The Risk-Minimizing Portfolio when Planning to Buy a Fixed Immediate Annuity*

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Abstract. The income paid out by a fixed immediate annuity usually varies directly with the level of interest rates prevailing when the annuity is purchased. A consumer planning to buy such an annuity may be able to protect against interest rate changes during the time before the purchase by holding fixed-income instruments whose duration is longer than cash, because the value of such instruments varies in the short run inversely with interest rates. We find the bond portfolios which minimized annuity-purchase risk for various lengths of time before purchase using data from 2003 to 2017.

JEL Codes: G11, J26, J32

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According to a Gallup poll, 64% of Americans surveyed say they are "very worried" or "moderately worried" about not having enough money for retirement (Gallup 2016). Such worries are well-founded because both private and public employers have been curtailing defined-benefit programs, shifting longevity risk to workers. Besides defined-benefit pensions and Social Security, the primary financial instrument that provides guaranteed retirement income is a single-premium fixed immediate annuity ("SPIA"), and accordingly was the focus of increased interest by, among others, the US Department of Labor [2010] during the Obama administration.[1]

The fixed monthly payout offered by fixed immediate annuities tends to fall as interest rates at the time of purchase fall. A man saving for a retirement age of 65 knew in November 2008 that each $100,000 of savings he accumulated could at that time buy a retirement income of roughly $680 per month, but by Nov. 2009, that had fallen to $605; by Nov. 2010, to $570; and by Nov. 2016, to $499. This paper describes how a retirement saver between 2003 and 2017 could have (partially) shielded the annuity income he or she would be able to afford from changes down, or up, in interest rates. Because annuities pay less when interest rates are low, a hedging instrument's value has to rise when interest rates fall. Bonds have that property, so we study how to construct a hedge for annuity income by using bonds of different sorts. Readers who think US interest rates have nowhere to go but up will have little use for this paper, thinking it was written more than a decade too late. Interest rates in major countries such as Germany and Japan are currently even lower than US rates, however, so some US readers—if not now then some years in the future—may be interested in ways of hedging their retirement savings against future declines in long-term interest rates. The hedges studied in this paper shield annuity income against changes in interest rates in either direction. They work by sacrificing "upside" potential so as to avoid "downside" risk.

[1] Background information on fixed immediate annuities is available at `www.immediateannuities.com/content_pages/lesson.htm` and at `https://www.actuarialfoundation.org/portfolio/making-your-money-last-a-lifetime/`. The "guarantee" of a fixed immediate annuity is only as good as the insurance company issuing the policy, with a limited state guarantee fund backup. Fixed immediate annuities, especially if the payments are fixed in *real* terms, are widely recommended at least in theory by economists, including Zvi Bodie and Jeremy Siegel (well known for their opposite opinions on how to best save for retirement) [National Association of Personal Financial Advisors, 2004]. Babbel and Merrill [2007a, p. 11–12] point out that if, as seems likely, there is some "minimum threshold of consumption tolerable to the individual" below which utility is $-\infty$, then the *only* utility-maximizing plan involves buying a "default-free annuity which continues throughout one's lifetime" to cover that minimum threshold.

1. Prior Work

We will not be able to find an instrument that will hedge annuity income perfectly. There is a large body of work on imperfect hedging, called "cross-hedging," but very little concerning hedging an annuity purchase. Certainly the hesitancy to purchase annuities when interest rates are low is well known; for example, in 2010 `morningstar.com` carried an interview in which Harold Evensky (president of Evensky & Katz Wealth Management) said:

> Because rates are historically low, we don't feel the pressure to recommend it [fixed immediate annuities] right now, but I think within the next few years as interest rates get more historically normal that it will become an extraordinarily important part of everyone's planning process. [Benz 2010]

However, hedging possibilities themselves have rarely been discussed. Cairns et al. [2006] simply remark that "the plan member's preference for a pension at retirement over a cash lump sum needs to be matched by a switch to long-dated bonds before retirement. . . rather than cash." Babbel and Merrill [2007b, p. 10] point out that "your accumulated assets need to be invested in something during the interim while awaiting the time to purchase a life annuity. . . the value erosion that typically accompanies rising interest rates may offset part or all of the gain that one hopes to garner by delaying the annuitization decision." Koijen et al. [2009] directly address hedging of annuity purchases, but their paper does not use actual annuity prices as this paper will, instead assuming annuities are (exactly) "fairly priced." They find that the intertemporal-utility-maximizing "hedging strategy holds long positions in 3-year nominal bonds and stocks, while 10-year nominal bonds and cash are shorted" (p. 23). This paper's simpler approach does not find the utility-maximizing percent of assets to be invested in annuities but takes that as given, and, to focus on easily-implemented strategies, it concentrates on long-only portfolios which use only two instruments ("long-only" in the sense of no asset borrowing, not "long-only" in the sense of hold holding long-term bonds).

Another line of research that resembles this paper's problem is asset-liability matching and liability-driven investing (see for example Brown and Jones [2011], Moore [2004], Vanguard [2007], and Amenc et al. [2009]), but annuity purchases have not been analyzed in that framework.

2. Data: Annuity Prices and Hedging Instruments

The web site `immediateannuities.com` publishes monthly "Comparative Annuity Reports," which give data on, among other things, the "Single-Premium

Immediate Annuity ["SPIA"] Payout Factor for Life with 10 Years Certain;" applying these factors to a $100,000 premium results in a monthly income (unindexed for inflation) this paper will simply call "payouts." The data set gives payouts offered by two to three dozen companies for eight age/gender categories of purchasers living in New Jersey: men and women aged 60, 65, 70 and 75 (abbreviated M60, F60, etc.). For each month, for each age/gender category, the payouts averaged over these companies were used. The data goes from 1/2003 to 7/2017, which is 175 observations, and are graphed in Figure 1 (solid lines, left-hand axis; that axis does not start at zero).[2]

In Figure 1 the monthly annuity income paid per $100,000 premium for a 65-year-old man varied $190, from a high of $680 to a low of $490, during this period. This $190 range was tied for the highest of the eight gender/age categories, but the others were not much less, with $177 being the lowest. This range as a percentage of its gender/age category's mean varied quite a bit however: it was 35% for F60 and M60, 32% for F65 and M65, 30% for F70, 29% for M70, 26% for F75, and 25% for F75, where gender/age categories are listed (as they always will be in this paper) from lowest to highest annuity payouts (lowest to highest lines in Figure 1, highest to lowest life expectancies). The varying interest rates affected, percentage-wise, older purchasers less than younger purchasers because more of older purchasers' annuity payout comes not from bond interest but from mortality "credits," which are unaffected by bond interest rate changes. The comparative insensitivity of older purchaser's annuity income to interest rate fluctuations means their hedging portfolios will be systematically different from those of younger purchasers.[3]

The nine bond mutual funds used, together with their ticker symbol, duration in mid-2017, and abbreviation this paper will use for them, were Vanguard's: Long-Term Investment-Grade (VWESX, 13.8, "LC" for "long corporate"); Long-Term Bond Index (VBLTX, 15.3, "LI"); Long-Term Treasury

[2]The "Comparative Annuity Reports" also provide graphs comparing annuity payouts with the yield on Moody's AAA Corporate Bonds, starting in 2003 and updated monthly. Similar data for Canada, using Canadian dollars and the Government of Canada 10 year bond yield and presented in tabular form, is available at `http://www.ifid.ca/payout.htm`.

[3]To the extent that persistent declines in *real* interest rates reflect bad economic conditions, they will co-vary with increases in rates of involuntary unemployment, in home foreclosures, in business and personal bankruptcies, and therefore with increases in the incidence of suicide, attractiveness of illicit drugs, and other things which cause mortality to rise. The rises in mortality in the USA since 2008 seem however to appear mostly in the under-65 population (perhaps because the USA has a stronger "safety net" of income and health care financing support for the elderly than for younger people). Therefore, the interest rate changes would not co-vary with over-65 mortality rates, which are the ones affecting annuities for the population studied in this paper.

