Section 1:

More than One Topic
Section 1.  
Answer all of the following three questions.

1. [11 points] Tell me everything you know about:
   (a) quasiconcavity; and
   (b) why it is important in consumer theory (you do not have to say
       anything about producer theory).

The essay you write should be understandable to an undergraduate 
student who has had no mathematics beyond the first year of calculus 
and one semester of linear algebra; so you will certainly have to define 
any terms you use which I taught this year.  
You do not have to discuss quasiconvexity, nor dual functions (so there 
are two reasons why you do not have to discuss the result that “the 
indirect utility function $v(p, m)$ is quasiconvex in $p$”).
A quasiconcave function has convex upper level sets.

(a) A convex set is one for which a line drawn between any two members of the set is itself in the set. i.e.,
\[
\begin{align*}
    x_1 & \in A \\
    x_2 & \in A
\end{align*}
\] \Rightarrow \lambda x_1 + (1 - \lambda) x_2 \in A \quad \text{for all} \quad \lambda \in [0, 1]

and for all \(x_1, x_2\) pairs in \(A\).

(I'm assuming that the quasiconcave function maps \(\mathbb{R}^n\) into \(\mathbb{R}^1\), which is the only case we consider this year.)

(b) An upper level set of a function \(f(x)\) is the set of all points in \(f\)'s domain whose image under \(f\) is \(\geq\) a particular number, say \(M\). i.e., if \(f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^1\) then

\[
\{ x \in \mathbb{R}^n \mid f(x) > M \}
\]

is the upper level set of \(f\) for \(M\).
A 3-dimensional example of a function which looks somewhat like a bowl.

The upper level set for \( M \); all these points in the domain (which is the \( x-y \) plane) get mapped by \( f \) into a value \( > M \).

A two-dimensional example (\( f \) is a "function of one variable").

The upper level set of \( f \) for \( M \); all these points in the domain (which is the \( x \) axis) get mapped by \( f \) into a value \( > M \).

Neither of the functions I've drawn on this page is quasi-convex because neither of their upper level sets is a convex set.

A line between the pair of \( x \)'s goes outside of the upper level set in both examples.
Consider the "bordered Hessian" of $f(x)$:

$$
\begin{bmatrix}
0 & \nabla f \\
\nabla^T f & \nabla^2 f
\end{bmatrix} =
\begin{bmatrix}
0 & f'_1 & f'_2 & f'_3 & \cdots & f'_n \\
& f''_1 & f''_2 & f''_3 & \cdots & f''_n \\
& f''_2 & f'''_2 & f'''_3 & \cdots & f'''_n \\
& f''_3 & f'''_3 & f'''_4 & \cdots & f'''_n \\
& \vdots & \vdots & \vdots & \ddots & \vdots \\
& f''_n & f'''_n & f'''_n & \cdots & f''_n
\end{bmatrix}
$$

Let $\delta_r$ be the $(r+1)^{st}$ leading principal minor of this bordered Hessian. A sufficient condition for $f$ to be quasiconcave is that $\delta_r$ alternate in sign beginning with $> 0$ for $r = 2, 3, \ldots, n$.

(Note that $\delta_1 = \begin{vmatrix} 0 & f'_1 \\ f'_1 & f''_{11} \end{vmatrix} = -(f'_{11})^2 < 0$ for any function $f$ so it never needs to be tested.)

In consumer theory, we would like to characterize utility functions $u(x)$ which give rise to the typical indifference curve shape studied in undergraduate texts: $x_2 \searrow x_1$. Consider the set of all $(x_1, x_2)$ which the consumer prefers to, say, $(\hat{x}_1, \hat{x}_2)$, and suppose the utility level at $(\hat{x}_1, \hat{x}_2) = \hat{u}$. This set of bundles looks like
In order for the indifference curve to have its "usual" shape, this "better than" set must be convex. This "better than" set is simply an upper level set for $u$. So, saying that the indifference curves have their usual shape is equivalent to saying that $u$ is quasiconcave. (This ensures a unique utility-maximizing bundle.)

If the indifference curves have a strange shape,

the upper level set is not convex, so $u$ is not quasiconcave.

Optional: Consider $\max_x f(x)$ s.t. $\mathbf{m} \cdot \mathbf{x} = c$ (i.e., a linear constraint).

