For Constant-Duration or Constant-Maturity Bond Portfolios, Initial Yield Forecasts Return Best near Twice Duration

previous title: "Initial Yield as a Forecast of Constant-Duration or Constant-Maturity Bond Portfolio Return near Twice Duration"

Gabriel A. Lozada

Department of Economics, University of Utah,
Salt Lake City, UT 84112
lozada@economics.utah.edu

March 2015

Abstract. Leibowitz and co-authors showed that if yield paths are linear in time, a constant-duration bond portfolio's initial yield forecasts its mean return near twice duration. We show that continuously/periodically-compounded returns match arithmetic/geometric mean returns and derive results similar to Leibowitz's for both cases. We also link positive/negative forecast error (realized returns minus initial yields) to the yield path's concavity/convexity. Sixty-two years of data on short, intermediate, and long bonds over various horizons reveal forecast errors at twice duration which are modest and well-explained by convexity and return formulas' nonlinearities.

It is well-known that when default-free bonds are bought and held for their duration, they will earn to a first-order approximation their initial yield-to-maturity, and thus they constitute over that horizon a negligible source of risk despite their short-run volatility. Some authors have asserted that even if bonds are not bought and held, but rather are regularly rolled over to maintain an approximately constant maturity or constant duration, they are still less risky than their short-run volatility suggests, and in fact that their return over a relatively long period will be close to their initial yield.

One group of authors takes this "relatively long period" to be the bonds' duration or maturity. Potts and Reichenstein (2004) show that cumulative return of a constant-maturity portfolio gets close to that predicted by initial yield at roughly the bonds' maturity; a similar assertion is made by John C. Bogle and others in Gay (2014). William McNabb, current CEO of the Vanguard Group, uses duration:

> There is a silver lining to rising [interest] rates. If your time horizon is longer than the duration of the bond funds you are invested in, you actually want interest rates to rise. [McNabb 2014]

By contrast, the important early paper of Langeteig, Leibowitz and Kogelman (1990) (henceforth LLK) uses simulation to suggest that cumulative return of a constant-duration portfolio gets closest to that predicted by initial yield at roughly *twice* the bonds' duration. This idea of a very long period seems to have been ignored until it was picked up again, and furnished with a theoretical explanation, in Leibowitz and Bova (2012) and especially in "Part I: Duration Targeting: A New Look at Bond Portfolios" in Leibowitz, Bova, Kogelman, and Homer (2013, henceforth LBKH); see Leibowitz, Bova, and Kogelman (2014) for a summary of both the theoretical and empirical arguments, and Bova (2013, pp. 4–8) and Leibowitz and Bova (2013) for empirical support.[1]

I will refer to the body of work developed by Leibowitz and his collaborators as "Leibowitz et al." Among the results of Section 1 are an extension of their theoretical framework to coupon bonds and the provision of a simple graphical interpretation of the main result. Section 1 implies that empirical work should investigate whether initial yield equals mean return near twice duration. Section 2 shows that Section 1's interpretation of "mean" return as the "arithmetic mean" is appropriate for continuous compounding, then develops initial-yield-versus-mean-return results for periodic compounding using its appropriate mean, the geometric. Section 2 also explains why mean return minus initial yield will tend to be negative if the path of yield through time is convex and positive if it is concave. Sections 3–5 report empirical results using US bond yields over six or more decades, using continuously-compounded yields and arithmetic mean returns because Section 2 shows that is the best way to find a close match between initial yield and mean

[1] Fridson and Xu (2014) point out that junk bonds' long-term return will fall short of their initial yield.

return. Section 3 treats many different horizons, while Section 4 focuses on Section 1's theoretically-important horizon of twice duration, illustrates the main empirical findings in Figures 4 and 6, and uses Section 2's convexity results to explain historical gaps between initial yields and mean returns. Section 5 explains those gaps more systematically. Overall, we confirm the basic conclusion of Leibowitz et al.: initial yield is a good forecast of constant-maturity or constant-duration bond return at twice duration, and not as good a forecast at much shorter or longer periods.

1. The Constant-Duration Framework and Results of Leibowitz et al., and Extensions

Supposing that at dates 1, 2, 3, ... a bond generates payments ("coupons") C_1, C_2, C_3, ..., denote by $PV(Y)$ the present value of the bond's future income flows discounted at rate Y, namely $\sum_{t=1}^{\infty} C_t e^{-Yt}$ or $\sum_{t=1}^{\infty} C_t/(1+Y)^t$ depending on whether discounting is, respectively, continuous or periodic. The bond's "modified duration" D is defined by $(-1/PV(Y)) \cdot d\,PV(Y)/dY$. Duration has units of time, and after the passage of D (respectively, $(1+Y)D$) periods, the future value of the bond is, to a first order approximation, the same irrespective of any change in its initial yield:

$$0 = \frac{\partial}{\partial Y}\left[PV(Y)\,e^{Yt}\right] \Rightarrow t = \frac{-1}{PV(Y)}\frac{\partial PV(Y)}{\partial Y} = D$$

and

$$0 = \frac{\partial}{\partial Y}\left[PV(Y)\,(1+Y)^t\right] \Rightarrow t = -\frac{1+Y}{PV(Y)}\frac{\partial PV(Y)}{\partial Y} = (1+Y)\,D$$

(the latter can easily be shown[2] to equal the "Macauley Duration of the periodically-compounded bond"). So if one holds on to the bond until date D (respectively, $(1+Y)D$), the return will be approximately the same as the initial yield. This paper addresses the question of whether, if one periodically sells one's bond holdings before date D, each time buying a new bond with duration D, one can expect the return of this "rolled bond" portfolio over some period of time to equal (or approximately equal) the initial yield of the first bond.

In this framework, an initial investment is made in a bond with duration D and initial yield Y_1, and at the end of each period, the bond is sold and the proceeds reinvested ("rolled") into a new bond with duration D, where $D > 1$, i.e., D is longer than the length of one period. (If D were less than one period the previous paragraph's result applies.) Without loss of generality, assume yield in period $t > 1$, denoted Y_t, evolves as $Y_t = Y_{t-1} + \Delta Y_{t-1}$. Make the following approximation.

[2] `http://en.wikipedia.org/wiki/Bond_duration`

Proposition 1. *["The Return Approximation"] An approximation of the one-period return R for a bond which is originally priced at yield Y but whose yield permanently changes to Y^* at the end of period is*

$$R_t \approx Y_t - (D_t - 1)\,\Delta Y_t$$

where D is the Modified Duration.

Proof. For the periodically-compounded case, the one-period return is $Y + PV(Y^*)/PV(Y) - 1$. Expanding $PV(Y^*)$ in a first-order Taylor Series around Y and using the definition of D gives $PV(Y^*)/PV(Y) = 1 - D\,\Delta Y$ for small ΔY, which leads to $R = Y - D\,\Delta Y$. However, the yield changes at the end of the period, when the duration of the bond becomes approximately $D - 1$. This is because the "Macauley duration for periodic compounding" at the beginning of the period, which is defined to be

$$\frac{\frac{C_1}{1+Y} + \frac{2C_2}{(1+Y)^2} + \frac{3C_3}{(1+Y)^3} + \cdots}{\frac{C_1}{1+Y} + \frac{C_2}{(1+Y)^2} + \frac{C_3}{(1+Y)^3} + \cdots} \tag{1}$$

will be approximately equal to "one plus the Macauley duration for periodic compounding" at the end of the period,

$$\begin{aligned} 1 + \frac{\frac{C_2}{1+Y} + \frac{2C_3}{(1+Y)^2} + \cdots}{\frac{C_2}{1+Y} + \frac{C_3}{(1+Y)^2} + \cdots} &= \left[\frac{\frac{C_2}{1+Y} + \frac{C_3}{(1+Y)^2} + \cdots}{\frac{C_2}{1+Y} + \frac{C_3}{(1+Y)^2} + \cdots} + \frac{\frac{C_2}{1+Y} + \frac{2C_3}{(1+Y)^2} + \cdots}{\frac{C_2}{1+Y} + \frac{C_3}{(1+Y)^2} + \cdots} \right] \cdot \frac{\frac{1}{Y}}{\frac{1}{Y}} \\ &= \frac{\frac{2C_2}{(1+Y)^2} + \frac{3C_3}{(1+Y)^3} + \cdots}{\frac{C_2}{(1+Y)^2} + \frac{C_3}{(1+Y)^3} + \cdots} \end{aligned}$$

as long as, writing the above in shorthand, $\frac{x+A}{x+B} \approx \frac{A}{B}$. If $C_1 = 0$, as for a zero-coupon bond, this approximation is exact because $x = C_1/(1+Y)$. To determine when this approximation is good for a coupon bond, assume C_1 is equal to the initial yield Y times the initial value $PV(Y)$. Note that $A > B$ since all the C's are positive. Define $f(x+A, x+B) = (x+A)/(x+B)$; then expand

$$\begin{aligned} f(x+A, x+B) &\approx f(A,B) + \left.\frac{\partial f(A,B)}{\partial A}\right|_{A,B} x + \left.\frac{\partial f(A,B)}{\partial B}\right|_{A,B} x \\ &= \frac{A}{B} + \frac{x}{B} - \frac{Ax}{B^2} = \frac{A}{B}\left(1 - \frac{x}{B}\right) - \frac{x}{B}\,. \end{aligned}$$

The approximation is good when $1 - x/B \ll 1$, which is equivalent to $x \ll B$, and when $A/B \gg x/B$, which is equivalent to $x \ll A$. Since $A > B$, only $x \ll B$ needs to be satisfied. One has

$$\begin{aligned} \frac{x}{B} &= \frac{C_1/(1+Y)}{\frac{C_2}{1+Y} + \frac{C_3}{(1+Y)^2} + \cdots} = \frac{C_1}{PV(Y) - \frac{C_1}{1+Y}} = \frac{Y \cdot PV(Y)}{PV(Y) - \frac{Y \cdot PV(Y)}{1+Y}} \\ &= \frac{Y(1-Y)}{1+Y-Y} = Y(1-Y) \end{aligned} \tag{2}$$

so $(x+A)/(x+B) \approx A/B$ is true to zeroth order when $Y\,(1-Y) \ll 1$, that is, when Y is small. Given that the "Macauley Duration for periodic compounding" changes by approximately one, the Modified Duration for this periodically-compounded bond will change by approximately $1/(1+Y)$, which is $1-Y$ to first order but simply 1 to zeroth order, which again is applicable for small Y.

If all yields and returns are continuously compounded,

$$\begin{aligned} \exp(R) - 1 &= (\exp(Y) - 1) + (\exp(\%\text{ capital gains}) - 1) \quad \Rightarrow \\ R &= \ln\left[e^Y - 1 + \frac{PV(Y^*)}{PV(Y)}\right] . \end{aligned} \tag{3}$$

Using the first-order Taylor Series expansion $e^Y \approx 1 + Y$ for small Y,

$$R \approx \ln\left[Y + \frac{PV(Y^*)}{PV(Y)}\right] .$$

Expanding the continuous-time $PV(Y^*)$ in a first-order Taylor Series around Y for small ΔY, and as before using the definition of D, gives $PV(Y^*)/PV(Y) = 1 - D\,\Delta Y$ for small ΔY, which leads to

$$R \approx \ln\left[Y + 1 - D\,(Y^* - Y)\right] .$$

Using the first-order Taylor Series expansion $\ln(1+x) \approx x$ for small "x" (small $Y - D\,\Delta Y$),

$$R \approx Y - D\,\Delta Y .$$

Duration should then be adjusted to its end-of-period value, which is approximately $D - 1$. This can be shown similarly to the periodically-compounded case, but it is easier than that because replacing $1/(1+Y)$ with e^{-Y} in (1) gives not only by definition the "Macauley Duration for continuous compounding" but also—for a proof see footnote 2's reference again—the Modified Duration for this continuously-compounded bond, so there is no need in the continuous-compounding case to use the $1/(1+Y) \approx 1$ approximation. ∎

To summarize, for periodic compounding this reflects one first-order approximation (small ΔY) and two zeroth-order approximations (small Y), whereas for continuous compounding this reflects three first-order approximations (small Y, ΔY, and $Y - D\,\Delta Y$) and one zeroth-order approximation (small Y).[3] The rest of this paper uses the Return Approximation with a constant duration, so from now on

$$R_t = Y_t - (D-1)\,\Delta Y_t . \tag{4}$$

[3]Leibowitz et al. (using somewhat different notation) use zero-coupon bonds, which simplifies the derivation of "$D - 1$" because as noted above, with zero-coupon bonds or any bonds that have $C_1 = 0$, the passage of one period reduces Macauley Duration by exactly one period. That means there is no need to make the first zeroth-order approximation of small Y. They do not consider the continuously-compounded case.

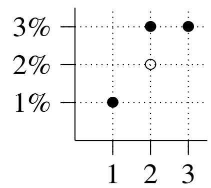

Figure 1. A fictional actual yield path (black dots), and its linear approximation ((1, 1%), (2, 2%), (3, 3%)).

