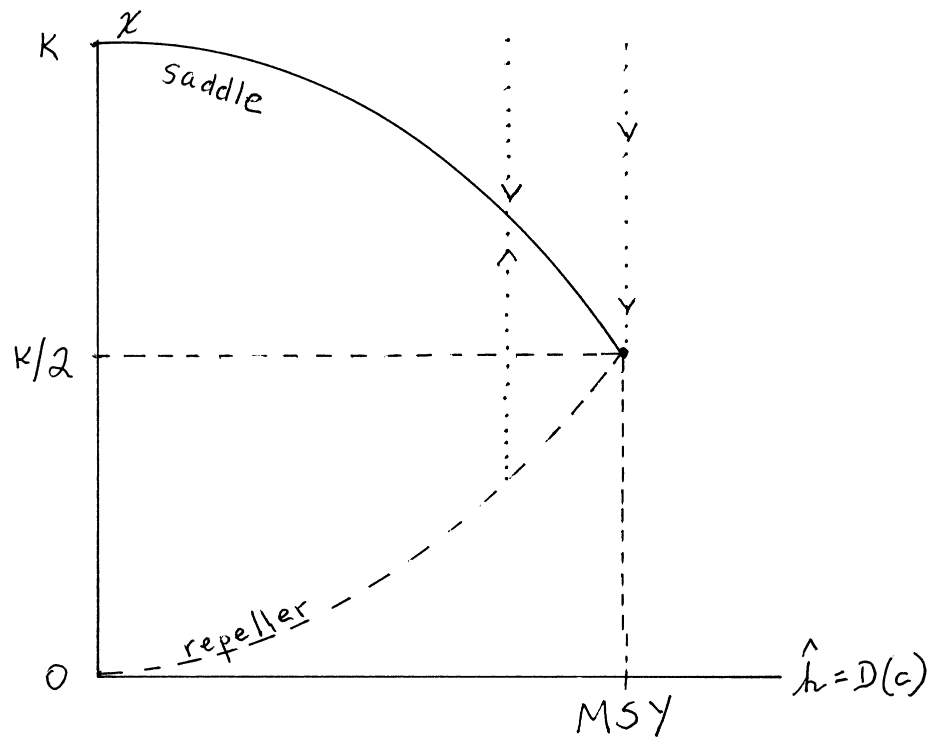

Figure 31. A summary of how steady-state points change with $\hat{h} = D(c)$ in Figures 28–29, which cover the case of $\delta > r$. The part to the left of MSY is from Figure 28; the part to the right of MSY is from Figure 29. The arrows are not phase paths, but do show how x, beginning at different values of x_0 would change with time, depending on where $\hat{h}$ lies. I have not determined behavior below the "repeller" line, or behavior to the right of MSY.

Section 7. Monopoly

In the case of monopoly,

$$\pi(x_t, h_t, t) = [\,\phi_t(h_t) - c(x_t, h_t, t)]\, h_t \tag{92}$$

is the most general form of the problem, where $\phi(h)$ is the market demand curve. (Compare with (91). Also note that before (74), imposing $p_t = \phi(h_t)$ was the last step, whereas here it is the first step.) Just as in the elementary case of a producible good, the monopolist has no supply curve; there is no relationship mapping p to h because the monopolist never takes p as given. Therefore, there is no steady-state analysis for the monopolist analogous to that giving rise to steady-state supply curves for the competitive industry.[16]

Substituting (92) into (21) gives the general solution for the monopolist. Instead of exhibiting this, consider the following special case of (92):

$$\pi(x_t, h_t) = [\,\phi(h_t) - c(x_t)]\, h_t\,. \tag{93}$$

This uses the same cost function as (35). Let *TR* be total revenue $\phi(h_t)\, h_t$ and *MR* be marginal revenue $d\,TR/dh$. One has $M\Pi = MR(h) - c(x)$, $\dot{M\Pi} = \dot{MR} - \dot{c}$, $\partial\pi/\partial x = c'(x)\, h$, and so substituting (93) and these results into (21) yields

$$\delta = F'(x_t) + \frac{\dot{MR} - \dot{c}(x_t) - c'(x_t)\, h_t}{MR - c(x_t)}\,.$$