It can be shown that if $f(x)$ is quasiconcave, then any $x^*$ satisfying this problem's first-order conditions will also satisfy its second-order conditions. An important such problem is maximizing utility of a price-taking consumer.
4. [17 points]

(a) Suppose a competitive profit-maximizing firm produces an output from two inputs \( \mathbf{x} = (x_1, x_2) \) according to the production function \( f(\mathbf{x}) \). By how much would the firm’s purchases of \( x_1 \) change if the price it paid for \( x_1 \) changed? Would the firm’s purchases of \( x_1 \) go up or go down if the price it paid for \( x_1 \) rose? Why?

(b) Suppose an expenditure-minimizing (not utility-maximizing) consumer obtains utility from two commodities \( \mathbf{x} = (x_1, x_2) \) according to the utility function \( u(\mathbf{x}) \). By how much would the consumer’s purchases of \( x_1 \) change if the price he paid for \( x_1 \) changed? Would the consumer’s purchases of \( x_1 \) go up or go down if the price he paid for \( x_1 \) rose? Why?

(c) Suppose a utility-maximizing consumer obtains utility from two commodities \( \mathbf{x} = (x_1, x_2) \) according to the utility function \( u(\mathbf{x}) \). By how much would the consumer’s purchases of \( x_1 \) change if the price he paid for \( x_1 \) changed? Would the consumer’s purchases of \( x_1 \) go up or go down if the price he paid for \( x_1 \) rose? Why?

(d) Explain in one sentence, with little or no mathematics, why some results you learned about in lectures make you unsurprised by the answers to parts (a), (b) and (c).
\[ \max_{x_1, x_2} \frac{f(x_1, x_2) - w_1 x_1 - w_2 x_2}{\text{profit } \pi} \]

F.O.C.:
\[
O = \frac{\partial \pi}{\partial x_1} = pf'_1 - w_1
\]
\[
O = \frac{\partial \pi}{\partial x_2} = pf'_2 - w_2
\]

Take differentials of the First Order Conditions:
\[
O = pf''_{11} dx_1 + pf''_{12} dx_2 - dw_1
\]
\[
O = pf''_{21} dx_1 + pf''_{22} dx_2 - dw_2
\]

\[
O = \begin{bmatrix} pf''_{11} & pf''_{12} \\ pf''_{21} & pf''_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} - \begin{bmatrix} dw_1 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} dw_1 \\ 0 \end{bmatrix} = \begin{bmatrix} pf''_{11} & pf''_{12} \\ pf''_{21} & pf''_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} pf''_{11} & pf''_{12} \\ pf''_{21} & pf''_{22} \end{bmatrix} \begin{bmatrix} \partial x_1 / \partial w_1 \\ \partial x_2 / \partial w_1 \end{bmatrix}
\]

so by Cramer's Rule,
\[
\frac{\partial x_1}{\partial w_1} = \frac{1 \cdot pf''_{12}}{pf''_{11} \cdot pf''_{22} - pf''_{12} \cdot pf''_{21}} = \frac{pf''_{12}}{pf''_{22} - pf''_{12} \cdot pf''_{21}} = \frac{1}{p} \frac{f''_{22}}{f''_{11} f''_{22} - (f''_{12})^2}
\]
The second-order sufficient conditions for a maximum:

\[ D_{2m+1} \text{ of } V^2L \text{ has the sign of } (-1)^{m+1} \]

then the larger \( D \)'s alternate in sign

until \( D_{m+n} \) is reached.

Here:

\[ m = 0 \]
\[ D_{2m+1} = D_1 \]
\[ (-1)^{m+1} = -1 \]
\[ D_{m+n} = D_2 > 0 \]

In this problem, \( D_2 \) of \( V^2L \) is

\[ p^2 f''_{11} f''_{22} - p^2 (f''_{12})^2 \]

This ought to be \( > 0 \),

so the denominator of \( \partial x_1/\partial w_1 \) should be \( > 0 \).

\( D_1 \) of \( V^2L \) is

\[ p f''_{11} \]

which ought to be \( < 0 \). Since \( p > 0 \), this implies \( f''_{11} < 0 \). Since

the numbering of the "first" or "second" input is arbitrary, this also implies \( f''_{22} < 0 \).