Any arbitrary path of yield through time can be decomposed into a linear component and a nonlinear component in various ways. The purpose of this section is to show that the linear component of this decomposition gives rise to a return which is predictable (over a particular horizon). Accordingly, from now on discard the nonlinear component of the yield path, and in particular, assume that yields follow linear paths through time, starting and ending at the actual yield. For example, if the actual path of yields for a bond is given by the solid dots in Figure 1, the linear approximation used in this paper begins and ends at the same points as the actual path and goes through the open circle. Such a linear path is in general not a first-order Taylor Series approximation to the original, nonlinear path (that is, not a best-fit trendline). Using the linear yield path, ΔY_t is the same "ΔY" for all t. Yields cannot actually follow linear paths in the long run because that would imply that they linearly rise or fall forever, or never change; nor can they follow linear paths in the short run because having such linear forward curves for multiple maturities would typically generate arbitrage opportunities. Once we establish that the linear component of the yield path gives rise to a predictable return, empirical deviations from that predicted return will have to be attributed to the nonlinear component of the yield path (or to Return Approximation errors), and the empirical sections of this paper will give examples of how large those deviations have been.

Using the Return Approximation and assuming linear time paths of yields, one has

$$Y_t = Y_1 + (t-1)\,\Delta Y \quad \text{and therefore} \tag{5}$$

$$R_t = Y_1 + (t-1)\,\Delta Y - (D-1)\,\Delta Y = Y_1 + (t-D)\,\Delta Y\,. \tag{6}$$

As shown by the solid lines with bullets in Figure 2, if D is an integer then for an arbitrary positive ΔY, the returns $R_1, R_2, \ldots, R_{D-1}$ are all less than Y_1; R_D is equal to Y_1; and $R_{D+1}, R_{D+2}, \ldots$ are all greater than Y_1, whereas for an arbitrary negative ΔY, the returns $R_1, R_2, \ldots, R_{D-1}$ are all greater than Y_1; R_D is equal to Y_1; and $R_{D+1}, R_{D+2}, \ldots$ are all less than Y_1. The task is to determine how many terms beyond D have to be taken in order for the cumulative mean return to be equal to Y_1. Understanding "cumulative mean return" as the "cumulative arithmetic mean return," Figure 2 suggests that for an arbitrary ΔY, the answer

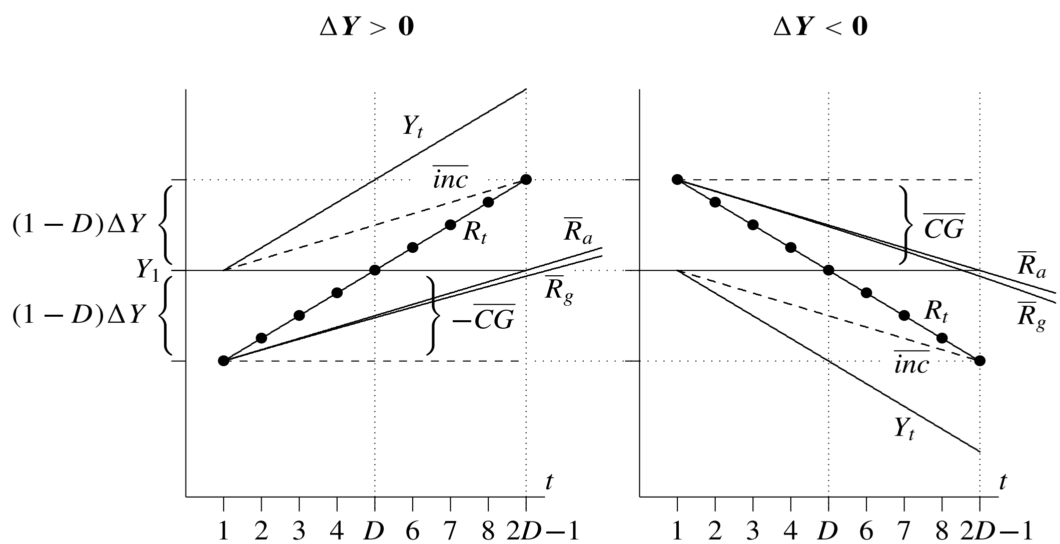

Figure 2. The solid lines with bullets are the instantaneous return paths R_t resulting from the linear paths of instantaneous yield Y_t with $D = 5$ and $\Delta Y > 0$ (left) and $\Delta Y < 0$ (right). As derived by Proposition 3 and its first corollary, the figure also shows, by a gap between the horizontal dashed line and the horizontal Y_1 line, the average (not instantaneous) capital gains up to time t, $\overline{CG}$ (on the left, average capital gains are negative, so they are negative one times the length of the vertical brace); and by a sloped dashed line, average interest income up to time t, $\overline{inc}$. Arithmetically-averaged return of the R_t's up to time t, $\overline{R}_a$, is the sum of the $\overline{inc}$ line and the $\overline{CG}$ gap, and is equal to Y_1 at $t = 2D - 1$. Section 2 derives the geometrically-averaged return of the R_t's, $\overline{R}_g$, which lies below $\overline{R}_a$ and thus equals Y_1 later than $2D - 1$ when $\Delta Y > 0$ and earlier than $2D - 1$ when $\Delta Y < 0$.

is $2D - 1$. Proposition 2 below proves that this is correct: regardless of the size of ΔY, initial yield will equal arithmetic mean return at date $2D - 1$. A slightly smaller value of ΔY in the left-hand graph would flatten the line marked Y_t, shrink the size of the gaps "$(1 - D)\Delta Y$" shown by the two vertical braces, and so flatten all the other rising lines, in a way that causes the line marked R_t (instantaneous return) to pivot around the point (D, Y_1), therefore making the line marked $\overline{R}_a$ (arithmetically-averaged return) pivot around the point $(2D-1, Y_1)$. So regardless of ΔY, $\overline{R}_a$ at time $2D - 1$ will be equal to Y_1.

Proposition 2. *If yields are linear in time, returns are approximated by (4), and twice duration is an integer, then the number of periods "N_a" which will make the arithmetic mean return equal to the initial yield is*

$$N_a = 2D - 1\,. \tag{7}$$

This yield path satisfies $R_t > -1$ for all $t \in [1, N_a]$ if

$$(D - 1) \cdot |\Delta Y| - Y_1 < 1\,. \tag{8}$$

This yield path satisfies $Y_t > 0$ for all $t \in [1, N_a]$ if

$$Y_1 > 0 \quad \textit{when } \Delta Y > 0 \textit{ and} \tag{9}$$

$$Y_1 + 2\,(D - 1)\Delta Y > 0 \quad \textit{when } \Delta Y < 0. \tag{10}$$

We construct a proof using Figure 2. Leibowitz et al., who prove the crux of the first sentence of Proposition 2, do not have that figure so they cannot appeal to its symmetry and they construct a very different proof.[4]

Proof of Proposition 2. The arithmetic mean return is

$$\overline{R}_a = \frac{1}{N_a}\sum_{t=1}^{N_a} R_t = \frac{1}{N_a}\sum_{t=1}^{N_a}\Big(Y_1 + (t - D)\,\Delta Y\Big) \tag{13}$$

[4]The gist of their method (LBKH (2013) p. 97; Leibowitz, Bova, and Kogelman (2014) p. 47 Column 2) is to set $\overline{R}_a = Y_1$ in (13), divide both sides by Y_1, and calculate

$$1 = \frac{1}{N_a}\sum_{t=1}^{N_a}\Big(1 + (t - D)\frac{\Delta Y}{Y_1}\Big) \tag{11}$$

$$= \frac{1}{N_a}\sum_{t=1}^{N_a}\Big(1 - D\,\frac{\Delta Y}{Y_1}\Big) + \frac{1}{N_a}\frac{\Delta Y}{Y_1}\sum_{t=1}^{N_a} t$$

$$= 1 - D\,\frac{\Delta Y}{Y_1} + \frac{1}{N_a}\frac{\Delta Y}{Y_1}\cdot\frac{N_a}{2}(N_a + 1) \tag{12}$$

$$= 1 + \left(\frac{N_a + 1}{2} - D\right)\frac{\Delta Y}{Y_1}\,,$$

so $(N_a + 1)/2 = D$ and (7) follows.

from (6). To confirm the intuition from Figure 2 that if $N_a = 2D - 1$ then $\overline{R}_a$ will equal Y_1, note that if one starts from D in Figure 2 and works outward, pairing $t = 4$ and $t = 6$, then $t = 3$ and $t = 7$, and so on, the R_t of each of these pairs will arithmetically average out to Y_1, so that the arithmetic average of all the pairs is just Y_1. Formally, if D is an integer then $\sum_{t=1}^{2D-1} f(t)$ for an arbitrary function f can be reordered as $f(D) + \sum_{s=1}^{D-1}[f(D-s) + f(D+s)]$; applying this reordering to the right-hand side of (13) (after substituting $2D - 1$ for N_a) turns it into

$$\frac{[Y_1 + (D-D)\Delta Y] + \sum_{s=1}^{D-1}\{[Y_1 + (D-s-D)\Delta Y] + [Y_1 + (D+s-D)\Delta Y]\}}{2D-1}$$
$$= \frac{Y_1 + \sum_{s=1}^{D-1}\{Y_1 + Y_1\}}{2D-1} = \frac{1 + \sum_{s=1}^{D-1} 2}{2D-1} \cdot Y_1 = \frac{2D-1}{2D-1} Y_1 = Y_1\,,$$

confirming the conjecture when D is an integer. If D is not an integer but $2D$ is an integer then $\sum_{t=1}^{2D-1} f(t)$ can be reordered as $\sum_{s=1}^{D-1/2}[f(D+1/2-s) + f(D-1/2+s)]$; applying this reordering to the right-hand side of (13) (after substituting $2D - 1$ for N_a) turns it into

$$\frac{\sum_{s=1}^{D-1/2}\{[Y_1 + (D+\frac{1}{2}-s-D)\Delta Y] + [Y_1 + (D-\frac{1}{2}+s-D)\Delta Y]\}}{2D-1}$$
$$= \frac{\sum_{s=1}^{D-1/2}\{Y_1 + Y_1\}}{2D-1} = \frac{\sum_{s=1}^{D-1/2} 2}{2D-1} \cdot Y_1 = \frac{2D-1}{2D-1} Y_1 = Y_1\,,$$

confirming the conjecture and completing this proof of (7).

From (6), if $\Delta Y > 0$ then R_t is increasing in t, so $R_t > -1$ for all t if $R_1 > -1$; imposing that leads to $(D - 1)\Delta Y - Y_1 < 1$. If $\Delta Y < 0$, R_t is decreasing in t; insisting that $R_{N_a} = R_{2D-1}$ be greater than -1 leads to $(D - 1)(-\Delta Y) - Y_1 < 1$, proving (8). Similarly, if $\Delta Y > 0$ the smallest Y_t is Y_1, proving the first part of (10); and if $\Delta Y < 0$ the smallest Y_t in $[1, N_a]$ occurs at $t = 2D - 1$, which, substituted into (5), proves the second part of (10). ∎

(8) and (9) are easily satisfied when using plausible parameter values, and are satisfied for all the examples in this paper. (10) can be violated with seemingly-plausible parameter values. "$2D - 1$" in (7) means "twice duration minus one rollover period," so that if bonds of duration 2 years (730 days) are rolled over annually, (7) gives $N_a = 2 \cdot 2 - 1 = 3$ years, but if the same bonds are rolled over daily, (7) gives $N_a = 2 \cdot 730 - 1 = 1459$ days $= 3.997$ years: the higher the rollover frequency, the closer "$2D - 1$" is to $2D$.

Although the Return Approximation is relatively innocuous period-by-period, Proposition 3 below shows that its errors unfortunately reinforce each other over time. The first term on the right-hand side of (14) and (15) represents average capital gains $\overline{CG}$ between Y_1 and Y_{N+1} using the Return Approximation, and the last term on the right-hand side of those equations represents average interest income $\overline{inc}$. Proposition 3 shows that the Return Approximation involves $Y_1 - Y_{N+1}$, which is likely to be large because those dates are so far apart; such large yield changes would undermine the applicability of the Return Approximation over long time periods, because when yield changes are large it is inappropriate to ignore the convexity of bond price's sensitivity to yield, especially for bonds of long duration. On the other hand, this effect is muted by division by N in (14); we will return to this point in Section 5. Proposition 3's Corollary 1 is used to construct Figure 2's $\overline{R}_a$ line as the sum of its $\overline{CG}$ line and its $\overline{inc}$ line. A third proof of Proposition 2 could be constructed by showing that $2D - 1$ is the date at which the $\overline{inc}$ line's distance from Y_1 equals the $\overline{CG}$ line's distance from Y_1, so that income's positive (or negative) deviation from Y_1 just offsets the negative (or positive) capital gains. Corollary 2, which gives conditions under which in the long run return is just equal to average yield, so capital gains become negligible, is used in Section 3.

Proposition 3. *Assuming returns are approximated by (4),*

$$\overline{R}_a(N) = \frac{(Y_1 - Y_{N+1}) \cdot (D-1)}{N} + \frac{1}{N}\sum_{t=1}^{N} Y_t \tag{14}$$

at date N.

Corollary 1. *Assuming returns are approximated by (4) and yields are linear in time,*

$$\overline{R}_a(N) = \Delta Y(1-D) + \left(Y_1 + \frac{N-1}{2}\,\Delta Y\right) \tag{15}$$

at date N.