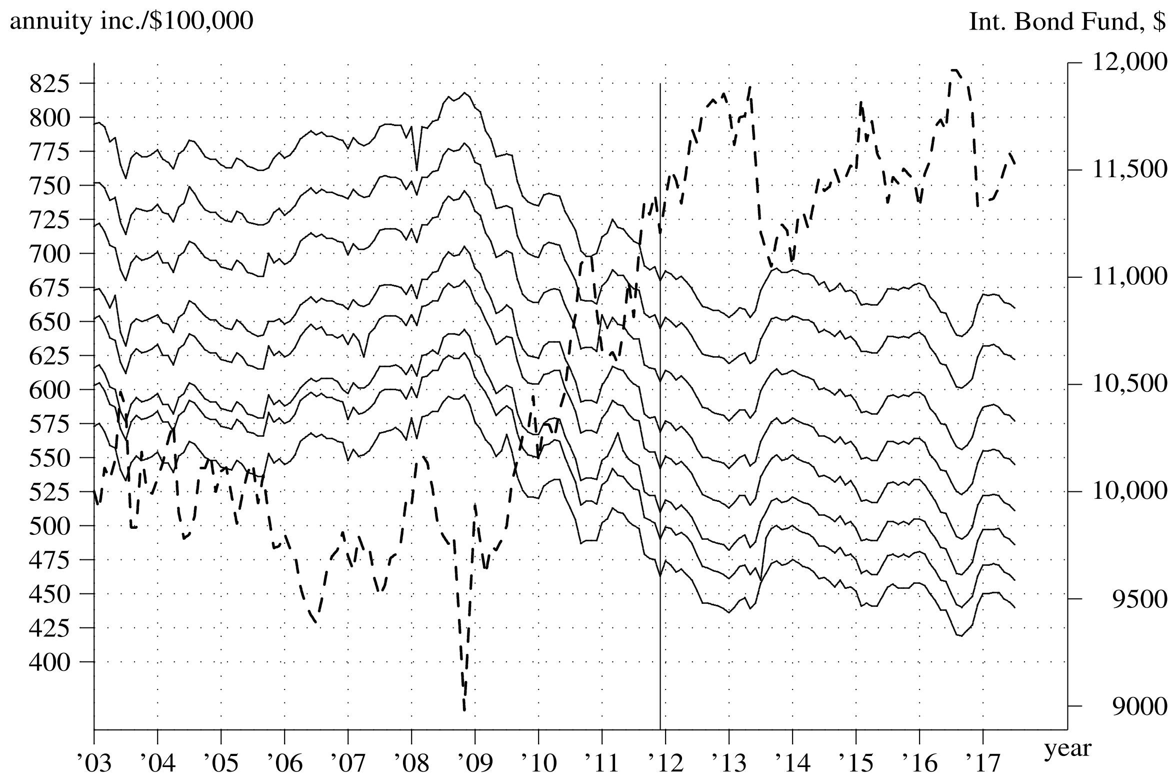

Figure 1. Thin solid lines: left-hand axis, monthly annuity income per $100,000 premium for, from bottom to top: females age 60, males age 60, females age 65, males age 65, females age 70, males age 70, females age 75, males age 75. Thick dashed line: right-hand axis, growth of an initial $10,000 invested in Vanguard's Intermediate-Term Bond Index Fund, reinvestment of capital gains only. Vertical line: December 1, 2011.

(VUSTX, 16.9, "LT"); Intermediate-Term Investment-Grade (VFICX, 5.5, "IC"); Intermediate-Term Bond Index (VBIIX, 7, "II"); Intermediate-Term Treasury (VFITX, 5.3, "IT"); Short-Term Investment-Grade (VFSTX, 2.6, "SC"); Short-Term Bond Index (VBISX, 2.8, "SI"); and Short-Term Treasury (VFISX, 2.3, "ST"). The duration of the actively-managed Vanguard funds do not change much over time, and the results in this paper should broadly apply to other simple (that is, mostly un-leveraged) bond mutual funds that hew to a well-defined style.

If interest rates in general rise, the value of an instrument which hedges annuities should fall; if interest rates fall, such an instrument's value should rise; and if interest rates are unchanged, such an instrument's value should be unchanged. Bond funds do not in general obey the last of these criteria because with unchanged interest rates their value increases over time as they earn interest. Discarding their income distributions, however, will create instruments whose value is unchanging if interest rates are unchanging (assuming the bonds' average premium or discount is negligible). Accordingly, for each of the bond funds we constructed a capital-gains-only return time series: we started with one share on 1/1/2003 and reinvested all the capital gains but ignored the dividend distributions. Fund values were calculated for the first day of each relevant month because the "Comparative Annuity Reports" are published at the beginning of each month. The data on distributions and prices came from a Bloomberg terminal. The other hedging instrument we used is a generic money market fund which, being stripped of its dividends, takes a constant value of one at every date (since U.S. *retail* money market funds still use "amortized cost"/"book-value" accounting instead of floating net asset value/"market-value" accounting). For the rest of this paper, all references to these funds will refer to their capital-gains-only versions, with their dividends discarded. Of course investors holding these instruments actually do receive interest earnings, which increase their owners' current consumption or increase their wealth and thus ability to afford more annuities later, but those benefits are irrelevant to their function as annuity hedges.

Figure 1 graphs the capital-gains-only "growth of $10,000" for the Intermediate-Term Bond Index Fund, "II." (The market disruptions of late 2008 are evident.) There is a very apparent negative correlation between II and affordable annuity income. For this period, the correlation between "monthly annuity income per $100,000 premium for men aged 65 ('M65')" and the bond funds (capital-gains-reinvested only, of course) were: LC, −0.92; LI, −0.95; LT, −0.91; IC, −0.92; II, −0.97; IT, −0.91; SC, −0.65; SI, −0.86; and ST, −0.84. The availability of instruments with strongly-negative correlation with "annuity income per $100,000 premium" bodes well for the individual investor's capacity to hedge changes in the cost of buying annuities. The correlation will however not be perfect. There were even six months (12/03, 7/06, 1/10, 2/13, 12/16, and the month 12/11 illustrated to the right of the vertical line in Figure 1) during which every age/gender annuity payment went up and every bond fund went up; but none of those months were contiguous, or even close to each other.

These irregularities limit how good a hedge against volatility bond funds will be, although they also mean consumers do not need interest-rate hedges as much as they would if interest rates and annuity payouts moved in lockstep.

3. Hedging for the Long Run

An initial \$100,000 permanently invested 100% in cash enabled a consumer to buy an amount of monthly annuity income given in Figure 1. However, if the initial \$100,000 was invested in a different portfolio, then the amount of annuity income the consumer could purchase would have varied not only with the respective F60, M60, etc. line of Figure 1, but also with the size of their portfolio. The term "affordable annuity income" will refer to the amount of monthly income a consumer could buy, at a particular date, had the consumer invested \$100,000 in a given portfolio on 1/1/2003 and rebalanced the portfolio to its initial allocations every month. The purpose of this section is to find which portfolio minimized the standard deviation of affordable annuity income over the period covered in our data for each of our age/gender combinations, using the bond funds described in Section 2.

Figure 2 gives our first result. Allowing a 65-year-old man to short-sell instruments and to hold as many instruments as we made available, the figure shows how much the standard deviation of affordable annuity income can fall compared to just looking a the annuity's cash price. The ten-instrument portfolio was LC 84%, LI −115%, LT 18%, IC −33%, II 284%, IT −101%, SC −47%, SI −275%, ST 241%, and MM 44% (the fractions sum to 100%). The extent to which this portfolio keeps affordable annuity income rather steady is impressive. However two objections can be raised to this result. First, individual investors would find it very difficult to hold such a complicated portfolio, and there would be short-selling costs if they did so, costs which we have not included. Second, the result is consequence of overfitting. Because there is no theoretical reason to suppose that in the future, for example, annuity hedging portfolios ought to go very short in SI and very long in ST, there is little confidence that the result gives useful guidance for the future. Accordingly, for the rest of this paper we rule out short sales and limit the purchaser to at most two, or in Section 4 three, instruments.