So

\[ \frac{\partial x_1}{\partial w_1} = \frac{1}{p} \frac{f''_{22}}{f''_{11} f''_{22} - (f''_{12})^2} < 0. \]

b)

\[ \min_{x_i, x_2} p, x_1 + p, x_2 \text{ s.t. } u(x_1, x_2) \geq \bar{u} \implies x = p, x_1 + p, x_2 + \lambda [\bar{u} - u(x_1, x_2)] \]

First-Order Conditions:

\[ 0 = \frac{\partial x}{\partial \lambda} = \bar{u} - u(x_1, x_2) \]
\[ 0 = \frac{\partial x}{\partial x_1} = p_1 - \lambda u_1' \]
\[ 0 = \frac{\partial x}{\partial x_2} = p_2 - \lambda u_2' \]

Differentials:

\[ 0 = 0 \text{ d} \lambda - u_1' \text{ d} x_1 - u_2' \text{ d} x_2 + \text{ d} u \]
\[ 0 = -u_1' \text{ d} \lambda - \lambda u''_{11} \text{ d} x_1 - \lambda u_{12} \text{ d} x_2 + \text{ d} p_1 - u_1' \text{ c} \]
\[ 0 = -u_2' \text{ d} \lambda - \lambda u''_{21} \text{ d} x_1 - \lambda u''_{22} \text{ d} x_2 + \text{ d} p_2 \]

set \( = 0 \)
\[
\begin{bmatrix}
0 \\
-dp_1 \\
0
\end{bmatrix} = \begin{bmatrix}
0 & -u_1' & -u_2' \\
-u_1' & -\lambda u_1'' & -\lambda u_2'' \\
-u_2' & -\lambda u_1'' & -\lambda u_2''
\end{bmatrix} \begin{bmatrix}
d\lambda \\
dx_1 \\
dx_2
\end{bmatrix}
\]

By Cramer's Rule,

\[
\frac{\partial x_1}{\partial p_1} = \frac{\begin{vmatrix}
0 & 0 & -u_1' \\
-u_1' & -\lambda u_1'' & -u_2'' \\
-u_2' & -\lambda u_2'' & 0
\end{vmatrix}}{|\nabla^2 \mathcal{L}|} = \frac{-\lambda (-1)^{2+2}}{|\nabla^2 \mathcal{L}|} = \frac{(u_2')^2}{|\nabla^2 \mathcal{L}|}
\]

The second-order sufficient conditions for a minimum:

\[D_{2m+1} \text{ of } \nabla^2 \mathcal{L} \text{ has the sign of } (-1)^m, \text{ as do all the other } D_i, \text{ up to } D_{m+n}.
\]

Here: \(m=1, (-1)^m = 0\), so \(D_{2m+1} = D_3 < 0; D_3 = \nabla^2 \mathcal{L};\)

\[
\frac{\partial x_1}{\partial p_1} = \frac{(u_2')^2}{|\nabla^2 \mathcal{L}|} < 0.
\]

c) \[\max \ u(x_1, x_2) \text{ s.t. } p_1 x_1 + p_2 x_2 = m \]

\[\mathcal{L} = u(x_1, x_2) + 1 \left( m - p_1 x_1 - p_2 x_2 \right)\]

F.O.C. \[0 = \partial \mathcal{L}/\partial x_1 = u_1' - \lambda p_1 \]
\[0 = \partial \mathcal{L}/\partial x_2 = u_2' - \lambda p_2 \]
Differentials:

\[
\begin{align*}
0 &= -p_1 \, dx_1 - p_2 \, dx_2 + dm - x_1 \, dp_1 - x_2 \, dp_2 \\
0 &= -p_1 \, d\lambda + u''_{11} \, dx_1 + u''_{12} \, dx_2 \\
0 &= -p_2 \, d\lambda + u''_{21} \, dx_1 + u''_{22} \, dx_2 - \lambda \, dp_1 \\
0 &= -p_2 \, d\lambda + u''_{21} \, dx_1 + u''_{22} \, dx_2 - \lambda \, dp_2
\end{align*}
\]

Take \( dm = 0 \) and \( dp_2 = 0 \)

\[
\begin{bmatrix}
x_1 \\
\lambda \\
x_2
\end{bmatrix}
\begin{bmatrix}
0 & -p_1 & -p_2 \\
-p_1 & u''_{11} & u''_{12} \\
-p_2 & u''_{21} & u''_{22}
\end{bmatrix}
\begin{bmatrix}
\frac{d\lambda}{dx_1} \\
\frac{d\lambda}{dx_2} \\
\frac{\partial x_1}{\partial \rho_1} \\
\frac{\partial x_2}{\partial \rho_1}
\end{bmatrix}
\]