Corollary 2. *Assuming returns are approximated by (4),*

$$\lim_{N\to\infty} \overline{R}_a(N) = \lim_{N\to\infty} \frac{\sum_{t=1}^{N} Y_t}{N} \quad \textit{if and only if} \quad \lim_{N\to\infty} \frac{Y_{N+1}}{N} = 0\,. \tag{16}$$

Proof of Proposition 3. From (13),

$$\begin{aligned}
N \cdot \overline{R}_a &= \sum_{t=1}^{N}[Y_t - (D-1)\Delta Y_t] \\
&= \sum_{t=1}^{N} Y_t - (D-1)\sum_{t=1}^{N} Y_{t+1} + (D-1)\sum_{t=1}^{N} Y_t \\
&= -D\sum_{t=1}^{N} Y_{t+1} + \sum_{t=1}^{N} Y_{t+1} + D\sum_{t=1}^{N} Y_t\,;
\end{aligned}$$

expanding the first and third terms and cancelling,

$$= D\,(Y_1 - Y_{N+1}) + \sum_{t=1}^{N} Y_{t+1}\,.$$

Add and subtract Y_1 and expand the last term, then divide by N. ∎

Proof of Corollary 1. First term: from the first term of (14), use $Y_{N+1} = Y_1 + N\,\Delta Y$. Second term: from the second term of (14),

$$\begin{aligned}\frac{1}{N}\sum_{t=1}^{N} Y_t &= \frac{1}{N}\sum_{t=1}^{N}[Y_1 + (t-1)\,\Delta Y] \\ &= (Y_1 - \Delta Y) + \frac{\Delta Y}{N}\sum_{t=1}^{N} t \\ &= Y_1 - \Delta Y + \Delta Y \frac{N+1}{2}\end{aligned}$$

and simplify. ▮

Proof of Corollary 2. Take the limit of (14) as $N \to \infty$. ▮

2. Quadratic Paths, Non-integral Durations, and the Geometric Mean

For yield paths which are not linear but quadratic, return at $2D - 1$ will not equal initial yield. Intuitively, since if yields follow a linear path then after $2D-1$ periods positive (or negative) different-from-Y_1 yields just balance negative (or positive) capital gains, resulting in an average return of Y_1, if yield falls below this linear path before it catches up—that is, if yield follows a convex path—then these lower yields cause overall return to be lower, below Y_1. Similarly, if yield rises above the linear path before it slows down to meet it—that is, if yield follows a concave path—then these higher yields cause overall return to be higher, above Y_1. This intuition is correct:

Proposition 4. *If yields follow the quadratic function $z_1 t^2 + z_2 t + z_3$ where t is time, if returns are approximated by (4), and if twice duration is an integer, then over a time period of length $N_a = 2D - 1$ the "forecast error"*

$$\overline{R}_a - Y_1 = \frac{2z_1}{3}(1 - D)\,D\,. \tag{17}$$

Corollary. *Under the conditions of Proposition 4, forecast error $\overline{R}_a - Y_1$ at period $2D - 1$ is negative if the yield path is convex and positive if the yield path is concave.*

Proof of Proposition 4. $\overline{R}_a = (1/N_a)\sum_{t=1}^{N_a}[Y_t - (D-1)\,\Delta Y_t]$. Writing Y_t as $z_1 t^2 + z_2 t + z_3$, one has: Y_t is convex if $z_1 > 0$ and concave if $z_1 < 0$; $Y_1 = z_1 + z_2 + z_3$; and $\Delta Y_t = Y_{t+1} - Y_t = 2z_1 t + z_1 + z_2$, so $\overline{R}_a = (1/N_a)\sum_{t=1}^{N_a}[z_1 t^2 + z_2 t + z_3 - (D-1)(2z_1 t + z_1 + z_2)]$. It can then be shown, either by tedious calculations using at one point $\sum_{t=1}^{T} t^2 = T^3/3 + T^2/2 + T/6$ and, as in (12), $\sum_{t=1}^{T} t = T\,(T+1)/2$, or by using a computer algebra system, that if $N_a = 2D-1$ then $\overline{R}_a - (z_1 + z_2 + z_3) = \frac{2}{3} z_1 (1-D)\,D$.
▮

	A	G
1. $\Delta Y > 0$ and $2D$ an integer	$N_a = 2D - 1$	$2D - 1 < N_g^+ < \infty$
2. $\Delta Y > 0$ and $2D$ not an integer	$N_a^+ = \lceil 2D - 1 \rceil$	$2\lfloor D \rfloor - 1 < N_g^+ < \infty$
3. $\Delta Y < 0$ and $2D$ an integer	$N_a = 2D - 1$	$D < N_g^+ \leq 2D - 1$
4. $\Delta Y < 0$ and $2D$ not an integer	$N_a^+ = \lceil 2D - 1 \rceil$	$D < N_g^+ \leq 2\lceil D \rceil - 1$

Table 1: Theoretical Results for Linear Yield Paths

Proof of Corollary. Since $D > 1$, the right-hand side of (17) has the opposite sign of c_1. ∎

(A fourth method of proving Proposition 2 is to set $c_1 = 0$ in (17).)

The rest of this section reverts to assuming linear yield paths.

Proposition 2 proved Column A Rows 1 and 3 of Table 1; the next extension is to prove its 2A and 4A. Leibowitz et al. do not discuss the complication that if $2D$ is not an integer then (7) cannot describe the integer N_a. In discussing it, it is helpful to use the standard mathematical notation for "the largest integer smaller than x," namely the "floor function" $\lfloor x \rfloor$, and for "the smallest integer larger than x," which is the "ceiling function" $\lceil x \rceil$. In case 2A, $\Delta Y > 0$, so $\overline{R}_a$ will be less than Y_1 if N_a is set to $\lfloor 2D - 1 \rfloor$ and larger than Y_1 if N_a is set to $1 + \lfloor 2D - 1 \rfloor$, which is $\lceil 2D - 1 \rceil$. Therefore $\overline{R}_a$ will never be exactly equal to initial yield, but $\overline{R}_a$ will be slightly less than initial yield in period $\lfloor 2D - 1 \rfloor$ and slightly more than initial yield in period $\lceil 2D - 1 \rceil$. Defining N_a^- as the largest date when $\overline{R}_a$ lies on the same side of Y_1 as it did at $t = 1$, and N_a^+ as the smallest date when $\overline{R}_a$ does not lie on the same side of Y_1 as it did at $t = 1$, we have $N_a^- = \lfloor 2D - 1 \rfloor$ and $N_a^+ = \lceil 2D - 1 \rceil$, proving 2A. In case 4A, $\Delta Y < 0$, so $\overline{R}_a$ will similarly never be exactly equal to initial yield, but it will be slightly more than initial yield in period $\lfloor 2D - 1 \rfloor$ and slightly less than initial yield in period $\lceil 2D - 1 \rceil$. Therefore in case 4A, as in case 2A, $N_a^- = \lfloor 2D - 1 \rfloor$ and $N_a^+ = \lceil 2D - 1 \rceil$. This completes the proof of Table 1's column A.

If a sequence of "returns" $\{R_t\}_{t=1}^T$ makes \$1 of wealth grow to $e^{R_1} e^{R_2} \cdots e^{R_T}$ then replacing each R_t by the arithmetic mean of the R's would lead to the same final value of wealth, making Section 1's use of the arithmetic mean appropriate if returns are continuously compounded. If $\{R_t\}_{t=1}^T$ makes \$1 of wealth grow to $(1 + R_1)(1 + R_2) \cdots (1 + R_T)$ then replacing each $1 + R_t$ by one plus the geometric mean of the R's—which is $\overline{R}_g = [\prod_{t=1}^{N_g} (1 + R_t)]^{1/N_g} - 1$ (not to be confused with the geometric mean $[\prod_{t=1}^{N_g} R_t]^{1/N_g}$ used in mathematics)—would lead to the same value of wealth, so studying periodically-compounded returns requires results for the geometric mean.

Proposition 5. *Assuming yields are linear in time and returns are approximated by (4), the number of periods "N_g" which will make the geometric mean return* *equal to the initial yield satisfies*

$$1 = \prod_{t=1}^{N_g}\Big(1 + (t - D)\frac{\Delta Y}{1 + Y_1}\Big) \tag{18}$$

$$= \begin{cases} \left(\frac{\Delta Y}{1+Y_1}\right)^{N_g} \dfrac{\Gamma\left(\frac{1+Y_1}{\Delta Y} - D + N_g + 1\right)}{\Gamma\left(\frac{1+Y_1}{\Delta Y} - D + 1\right)} & \text{if } \Delta Y > 0 \\ \left(\frac{-\Delta Y}{1+Y_1}\right)^{N_g} \dfrac{\Gamma\left(-\frac{1+Y_1}{\Delta Y} + D\right)}{\Gamma\left(-\frac{1+Y_1}{\Delta Y} + D - N_g\right)} & \text{if } \Delta Y < 0 \end{cases} \tag{19}$$

(where Γ means the gamma function of mathematics, not to be confused with the Gamma of option price theory).

Proof. The geometric mean return is

$$\overline{R}_g = \left[\prod_{t=1}^{N_g}(1 + R_t)\right]^{1/N_g} - 1 = \left[\prod_{t=1}^{N_g}\Big(1 + Y_1 + (t - D)\,\Delta Y\Big)\right]^{1/N_g} - 1\,. \tag{20}$$

Setting $\overline{R}_g = Y_1$ and adding one to both sides of the equation means that N_g satisfies

$$1 + Y_1 = \left[\prod_{t=1}^{N_g}\Big(1 + Y_1 + (t - D)\,\Delta Y\Big)\right]^{1/N_g} \quad \text{or}$$

$$(1 + Y_1)^{N_g} = \prod_{t=1}^{N_g}\Big(1 + Y_1 + (t - D)\,\Delta Y\Big) \tag{21}$$

which leads to (18).

Define $A = 1 - D\,\Delta Y/(1 + Y_1)$ and $B = \Delta Y/(1 + Y_1)$, so that (18) becomes $1 = \prod_{t=1}^{N_g}(A + Bt)$. Since $A + Bt = (1 + R_t)/(1 + Y_t)$, we know that $0 < A + Bt$ for all t. The latter implies $0 < A + Bt = B \cdot (\frac{A}{B} + t)$, so either

$\frac{A}{B} + t > 0$ for all t and $B > 0$, i.e., $\Delta Y > 0$, or
$\frac{A}{B} + t < 0$ for all t and $B < 0$, i.e., $\Delta Y < 0$.

For the $\Delta Y > 0$ case,

$$\exp\ln\prod_{t=1}^{N_g}(A + Bt) = \exp\sum_{t=1}^{N_g}\ln(A + Bt) = \exp\sum_{t=1}^{N_g}\big(\ln B + \ln(\tfrac{A}{B}+t)\big)\,;$$

using the identity $\sum_{t=1}^{T} f(t+C) = \sum_{j=1+C}^{T+C} f(j)$ and setting its C equal to $\frac{A}{B}$ and its f equal to ln,

$$= \exp\{N_g \ln B + \sum_{j=\frac{A}{B}+1}^{\frac{A}{B}+N_g} \ln j\}\,; \tag{22}$$

from the Lemma given after this proof,

$$= \exp\{N_g \ln B + \ln\Gamma(\tfrac{A}{B}+N_g+1) - \ln\Gamma(\tfrac{A}{B}+1)\} \tag{23}$$

$$= \exp\ln\bigl[B^{N_g}\,\Gamma(\tfrac{A}{B}+N_g+1)\big/\Gamma(\tfrac{A}{B}+1)\bigr]$$

$$= B^{N_g}\,\Gamma(\tfrac{A}{B}+N_g+1)\big/\Gamma(\tfrac{A}{B}+1)\,, \tag{24}$$

as was to be shown. For the $\Delta Y < 0$ case, using the identity $\sum_{t=1}^{T} f(-t+C) = \sum_{j=-T+C}^{-1+C} f(j)$ one has

$$\begin{aligned}\exp\ln\prod_{t=1}^{N_g}(A+Bt) &= \exp\sum_{t=1}^{N_g}\ln(A+Bt) = \exp\sum_{t=1}^{N_g}\bigl(\ln(-B)+\ln(-\tfrac{A}{B}-t)\bigr)\\ &= \exp\{N_g\ln(-B) + \sum_{j=-\frac{A}{B}-N_g}^{-\frac{A}{B}-1}\ln j\}\\ &= \exp\{N_g\ln(-B) + \ln\Gamma(-\tfrac{A}{B}) - \ln\Gamma(-\tfrac{A}{B}-N_g)\}\\ &= \exp\ln\bigl[(-B)^{N_g}\,\Gamma(-\tfrac{A}{B})\big/\Gamma(-\tfrac{A}{B}-N_g)\bigr]\\ &= (-B)^{N_g}\,\Gamma(-\tfrac{A}{B})\big/\Gamma(-\tfrac{A}{B}-N_g)\,.\end{aligned}$$

▮

Lemma. *If a and b are positive real numbers and $b-a$ is a positive integer then*[‡]

$$\sum_{j=a}^{b}\ln j = \ln a + \ln(a+1) + \cdots + \ln(b) = \ln\Gamma(b+1) - \ln\Gamma(a)\,.$$

Proof. Since $\Gamma(n) = (n-1)!$ when n is a positive integer, if a and b are both positive integers then this is merely the claim that $\sum_{j=a}^{b}\ln j = \ln[b!] - \ln[(a-1)!]$, which can be proven by writing out $b!$. To construct a proof for non-integer a and b, recall that a basic property of the gamma function is $\Gamma(x+1) = x\,\Gamma(x)$. Letting Δ denote the difference operator in this paragraph only, $\Delta\ln\Gamma(x) = \ln\Gamma(x+1) - \ln\Gamma(x) = \ln\bigl(x\,\Gamma(x)\bigr) - \ln\Gamma(x) = \ln x$. If $\Delta f(x) = g(x)$ then[5]

$$\sum_{x=a}^{b} g(x) = \sum_{x=a}^{b}\Delta f(x) = \sum_{x=a}^{b}[f(x+1) - f(x)]$$

[‡] Referee: I include the sum's definition because some people interpret such a sum instead as running over all integers between a and b. See `http://math.stackexchange.com/questions/35080/upper-limit-of-summation-index-lower-than-lower-limit`.