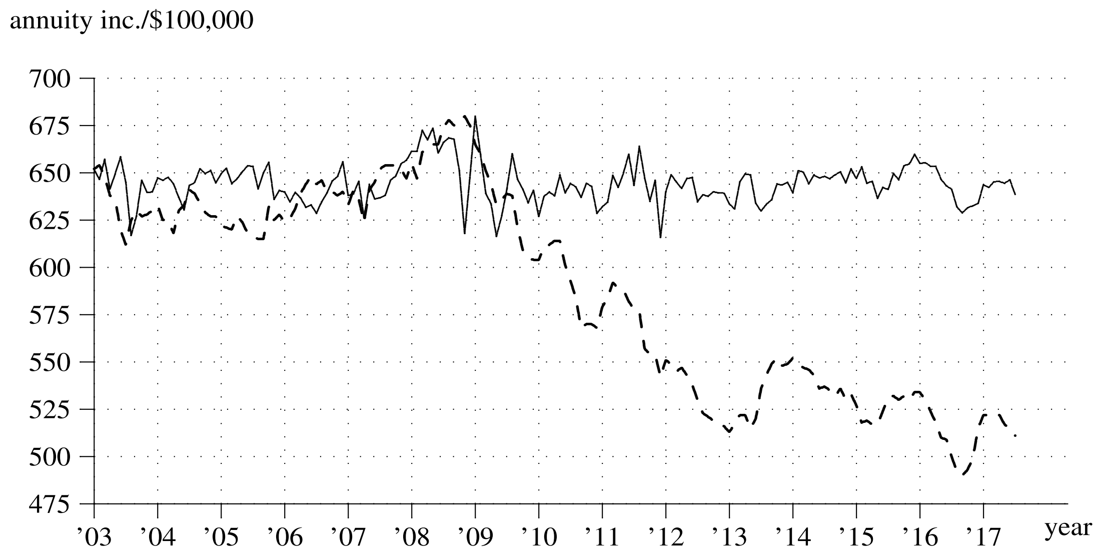

Figure 2. Thick dashed line: male age 65 monthly annuity income per $100,000. Its standard deviation is 54.3. Thin solid line: male age 65 annuity income per $100,000 initially invested, always in the "shorting allowed" portfolio given in the text. Its standard deviation is 10.4.

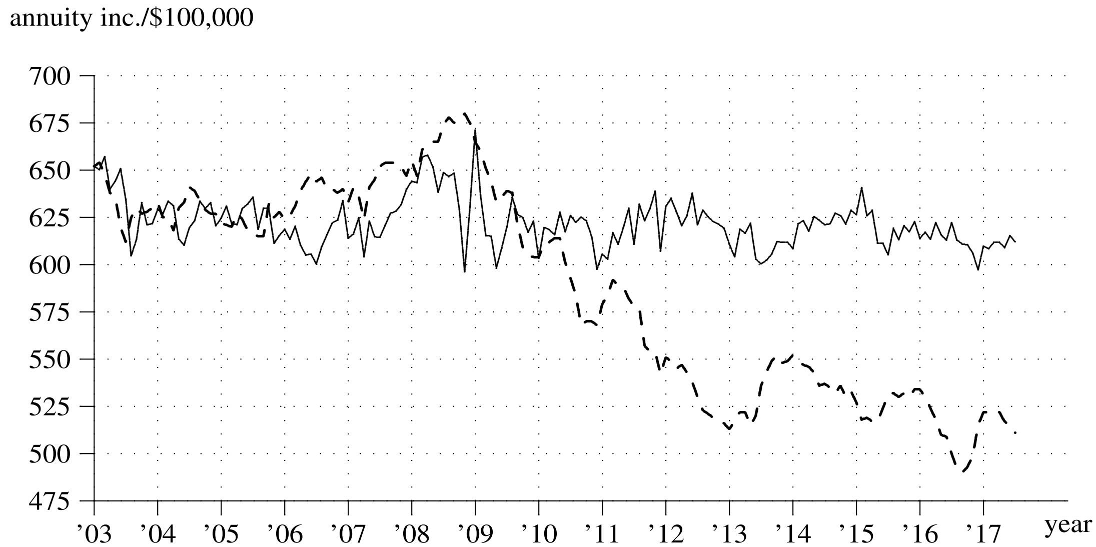

Figure 3. Thick dashed line: same as in Figure 2 (male age 65 monthly annuity income per \$100,000). Thin solid line: male age 65 annuity income per \$100,000 initially invested, always in 51% LC, 49% IT. Its standard deviation is 13.1.

With these added restrictions, the problem becomes one of, first, calculating the standard-deviation-minimizing shares of each instrument in each possible two-instrument portfolio—which number "ten things taken two at a time," namely $10!/(2!(10-2)!) = 45$—and, second, then choosing the best one of these 45. For the 65-year-old man of Figure 2, with these restrictions the best we could have achieved is Figure 3, with a portfolio 51% LC and 49% IT. This is not as good as in Figure 2, but its standard deviation of 13.1 represents a very large improvement over 100% cash's affordable annuity income's standard deviation of 54.3.

While Figure 3 shows generally higher affordable annuity income with the LC/II portfolio than with cash, that is only due to interest rates having fallen during the period studied. Had they risen, the LC/II portfolio would show generally lower affordable annuity income. The point of the LC/II portfolio and of all the other portfolios in this paper is not higher affordable annuity income, it is lower standard deviation of affordable annuity income. The 59% LC/49% II portfolio is the M65 efficient minimum-risk portfolio. A man contemplating annuity purchase at age 65 and willing to forego some risk-minimization in return for potentially higher annuity income will want to deviate from the LC/II

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
51.8										100	0.1	1.0
13.0	29		21		43	7					11	0.0
13.5		**55**			**45**						11	0.0
13.6			43	57							10	−0.1
13.9		66				34					12	0.3
14.1	52					48					10	0.0
18.7					100						6	0.0

Table 1. Selected results for females age 60 (long run).

portfolio in whatever direction he judges will have higher future returns. Such deviations are not discussed in this paper.

Tables 1–8 show the results for all the age/gender combinations, with Table 4 showing the results of all 45 two-instrument combinations and the other tables showing just the top few. The tables' first column is the "criterion," which is standard deviation. Their second-to-last column reports the portfolio's duration, but only reflects its mid-2017 status and so should only be taken as a general reflection of what its duration was over the entire period (though as mentioned above, these funds' durations do not tend to change very much). The tables' last column is an even rougher measure, an index of portfolio quality, where the corporate bond funds were assigned a rating of -1, the Treasury funds were assigned a rating of $+1$, and the index funds and money market fund were assigned a rating of zero. In all the tables, the first line is special: it shows the results of using 100% cash. In all the tables, the second line is also special: it shows the risk-minimizing portfolio if any number of instruments could be used, but no shorting is allowed. In all the abbreviated tables, the last line is special as well, showing which one-instrument portfolio had the smallest standard deviation. (The reason some numbers are set in bold is given in Section 4.)

Table 4 shows patterns which are typical of the other age/gender categories. The best portfolios cluster in the upper left, using long-term and intermediate-term bonds. The worst portfolios cluster in the lower-right, using short-term bonds. The worst portfolio of all is the 100% cash portfolio (the top line). All the good portfolios have high durations, but not all the high duration portfolios are good—quite a few of them are only middling. For example, portfolios of duration 10 appear both at the top and in the middle of the table. The ones at the very top achieve their duration of 10 by mixing only long-term and intermediate-term bonds. The others that do worse achieve their duration of 10 by mixing long-term and short-term bonds. As far as quality goes, the extremes (+1 and

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
54.8										100	0.1	1.0
14.0	31		22	5	28	15					11	0.0
14.4			44	56							10	−0.1
14.7		**56**			**44**						11	0.0
15.0		66				34					12	0.3
15.2	52					48					10	0.0
20.2					100						6	0.0

Table 2. Selected results for Males age 60 (long run).

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
51.0										100	0.1	1.0
12.2	34		8		38	20					10	−0.1
12.3	51					49					10	0.0
12.7		**37**			**63**						10	0.0
12.9			35	65							10	−0.3
13.5	31				69						9	−0.3
15.9					100						6	0.0

Table 3. Selected results for females age 65 (long run).