(Ramer's Rule):

\[
\frac{\partial x_1}{\partial \rho_1} = \frac{\begin{vmatrix}
0 & x_1 & -p_2 \\
-p_1 & \lambda & u''_{12} \\
-p_2 & 0 & u''_{22}
\end{vmatrix}}{\begin{vmatrix}
0 & -p_1 & -p_2 \\
-p_1 & u''_{11} & u''_{12} \\
-p_2 & u''_{21} & u''_{22}
\end{vmatrix}} = \frac{x_1 (-1)^{1+2} - p_1 \, u''_{12} + \lambda (-1)^{2+2} \, p_2 \, u''_{22}}{\begin{vmatrix}
0 & -p_1 & -p_2 \\
-p_1 & u''_{11} & u''_{12} \\
-p_2 & u''_{21} & u''_{22}
\end{vmatrix}}
\]

\[
\frac{\partial x_2}{\partial \rho_1} = \frac{x_1 (-1) \, [ -p_1 \, u''_{22} + p_2 \, u''_{12} ] + \lambda (-1) \, [ 0 + p_2 ]}{(D_3 \text{ of } \nabla^2 \mathcal{L})}
\]

\[
\frac{\partial x_1}{\partial \rho_1} = \frac{x_1 (p_1 \, u''_{22} - p_2 \, u''_{12}) + \lambda \, p_2}{(D_3 \text{ of } \nabla^2 \mathcal{L})}
\]

The S.O.C. for a maximum are given in part a. Here \( m = 1 \), \( D_{mn} = D_3 \), \((-1)^{m+1} = (-1)^2 > 0\), and \( D_{m+n} = D_3 \). So the denominator of \( \frac{\partial x_1}{\partial \rho_1} \) is
positive. The numerator, however, cannot be signed. \( u_{12} \) is especially hard to
say anything about. \( \) (Note that by the Envelope Theorem, \( \partial u^*/\partial m = \)
\[ \partial y/\partial m = \lambda, \text{ and it's clear that } \partial u^*/\partial m \geq 0, \]
so \( \lambda \geq 0. \)

d) Firms have downward-sloping input demand curves (part a), and
consumers have downward-sloping Hicksian (compensated) demand
curves (part b), but because Giffen goods could exist, consumer's
Marshallian demand curves might not be downward sloping (part c).
6710 Section

Answer one of the following two questions.

1. (a) A consumer consumes $n$ commodities labeled 1, 2, 3, $\ldots$, $n$. This consumer takes the prices of commodities 2, 3, 4, $\ldots$, $n$ as given, but this consumer can affect the price he pays for commodity 1 because the more of commodity 1 he buys, the lower the price he has to pay for commodity 1.

For what commodities $i$, if any, is it true that this consumer’s commodity demands obey

$$x_i = \frac{\partial v/\partial p_i}{\partial v/\partial m} ?$$

The letter $v$ denotes the indirect utility function.

(b) A firm has market power over commodities 1, 2, 3, $\ldots$, $m$, but this firm has no market power over commodities $m+1$, $m+2$, $\ldots$, $n$. Suppose that $m < n$.

For what commodities $i$ and $j$, if any, is it true that this firm’s supply and demands obey

$$\frac{dy_i}{dp_j} = \frac{dy_j}{dp_i} ?$$

For what commodities $i$, if any, is it true that this firm’s supply and demands obey

$$\frac{dy_i}{dp_i} \geq 0 ?$$
Optimal Question 1.

(a) \( v(p_1, p_2, \ldots, p_n, m) = \max_{x} u(x) \text{ s.t. } p_1 x_1 + p_2 x_2 + \ldots + p_n x_n = m \).

\[ L = u(x) + \lambda \left[ m - p_1 x_1 - p_2 x_2 - p_3 x_3 - \ldots - p_n x_n \right] \]

Envelope Theorem: \( \frac{\partial v}{\partial p_i} = \frac{\partial L}{\partial p_i} \quad \forall i = 2, 3, \ldots, n \)

\( \frac{\partial v}{\partial m} = \frac{\partial L}{\partial m} \).

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Answer 1

Therefore \( \frac{\partial v}{\partial p_i} = -\lambda x_i \quad \forall i = 2, 3, \ldots, n \)

\( \frac{\partial v}{\partial m} = \lambda \)

and dividing one by the other, then multiplying both sides by negative one,

\[ x_i = -\frac{\partial v/\partial p_i}{\partial v/\partial m} \quad \forall i = 2, 3, \ldots, n. \]

There is no similar result for \( x_1 \). Note that \( p_1 \) is not an exogenous variable in this problem, so \( \partial v/\partial p_1 \) cannot be expected to be equal to \( \partial L/\partial p_1 \) in general.