[5]This method is called "additive telescoping" on p. 5 of Naik (no date), who writes: "This is just like the fundamental theorem of calculus. Here, f is the discrete analogue of an antiderivative for g, and to add the g-values over an interval, we evaluate f at the endpoints and take the difference. However, the discrete nature of the situation makes things slightly different: instead of $f(b)-f(a)$, we get $f(b+1) - f(a)$." Gleich (2005, pp. 7–8) refers to this as the definition of the discrete antiderivative, the definition of the discrete definite integral, and the Fundamental Theorem of Finite Calculus.

$$
\begin{array}{rl}
= & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + f(a{+}1) - f(a) \\
 & \qquad\qquad\qquad\qquad\qquad\quad + f(a{+}2) - f(a{+}1) \\
 & \qquad\qquad\qquad\quad + f(a{+}3) - f(a{+}2) \\
 & \qquad\qquad\quad + \cdots \\
 & + f(b{+}1) - f(b) \\
= & f(b{+}1) + \ 0 \ + \cdots + \ 0 \ + \ 0 \ + \ 0 \ - f(a)\,.
\end{array}
$$

Identify $\ln\Gamma(x)$ with f and $\ln x$ with g. ▮

(18) has some resemblance to (11), but unlike (11), it is not possible to solve (18) or (19) for N_g as a function of D, ΔY, and Y_1, even though one can write (18) in closed form using the gamma function. Furthermore, although N_a did not depend on Y_1 or ΔY, N_g does depend on them, which means it is time-varying and it is not possible to calculate in advance.[6]

There may be no integer N_g making the right-hand side of (18) exactly equal to one, but if not, there will exist some integer N_g^+ such that the right-hand side of (18) switches from being less than one to being more than one (or vice versa) when the upper limit of the product switches from being $N_g^+ - 1$ to being N_g^+. We denote $N_g^+ - 1$ by N_g^-. In numerical examples, N_g^+ and N_g^- are easily found by trial and error, calculating the right-hand side of (18) or (19) with $N_g = 2, 3, \ldots$ until it crosses one.

$\overline{R}_a$ equals $\overline{R}_g$ to first order, so N_a will be close to N_g in most cases.[7] Leibowitz, Bova and Kogelman (2014, Table 1) work an example with duration $D = 5$ years, $\Delta Y = 0.5\%$, and $Y_1 = 3\%$, giving $N_a = 9$. Substituting these values into (18) gives a right-hand side which is less than one for $N_g \leq 9$ and greater than one for $N_g \geq 10$, leading to the N_g^- and N_g^+ reported in the first row of Table 2. Keeping D and Y_1 the same and changing the sign of ΔY results in an example which violates (10). The next six lines show cases for various positive and negative values of ΔY in which (10) is not violated; the ones with $\Delta Y > 0$ have $N_g^- = N_a$ and the ones with $\Delta Y < 0$ have $N_g^+ = N_a$. Attempting to enlarge the difference between N_g and N_a by increasing D from 5 to 20 requires decreasing ΔY to meet (10) in the $\Delta Y < 0$ case; doing so still gives $N_g^- = N_a$ for $\Delta Y > 0$ and $N_g^+ = N_a$ for $\Delta Y < 0$. The last line of Table 2 shows that the arithmetic and geometric means can give exactly the same results when duration is not an integer. Figure 3 uses exaggerated

[6] If, for fixed N, N-period periodically-compounded returns *exactly* equalled initial yields period after period, an implausible periodicity occurs. Assuming an arbitrary (not necessarily linear) time path of yields, if $1 + R_1 = \prod_{t=1}^{N}(1 + R_t)^{1/N}$ and $1 + R_2 = \prod_{t=2}^{N+1}(1 + R_t)^{1/N}$ then $(1 + R_1)/(1 + R_2) = ((1 + R_1)/(1 + R_{N+1}))^{1/N}$, i.e., R_{N+1} is completely determined by R_1 and R_2. Similarly R_{2N+1} would be completely determined by R_{N+1} and R_{N+2}, which in turn would be determined by R_1, R_2 and R_3; and R_{N+3} would be completely determined by R_1, R_2, R_3 and R_4; and so on *ad infinitum.*

[7] $(1 + \overline{R}_g)^N = \prod_{t=1}^{N}(1 + R_t) \iff \ln(1 + \overline{R}_g)^N = \sum_{t=1}^{N} \ln(1 + R_t)$. Applying the first-order Taylor Series approximation $\ln(1 + x) \approx x$ to both sides and dividing by N, $\overline{R}_g \approx (1/N)\sum_{t=1}^{N} R_t = \overline{R}_a$. For higher-order approximations of $\overline{R}_g$, see Mindlin (2011) and Yogaranpan (2005).

ΔY	Y_1	D	N_a^-	N_a^+	N_g^-	N_g^+
+0.5%	3%	5	9	9	9	10
+0.3%	3%	5	9	9	9	10
+0.2%	3%	5	9	9	9	10
+0.07%	3%	20	39	39	39	40
−0.3%	3%	5	9	9	8	9
−0.2%	3%	5	9	9	8	9
−0.07%	3%	20	39	39	38	39
+0.5%	3%	5.4	9	10	9	10

Table 2: Results for selected linear yield paths. All satisfy (10).

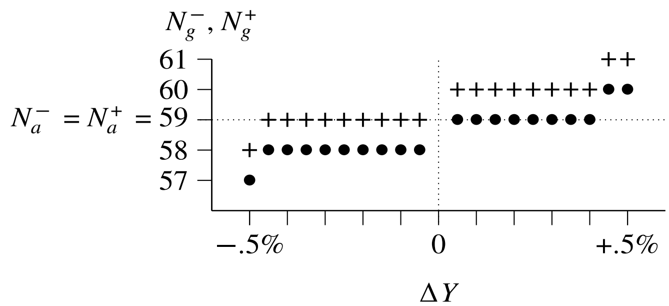

Figure 3. For $D = 30$, $Y_1 = 30\%$, and linear yield paths with different values of ΔY, the plus signs denote N_g^+ and the bullets denote N_g^-.

values of Y_1 and D to show that it is theoretically possible for N_g to be completely different from N_a while obeying (10).

In Table 2 and Figure 3, when D is an integer and $\Delta Y > 0$, $N_g^+ > 2D - 1$, whereas when $\Delta Y < 0$, $N_g^+ \leq 2D - 1$. We need to prove the claims in Table 1 Column G Rows 1 and 3 that this is a general asymmetry: that it would take longer to get the geometric mean return up to Y_1 after suffering a capital loss resulting from $\Delta Y = +x\%$ than it would take to bring the geometric mean return down to Y_1 after enjoying a capital gain resulting from $\Delta Y = -x\%$. This is because the geometric mean is less than or equal to the arithmetic mean, as depicted in Figure 2.

Proposition 6. *Assuming yields are linear in time and returns are approximated by (4), N_g^+ satisfies 1G, 2G, 3G, and 4G of Table 1.*

Proof. Denote the geometric mean return up to time N as $\overline{R}_g(N) = [\prod_{t=1}^{N}(1 + R_t)]^{1/N} - 1$ and define "one plus R" $OPR(N) = \overline{R}_g(N) + 1$.

Case 1G's first inequality: From Proposition 2, $N_a = 2D - 1$; that is, as shown in the left-hand panel of Figure 2, $\overline{R}_a = Y_1$ when $t = 2D - 1$. Since the geometric mean is less than or equal to the arithmetic mean,[8] as long as we can show that the geometric mean is increasing in N, it is clear that the geometric mean will take longer than $2D - 1$ periods to reach Y_1.

According to (6), R_t is strictly increasing in t. Hence

$$OPR(N) = \prod_{t=1}^{N}(1+R_t)^{1/N} < \prod_{t=1}^{N}(1+R_N)^{1/N} = 1 + R_N\,. \tag{25}$$

To prove that $\overline{R}_g(N)$ is increasing in N, it suffices to show that the following is positive:

$$\begin{aligned}
\ln OPR(N+1) - \ln OPR(N) &= \frac{1}{N+1}\sum_{t=1}^{N+1}\ln(1+R_t) - \frac{1}{N}\sum_{t=1}^{N}\ln(1+R_t)\\
&= \frac{1}{N+1}\ln(1+R_{N+1}) + \left[\frac{1}{N+1} - \frac{1}{N}\right]\sum_{t=1}^{N}\ln(1+R_t)\\
&= \frac{1}{N+1}\ln(1+R_{N+1}) + \frac{-1}{N(N+1)}\sum_{t=1}^{N}\ln(1+R_t)\\
&= \frac{1}{N+1}\left[\ln(1+R_{N+1}) - \frac{1}{N}\sum_{t=1}^{N}\ln(1+R_t)\right]\\
&= \frac{1}{N+1}\left[\ln(1+R_{N+1}) - \ln OPR(N)\right] \qquad (26)\\
&> \frac{1}{N+1}\left[\ln(1+R_N) - \ln OPR(N)\right].
\end{aligned}$$

This is positive by (25).

Case 1G's second inequality: When $\Delta Y > 0$, $\overline{R}_g(1)$ is less than $(1 + Y_1)$; if we can show that $\overline{R}_g(N)$ becomes larger than Y_1 as $N \to \infty$, this will prove that $N_g^+ < \infty$.

To prove that $\overline{R}_g(N)$ becomes larger than Y_1 as $N \to \infty$, recall that $R_D = Y_1$ and R_t is increasing. Pick a date τ such that $\tau > D$, and let $Y_2 = R_\tau$. We know that $Y_2 > Y_1$; let their difference be K. The left-hand side of the following inequality is the *OPR* of R_t up to time $N > \tau$; the right-hand side is the *OPR* of a return path which equals R_t up to time τ, then becomes Y_2 forever (the "kinked path"):

$$\prod_{t=1}^{N}(1+R_t)^{1/N} > \prod_{t=1}^{\tau}(1+R_t)^{1/N}\prod_{t=\tau+1}^{N}(1+Y_2)^{1/N}\,. \tag{27}$$

[8] A quick way to show that the famous "arithmetic mean-geometric mean inequality" of mathematics, $[\prod_{i=1}^{n} x_i]^{1/n} \le \sum_{i=1}^{n} x_i/n$, applies also to the geometric mean of finance, $[\prod_{t=1}^{N_g}(1+x_t)]^{1/N_g} - 1$, is to rewrite the latters' "-1" term as $-\sum_{i=1}^{n} 1/n$ and add its opposite to both sides of the inequality.

The ratio of the *OPR* of the kinked path to the *OPR* an $R_t \equiv Y_2$ path is

$$\left(\prod_{t=1}^{\tau} \frac{1 + R_t}{1 + Y_2} \right)^{1/N} . \tag{28}$$

The fraction in (28) is less than one. The limit of (28) as $N \to \infty$ is one. Hence for sufficiently large N, the geometric mean of the kinked path can be made arbitrarily close to the geometric mean of the "always Y_2" path—and in particular, for sufficiently large N the geometric mean of the kinked path can be made closer than K to the geometric mean of the "always Y_2" path, which is Y_2. For such N, the geometric mean of the kinked path must be larger than Y_1; and hence from (27), the geometric mean of the original path must be larger than Y_1.

Case 3G's second inequality: From Proposition 2, $N_a = 2D - 1$; that is, as shown in the left-hand panel of Figure 2, $\overline{R}_a = Y_1$ when $t = 2D - 1$. Since the geometric mean is less than or equal to the arithmetic mean, as long as we can show that the geometric mean is decreasing in N, it is clear that the geometric mean will be equal to Y_1 at an earlier date than $2D - 1$.

According to (6), R_t is strictly decreasing in t. Hence

$$OPR(N) = \prod_{t=1}^{N} (1 + R_t)^{1/N} > \prod_{t=1}^{N} (1 + R_N)^{1/N} = 1 + R_N . \tag{29}$$

To prove that $\overline{R}_g(N)$ is decreasing in N, it suffices to show that the following is negative (using (26)):

$$\begin{aligned} \ln OPR(N+1) - \ln OPR(N) &= \frac{1}{N+1} \Big[\ln(1 + R_{N+1}) - \ln OPR(N) \Big] \\ &< \frac{1}{N+1} \Big[\ln(1 + R_N) - \ln OPR(N) \Big] . \end{aligned}$$

This is negative by (29).

($N_g^+ = 2D - 1$ is possible in the latter case because although $\overline{R}_g(2D-1)$ is too small, $\overline{R}_g(2D-2)$ may be too large, making $N_g^+ \le 2D - 1$ and $N_g^- = 2D - 2$.)

Case 3G's first inequality: When $\Delta Y < 0$, $\overline{R}_g(D)$ is greater than Y_1:

$$OPR(D) = \sum_{t=1}^{D} (1 + R_t)^{1/D} > \sum_{t=1}^{D} (1 + R_D)^{1/D} = 1 + R_D = 1 + Y_1 .$$

Since $\overline{R}_g(N)$ is decreasing in N, more time must pass before $\overline{R}_g$ is equal to Y_1.