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
54.3										100	0.1	1.0
13.0	37		11	15	4	34					10	−0.1
13.1	51					49					10	0.0
13.6			36	64							10	−0.3
13.7		**39**			**61**						10	0.0
14.6	32				68						9	−0.3
14.7			18		82						8	0.2
14.8		69							31		11	0.3
14.9		71						29			12	0.0
15.4		60		40							11	−0.4
15.5		75					25				12	−0.3
15.8		78								22	12	0.2
16.1		58				42					11	0.4
17.3					100					0	6	0.0
17.3					100		0				6	0.0
17.3					100			0			6	0.0
17.3					100				0		6	0.0
17.3					100	0					6	0.0
17.3				0	100						6	0.0
17.6	67								33		10	−0.3
17.9			52				48				10	0.0
19.3	70							30			11	−0.7
19.6				45		55					5	0.1
19.9	33	67									15	−0.3
20.7	76		24								15	−0.5
20.8			48					52			10	0.5
20.9		100	0								15	0.0
21.3	59			41							10	−1.0
21.6	82									18	11	−0.6
22.1	81						19				12	−1.0
22.4			57							43	10	1.0
22.4			46						54		9	1.0
23.1			20			80					8	1.0
24.5						91	9				5	0.8
24.8						100		0			5	1.0
24.8						100				0	5	1.0
24.8						100			0		5	1.0
27.0				90					10		5	−0.8
27.1				100						0	6	−1.0
27.1				100			0				6	−1.0
27.1				100				0			5	−1.0
39.2									100	0	2	1.0
39.2								0	100		2	1.0
39.2							0		100		2	1.0
40.6								100		0	3	0.0
40.6							0	100			3	0.0
46.3							100			0	3	−1.0

Table 4. All results for males age 65 (long run).

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
51.3										100	0.1	1.0
12.0	31		4	6	31	20			9		9	0.0
12.4	21				79						8	−0.2
12.6		**20**			**80**						8	0.0
12.7	46					54					9	0.1
13.1			9		91						7	0.1
13.9					100						6	0.0

Table 5. Selected results for females age 70 (long run).

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
55.2										100	0.1	1.0
12.5	26		8	13	21	25			7		8	0.0
13.0	44					56					9	0.1
13.0		**21**			**79**						8	0.0
13.3	19				81						8	−0.2
13.4			10		90						8	0.1
14.6					100						6	0.0

Table 6. Selected results for males age 70 (long run).

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
52.2										100	0.1	1.0
11.8	23		2		48	5			22		7	0.1
12.6	7				93						7	−0.1
12.7					97					3	6	0.0
12.7				7	93						6	−0.1
12.7					97		3				6	0.0
12.8					97				3		6	0.0
12.8		**2**			**98**						7	0.0
12.8					98			2			6	0.0
12.8					100						7	0.0

Table 7. Selected results for females age 75 (long run).

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
53.8										100	0.1	1.0
11.9	16	1	2		58				23		7	0.1
12.4					95					5	6	0.1
12.5					93				7		6	0.1
12.6					95		5				6	−0.1
12.6					94			6			6	0.0
12.7				7	93						6	−0.1
12.8	2				98						7	0.0
12.8		**0**			**100**						6	0.0

Table 8. Selected results for males age 75 (long run).

−1) appear at the lower part of the table, while the upper part has values all close to zero. The best single-instrument portfolio, 100% II, has a standard deviation of 17.3, which is not as good as the 13.1 achieved by 51% LC/49% IT, but is much better than many other portfolios.

Rounding is the explanation for any numerical anomalies in the table. For example, consider the three rows with a standard deviation of 27.1. All report 100% IC and 0% of another instrument, but these figures are rounded-off not exact. The duration of IC is 5.5, so, rounding up, one would expect a “6” in the column for rounded duration. The reason the third line has a “5” instead of a “6” is that the actual risk-minimizing portfolio is 99.999999999941% IC/ 0.000000000058% SI. Since SI’s duration is less than 5.5, this portfolio’s duration falls just shy of 5.5, which then rounds to 5 instead of to 6. The other two portfolios’ durations round to 6 because they are exactly 100% in IC.

The top-four portfolios had durations of between 10 and 12 years for 60-year-old purchasers, 9 and 10 years for 65-year-olds, 7 to 9 years for 70-year-olds, and 6 and 7 years for 75-year-olds. This was to be expected based on Section 2’s observation that older purchasers’ annuity income is less sensitive to interest rate fluctuations than younger purchasers’. An even stronger reflection of that is that the tables for both 75-year-old purchasers are quite different from the others, with greater-than-92% allocations to II in all of their top portfolios and little or, more commonly, nothing in long-term bonds, whereas for all of the under-75-year-old purchasers, the top-four portfolios combined a long-term bond fund with an intermediate-term bond fund.

Using 51% LC/49% IT (duration 10 years) minimizes M65 standard deviation but its resulting annuity income still has a 0.38 correlation with unhedged (100% cash) annuity income, shown in the left-hand part of Figure 4. By con-

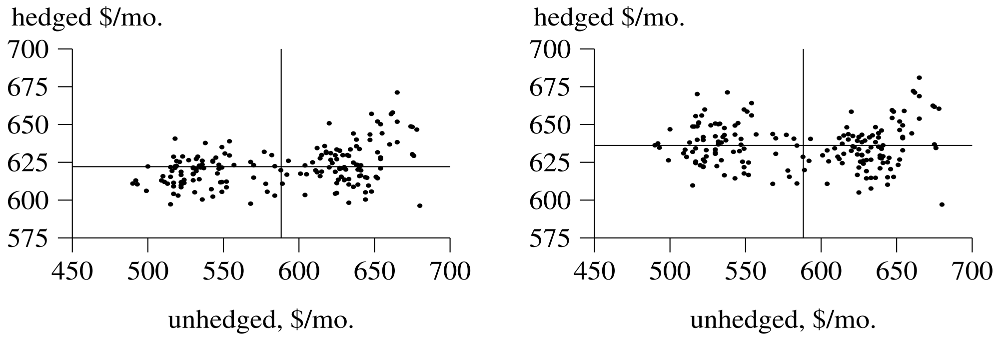

Figure 4. Left: Scatter diagram from Figure 3. Horizontal axis is annuity income with 100% cash; vertical axis is 51% LC/49% IT; correlation coefficient 0.38. Right: Scatter diagram from Figure 5. Same horizontal axis; vertical axis 59% LI/41% II; correlation coefficient zero. Vertical and horizontal lines denote means. Horizontal and vertical axes scaled equally.

trast the portfolio on the right-hand part of Figure 4 and in Figure 5, 59% LI/ 41% II (duration 12 years), makes M65 affordable annuity income have a zero correlation coefficient with cash M65. Some, but perhaps not many, retirement savers may want to accept greater standard deviation in return for such a zero correlation of their affordable annuity income with a 100% cash holder's affordable annuity income. Assuming LI/II portfolios, the zero-correlation proportion of LI, and the duration in years, is, for the other age/gender categories: F60, 93%, 15; M60, 96%, 15; F65, 57%, 12; F70, 30%, 9; M70, 33%, 9; F75, 6%, 7; and M75, 0%, 7. For younger purchasers these represent much higher durations than the standard-deviation-minimizing portfolios, but for older purchasers the differences are negligible.

Minimization of standard deviations reflects an older meaning of the term "hedge": it focuses on risk minimization, on trying to "lock in" a future annuity income. Attaining zero correlation reflects a newer meaning of the term "hedge," as in "hedge fund;" it does not shy away from volatility as long as the resulting changes in affordable annuity income are "market neutral"—"annuity-market neutral" in this paper, which is roughly the same as "bond-market neutral." Achieving bond market neutrality would be like having zero duration overall, in that the overall assets-plus-liabilities portfolio would not systematically vary with interest rates, but it would be volatile for other reasons.[4] A perfect hedge

[4]Technically, since zero correlation does not mean independence, zero correlation would still allow nonlinear dependence on interest rates. In this way zero correlation also resembles zero