(b) \( v = \max_{y} p \cdot y \text{ s.t. } y \in Y \), or more explicitly,

\[ \pi(p_{m+1}, p_{m+2}, \ldots, p_n) = \max_{y_i, \text{isism}} p_i(y_i) \cdot y_i + \max_{y_i, \text{misism}} p_i(y_i) \text{ s.t. } y \in Y \]

Envelope Theorem: \( \frac{\partial \pi}{\partial p_i} = \frac{\partial L}{\partial p_i}, \quad m+1 \leq i \leq n \).
\[ \mathcal{L} = \sum_{i=1}^{m} p_i (y_i - y_i) + \sum_{i=m+1}^{n} p_i y_i \]

so

\[ \frac{\partial \mathcal{L}}{\partial p_i} = y_i \text{ for } m+1 \leq i \leq n \] (since the profit function is only a function of the i's between m+1 and n).

Let the vector \((p_{m+1}, p_{m+2}, \ldots, p_n)\) be denoted \(\vec{p}\) and let the vector \((y_{m+1}, y_{m+2}, \ldots, y_n)\) be denoted by \(\vec{y}\). Then

\[ \nabla \mathcal{L} (\vec{p}) = \vec{y}, \]

and differentiating both sides,

\[ \nabla^2 \mathcal{L} = \nabla^2 \mathcal{L} \vec{y}. \]

\[ \begin{bmatrix} \frac{\partial^2 y_{m+1}}{\partial p_{m+1}^2} & \frac{\partial^2 y_{m+1}}{\partial p_{m+1} \partial p_{m+2}} & \cdots & \frac{\partial^2 y_{m+1}}{\partial p_{m+1} \partial p_n} \\ \frac{\partial^2 y_{m+2}}{\partial p_{m+1} \partial p_{m+2}} & \frac{\partial^2 y_{m+2}}{\partial p_{m+2}^2} & \cdots & \frac{\partial^2 y_{m+2}}{\partial p_{m+2} \partial p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 y_n}{\partial p_{m+1} \partial p_n} & \frac{\partial^2 y_n}{\partial p_{m+2} \partial p_n} & \cdots & \frac{\partial^2 y_n}{\partial p_n^2} \end{bmatrix} \]

This is

\[ \begin{bmatrix} \frac{\partial^2 y_{m+1}}{\partial p_{m+1}^2} & \frac{\partial^2 y_{m+1}}{\partial p_{m+1} \partial p_{m+2}} & \cdots & \frac{\partial^2 y_{m+1}}{\partial p_{m+1} \partial p_n} \\ \frac{\partial^2 y_{m+2}}{\partial p_{m+1} \partial p_{m+2}} & \frac{\partial^2 y_{m+2}}{\partial p_{m+2}^2} & \cdots & \frac{\partial^2 y_{m+2}}{\partial p_{m+2} \partial p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 y_n}{\partial p_{m+1} \partial p_n} & \frac{\partial^2 y_n}{\partial p_{m+2} \partial p_n} & \cdots & \frac{\partial^2 y_n}{\partial p_n^2} \end{bmatrix} \]

The left-hand side is a Hessian, which by Young's theorem is symmetric, so

\[ \frac{\partial y_i}{\partial p_i} = \frac{\partial y_j}{\partial p_j} \text{ for all } i \text{ and } j \text{ between } m+1 \text{ and } n. \]

I claim that the profit function \(\mathcal{L} (\vec{p})\) is convex. The proof is that

\[ \mathcal{L} (\lambda \bar{p}_a + (1-\lambda) \bar{p}_b) = \max_{\vec{y}} \left[ (\lambda \bar{p}_a + (1-\lambda) \bar{p}_b) \cdot \vec{y} \right] \]
\[
\begin{align*}
\max_y & \left[ (p_1, \ldots, p_m) \cdot (y_1, \ldots, y_m) + \lambda \bar{p}_a \cdot \bar{y} + (1-\lambda) \bar{p}_b \cdot \bar{y} \right] \\
\leq & \max_y \left[ \lambda (p_1, \ldots, p_m) \cdot (y_1, \ldots, y_m) + \lambda \bar{p}_a \cdot \bar{y} \right] \\
& + \max_y \left[ (1-\lambda) (p_1, \ldots, p_m) \cdot (y_1, \ldots, y_m) + (1-\lambda) \bar{p}_b \cdot \bar{y} \right] \\
\end{align*}
\]