Case 2G's first inequality: To prove 2G, note that D cannot be an integer when $2D$ is not an integer. Thus

$$\prod_{t=1}^{2\lfloor D \rfloor - 1} [1 + Y_1 + (t - D)\Delta Y] < \prod_{t=1}^{2\lfloor D \rfloor - 1} [1 + Y_1 + (t - \lfloor D \rfloor)\Delta Y] < (1 + Y_1)^{2\lfloor D \rfloor - 1}$$

using Case 1G. Similarly for 4G:

$$\prod_{t=1}^{2\lceil D\rceil-1}[1+Y_1+(t-D)\Delta Y] < \prod_{t=1}^{2\lceil D\rceil-1}[1+Y_1+(t-\lceil D\rceil)\Delta Y] < (1+Y_1)^{2\lceil D\rceil-1}.$$

∎

Since $N_g^+ > 2D-1$ for $2D$ an integer and positive ΔY, while $N_g^+ \leq 2D-1$ for negative ΔY, when ΔY is very small in absolute value, N_g^+ will be close to $2D-1$, which is N_a, as illustrated in Figure 3. For realistic parameter values, there is little or no difference between the arithmetic mean and the geometric mean in this analysis. Nevertheless, it is still better to use continuously-compounded returns, because then the relevant mean is the arithmetic, so Propositions 2 and 4 and its corollary and the results of Column A of Table 1 govern, and the date at which mean return will be equal to initial yield, being $N_a = 2D-1$, can, unlike N_g, be calculated in advance (because it does not depend on ΔY) and is not, unlike N_g, time-varying (because it does not depend on Y_1).

3. Empirical Results for Various Horizons

Under the assumptions of Proposition 2, historical returns would equal initial yields after the passage of an amount of time equal to twice duration minus one rollover period. However, the assumptions of Proposition 2 do not hold historically, and therefore historical evidence on performance of different horizons is helpful in deciding how useful Proposition 2's horizon actually is. In this section we study fifteen different horizons and six types of bonds to see how well initial yield predicted realized return over those horizons. For the reasons explained in Section 2, in this section all computed yields and returns will be continuously compounded.

While Section 1 and 2's theoretical results were for constant-duration portfolios, it is of interest to see if they have relevance for constant-maturity portfolios, which are more widely discussed among retail investors[9] and which sometimes do not differ much from constant-duration portfolios. Constant-maturity bond yield data in Excel spreadsheet format is easily downloaded from the "Federal Reserve Economic Data" ("FRED") web site of the St. Louis branch of the Federal Reserve System (`https://research.stlouisfed.org/`). The series used and their FRED designations were:

- 3-Year Treasury Constant Maturity Rate (GS3)
- 5-Year Treasury Constant Maturity Rate (GS5)
- 10-Year Treasury Constant Maturity Rate (GS10)

[9]For example, the phrase `"constant maturity"` bond gets three times more hits on a Google web search than `"constant duration"` bond, and a search for "constant duration" on the web site of Fidelity Investments returns no hits (either using `"constant duration" site:fidelity.com` on Google or going to that site and using its search box).

- 20-Year Treasury Constant Maturity Rate (GS20)[10]
- Moody's Seasoned Baa Corporate Bond Yield (BAA). This series does not have a completely constant maturity; its average maturity is approximately 25 years (Hall (2001 p. 1200); Ivaschenko (2003 p. 17)). The calculations below assume this series has a constant maturity of 25 years, unavoidably introducing some error.

All these are monthly series, not seasonally adjusted, and for all of them we will assume that their bonds are rolled over monthly. The first four series have data starting in April 1953; we used data for all five from April 1953 until April 2014, a period of 62 years 1 month. Since for each month and each Treasury series we can calculate duration, it is possible to synthesize constant-duration yield time series. We synthesized a 75-month constant-duration Treasury yield series by linear interpolation using the duration and yield data for the 5-year and 10-year series when 75 months fell between their durations, and for the 10-year and 20-year series during the much less frequent times when 75 months fell between their durations. Seventy-five months was almost the longest-duration series we could synthesize by interpolation because the minimum duration of the 20-year series was 78.4 months. We did not synthesize other, shorter-duration yield series because the difference between "constant maturity" and "constant duration" is most evident with longer-term bonds (see for example how the range of the durations varies with maturity in the fifth row of Table 4 below), and because even using the largest duration possible, the results for the constant-duration series did not differ very much from the results of the 10-year constant-maturity series.[11]

Table 3 summarizes returns and yield changes for the full data set, and separately for the "early period" before the September 1981 peak of decades of generally rising interest rates, and for the "late period" of generally falling interest rates after September 1981. The early period is 28 years 6 months long and the late period is 32 years 7 months long. In the early period, with rising yields, average (nominal) returns were lower than average yields, while the opposite was true in the late period; over the full period, average returns were quite close to average yields. This may not be unexpected over a sufficiently long time period because of

[10]The 20-year series has 81 months of missing data, from January 1987 until September 1993. The average 20-year yield for the previous 40 months and succeeding 40 months was 8.801%; the average 30-year yield for the previous 40 months and succeeding 40 months was, from another FRED data set, almost the same, at 8.691%, reflecting a yield curve that was just barely inverted at the long end. We approximated the 20-year yield for the missing months as 8.801/8.691 = 1.013 times the corresponding 30-year yield. (Otherwise we did not use FRED's 30-year constant-maturity Treasury data because it only starts in February 1977 and 2003–2006 is missing.)

[11]Leibowitz et al. also investigate the empirical fit of their theory. They use shorter data sets than ours, which is appropriate because in their models, "For clarity of illustration, all returns are in a simple-interest format that ignores the price effects of compounding and multi-year compounding" (Leibowitz, Bova and Kogelman 2014 pp. 34–35). The only use of compound returns in LBKH (2013) is in its Chapter 4, in an empirical context. Chapter 5 of that book has an interesting investigation of the convergence of mean return to initial yield for laddered portfolios.

	3 year Treasury	5 year Treasury	75 month duration Treasury	10 year Treasury	20 year Treasury	Long-Term Corporate
Full Period (3.59% avg. inflation)						
Change in yields	−1.6%	−0.9%	−0.6%	−0.1%	+0.2%	+1.2%
Avg. annual yield	5.5%	5.7%	5.8%	6.0%	6.2%	7.8%
Nominal Avg. annual return	5.4%	5.6%	5.9%	5.8%	6.0%	7.3%
Real Avg. annual return	1.8%	2.1%	2.3%	2.3%	2.4%	3.7%
Early Period, April 1953 to September 1981 (4.41% avg. inflation)						
Change in yields	+13.1%	+12.7%	+11.9%	+12.0%	+11.5%	+12.6%
Avg. annual yield	5.5%	5.6%	5.7%	5.7%	5.7%	7.0%
Nominal Avg. annual return	4.3%	3.8%	3.0%	2.7%	1.5%	2.4%
Real Avg. annual return	−0.1%	−0.6%	−1.4%	−1.7%	−2.9%	−2.0%
Late Period, October 1981 to April 2014 (2.87% avg. inflation)						
Change in yields	−14.1%	−13.2%	−12.5%	−11.9%	−11.3%	−11.6%
Avg. annual yield	5.4%	5.7%	6.0%	6.2%	6.7%	8.5%
Nominal Avg. annual return	6.4%	7.3%	8.4%	8.6%	9.9%	11.6%
Real Avg. annual return	3.6%	4.4%	5.5%	5.7%	7.0%	8.7%

Table 3: All rates, including inflation, are continuously-compounded annual percentages. Inflation is calculated from the FRED database “Consumer Price Index for All Urban Consumers: All Items, Index 1982–84 = 100, Monthly, Not Seasonally Adjusted” (CPIAUCNS). Inflation and real returns play no formal role in the analysis but inform its historical perspective.

Corollary 2 of Proposition 3, which says (accepting (4)) that the limit of $\overline{R}_a$ is average yield (and capital gains approach zero) if and only if $\lim_{N\to\infty} Y_{N+1}/N = \Delta Y$ is equal to zero, which is close to what happened over the full period. Similarly, over the early and late periods, ΔY was not close to zero, and returns were not similar to average yields.

The linear yield model predicts that average return will come closest to equaling initial yield over a horizon of length $2D - 1$. To examine a wide variety of horizons on both sides of that prediction, we used horizons of length "F" times duration, with F varying from 0.75 to 2.5. A horizon of $2D - 1$ implies an F of $2 - 1/D$. Using the smallest and largest values of D observed in the data (Table 4), 2.5 years and 16.9 years, converted to months since that is our rollover period, leads to an F for the linear yield model of between $2 - 1/(2.5 \cdot 12) = 1.964$ and $2 - 1/(16.9 \cdot 12) = 1.995$, a small range; comparing F for the *average* durations of the 3-year and 20-year or long-term corporate would generate an even smaller range. Rounding to one decimal place, the linear yield model thus predicts that the best horizon over which initial yield forecasts return will be at $F = 2$ for all of our bond series.

For the rest of this Section, the "early period" data will only include bonds whose "purchase date plus 2.5 times their initial duration" occurred on or before the September 1981 (2.5 being the maximum F). Since bonds whose purchase dates are before 9/1/81 but whose "purchase date plus 2.5 $*$ initial duration" are after 9/1/81 are excluded from the early and late periods but are included in the full period, the full period has more purchase dates, and thus has more observations, than the union of the early period and the late period, as shown in the last line of Table 4. Similarly, data for the "late period" and the "full period" will only include bonds whose "purchase date plus 2.5 times their initial duration" occurred on or before the April 2014. This ensures that comparisons are only made using the *same* bonds with different values of F. The consequences of this choice are illustrated in Table 5. For example, the 10-year bond purchased on 6/1/95 has a rounded initial duration of 91 months. With $F = 2$, its rounded "purchase date plus F times initial duration" is 8/1/10, which is in the data set. Hence this bond's experience could have been included in the analysis. It was not, because the outcome of this same bond with $F = 2.5$ is not known: its rounded "2.5 times initial duration" corresponds to 5/1/14, which is outside the data set. This is why the "last date of purchase" in the 10-year Treasury column of Table 4 is before 6/1/95. The 10-year bond purchased one month earlier, on 5/1/95, has a rounded "2.5 times initial duration" corresponding to 12/1/13, and so it is included in the analysis. This explains why in Table 4 the "last date of purchase," and therefore the number of observations, both fall at roughly 2.5 times the rate of increasing duration.[12]

[12] First example: For the full period and the late period, there are roughly 4½ years fewer observations for the 5-year Treasury than for the 3-year Treasury: the 55 months fewer observations is exactly 2.5 times the difference between the 8/1/02 5-year's 56 month duration and the 3/1/07 3-year's 34 months duration. Second example: For the full period and the late period, there are almost

	3 year Treasury	5 year Treasury	75 month duration Treasury	10 year Treasury	20 year Treasury	Long-Term Corporate
FULL PERIOD						
First Date of purchase	4/1/53	4/1/53	4/1/53	4/1/53	4/1/53	4/1/53
Last Date of 2.5 $*$ initial duration	4/1/14	3/1/14	4/1/14	12/1/13	3/1/14	10/1/13
Last Date of purchase	3/1/07	8/1/02	8/1/98	5/1/95	6/1/89	1/1/91
No. of Observations	648	593	545	506	435	454
Range of initial duration (years)	2.5–2.9	3.6–4.8	6.3	5.2–9.0	6.5–15.8	6.0–16.9
EARLY PERIOD						
First Date of purchase	4/1/53	4/1/53	4/1/53	4/1/53		
Last Date of 2.5 $*$ initial duration	9/1/81	9/1/81	9/1/81	9/1/81		
Last Date of purchase	1/1/74	11/1/70	1/1/66	10/1/60		
No. of Observations	260	212	154	91		
Range of initial duration (years)	2.7–2.9	4.2–4.8	6.3	8.1–9.0		
LATE PERIOD						
First Date of purchase	10/1/81	10/1/81	10/1/81	10/1/81	10/1/81	10/1/81
Last Date of 2.5 $*$ initial duration	4/1/14	3/1/14	4/1/14	12/1/13	3/1/14	10/1/13
Last Date of purchase	3/1/07	8/1/02	8/1/98	5/1/95	6/1/89	1/1/91
No. of Observations	306	251	203	164	93	112
Range of initial duration (years)	2.5–2.9	3.6–4.7	6.3	5.3–7.8	6.5–10.8	6.0–9.7
EARLY PERIOD plus LATE PERIOD						
No. of Observations	566	463	357	255	93	112

Table 4: Details for the different time periods. The number of observations and the range of initial duration only span the time between the first date of purchase and the last date of purchase.

Period		4/53	10/60	7/77	9/81	10/81	5/95	3/10	12/13	4/14
Early	Purchases	X								
	Data used, $F = 2$	X	X							
	Data used, $F = 2.5$	X	X	X						
Full	Purchases	X	X	X	X	X				
	Data used, $F = 2$	X	X	X	X	X	X			
	Data used, $F = 2.5$	X	X	X	X	X	X	X		
Late	Purchases					X				
	Data used, $F = 2$					X	X			
	Data used, $F = 2.5$					X	X	X		

Table 5: Time periods for the example of the 10-year Constant-Maturity Treasury series, $F = 2$, and $F = 2.5$. All dates refer to the first of the month.