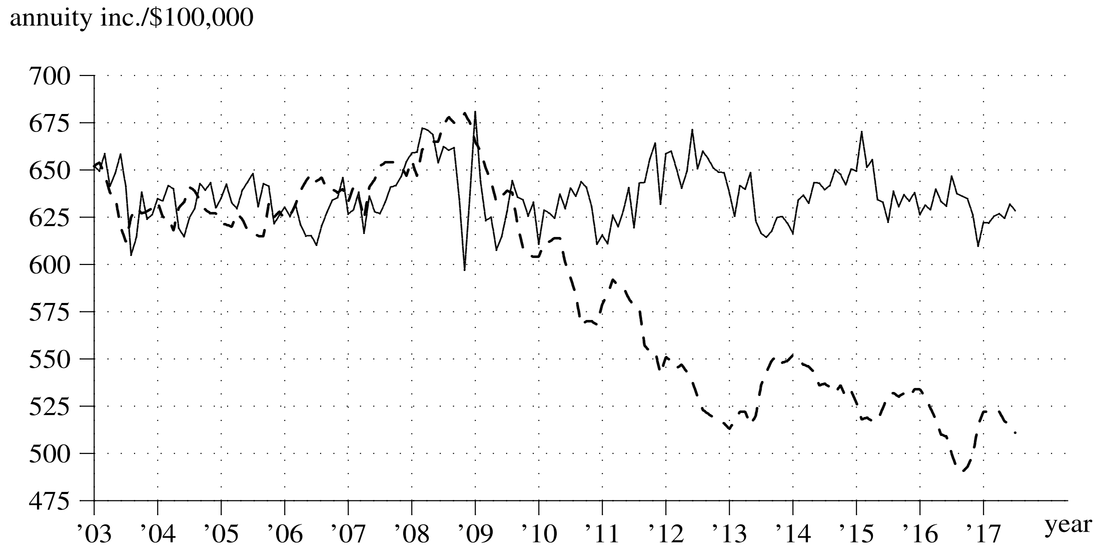

Figure 5. Thick dashed line: same as in Figure 2 (male age 65 monthly annuity income per $100,000). Thin solid line: male age 65 annuity income per $100,000 initially invested, always in 59% LI, 41% II. Its correlation with the thick dashed line is zero, but it does not minimize standard deviation of income, which Figure 3’s portfolio did.

in the standard-deviation sense would achieve a standard deviation of zero and therefore it would also achieve a correlation of zero and be a perfect hedge in the correlation-sense. Imperfect hedging (cross-hedging), as in this paper, entails a tradeoff between the two notions of a "hedge," as we have seen. Except for a brief discussion at the end of Section 4, we will ignore hedging in the zero-correlation sense for the rest of this paper.

4. Hedging for the Short Run

A hedging portfolio good for keeping affordable annuity income rather steady over a 14-year period may not be the best for keeping affordable annuity income rather steady over a much shorter period. For example, over all rolling 12-month periods, for M65, the largest variation in annuity income was \$71 (11%), the smallest was \$14 (2%), and the mean variation was \$31 (5%). This compared with a 32% variation for the entire period. In this section we find what hedging portfolios would have been the best for all rolling 12-month periods.[5]

In the previous section we used standard deviation as the measure of volatility. Here we want to minimize how much the subsequent 11 annuity payouts differ from the first. There are many possible ways to measure how much the last eleven payouts "vary from the first payout;" this paper uses $\sum_{i=1}^{12}(a_i - a_1)^2/12$, the (annual) "mean squared deviation from a_1." This is related to the standard deviation, but standard deviation measures deviations from the mean of the a_i's, not from the initial level a_1, which is the target of the hedge.[6]

Because this equation pertains to a single twelve-month period, there are $175 - 11 = 164$ values of it that have to be combined into one summary measure of volatility. There are many possible ways to do this; this section uses the square root of average of the 164 numbers,

$$\sqrt{\sum_{j=1}^{164}\left(\sum_{i=j}^{11+j}\frac{(a_i - a_1)^2}{12}\right)\Big/164}\quad, \tag{1}$$

as its criterion. The units of (1) are the same as for Section 3's criterion of standard deviation.

duration, which does not mean "no effect of changing interest rates on portfolio value," merely no linear effect, allowing nonzero nonlinear "convexity" effects.

[5] Using rolling periods rather than simple Monte Carlo techniques has the disadvantage of oversampling the months in the middle of the data set but the advantage of preserving the historically-interrelated time paths of bond prices and SPIA payouts. Using sophisticated Monte Carlo techniques would require econometric estimation of the (potentially nonstationary) time series properties of our data and is beyond the scope of this paper.

[6] For different risk measures see Hoe et al. [2010], Huang [2010], and Polak [2010].

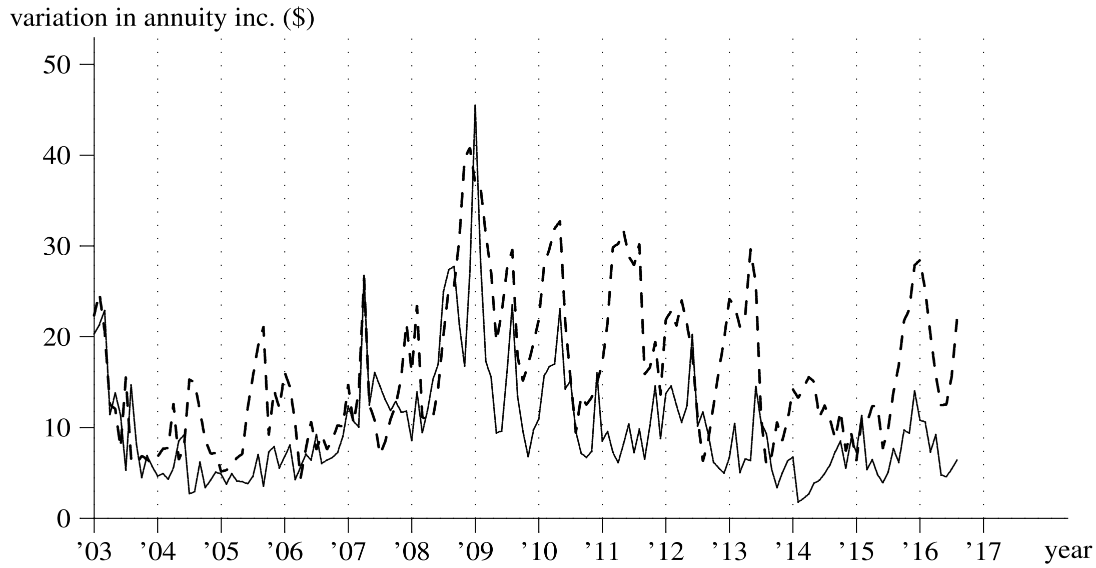

Figure 6. Variation in male age 65 affordable monthly annuity income in rolling 12-month periods. Thick dashed line: same as in Figure 9 (portfolio, initially $100,000, always 100% in cash). Thin solid line: portfolio, initially $100,000, always in 41% LI, 59% cash; average variation 11.8.

The analysis the proceeds in a similar way to Section 3. For example, the risk-minimizing portfolio for M65 is found by computing the risk-minimizing portfolio for all 45 pairs of instruments and then taking the best one. It is 41% LI/59% cash. Its duration, 6 years, is much lower duration than the 10-year duration of the 51% LC/39% IT portfolio that was best for M65 in the long run. That is no doubt because, as noted above, percent variations in annuity income per dollar of premium were so much smaller over 12-month periods than over the 14-year period.

Figure 6 illustrates this result. It graphs the variation measures

$$\sqrt{\sum_{i=j}^{11+j} \frac{(a_i - a_1)^2}{12}} \quad , \quad j = 1, 2, \ldots, 164 \qquad (2)$$

for the 100% cash portfolio and for the 41% LI/59% cash portfolio. To make the improvement of the latter over the former easier to see—it is not as dramatic as the improvement in Figure 3—Figure 7 presents the same data, sorted.

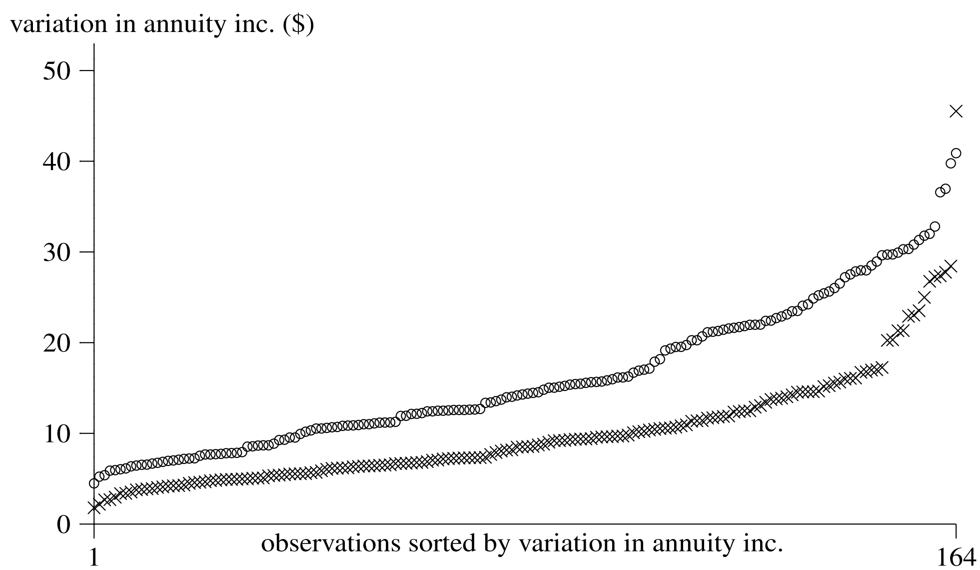

Figure 7. Sorted data of Figure 6. Variation in male age 65 affordable monthly annuity income in rolling 12-month periods. Circles: same as in Figure 10 (portfolio, initially $100,000, always 100% in cash). X's: portfolio, initially $100,000, always in 41% LI, 59% cash.