This completes the proof that \( \tau(\bar{p}) \) is convex. It follows that \( \nabla^2 \tau \) is positive semi-definite, and hence that its diagonal terms, which are equal to \( \partial^2 y_i / \partial p_i \) for all \( i \) between \( m+1 \) and \( n \), are \( \geq 0 \).
3. One form of the Envelope Theorem is: if

\[ M(a) = \max_{x_1, x_2} g(x_1, x_2, a) \quad \text{such that} \quad h(x_1, x_2, a) = 0 \]

and if \( \mathcal{L} = g - \lambda h \) then

\[ \frac{dM(a)}{da} = \frac{\partial \mathcal{L}^*}{\partial a}. \]

Another form of the Envelope Theorem is: if

\[ M(a) = \max_{x_1, x_2} g(x_1, x_2, a) \]

then

\[ \frac{dM(a)}{da} = \frac{\partial g^*}{\partial a}. \]

Use the Envelope Theorem to prove the following four results. Explain what each symbol means.

(a) \( h_i(p, u) = \partial e(p, u)/\partial p_i. \)
(b) \( x_i(p, m) = -[\partial v(p, m)/\partial p_i] \div [\partial v(p, m)/\partial m] \) [Roy’s Identity].
(c) \( y_i(p) = \partial \pi(p)/\partial p_i \) [Hotelling’s Lemma].
(d) \( x_i(w, y) = \partial c(w, y)/\partial w_i \) [Shephard’s Lemma].
(3) a) \[ \max_{x} \ p \cdot x \quad \text{s.t.} \quad u(x) \geq u \quad = e(p, u) \]

\[ \ell = p \cdot x + \lambda (u(x) - u) \]

\[ \frac{dM(a)}{da} = \frac{\partial \ell}{\partial a} \Rightarrow \frac{\partial e(p, u)}{\partial p_i} = \frac{\partial \ell_x}{\partial p_i} = x_i \]

\( p \): price
\( x \): consumer demand
\( u(\cdot) \): utility function
\( e \): expenditure function
\( \lambda \): Hicksian demand curve

b) \[ \max_{x} u(x) \quad \text{s.t.} \quad p \cdot x \leq m \quad = v(p, m) \]

\( m \): income

\[ \lambda = u(x) + \lambda \left[ m - p \cdot x \right] \]

\[ \frac{dM(a)}{da} = \frac{\partial \ell}{\partial a} \Rightarrow \begin{cases} \frac{\partial \ell_x}{\partial p_i} = \frac{\partial x}{\partial p_i} = -\lambda x_i = - \frac{\partial v}{\partial m} x_i ; \text{solve for } x_i \\ \frac{\partial \ell_x}{\partial m} = \frac{\partial x}{\partial m} = \lambda \end{cases} \]
c) profit function

\[ \pi(p) = \max_{x,y} p \cdot x - c(x,y) \sim \text{firms' inputs and outputs} \]

\[ \frac{dM(a)}{da} = \frac{\partial g^*}{\partial a} \Rightarrow \frac{\partial \pi(p)}{\partial p_i} = \frac{\partial p \cdot y}{\partial p_i} = y_i(p) \]

firm's input supply
or output demand function

\[ \min_w w \cdot x \quad \text{s.t.} \quad f(x) \geq y = c(w,y) \quad \therefore L = w \cdot x + \lambda [f(x)-y] \]

input
inputs
production function
output

\[ \frac{dM(a)}{da} = \frac{\partial L^*}{\partial a} \Rightarrow \frac{\partial c(w,y)}{\partial w_i} = \frac{\partial L^*}{\partial w_i} = x_i(w,y) \]

input demand function
621 Section (must answer one)

Question 1. What is the Envelope Theorem? (You do not have to prove the Envelope Theorem.)