But

$$\begin{aligned} MR &= \tfrac{d}{dh}[\phi(h)\cdot h] = \phi'(h)\cdot h + \phi \;\Rightarrow \\ \dot{MR} &= \phi''\dot{h}h + \phi'\dot{h} + \phi'\dot{h} \\ &= \phi''\dot{h}h + 2\phi'\dot{h} \end{aligned}$$

and

$$\dot{c} = \tfrac{d}{dt}c(x) = c'(x)\cdot\dot{x}\,,$$

so

$$\delta = F' + \frac{\phi''\dot{h}h + 2\phi'\dot{h} - c'\dot{x} - c'h}{\phi' h + \phi - c}\,;$$

[16] There is a relationship mapping $\phi(h)$ to h for the monopolist, but this is not a supply curve (which maps $\mathbf{R}^1$ to $\mathbf{R}^1$) but a supply functional (mapping a function space into $\mathbf{R}^1$). Any given demand curve $\phi_1(h)$ does have a point (call it (p_1, h_1)) which the monopolist would choose, but this is not a point on a supply curve because one could draw another demand curve $\phi_2(h)$ through the same (p_1, h_1) and with ϕ_2 the monopolist might not choose to produce at (p_1, h_1). All this is the same as in the case of a static producible resource.

gathering the $\dot{h}$ terms and noting that $-c'\dot{x} - c'h = -c'F$ from (17) yields

$$\delta = F' + \frac{(\phi'' h + 2\phi')\dot{h} - c'F}{\phi' h + \phi - c} \,.$$

Solving this last equation for $\dot{h}$ gives the following dynamic system:

$$\dot{h}_t = \frac{[\delta - F'(x_t)][\phi'(h_t)h_t + \phi(h_t) - c(x_t)] + c'(x_t)\,F(x_t)}{\phi''(h_t)h_t + 2\phi'(h_t)} \tag{94}$$

$$\dot{x}_t = F(x_t) - h_t \,. \tag{17}$$

(Compare this with the system formed by (74) and (17) in Section 5.)

Exercise. Re-work "Example 1, dynamics" for the case of a monopolist. Use the same functional forms $c(x)$ and $F(x)$ as in "Example 1, dynamics," together with the same parameters q, γ, r, K, and δ. Also use the same demand curves, (63)–(67). Once you have found the steady-state price-quantity combination, sketch a graph showing each demand curve and marking the steady-state price-quantity combination on it (obviously there is no steady-state monopoly supply curve). Compare this graph with Figure 5 and especially with the graph from Exercise 2 of Section 5.

Exercise. In (74), show that the $\dot{h} = 0$ isocline is characterized by $dh/dx > 0$ and $d^2h/dx^2 < 0$. Is this true for the $\dot{h} = 0$ isocline under (94)?

Section 8. Competitive, Open Access Fishery: Steady States

In open access steady-state equilibrium, total profit is zero: $0 = \pi(x, h) = ph - c(x, h)\,h$ (so $p = c(x, h)$). Also, $0 = \dot{x} = F(x) - h$. Combining these equations yields

$$\begin{aligned} p &= c(x, F(x)) \\ h &= F(x). \end{aligned} \tag{95}$$

If, as a special case, we have constant average cost (hence constant returns to scale and average cost equals marginal cost), then $c(x, h) = c(x)$ and (95) becomes

$$\begin{aligned} p &= c(x) \\ h &= F(x). \end{aligned} \tag{96}$$

(95) and especially (96) should be compared with the system formed by (42) and (41). As in the previous system, each value of x will give a value for price p and for quantity h from (95) or (96). Running through values of x will thus give rise to price–quantity combinations; the graph of these combinations is the steady-state supply curve. Alternatively, the steady-state supply curve can be derived graphically, using the same technique illustrated

in Figure 3. (Note that in the limit as $\delta \to \infty$, the system formed by (42) and (41) approaches (96).)

The traditional way of deriving this supply curve is completely different. In this traditional approach, harvest h (also called yield Y) is a function of x and of "fishing effort" E: $Y = h(E, x)$. In the steady state, $\dot{x} = 0$ so $Y = F(x)$ from (17). Hence, given E_{ss} (whose subscript denotes steady state), x_{ss} solves the equation $h(E_{ss}, x_{ss}) - F(x_{ss}) = 0$. Write the solution of this equation as $x_{ss}(E_{ss})$. Then the "yield-effort curve" is defined by $Y(E_{ss}) = F(x_{ss}(E_{ss}))$. Total cost incurred by the firm is some function $TC(E)$. Total revenue is $p\,Y(E_{ss})$. In open-access equilibrium, total revenue equals total cost, so $TC(E_{ss}) = p\,Y(E_{ss})$. This yields E_{ss} as an implicit function of p; call this $E_{ss}(p)$. Then steady-state harvest is $Y(E_{ss}(p))$, which is the supply curve.