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
17.6										100	0.1	1.0
11.4	25		10		23					42	7	0.3
11.6		**46**								**54**	7	0.5
11.9		40						60			8	0.0
12.0	42								58		7	0.2
12.0	43									57	6	0.1
12.7					100						6	0.0

Table 9. Selected results for females age 60 (short run).

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
18.0										100	0.1	1.0
11.9	24		9		24					43	6	0.3
12.1		**45**								**55**	7	0.6
12.4	42									58	6	0.2
12.4	41								59		7	0.2
12.4		38						62			8	0.0
13.2					100						6	0.0

Table 10. Selected results for Males age 60 (short run).

Tables of numerical results for all of the age/gender categories follow, in Tables 9 to 16, using the same format as the tables of Section 3, except with the criterion “crit.” here being (1). As discussed for M65 above, these portfolios are very different from the ones of Section 3, with very heavy allocations here to cash, while also incorporating long- or intermediate-term bonds. Durations in the top 4 are between 6 and 8 for age 60, 6 and 7 for age 65, 4 to 6 for age 70 and for age 75, so again we see risk-minimizing duration falling with age, although not as strongly as in the long run. As in the long run, top-4 portfolio quality is never 100% corporate, nor ever is it 100% Treasury.

The tables bear out what we saw in comparing Figure 7 with Figure 3, namely that the improvements gained in the short run are not as dramatic as the improvements gained in the long run. For M65, Table 12’s best two-instrument portfolio achieved a criterion ((1)) of 11.8, versus 18.0 for a 100% cash portfolio. This is an improvement by a factor of about 1.5. The long-run table, Table 4, had a best two-instrument criterion whose improvement over 100% cash was by a factor of 54.3/13.1 = 4.1. The same general results occur for the other age/gender categories. One discovers why when calculating the average of

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
17.7										100	0.1	1.0
11.3	22		8		28					43	6	0.3
11.6		**43**								**57**	7	0.6
11.8	39								61		7	0.2
11.8		37						63			7	0.0
11.9	41									59	6	0.2
12.8					100						6	0.0

Table 11. Selected results for females age 65 (short run).

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
18.0										100	0.1	1.0
11.5	22		6		26					45	6	0.3
11.8		**41**								**59**	6	0.6
12.0	39									61	6	0.2
12.1	38								62		7	0.2
12.2	33							67			6	−0.3
13.3					100						7	0.0

Table 12. Selected results for males age 65 (short run).

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
17.6										100	0.1	1.0
11.1	19		4		31					45	5	0.3
11.6		**39**								**61**	6	0.6
11.7	37									63	5	0.3
11.7					69					31	5	0.3
11.7	36								64		6	0.3
13.1					100						6	0.0

Table 13. Selected results for females age 70 (short run).

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
17.9										100	0.1	1.0
11.0	18		3		33					46	5	0.3
11.6					**67**					**33**	4	0.3
11.6	36									64	5	0.3
11.6	29							71			6	−0.3
11.7	34								66		6	0.3
11.7		37								63	6	0.6
13.4					100						6	0.0

Table 14. Selected results for males age 70 (short run).

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
17.3										100	0.1	1.0
10.8	15				36					48	5	0.3
11.1					**62**					**38**	4	0.4
11.3	25							75			6	−0.3
11.4	33									67	5	0.3
11.5	31								69		6	0.4
11.6		33								67	5	0.7
13.8					100						6	0.0

Table 15. Selected results for females age 75 (short run).

crit.	LC	LI	LT	IC	II	IT	SC	SI	ST	MM	dur.	qual.
17.9										100	0.1	1.0
11.0	10		1	4	40					46	4	0.3
11.2					**62**					**38**	4	0.4
11.6	24							76			5	−0.2
11.8	30								70		6	0.4
11.9	32									68	4	0.4
11.9		23						77			6	0.0
12.0					53			47			5	0.0
12.1			23				77				6	−0.5
12.1		32								68	5	0.7
14.2					100						6	0.0

Table 16. Selected results for males age 75 (short run).

the 164 12-month correlation coefficients between, for example, M65 annuity income and the bond funds, which were: LC, −0.76; LI, −0.73; LT, −0.60; IC, −0.71; II, −0.73; IT, −0.57; SC, −0.48; SI, −0.57; and ST, −0.49. The long-run coefficients were all between 0.16 and 0.34 lower (closer to −1) than these. Weaker negative correlation between the target and its hedges explains less-effective hedging.

The three age/gender categories with the lowest life expectancies (highest payouts) all had "II/MM" as the top two-instrument portfolio. The other age/gender categories all had "LI/MM" as the top two-instrument portfolio. In the long run, the risk-minimizing age-75 portfolios were close to 100% II. For age 75, then, moving from the long-run hedging portfolio to the short-run hedging portfolio would just be a matter of moving from 100% II to some combination of II and MM. The simplest transition for younger purchasers would involve moving from an "LI plus intermediate" portfolio in the long run to an LI/MM portfolio in the short run. In every one of the long run tables, the best two-instrument portfolio which contains LI has II as its other component. These LI/II long-run portfolios are, for the under-75 age group, quite good, taking first place once, second place four times, and third place once (recall these rankings are out of 45 two-instrument portfolios). To simplify transition from the long run to the short run, then, the most interesting portfolios are LI/II in the long run and, in the short run, LI/MM for younger purchasers and II/MM for M70, F75, and M75 (although LI/MM has ranks 5, 5, and 8, which are not bad, even for the oldest groups). These LI/II, LI/MM, and II/MM portfolios are the ones in bold in the tables. Portfolios which only use LI, II and MM should be sufficient for prudent annuity-hedging investors in the future: they span the needed duration space, are desirable in that they are neither 100% corporate nor 100% Treasury, and, being index funds, should have correlations with annuity prices in the future which are especially similar to those they have had in the past.

Given the results of this section it goes without saying that portfolios risk-minimizing in the short run are not risk-minimizing in the long run and vice versa, but illustrations of the extent to which they fall short are still of interest. Figure 8 shows that the short-run risk-minimizing portfolio does help somewhat in the long run, but clearly not as much as the long-run risk-minimizing portfolio shown in Figure 3. Similarly, the long-run risk-minimizing portfolio's performance in the short run is shown in Figures 9 and 10: helpful compared to 100% cash, but not as good as the short-run risk-minimizing portfolio of Figures 6 and 7.

As in Section 3, the risk-minimizing portfolios in this section retain some average (over 164 periods) correlation with unhedged annuities. For the M65 41% LI/59% MM portfolio (duration 6 years), the mean correlation coefficient annuity inc./$100,000

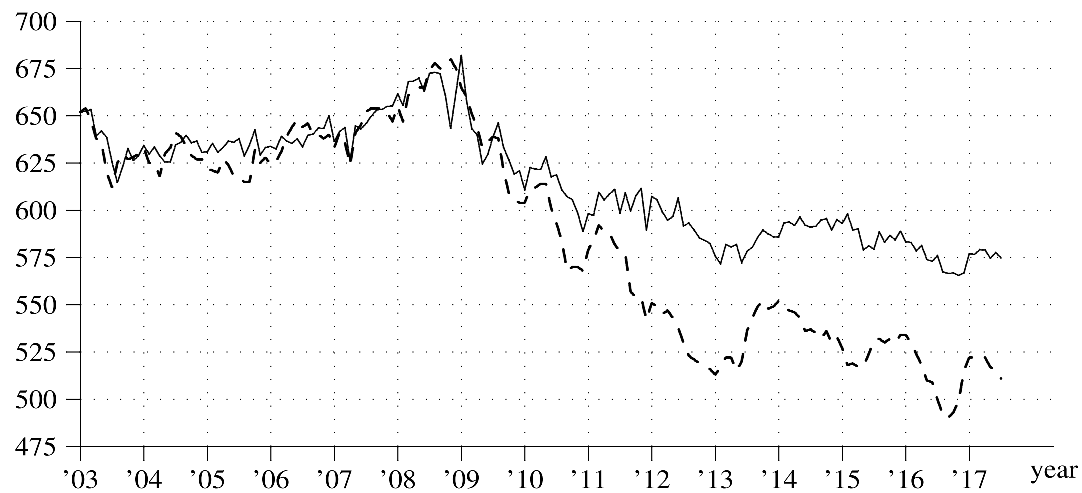

Figure 8. Thick dashed line: same as in Figure 2 (male age 65 monthly annuity income per $100,000). Thin solid line: male age 65 annuity income per $100,000 initially invested, always in 41% LI, 59% cash. Its standard deviation is 28.9.