The fall in the number of observations with increasing maturity causes there to be no observations at all of the two longest-maturity bonds during the early period, which is the shortest period. Finally, the fact that bonds purchased just one month apart (March versus June 1995) can have "2.5 times initial duration" quite a bit apart (December 2013 versus May 2014) explains why the last column of Table 5 is blank and why not all of the "last date of $2.5 *$ initial duration" entries in Table 4 are 4/1/14.

For each month, the capital gain was the difference between 100 and the price of coupon bond with par value of 100, coupon interest rate equal to the semiannually-compounded constant-maturity yield at the beginning of the month, maturity of one month less than it had at the beginning of the month, and current yield equal to that of the first day of the next month, as calculated by Excel's "price" function. (This assumes that the yield curve is flat between 35 and 36 months for 3-year bonds; flat between 59 and 60 months for 5-year bonds; and so on.) This capital gain was converted to a monthly percent, then to a monthly continuously-compounded percent; the start-of-period yield was also converted to a monthly continuously-compounded percent; then the capital gain was combined with the yield as in (3)) to obtain the monthly continuously-compounded total return. From this a monthly "growth of $10,000" series was generated. The initial Modified Duration of each month's bond was next calculated using Excel, multiplied by the factor F, rounded to the nearest integer; then the annual continuously-compounded return was calculated for that forward time span.[13] "Forecast Error" was defined as this value

6 years fewer observations for the 20-year Treasury than for the 10-year Treasury: 71 months fewer observations is 4 months less than 2.5 times the difference between the 6/1/89 20-year's 119 month duration and the 5/1/95 10-year's 89 months duration, with 3 months of the difference due to their differences in "last date of $2.5 *$ initial duration."

[13]Excel is not designed to calculate duration for continuous compounding, but it can be manipulated into doing so by using its Macauley Duration function for periodic compounding, (1), with yields that are equal to exp(continuous-compounding Y) $- 1$.

minus the annual continuously-compounded initial yield. Formally, if $\overline{R}_{amFt}$ denotes the arithmetic mean realized annual continuously-compounded return for "a bond purchased at date t with constant maturity (or duration) m then rolled over monthly" over the forward period of length "$F *$ initial duration," then since its initial yield Y_{mt} is its predicted annual return, its forecast error is $\overline{R}_{amFt} - Y_{mt}$.

With fifteen values of F and sixteen series of bonds (six each for the full and late periods and four for the early period) there were 240 time series of forecast errors. Section 4 presents a fairly complete empirical analysis, including graphs, of the $F = 2 - 1/D \approx 2$ horizon, but space prohibits a similarly complete analysis of all the values of F here. Instead, each time series here will be described by no graphs and just a few of the numbers of interest:

Centered R^2 with slope 1 and intercept 0: This is the R^2 measure of goodness of fit for the equation $\overline{R}_{amFt} = 1 * Y_{mt} + 0$. Many software packages, such as Excel,[14] report "centered R^2" as their goodness-of-fit measurement for regressions with a constant term, and "uncentered R^2" as their goodness-of-fit measurement for regressions without a constant term. We fit no regressions because we are not interested in what line has historically best related Y_1 to $\overline{R}$: we are only interested in the line which has a slope of one and an intercept of zero. Nevertheless, both R^2 measures can be calculated. I agree with Wooldridge (2012 p. 237) that it is better to use centered R^2, which is the scale in most people's minds when thinking about R^2 because most regressions have constant terms. For an opposite opinion and survey of this "long dispute" in statistics, see Eisenhauer (2003). Unfortunately, centered R^2 can be negative if there is no constant term and the fit is poor, and this occurs several times in the second part of Table 6 below; when it is negative it is admittedly not "the scale in most people's minds," and it is hard to think of it as something "squared." Uncentered R^2 is always positive, always greater than centered R^2, and could make the reader think the fit is better than it actually is.

Root mean square ('RMS') forecast error: Given n purchase dates for bonds of a fixed maturity or duration, and a fixed choice of F, this is $(\sum_t(\overline{R}_{amFt} - Y_{mt})^2/n)^{1/2}$ over the relevant purchase dates.[15]

Average forecast error: $\sum_t(\overline{R}_{amFt} - Y_{mt})/n$. While all our other measures of goodness of fit would rank forecast errors of $\{-2, +2, -2, +2\}$ worse than $\{+1/2, +1/2, +1/2, +1/2\}$, this one will rank the latter worse than the former, and it is possible investors would have such a preference (for example, that they would care about some moving average of the errors).

[14] http://office.microsoft.com/en-us/excel-help/linest-HP005209155.aspx

[15] LBKH (2013, p. 4-13) call this the "tracking error" ("TE"); so do Leibowitz, Bova, and Kogelman (2014 p. 49).

Frequency of Absolute Value of Forecast Error ("FFE") less than $x\%$: This is the value of the cumulative distribution function of the absolute value of forecast errors for various arbitrary values of x.

Initial yield predicts future return better the smaller the RMS forecast error; the smaller the absolute value of the average forecast error; the larger the centered R^2; and the larger "frequency of absolute value of forecast error $< x\%$" is for any given x. Of these criteria, RMS error is probably the best, since R^2 is controversial, average forecast error treats negative and positive errors unconventionally, and the cumulative distribution function of forecast errors requires an arbitrary specification of x.

Table 6 reports the 2160 results for these criteria. The remainder of this section gives one interpretation of this table.

A star in Table 6 denotes a value within a factor of 1.03 of the best value for that row. Stars were not assigned to "FFE $< x\%$" rows having $x \geq 2\%$ because those rows would have had almost all columns being awarded stars. The number of stars in each column is summed in the last row of the table's last part, and can be used to summarize which F was best for forecasting in this data set. The boldface "G" (for "good") and "B" (for "bad") in the left-hand column designate rows whose best values satisfy:

	Good	Bad
RMS FE	$\leq 0.50\%$	$\geq 1\%$
Cent R^2	≥ 0.90	≤ 0.80
FFE $< .5\%$	$\geq 70\%$	$\leq 50\%$
FFE $< 1\%$	$\geq 90\%$	$\leq 70\%$

The sum of the number of stars shows that overall, while bond returns are somewhat predictable at all of the F values used, the predictions are better for F values near the middle of the given range. Strictly speaking, conventional wisdom is not wrong in saying that initial yield is an approximation of realized return over the bonds' initial duration, but that is a poorer approximation than using a longer period. On the other hand, Section 1 and 2's theoretical model's $F \approx 2$, while certainly better than $F = 1$, did more poorly overall than F's closer to $1\,3/4$. The only bonds for which $F > 2$ did well, the early period's 75-month and 10-year, had among the smallest number of observations; and among the series with the largest number of observations were the 20-year and Long-term Corporate for the full period, for which F's as low as 1.7 or even 1.6 did better than $F = 2$.

We know from the theoretical model of Sections 1 and 2 that the reason $F = 2$ did not perform the best is because of the nonlinear component of the yield paths or from Return Approximation errors, and we will quantify these sources of errors empirically in Section 5. Those two error sources contribute unpredictably to return. The predictable part of return derives from the linear component of the yield path and the Return Approximation, and reveals itself at $F \approx 2$, so Section 4

F	0.75	1	1.25	1.5	1.6	1.7	1.75	1.8	1.9	2	2.1	2.2	2.3	2.4	2.5
3 YEAR															
Full															
RMS FE	1.74%	1.30%	1.05%	0.92%★	0.90%★	0.92%★	0.89%★	0.92%★	0.94%	0.96%	0.98%	0.99%	1.01%	1.02%	1.05%
Avg FE	-0.10%	-0.09%	-0.05%	-0.05%	-0.03%★	-0.04%	-0.04%	-0.04%	-0.04%	-0.03%★	-0.04%	-0.04%	-0.04%	-0.04%	-0.05%
Cent R^2	0.77	0.84	0.88★	0.90★	0.90★	0.89★	0.89★	0.89★	0.88★	0.87★	0.86	0.85	0.85	0.84	0.83
FFE < .5% **B**	25%	32%	37%	41%	41%	43%★	42%★	42%★	40%	40%	38%	38%	42%★	42%★	40%
FFE < 1%	45%	57%	68%	73%★	73%★	72%★	74%★	72%★	72%★	70%	70%	71%	67%	66%	68%
FFE < 2%	77%	87%	94%	98%	98%	97%	98%	97%	96%	96%	95%	95%	95%	94%	93%
FFE < 3%	92%	98%	99%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 4%	96%	99%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	99%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
5 YEAR															
Full															
RMS FE	1.83%	1.32%	1.08%	0.98%	0.97%	0.93%★	0.91%★	0.93%★	0.94%★	0.94%★	0.97%	0.96%	0.97%	0.98%	1.00%
Avg FE	-0.17%	-0.17%	-0.12%	-0.08%	-0.06%	-0.07%	-0.07%	-0.05%	-0.04%	-0.03%	-0.01%★	-0.02%	-0.02%	-0.01%★	-0.01%★
Cent R^2	0.75	0.84	0.88★	0.89★	0.89★	0.89★	0.89★	0.89★	0.88★	0.88★	0.86★	0.86★	0.85	0.84	0.84
FFE < .5%	28%	38%	42%	49%	50%	53%★	54%★	52%	50%	50%	49%	47%	44%	44%	44%
FFE < 1%	50%	63%	69%	76%	78%★	79%★	80%★	80%★	80%★	78%★	76%	75%	74%	75%	74%
FFE < 2%	79%	89%	94%	94%	94%	94%	94%	93%	94%	94%	95%	95%	95%	93%	92%
FFE < 3%	90%	96%	98%	99%	99%	99%	100%	99%	99%	99%	97%	98%	99%	100%	100%
FFE < 4%	95%	98%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	98%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
75 MONTH															
Full															
RMS FE **B**	2.14%	1.60%	1.28%	1.13%	1.10%	1.06%	0.94%★	1.03%	1.02%	1.03%	1.05%	1.07%	1.13%	1.17%	1.21%
Avg FE	-0.21%	-0.12%	-0.07%	-0.04%	-0.03%	0.00%★	-0.13%	0.01%	0.03%	0.05%	0.07%	0.09%	0.11%	0.13%	0.14%
Cent R^2	0.74	0.81	0.86	0.87★	0.88★	0.88★	0.89★	0.88★	0.88★	0.87★	0.86	0.85	0.82	0.80	0.78
FFE < .5%	32%	32%	43%	47%	51%	51%	51%	49%	51%	54%★	52%★	48%	44%	45%	42%
FFE < 1%	47%	55%	66%	74%	72%	74%	77%★	77%★	73%	70%	70%	69%	67%	68%	66%
FFE < 2%	76%	84%	90%	91%	89%	91%	94%	91%	92%	92%	91%	91%	88%	86%	86%
FFE < 3%	88%	92%	94%	97%	97%	99%	99%	99%	99%	100%	100%	100%	100%	100%	99%
FFE < 4%	92%	95%	99%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	94%	99%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
10 YEAR															
Full															
RMS FE	2.04%	1.74%	1.29%	1.00%	1.07%	1.02%	0.96%	0.93%★	0.92%★	0.92%★	0.96%	1.03%	1.10%	1.17%	1.27%
Avg FE	-0.44%	-0.23%	-0.13%	-0.06%	-0.07%	-0.04%★	-0.04%★	0.03%★	0.11%	0.18%	0.24%	0.30%	0.32%	0.37%	0.42%
Cent R^2 **G**	0.76	0.81	0.87	0.91★	0.89★	0.90★	0.90★	0.91★	0.91★	0.90★	0.89★	0.87	0.83	0.78	0.74
FFE < .5%	18%	27%	29%	36%	41%	42%	47%	50%★	50%★	50%★	52%★	42%	35%	28%	26%
FFE < 1%	43%	52%	60%	76%★	72%	72%	71%	73%	72%	71%	72%	68%	62%	58%	55%
FFE < 2%	77%	82%	89%	94%	92%	92%	95%	96%	97%	97%	95%	94%	93%	92%	88%
FFE < 3%	90%	89%	95%	99%	98%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 4%	93%	95%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	95%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
20 YEAR															
Full															
RMS FE **B**	2.98%	1.91%	1.43%	1.29%	1.23%★	1.21%★	1.24%★	1.29%	1.31%	1.53%	1.63%	1.66%	1.77%	1.85%	1.98%
Avg FE	-0.77%	-0.66%	-0.10%★	0.49%	0.53%	0.63%	0.75%	0.87%	0.95%	1.19%	1.25%	1.26%	1.34%	1.38%	1.45%
Cent R^2	0.70	0.82	0.88★	0.88★	0.88★	0.87★	0.85	0.82	0.79	0.61	0.51	0.41	0.28	0.10	-0.22
FFE < .5% **B**	6%	3%	8%	14%	20%	23%	20%	29%	32%★	20%	16%	15%	13%	10%	10%
FFE < 1% **B**	9%	11%	30%	48%★	48%★	43%	43%	45%	43%	30%	26%	27%	25%	23%	19%
FFE < 2%	48%	74%	87%	90%	93%	95%	96%	92%	93%	83%	74%	75%	63%	57%	52%
FFE < 3%	76%	92%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	99%	99%
FFE < 4%	83%	98%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	90%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
LT CORP															
Full															
RMS FE **B**	3.13%	2.10%	1.39%★	1.41%★	1.42%	1.37%★	1.40%★	1.53%	1.77%	1.89%	1.99%	2.04%	2.18%	2.21%	2.19%
Avg FE	-0.76%	-0.66%	0.10%★	0.57%	0.71%	0.94%	1.04%	1.20%	1.44%	1.43%	1.47%	1.43%	1.49%	1.45%	1.44%
Cent R^2 **G**	0.72	0.82	0.91★	0.87	0.87	0.85	0.83	0.78	0.64	0.51	0.34	0.21	-0.06	-0.22	-0.25
FFE < .5% **B**	7%	4%	12%	26%	17%	34%★	29%	27%	18%	16%	15%	14%	10%	10%	10%
FFE < 1% **B**	14%	10%	37%	46%	39%	49%★	46%	44%	30%	25%	25%	23%	22%	19%	22%
FFE < 2%	39%	60%	87%	86%	88%	82%	81%	74%	70%	60%	55%	55%	50%	50%	52%
FFE < 3%	68%	89%	100%	100%	100%	100%	100%	100%	95%	93%	91%	89%	83%	79%	81%
FFE < 4%	83%	97%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	87%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%

Table 6, first part: Measures of goodness-of-fit for the full period. The abbreviations are: "RMS FE," root mean square forecast error; "Avg FE," average forecast error; "Cent R^2," centered R^2 of $\overline{R}_{amFt} = 1Y_{mt} + 0$; "FFE $<$ $x\%$," frequency of absolute value of forecast error less than the given percent. The multiple of initial duration over which errors are computed is F. See the text for explanations of the stars and boldface "G" and "B." This table continues in two more parts.