What four important results is the Envelope Theorem used to prove? What are the proofs?
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Answer 1

The Envelope Theorem: Variational

Theorem

\[ f(x_1, x_2, a) = \max_{x_1, x_2} g(x_1, x_2, a) \]
\[ \text{s.t. } x_1 \geq 0, x_2 \geq 0 \]
\[ x_1 + x_2 = a \]

Proof:

\[ \frac{\partial f}{\partial a} = \frac{\partial}{\partial a} \left( \max_{x_1, x_2} g(x_1, x_2, a) \right) \]
\[ = \max_{x_1, x_2} \left( \frac{\partial g}{\partial a} \right) \]
\[ = \max_{x_1, x_2} \left( \frac{\partial g}{\partial x_1} \right) \]

\[ \text{Theorem } \]
Qualifying Exam 1998

Answer 1 cont...

\[ M(a) = \max_{x, x'} \left\{ g(x, x', a) \right\} \]

\[ \Rightarrow \quad \frac{\partial M(a)}{\partial a} = \frac{\partial g}{\partial a} \]

Hoeffding's Lemma

\[ \tilde{g}(p) = \tilde{y} * \text{rate} \]

Profit Function

\[ \tilde{y}(p) = \text{profit} \]

Analogous

\[ M \in \mathbb{R}^n \]

\[ x, x' \]

\[ y \]

\[ p \]

\[ \tilde{y}(p) \]

\[ \tilde{y}(p) = \text{profit} \]

\[ \text{rate} \]

\[ \text{profit} \]
2. [8 points] Illustrate the use of the Envelope Theorem in a problem of economic interest. Explain how the Envelope Theorem is being used.
The Envelope Theorem states that if

\[ M(a) = \max_{x} g(x, a) \quad \text{s.t.} \quad h(x, a) = 0 \]

then

\[ \frac{dM(a)}{da} = \frac{\partial \tilde{L}^{*}}{\partial a} \]

where \( \tilde{L}^{*} \) is the maximized Lagrangian of the optimization problem.

There are two classic Envelope Theorem results for the form, namely

a) \( x_i = \frac{\partial c}{\partial w_i} \), Shepard's Lemma

b) \( y_i = \frac{\partial \pi}{\partial p_i} \), Hotelling's Lemma,
and two for the consumer:

c) \( h_i = \frac{\partial e}{\partial p_i} \) \quad resembles \text{Shepherd's lemma} \\

d) \( x_i = -\frac{\partial v/\partial p_i}{\partial v/\partial m} \quad \text{Roy's identity} \\

Their proofs are as follows.

a) \( c = \max_{\mathbf{x}} \frac{1}{2} \mathbf{x}^\top \mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) = y \quad \text{(cost minimization)} \)

\[ L = \frac{1}{2} \mathbf{x}^\top \mathbf{x} + \lambda \left[ y - f(\mathbf{x}) \right] \]

\[ \frac{\partial L}{\partial \mathbf{w}} = \frac{\partial \mathbf{x}^\top \mathbf{x}}{\partial \mathbf{w}} = \mathbf{x}^* \]

b) \( \tau = \max_{\mathbf{y}} \mathbf{p} \cdot \mathbf{y} \quad \text{(profit maximization)} \)

\[ \frac{\partial \tau}{\partial \mathbf{p}} = \frac{\partial \mathbf{y}^*}{\partial \mathbf{p}} = \frac{\partial (\mathbf{p} \cdot \mathbf{y})^*}{\partial \mathbf{p}} = \mathbf{y}^* \]

c) \( e = \min_{\mathbf{x}} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u_0 \quad \text{(expenditure minimization)} \)

\[ L = \mathbf{p} \cdot \mathbf{x} + \lambda \left[ u_0 - u(\mathbf{x}) \right] \]

\[ \frac{\partial e}{\partial \mathbf{p}} = \frac{\partial \mathbf{x}^*}{\partial \mathbf{p}} = \mathbf{x}^* \quad \text{which is usually called} \: h_i^* \quad \text{(the Hicksian demand)} \]
d) \( v = \max u(x) \) s.t. \( p \cdot x \leq m \) (utility maximization)

\[ x = u(x) + \lambda [m - p \cdot x] \]

\[ \frac{\partial v}{\partial p_i} = \frac{\partial u}{\partial p_i} = -\lambda x^*_i \]

\[ \frac{\partial v}{\partial m} = \frac{\partial u}{\partial m} = \lambda \]

There are other possible answers.
4. (a) Suppose a consumer's utility function is given by \( u(x_1, x_2) = x_1^a x_2^b \). The consumer's income is fixed at \( m \) and the prices of goods 1 and 2 are fixed at \( p_1 \) and \( p_2 \), respectively. Find the consumer's indirect utility function.