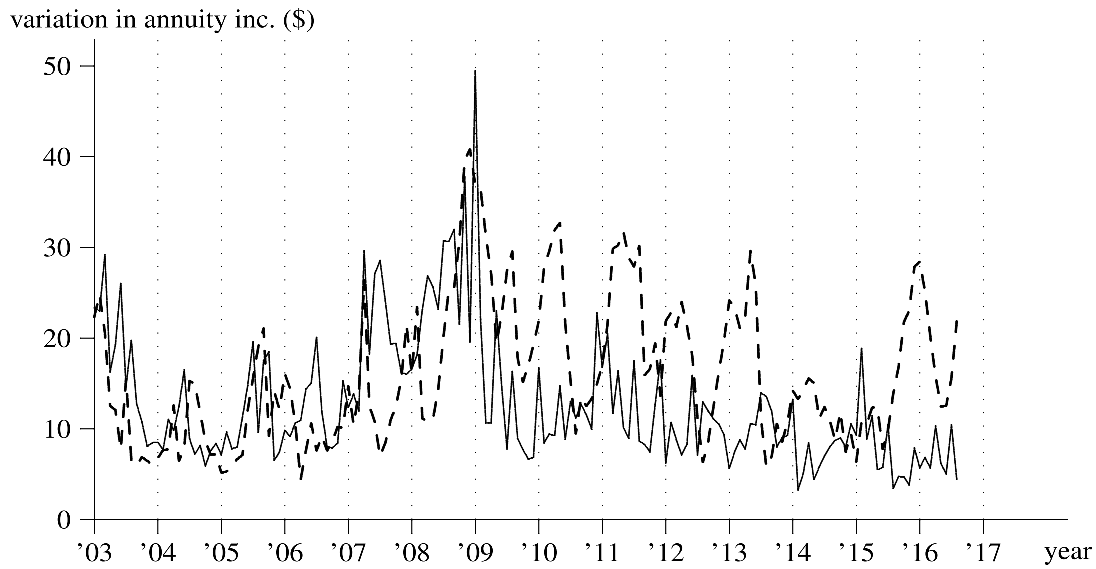

Figure 9. Variation in male age 65 affordable monthly annuity income in rolling 12-month periods. Thick dashed line: portfolio, initially $100,000, always 100% in cash; average variation 18.0. Thin solid line: portfolio, initially $100,000, always in 51% LC, 49% IT; average variation 14.8. (For the definition of 'variation,' see text.)

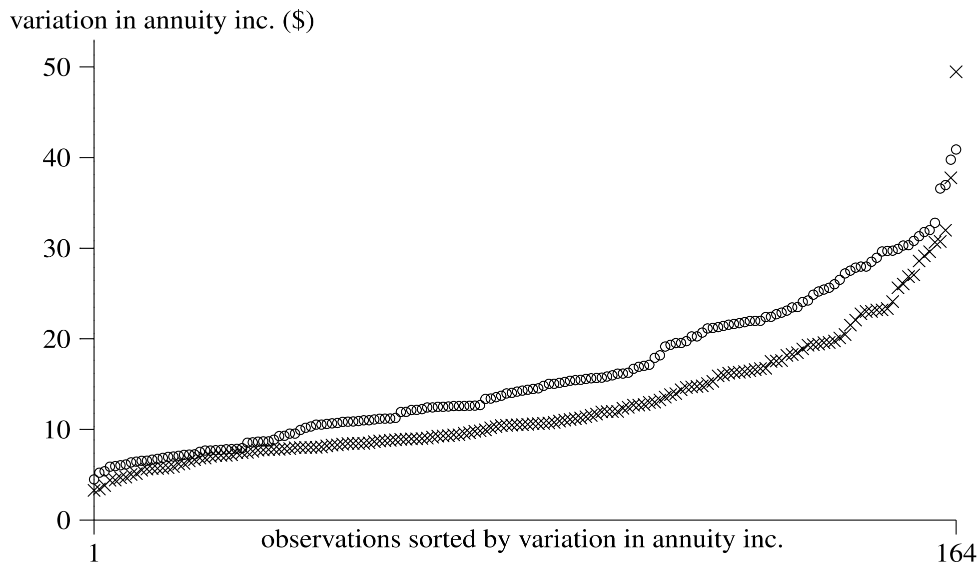

Figure 10. Sorted data of Figure 9. Variation in male age 65 affordable monthly annuity income in rolling 12-month periods. Circles: portfolio, initially $100,000, always 100% in cash. X's: portfolio, initially $100,000, always in 51% LC, 49% IT.

with unhedged M65 was 0.50. An M65 portfolio of 27% LI/75% II (duration 9 years) would have had zero mean correlation with unhedged M65. Assuming LI/II portfolios, the zero-mean-correlation proportion of LI, and the duration in years, is, for the other age/gender categories: F60, 45%, 10; M60, 43%, 10; F65, 32%, 9; F70, 16%, 8; and M70, 11%, 7. For 75-year-olds, no LI/II portfolio can attain zero mean correlation without short selling, but these II/MM portfolios do: F75, 99% II/1% MM, duration 6 years; M75, 94% II/6% MM, duration 6 years. As in Section 3, for younger purchasers these represent much higher durations than the standard-deviation-minimizing portfolios, but for older purchasers the differences are negligible.

5. Hedging for the Medium Run, and Overall

Near the end of Section 3 we suggested that a prudent time path for an annuity-hedging portfolio would be from the long-run LI/II to the short-run II/MM for M70, F75, and M75, and from the long-run LI/II to the short-run LI/MM for the other groups. The long run we studied was more than 14 years; the short run was 12 months. This raises the question of how the transition from the long run to the short run should occur.

In this section we investigate this limiting ourselves to the three instruments of greatest interest, LI, II, and MM. We use the same procedure as for the short-run of 12 months, but apply it to 30 months, then 60 months, then 90 months, and finally 120 months. The results are shown in Tables 17–20, which follow the same format as earlier tables but omit their "100% cash" line. LI/II or, for some older purchasers, II/MM portfolios are shown in bold.

We expect from our study of the long run and the short run that the risk-minimizing duration will be shorter for older purchasers than for younger purchasers. This holds within each of the new tables of this section.

We expect from our study of the long run and the short run that the risk-minimizing duration will rise as the hedging time period rises. This is true for the transitions from the short run to 30 months, from 30 to 60 months, from 60 to 90 months, and from 120 months to the long run, but it is not true of the transition from 90 to 120 months, for all age/gender categories. (At best, depending on which measure of duration one uses, some of the older age groups' duration is the same for 90 and 120 months.) The effect, fortunately, is not very large.