F	0.75	1	1.25	1.5	1.6	1.7	1.75	1.8	1.9	2	2.1	2.2	2.3	2.4	2.5
3 YEAR															
Early															
RMS FE	1.13%	0.83%	0.68%	0.64%	0.61%★	0.61%★	0.59%★	0.60%★	0.61%★	0.65%	0.65%	0.63%	0.65%	0.65%	0.65%
Avg FE	-0.46%	-0.34%	-0.24%	-0.20%	-0.17%	-0.15%	-0.15%	-0.14%	-0.12%	-0.10%	-0.08%	-0.05%	-0.03%	-0.01%	0.00%★
Cent R^2	0.74	0.83★	0.85★	0.84★	0.85★	0.85★	0.85★	0.85★	0.84★	0.80	0.80	0.81	0.78	0.77	0.76
FFE < .5%	31%	42%	49%	55%	61%	61%	63%★	62%★	56%	57%	55%	58%	61%	63%★	61%
FFE < 1% **G**	58%	77%	88%	90%★	90%★	91%★	91%★	91%★	92%★	92%★	92%★	91%★	88%	90%★	92%★
FFE < 2%	92%	99%	100%	100%	100%	100%	100%	100%	99%	99%	99%	100%	99%	98%	98%
FFE < 3%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
%FFE < 4%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
%FFE < 5%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
5 YEAR															
Early															
RMS FE **G**	1.11%	0.87%	0.71%	0.56%	0.51%	0.48%★	0.50%★	0.51%	0.53%	0.55%	0.58%	0.64%	0.65%	0.66%	0.70%
Avg FE	-0.64%	-0.54%	-0.44%	-0.28%	-0.22%	-0.16%	-0.16%	-0.12%	-0.10%	-0.07%	-0.04%	-0.03%	-0.02%	0.01%★	0.01%★
Cent R^2	0.64	0.68	0.73	0.82	0.85★	0.87★	0.85★	0.85★	0.82	0.81	0.77	0.70	0.69	0.67	0.59
FFE < .5% **G**	38%	53%	54%	59%	70%★	71%★	72%★	67%	62%	61%	64%	64%	59%	60%	60%
FFE < 1% **G**	62%	74%	81%	93%★	95%★	95%★	95%★	94%★	93%★	92%	89%	88%	89%	89%	89%
FFE < 2%	94%	98%	100%	100%	100%	100%	100%	100%	100%	100%	100%	99%	100%	100%	98%
FFE < 3%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
%FFE < 4%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
%FFE < 5%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
75 MONTH															
Early															
RMS FE **G**	1.41%	1.06%	0.80%	0.69%	0.66%	0.60%	0.52%	0.52%	0.46%	0.45%	0.43%	0.41%	0.40%★	0.39%★	0.44%
Avg FE	-1.02%	-0.78%	-0.55%	-0.45%	-0.42%	-0.36%	-0.39%	-0.28%	-0.20%	-0.15%	-0.09%	-0.04%	-0.01%★	0.02%	0.03%
Cent R^2 **B**	-0.51	-0.45	-0.34	-0.40	-0.45	-0.01	0.35	0.48	0.68	0.69	0.70★	0.72★	0.72★	0.72★	0.60
FFE < .5% **G**	45%	50%	62%	62%	67%	71%	68%	69%	73%	78%★	75%	75%	75%	77%★	75%
FFE < 1% **G**	52%	64%	79%	85%	88%	89%	93%	92%	95%	95%	96%	100%★	99%★	100%★	99%★
FFE < 2%	86%	92%	97%	100%	99%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 3%	95%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 4%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
10 YEAR															
Early															
RMS FE **G**	0.75%	0.72%	1.04%	0.79%	0.71%	0.62%	0.57%	0.51%	0.48%	0.38%	0.25%★	0.26%★	0.28%	0.41%	0.59%
Avg FE	-0.64%	-0.62%	-0.87%	-0.69%	-0.61%	-0.54%	-0.51%	-0.38%	-0.28%	-0.16%	-0.04%★	0.03%★	0.08%	0.15%	0.08%
Cent R^2 **B**	0.26	-1.32	-11.27	-0.37	0.07	0.28	0.31	0.35	0.12	0.35	0.78	0.80★	0.70	0.07	-2.51
FFE < .5% **G**	41%	60%	38%	35%	38%	44%	46%	60%	67%	76%	97%★	91%	92%	71%	51%
FFE < 1% **G**	88%	79%	59%	81%	84%	93%	99%★	99%★	96%	100%★	100%★	100%★	100%★	100%★	96%
FFE < 2%	100%	100%	97%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 3%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
%FFE < 4%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
%FFE < 5%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%

Table 6, second part: the early period.

F	0.75	1	1.25	1.5	1.6	1.7	1.75	1.8	1.9	2	2.1	2.2	2.3	2.4	2.5
3 YEAR															
Late															
RMS FE	1.74%	1.31%	1.06%	0.96%⋆	0.96%⋆	0.96%⋆	0.95%⋆	0.99%	1.00%	1.01%	1.03%	1.04%	1.04%	1.06%	1.08%
Avg FE	0.50%	0.40%	0.32%	0.18%	0.13%	0.07%	0.07%	0.01%	-0.06%⋆	-0.11%	-0.18%	-0.24%	-0.29%	-0.33%	-0.38%
Cent R^2	0.78	0.84	0.88⋆	0.88⋆	0.87⋆	0.86⋆	0.85⋆	0.84	0.83	0.82	0.80	0.79	0.78	0.77	0.76
FFE < .5% **B**	24%	29%	34%⋆	31%	29%	31%	30%	28%	29%	30%	30%	28%	34%⋆	33%⋆	29%
FFE < 1% **B**	40%	49%	61%	65%⋆	66%⋆	65%⋆	66%⋆	64%⋆	62%	60%	61%	64%⋆	61%	59%	61%
FFE < 2%	73%	86%	94%	99%	99%	98%	99%	98%	98%	98%	96%	96%	96%	96%	94%
FFE < 3%	93%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 4%	98%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
5 YEAR															
Late															
RMS FE	1.79%	1.25%	0.98%	0.76%	0.74%	0.68%	0.64%⋆	0.67%	0.65%⋆	0.68%	0.77%	0.80%	0.84%	0.84%	0.88%
Avg FE	0.89%	0.58%	0.32%	0.08%	0.01%⋆	-0.08%	-0.07%	-0.15%	-0.17%	-0.19%	-0.24%	-0.31%	-0.38%	-0.43%	-0.48%
Cent R^2 **G**	0.71	0.83	0.87	0.89⋆	0.89⋆	0.90⋆	0.90⋆	0.90⋆	0.90⋆	0.89⋆	0.86	0.84	0.83	0.82	0.81
FFE <	27%	31%	36%	53%⋆	49%	53%⋆	53%⋆	51%	53%⋆	53%⋆	49%	45%	45%	44%	45%
FFE < 1% **G**	50%	61%	71%	77%	80%	83%	87%⋆	88%⋆	90%⋆	85%	82%	77%	76%	78%	78%
FFE < 2%	77%	88%	96%	100%	100%	100%	100%	100%	100%	100%	99%	98%	98%	97%	95%
FFE < 3%	91%	98%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 4%	96%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
75 MONTH															
Late															
RMS FE	2.02%	1.14%	0.78%	0.69%	0.68%	0.66%	0.60%⋆	0.61%⋆	0.67%	0.74%	0.72%	0.74%	0.89%	1.02%	1.11%
Avg FE	1.36%	0.90%	0.56%	0.23%	0.14%	0.04%⋆	-0.15%	-0.06%	-0.19%	-0.29%	-0.37%	-0.42%	-0.51%	-0.58%	-0.68%
Cent R^2 **G**	0.64	0.77	0.89⋆	0.91⋆	0.90⋆	0.90⋆	0.90⋆	0.91⋆	0.88⋆	0.83	0.84	0.83	0.73	0.62	0.54
FFE < .5%	28%	29%	48%	51%	54%	52%	56%	50%	54%	63%	68%⋆	61%	47%	46%	40%
FFE < 1% **G**	43%	60%	77%	90%	80%	88%	92%⋆	94%⋆	89%	82%	85%	83%	78%	73%	68%
FFE < 2%	74%	91%	100%	100%	100%	100%	100%	100%	100%	99%	97%	99%	96%	91%	90%
FFE < 3%	91%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	99%
FFE < 4%	95%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	95%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
10 YEAR															
Late															
RMS FE **G**	2.13%	1.13%	0.85%	0.60%	0.65%	0.57%	0.47%⋆	0.53%	0.49%⋆	0.54%	0.66%	0.82%	0.92%	0.91%	0.98%
Avg FE	1.47%	0.91%	0.67%	0.35%	0.24%	0.03%⋆	-0.07%	-0.20%	-0.17%	-0.17%	-0.34%	-0.45%	-0.54%	-0.56%	-0.67%
Cent R^2 **G**	0.60	0.78	0.84	0.92⋆	0.91⋆	0.92⋆	0.94⋆	0.92⋆	0.94⋆	0.92⋆	0.87	0.76	0.69	0.68	0.64
FFE < .5% **G**	27%	30%	38%	46%	57%	58%	76%⋆	70%	73%	71%	58%	42%	35%	37%	35%
FFE < 1% **G**	45%	60%	73%	96%⋆	89%	92%	94%⋆	91%	95%⋆	92%	89%	80%	73%	74%	71%
FFE < 2%	74%	97%	100%	100%	100%	100%	100%	100%	100%	100%	100%	99%	96%	98%	95%
FFE < 3%	90%	99%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 4%	93%	99%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	94%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
20 YEAR															
Late															
RMS FE **G**	2.63%	1.49%	1.25%	0.79%	0.69%	0.54%	0.57%	0.52%	0.48%⋆	0.50%⋆	0.65%	0.65%	0.68%	0.82%	0.93%
Avg FE	2.09%	1.39%	1.02%	0.68%	0.53%	0.27%	0.31%	0.16%	0.03%⋆	-0.25%	-0.32%	-0.39%	-0.47%	-0.47%	-0.55%
Cent R^2 **G**	0.38	0.57	0.69	0.81	0.85	0.90⋆	0.87	0.90⋆	0.92⋆	0.90⋆	0.81	0.82	0.80	0.68	0.49
FFE < .5%	15%	4%	27%	33%	48%	59%	54%	63%	67%⋆	68%⋆	55%	59%	44%	31%	33%
FFE < 1% **G**	19%	26%	61%	80%	84%	96%⋆	95%⋆	97%⋆	96%⋆	96%⋆	85%	87%	86%	77%	63%
FFE < 2%	52%	87%	87%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 3%	81%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 4%	92%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	95%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
LT CORP															
Late															
RMS FE **G**	3.07%	1.66%	1.27%	0.64%	0.61%	0.47%⋆	0.53%	0.46%⋆	0.52%	0.77%	0.68%	0.89%	1.01%	1.13%	1.10%
Avg FE	2.57%	1.49%	1.01%	0.37%	0.41%	0.26%	0.33%	0.20%	0.03%⋆	-0.35%	-0.50%	-0.72%	-0.86%	-0.94%	-0.93%
Cent R^2 **G**	0.23	0.58	0.76	0.92	0.92	0.95⋆	0.93⋆	0.95⋆	0.94⋆	0.87	0.87	0.78	0.68	0.58	0.56
FFE < .5% **G**	1%	13%	26%	62%	47%	72%⋆	61%	68%	63%	55%	54%	42%	30%	29%	29%
FFE < 1% **G**	7%	24%	40%	91%	96%	97%⋆	95%	100%⋆	96%	81%	79%	70%	61%	50%	59%
FFE < 2%	47%	77%	89%	99%	100%	100%	100%	100%	100%	98%	100%	99%	100%	97%	97%
FFE < 3%	75%	96%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 4%	88%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
FFE < 5%	90%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%
Total Stars	0	1	11	20	24	39	40	32	30	20	12	9	7	11	5

Table 6, last part: the late period.

gives more detail about the performance of initial yield as a predictor of return at that F.