Exercise. Find the open-access steady-state supply curve for: (a) logistic growth; (b) depensation; (c) critical depensation. Use $c(x) = \gamma/(qx)$.

Note that between the cases of private property and open access there are intermediate cases where enforcement of property rights exists but is imperfect.

Section 9. Competitive, Open Access Fishery: Dynamics

Along with the biological equation (17), we require an equation describing the dynamic behavior of firms. Open-access firms are usually not modeled as solving an explicit intertemporal maximization problem because there is no benefit to them of leaving fish in the ocean (because someone else will fish them out today). For open-access dynamics, the most common *ad hoc* assumption is that, if E is fishing effort, then $\dot{E}$ is proportional to profit:

$$\dot{E} = k\pi = k[\,ph(E, x) - TC(E)]. \tag{97}$$

(The constant of proportionality k is not to be confused with the usual notation for carrying capacity K.) In order to compare this with our previous results, we would like to change from (97) to an equation giving $\dot{h}$. To do this, note that since it is natural to assume $\partial h(E, x)/\partial E \neq 0$ (and in fact that $\partial h(E, x)/\partial E > 0$ since x is held constant in this derivative), the Implicit Function Theorem assures us that it is possible for the equation $h(E, x)$ to be inverted to give effort as a function of h and x: $E(x, h)$. Then (97) becomes

$$\frac{d}{dt}E(x, h) = k[\,ph - TC(E(x, h))] \quad \text{or}$$
$$\frac{\partial E}{\partial h}\dot{h} + \frac{\partial E}{\partial x}\dot{x} = k[\,ph - TC(E(x, h))]$$

so

$$\dot{h}_t = \frac{k[\, p_t h_t - TC(E(x_t, h_t))] - \frac{\partial E}{\partial x}[F(x_t) - h_t]}{\partial E/\partial h}\,. \tag{98}$$

Combined with (17), (98) gives a dynamical system in (x, h) as in previous sections above. The *final* step would be to impose equilibrium, namely $p_t = \phi(h_t)$. (Setting $p_t = \phi(h_t)$ was also the final step in Section 5, because it also dealt with competition, and it was the first step in Section 7, because that was about monopoly.) The final dynamical system would be

$$\begin{aligned} \dot{h}_t &= \frac{k[\, \phi(h_t)h_t - TC(E(x_t, h_t))] - \frac{\partial E}{\partial x}[F(x_t) - h_t]}{\partial E/\partial h} \\ \dot{x}_t &= F(x_t) - h_t \end{aligned} \tag{99}$$

Note that if in (99) $\dot{x} = 0$ together with $\dot{h} = 0$, then we do get (95).

In deriving the explicit dynamic equation for private-property competition, (74), and for monopoly, (94), I assumed that the total cost incurred by the firm had the special form $c(x_t)\, h_t$ (which is constant average cost). It is possible to make special assumptions on $TC(E)$ here which also imply that total cost is $c(x_t)\, h_t$:

Proposition 4. *Suppose that $TC(E)$ is proportional to E (so that $TC(E)$ can be written as αE for some $\alpha > 0$). Suppose that $E(x, h)$ is linear in h and is separable in x and h (so that $E(x, h)$ can be written as $E_1(x) \cdot h$ for some function $E_1(x)$). Then if a new function $c(x_t)$ is defined as $\alpha E_1(x_t)$ and a new constant j is defined as $k \cdot \alpha$, total cost $TC(E)$ can be written $c(x) \cdot h$, and in addition (99) becomes*

$$\begin{aligned} \dot{h}_t &= \frac{j[\, \phi(h_t) - c(x_t)] - c'(x_t)[F(x_t) - h_t]}{c(x_t)}\, h_t \\ \dot{x}_t &= F(x_t) - h_t\,. \end{aligned} \tag{100}$$

Proof. To prove that total cost can be written $c(x)\, h$:

$$\begin{aligned} TC(E(x, h)) &= \alpha\, E(x, h) \quad \text{since } TC(E) = \alpha E \\ &= \alpha\, E_1(x)\, h \quad \text{since } E(x, h) = E_1(x)\, h \\ &= c(x)\, h \quad \text{by the definition of } c(x). \end{aligned}$$