Table 21 is a *subjective* synthesis of the general trends this paper has uncovered. It uses only LI, II, and MM. It groups the 60, 90, and 120-month horizons together to smooth over the anomaly discussed in the previous paragraph. The allocations given in the top part of the table generate the durations in the bottom

crit.	LI	II	MM	dur.	qual.	crit.	LI	II	MM	dur.	qual.
F60						M60					
16.4	33	53	14	9	0.1	17.1	31	54	15	8	0.2
16.7	**25**	**75**		9	0.0	17.4	**22**	**78**		8	0.0
17.5		100		6	0.0	18.1		98	2	6	0.0
17.5	62		38	10	0.4	18.2	60		40	9	0.4
F65						M65					
16.6	29	51	20	8	0.2	17.0	28	51	21	8	0.2
17.2	**17**	**83**		8	0.0	17.8	**15**	**85**		8	0.0
17.6		93	7	6	0.1	18.0		91	9	6	0.1
17.8	57		43	9	0.4	18.3	56		44	9	0.4
F70						M70					
16.5	22	54	25	7	0.2	16.2	19	56	25	7	0.2
17.2		**85**	**15**	6	0.1	16.8		**84**	**16**	5	0.2
17.7	7	93		7	0.0	17.5	4	96		7	0.0
18.0	51		49	8	0.5	18.1	50		50	8	0.5
F75						M75					
16.3	12	59	29	6	0.3	16.0	9	63	28	6	0.3
16.6		**77**	**23**	5	0.2	16.1		**76**	**24**	5	0.2
18.3		100		6	0.0	18.1		100		6	0.0
18.5	44		56	7	0.6	18.8	43		57	7	0.6

Table 17. Results for 30 months.

crit.	LI	II	MM	dur.	qual.	crit.	LI	II	MM	dur.	qual.
F60						M60					
18.9	**41**	**59**	0	10	0.0	19.6	**40**	**60**	0	10	0.0
18.9	41	59		10	0.0	19.6	40	60		10	0.0
21.3	79		21	12	0.2	22.1	78		22	12	0.2
21.8		100		6	0.0	22.5		100		6	0.0
F65						M65					
18.9	34	62	4	9	0.0	19.7	34	60	6	9	0.1
19.0	**31**	**69**		9	0.0	19.8	**29**	**71**		9	0.0
20.9		100		6	0.0	21.8		100		6	0.0
21.4	73		27	11	0.3	22.2	72		28	11	0.3
F70						M70					
18.7	26	63	11	8	0.1	18.5	23	65	11	8	0.1
19.2	**17**	**83**		8	0.0	19.1	**14**	**86**		8	0.0
19.9		99	1	6	0.0	19.7		98	2	6	0.0
21.7	66		34	10	0.3	22.3	64		36	10	0.4
F75						M75					
18.2	14	70	16	7	0.2	18.0	11	72	17	6	0.2
18.7		**89**	**11**	6	0.1	18.3		**88**	**12**	6	0.1
19.5		100		7	0.0	19.6		100		6	0.0
22.8	58		42	9	0.4	23.4	56		44	9	0.4

Table 18. Results for 60 months.

crit.	LI	II	MM	dur.	qual.	crit.	LI	II	MM	dur.	qual.
F60						M60					
20.6	**49**	**51**	0	11	0.0	21.3	**48**	**52**	0	11	0.0
20.6	49	51		11	0.0	21.3	48	52		11	0.0
22.9	87		13	13	0.1	23.8	87		13	13	0.1
24.8		100		6	0.0	25.6		100		6	0.0
F65						M65					
20.1	37	61	2	10	0.0	21.0	38	59	3	10	0.0
20.1	**35**	**65**		10	0.0	21.0	**34**	**66**		9	0.0
22.5	80		20	12	0.2	23.4	79		21	12	0.2
22.7		100		6	0.0	23.6		100		6	0.0
F70						M70					
19.9	29	61	9	8	0.1	19.9	26	66	8	8	0.1
20.4	**17**	**83**		8	0.0	20.3	**15**	**85**		8	0.0
21.2		100		6	0.0	21.1		100		6	0.0
22.7	72		28	11	0.3	23.6	72		28	11	0.3
F75						M75					
19.4	18	68	15	7	0.1	19.4	14	71	15	7	0.1
20.1		**92**	**8**	6	0.1	19.8		**90**	**10**	6	0.1
21.0		100		6	0.0	21.2		100		6	0.0
23.7	65		35	10	0.4	24.5	63		37	10	0.4

Table 19. Results for 90 months.

crit.	LI	II	MM	dur.	qual.	crit.	LI	II	MM	dur.	qual.
F60						M60					
16.8	**44**	**56**	0	10	0.0	17.8	**43**	**57**	0	10	0.0
16.8	44	56		10	0.0	17.8	43	57		10	0.0
19.5	86		14	13	0.1	20.7	86		14	13	0.1
20.4		100		6	0.0	21.4		100		6	0.0
F65						M65					
16.6	37	55	8	9	0.1	17.2	36	55	9	9	0.1
17.0	**27**	**73**		9	0.0	17.8	**25**	**75**		9	0.0
18.5		100		6	0.0	19.3		100		6	0.0
18.9	77		23	12	0.2	19.8	76		24	12	0.2
F70						M70					
16.4	30	55	15	8	0.2	16.7	27	60	12	8	0.1
17.9		94	6	6	0.1	18.0	**11**	**89**		7	0.0
18.1	**9**	**91**		7	0.0	18.2		96	4	6	0.0
19.1	69		31	11	0.3	20.4	71		29	11	0.3
F75						M75					
16.3	21	59	20	7	0.2	16.7	18	63	19	7	0.2
17.2		**87**	**13**	6	0.1	17.5		**87**	**13**	6	0.1
19.8		100		6	0.0	20.4		100		6	0.0
20.2	63		37	10	0.4	21.3	63		37	10	0.4

Table 20. Results for 120 months.

time till purchase	F60	M60	F65	M65	F70	M70	F75	M75
long run	55/45/−		40/60/−		20/80/− *simpler: −/100/−*		−/100/−	
120/90/ 60 mo.	45/55/−		30/70/−		15/85/− *simpler: −/100/−*		−/90/10	
30 mo.	25/75/−	20/80/−	15/85/−		−/75/25			
12 mo.	45/−/55 *simpler: −/100/−*			40/−/60	−/70/30	−/65/35	−/60/40	

time till purchase	F60	M60	F65	M65	F70	M70	F75	M75
long run	11.3		10.0		8.3 *simpler: 7.0*		6.5	
120/90/ 60 mo.	10.5		9.1		7.8 *simpler: 7.0*		5.9	
30 mo.	8.7	8.3	7.8		4.9			
12 mo.	7.0 *simpler: 7.0*			6.2	4.6	4.3	4.0	

Table 21. Top: rounded, smoothed results from LI/II/MM boldface allocations in previous tables. Bottom: the resulting durations.

part of the table. They follow the pattern identified in all of this paper's results (except the 90- to 120-month transition), falling from left to right and falling from top to bottom. Using the "simpler" variants increases standard deviation slightly but means the F60/M60/F65/M65 portfolios only use LI and II, never MM, and the other portfolios only use II and MM, never LI, so one would have two-instrument portfolios all the way through time across the board.

Conclusion

For someone planning on buying a single-premium immediate fixed annuity, holding funds for that future purchase in cash means exposing oneself, for good or for ill, to the full vicissitudes of the fixed-income market. Holding those funds in short-term bonds is almost as risky. A risk-averse person—as many annuity purchasers probably are—will want some shield, if only partial, from such vicissitudes. This paper showed what risk-minimizing portfolios looked like for the future annuity buyer during the period from January 2003 to July 2017. If risk-minimization is the only goal, it was fine to limit oneself to only using bond index funds and money-market accounts. Risk-minimizing bond portfolio durations for purchases well in the future ranged from 11, for those planning to buy at age 60, to 6 or 7 for those planning to buy at age 75. Risk-minimizing bond portfolio durations for purchases expected to be made within the next twelve months ranged from 7, for those planning to buy at age 60, to 4, for those planning to buy at age 75.

Unfortunately, there was no data available to study inflation-indexed SPIAs; future work on hedging purchases of such annuities would be welcome. Data is also lacking on single-premium fixed deferred annuities, in which payments start at a later date. They "lock in" future annuity income with certainty (conditional on solvency of the insurer), so the open question would be how much more expensive they are than an actuarially-fair policy, a question which requires some way to measure the opportunity costs of buying such a deferred annuity instead of investing the funds oneself and buying an immediate annuity later.

Another limitation is that we only studied the risk-minimizing point on the future-annuity-purchase efficient frontier; no effort was made to locate the other points on that frontier, points representing higher expected annuity incomes but also greater annuity purchase risk. Retirement savers desiring not to have a risk-minimizing portfolio will want to deviate from the points computed in this paper according to their tolerance for risk and, perhaps, predictions of future interest rates. Even they, however, may benefit from realizing to what extent cash can be a risky fixed-income instrument and long-term bonds can be a safe one.

Those familiar with liability-matching portfolios will find that last point a triviality, but may still be interested in the extent to which our intuition about what annuity hedging portfolios ought to look at—long for young purchasers and for purchases far in the future, shorter for old purchasers and purchases to be made soon—are confirmed by the data we have.

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