(b) Suppose instead that the income of the consumer in part (a) is a random variable; with probability 1/4 his income is \( m_1 \), and with probability 3/4 his income is \( m_2 \). Suppose this consumer's preferences obey the Expected Utility Hypothesis. Find the consumer's indirect utility function.
\[ u = x_1^\alpha x_2^\beta \]

a) \[ L = x_1^\alpha x_2^\beta + \lambda (m - p_1 x_1 - p_2 x_2) \]

F.O.C. \[ \frac{\partial L}{\partial x_1} = \alpha \frac{u}{x_1} - \lambda p_1 \]

\[ 0 = \frac{\partial L}{\partial x_1} = \alpha \frac{u}{x_1} - \lambda p_1 \]

\[ \frac{\partial L}{\partial x_2} = \beta \frac{u}{x_2} - \lambda p_2 \]

\[ 0 = \frac{\partial L}{\partial x_2} = \beta \frac{u}{x_2} - \lambda p_2 \]

\[ \frac{\alpha u}{x_1} \div \frac{\beta u}{x_2} = \lambda p_1 \div \lambda p_2 \]

\[ \frac{\alpha u}{x_1} \cdot \frac{x_2}{\beta u} = \frac{\lambda p_1}{\lambda p_2} \Rightarrow \frac{\alpha x_2}{\beta x_1} = \frac{p_1}{p_2} \]

Therefore \[ x_2 = \frac{\beta}{\alpha} \frac{p_1}{p_2} x_1 \]. Substituting into the budget constraint, \[ m = p_1 x_1 + p_2 \left( \frac{\beta}{\alpha} \frac{p_1}{p_2} x_1 \right) = \left( p_1 + \frac{\beta}{\alpha} \frac{p_1}{p_2} \right) x_1 = (1 + \frac{\beta}{\alpha}) p_1 x_1 \]

\[ = \frac{\alpha + \beta}{\alpha} p_1 x_1 \Rightarrow x_1^* = \frac{\alpha}{\alpha + \beta} \frac{m}{p_1} \]

Then \[ x_2^* = \frac{\beta}{\alpha} \frac{p_1}{p_2} x_1^* = \frac{\beta}{\alpha} \frac{p_1}{p_2} \frac{\alpha}{\alpha + \beta} \frac{m}{p_1} = \frac{\beta}{\alpha + \beta} \frac{m}{p_2} \]

Finally, \[ v(p,m) = (x_1^*)^\alpha (x_2^*)^\beta = \left( \frac{\alpha}{\alpha + \beta} \frac{m}{p_1} \right)^\alpha \left( \frac{\beta}{\alpha + \beta} \frac{m}{p_2} \right)^\beta \]
b) For the case where utility is assumed to be just a function of wealth, \( u(w) \),
then if \( w = \pi \cdot w_1 + (1-\pi) \cdot w_2 \), the Expected Utility Hypothesis gives
utility equal to \( \pi \cdot u(w_1) + (1-\pi) \cdot u(w_2) \). In this problem, the indirect
utility function \( v(p, m) \) plays the role of \( u(w) \), where \( m \) (income)
corresponds to \( w \) (wealth) and the prices \( p \) are constant.

Therefore,

\[
v(p, \frac{1}{4} \cdot m_1 + \frac{3}{4} \cdot m_2) = \frac{1}{4} \cdot v(p, m_1) + \frac{3}{4} \cdot v(p, m_2) \]

\[
= \frac{1}{4} \left( \frac{\alpha}{p_1} \right)^{\alpha} \left( \frac{\beta}{p_2} \right)^{\beta} \left( \frac{m_1}{\alpha + \beta} \right)^{\alpha + \beta} + \frac{3}{4} \left( \frac{\alpha}{p_1} \right)^{\alpha} \left( \frac{\beta}{p_2} \right)^{\beta} \left( \frac{m_2}{\alpha + \beta} \right)^{\alpha + \beta}
\]

\[
= \left( \frac{\alpha}{p_1} \right)^{\alpha} \left( \frac{\beta}{p_2} \right)^{\beta} \left( \frac{1}{\alpha + \beta} \right)^{\alpha + \beta} \left[ \frac{m_1}{4} + \frac{3m_2}{4} \right]
\]

\[
= \frac{1}{4} \left( \frac{\alpha}{p_1} \right)^{\alpha} \left( \frac{\beta}{p_2} \right)^{\beta} \left( \frac{1}{\alpha + \beta} \right)^{\alpha + \beta} \left( m_1^{\alpha + \beta} + 3m_2^{\alpha + \beta} \right).
\]