It follows that

$$\frac{\partial E}{\partial h} = E_1(x) = c(x)/\alpha$$

$$\frac{\partial E}{\partial x} = h\frac{\partial E_1(x)}{\partial x} = h\frac{\partial [c(x)/\alpha]}{\partial x} = \frac{h}{\alpha}c'(x) \, .$$

Making these substitutions into (99), along with $j = k\alpha$, results in (100). ▮

Example. Suppose $TC(E) = \gamma \cdot E$ and $h = qEx$ (so that $E(x,h) = h/(qx)$). Then the conditions of the above proposition are satisfied with $\alpha = \gamma$ and $E_1(x) = 1/(qx)$, so $c(x) = \gamma/(qx)$ and $j = k \cdot \gamma$. From (100),

$$\dot{h}_t = \frac{k \cdot \gamma[\,\phi(h_t) - \gamma/(qx)] + (\gamma/(qx^2))[F(x_t) - h_t]}{\gamma/(qx)}\, h_t \, .$$

Simplifying yields the dynamic system

$$\begin{aligned} \dot{h} &= \left[kq\,\phi(h_t)\,x_t - k\,\gamma + \frac{F(x_t) - h_t}{x_t}\right] h_t \\ \dot{x}_t &= F(x_t) - h_t \, . \end{aligned} \tag{101}$$

Exercise. (See the Exercise in Section 7.) Re-work "Example 1, dynamics" for the case of open access. Use the same functional forms $c(x)$ and $F(x)$ as in "Example 1, dynamics" (which, because of the above comments on costs, means (100) holds), together with the same parameters q, γ, r, K, and δ. Also use the same demand curves, (63)–(67). Once you have found the steady-state price-quantity combination, sketch a graph showing each demand curve and marking the steady-state price-quantity combination on it. Compare this graph with Figure 5 and especially with the graph from Exercise 2 of Section 5.

Section 10. Summary

Throughout this section assume that total costs have the form $c(x) \cdot h$ (constant returns to scale).

The steady-state supply curve for a private-property competitive fishery is

$$p = c(x) - \frac{c'(x)\,F(x)}{\delta - F'(x)} \tag{42}$$

$$h = F(x) \tag{41}$$

whereas the steady-state supply curve for an open-access fishery is

$$\begin{aligned} p &= c(x) \\ h &= F(x). \end{aligned} \tag{96}$$

There is no steady-state supply curve for a monopolist.

The dynamic system for a private-property competitive fishery is

$$\dot{h}_t = \frac{[\delta - F'(x_t)][\phi(h_t) - c(x_t)] + c'(x_t)\,F(x_t)}{\phi'(h_t)} \, . \tag{74}$$

$$\dot{x}_t = F(x_t) - h_t \,. \tag{17}$$

The dynamic system for a monopolist is

$$\dot{h}_t = \frac{[\delta - F'(x_t)][\phi'(h_t)h_t + \phi(h_t) - c(x_t)] + c'(x_t)\,F(x_t)}{\phi''(h_t)h_t + 2\phi'(h_t)} \tag{94}$$

$$\dot{x}_t = F(x_t) - h_t \,. \tag{17}$$

The dynamic system for an open-access fishery is

$$\begin{aligned} \dot{h}_t &= \frac{j[\,\phi(h_t) - c(x_t)] - c'(x_t)[F(x_t) - h_t]}{c(x_t)}\; h_t \\ \dot{x}_t &= F(x_t) - h_t \,. \end{aligned} \tag{100}$$

By the First Theorem of Welfare Economics, competitive equilibria are socially optimal. Therefore it would have been possible to find the competitive equilibrium by first solving the social planner's problem and then appealing to the First Theorem of Welfare Economics. That would have been easier than what we did above, which was to find the competitive equilibrium path of prices; but it is good to know how to find competitive equilibria.

Section 11. Outline of Regulatory History

1. Mesh size
2. season length
3. Alaskan halibut
4. ITQ's
5. Lobsters: Australia vs. New England
6. Brine Shrimp
7. Ecuador
8. Iceland
9. Cod, Canada, EU
10. Tuna & the EU
11. Somali pirates: Tom Hanks, "Captain Phillips"
12. Brexit
13. Marine Reserves