4. Empirical Results for an Horizon of Twice Duration

This section provides an in-depth analysis of the $F = 2$ case. Figure 4 begins by showing both the initial yield and the forward realized return for the bonds. For example, the 10-year bond bought in February 1976 had "two times initial duration" ending on April 1990; its initial yield was 7.64%, but its actual annualized return over that period was 8.86%; so its red dotted line is 1.22% higher than its solid line, illustrated by the vertical dashed line. The yield at the end of the period was 8.56%; the figure's dashed sloped line joins the initial yield to the end-of-period yield. This bond did better than initially expected primarily because despite the fact that yields rose, which caused a small capital loss, the sharply higher yields in the middle of the period caused its interest earnings to be much higher than its initial yield. Section 5 analyzes such reasons for forecast errors.

The gap between the two lines in Figure 4 is graphed with a black solid line in Figure 5. (Ignore the figure's blue dash-dot lines until Section 5.) The "G" and "B" code used in Table 6 is also used in this figure, with a dash representing neither G nor B, with the five criteria listed in the order they appear in Table 6 (RMS FE, Avg FE, etc.), and with "Avg FE" obtaining a "G" if its absolute value is less than 0.20% and a "B" if its absolute value is greater than 0.50%.[16] Because here these codes apply only to $F = 2$, the pattern can differ from that in Table 6 despite keeping Table 6's definitions of each period's dates. The G/B/- code next to the name of the bond series pertains to the full period, i.e., all purchase dates lying to the left of the vertical dashed line; that near the upper left-hand corner pertains to the early period, whose purchase dates lie between the vertical axis and the first solid vertical line; and the code in the upper center-right part of the graph pertains to the late period, whose purchase dates lie between the 9/1/81 solid vertical line and the dashed vertical line. It is impossible to calculate forecast errors for purchase dates to the right of the dashed vertical line when $F = 2.5$ and so those dates are not included in any of this Section's analyses, but some of them are feasible for $F = 2$ and are depicted in Figure 5. (They brought Figure 5's number of observations, listed in the order given in Table 4, to 664, 624, 583, 545, 478, and 478.)

The sharp drop near the end of the Long-term Corporate series reflects the late-2008 market disruptions.[17]

The best fit is for the 10-year bonds' late period, where three of the five criteria deem the fit to be good. For overall performance, the 10-year is somewhat

[16] This was not used in Table 6 because if it had been, every "Avg FE" row in that table except one would have gotten a "G."

[17] Long-term Corporate data for (purchase date, last date of twice initial duration, forecast error) for twice-duration periods ending from October 2008 to January 2009 is: (10/89, 11/08, −1.78%); (11/89, 12/08, −1.38%); (12/89, 1/09, −1.23%); (1/90, 12/08, −1.45%); (2/90, 10/08, −1.79%); (6/90, 12/08, −1.62%); (8/90, 11/08, −2.14%); (11/90, 11/08, −2.27%).

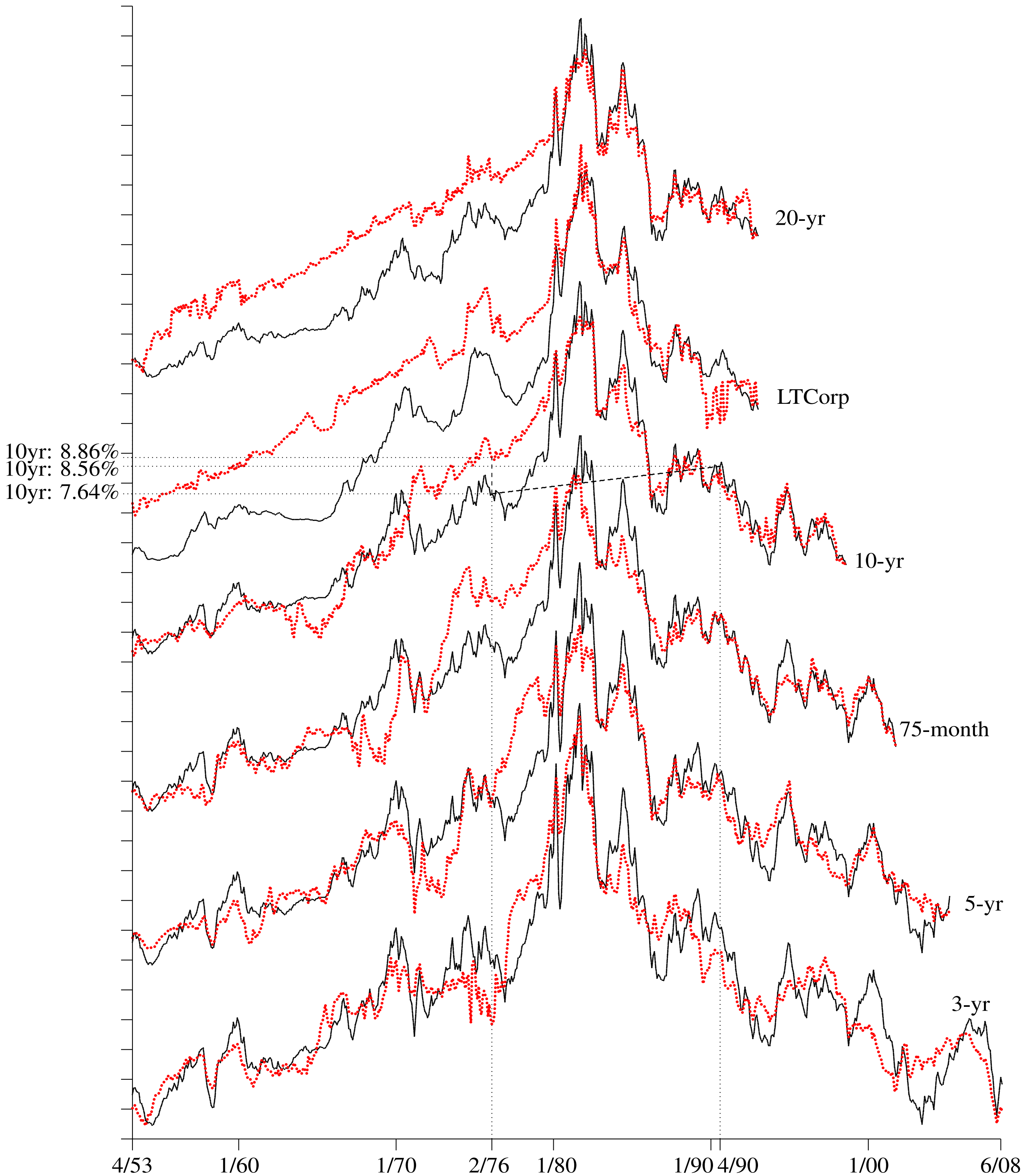

Figure 4. Initial yield (solid line) and forward return (dotted red line) for $F = 2$. Horizontal axis: purchase date. Vertical axis: continually-compounded annualized return, with absolute yields not shown so that the graphs can be separated for legibility, but vertical tick marks given every 1%.

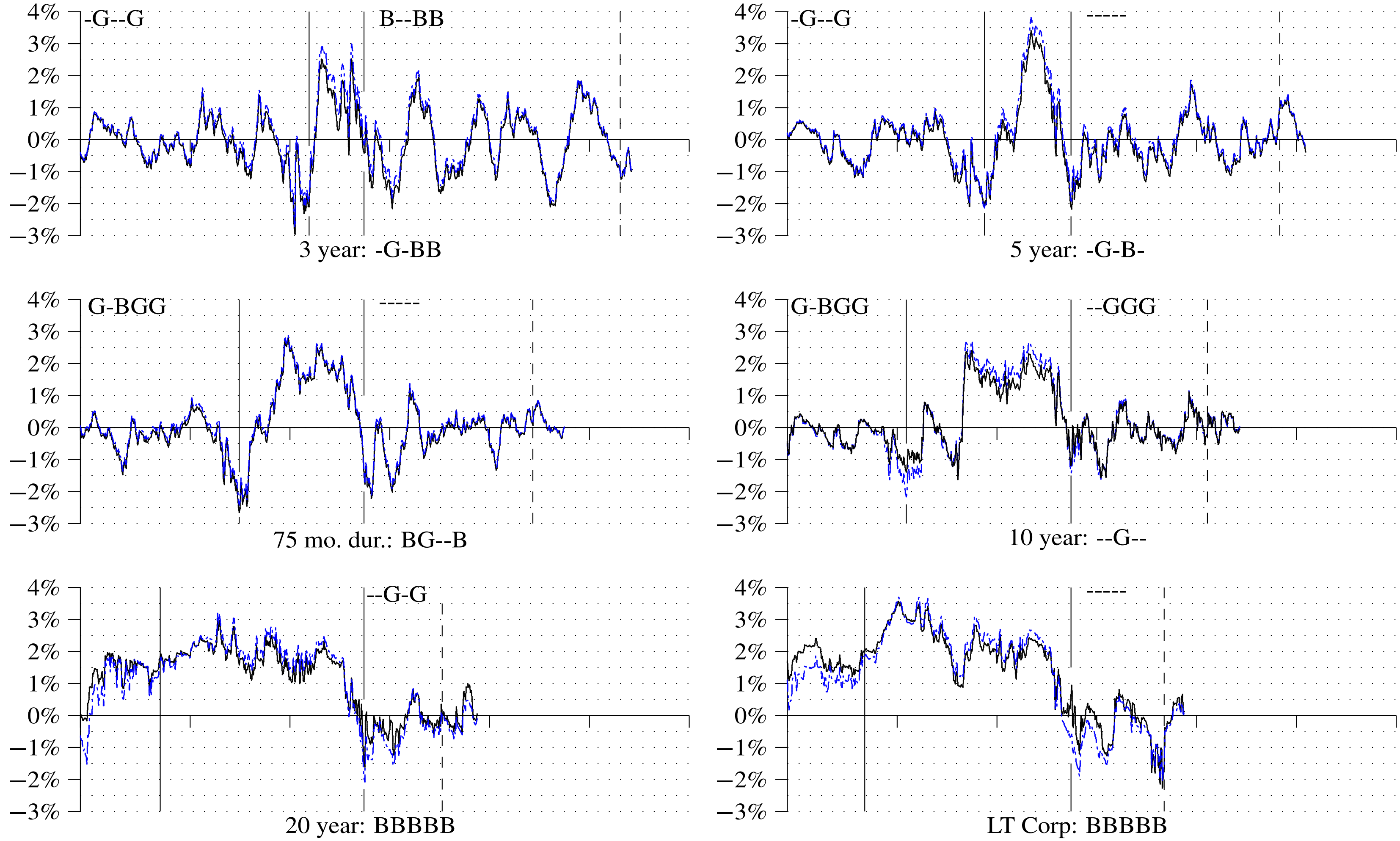

Figure 5. The solid lines are the forecast errors for each series when $F = 2$, graphed with identical axes. The dash-dot blue lines are Section 5's "Nonlinearity Measure" for each "two times initial duration" period. The horizontal axes are purchase dates. They start on 4/1/53, end on 4/1/14, have a tick mark every decade, and have a solid vertical line marking 9/1/81. There is another solid vertical line marking each "last date of purchase" of the early period for $F = 2$. The solid vertical lines are sometimes interrupted to improve legibility. The dashed vertical lines show the last observation possible when $F = 2.5$. The text explains the G, B and - codes, which were calculated for the same "early," "late," and "full" dates used when $F = 2.5$.

better than the 3-year's, 5-year's, and 75-month constant duration's, while the 20-year and the long-term corporate have the worst performance, with all five criteria deeming their fit poor, due to persistent positive forecast errors for bonds purchased before the early 1980's.

There is no data for the 20-year bonds before 4/1953, but for the long-term corporate bonds there is data going back to 1919, which is useful in obtaining a longer historical perspective on the persistent forecast errors for long bonds. The yield, forward return, and duration are plotted in Figure 6. A very prominent feature of Figure 6 graph is its the two interest rates peaks, in 1932 and 1981.[18] From 1932 until the late-1950's, the yield path on a scale appropriate to long-term bonds—approximately two decades—has an overall convex shape; then it reaches an inflection point and becomes concave. From the Corollary to Proposition 4, this would give rise to negative forecast errors during the earlier period, which are observed at least from 1934 until the late 1940's, and positive errors afterwards, which are observed until the early-1980's.

For concreteness, Figure 6 has three straight lines, each of which has a length equal to twice the duration of the bond on the left-hand endpoint of the line. One of these lines connects the 4/35 yield with the 4/61 yield, which was lower. The intervening yields lie mostly below this line, so the shape is generally convex. That explains the 4/35 negative forecast error in accordance with the Corollary of Proposition 4. The next line connects the 2/40 yield with the 7/69 yield, which was higher. The intervening yields lie mostly below this line, so the shape is again generally convex, again leading to a negative forecast error. The yields between the start and end of the line from 9/67 to 1/93 lie mostly above the line, so the shape is generally concave, and the forecast error positive. The lines going to the right from 2/40 and from 9/67 have similar positive slopes, but they have very different forecast errors—the first negative, the second positive—because their associated yield paths differ in convexity. These results are all in complete accordance with the Corollary of Proposition 4 that it is not the direction of the yield path but rather its concavity or convexity that determines the sign of the forecast error.[19]

Table 7 gives details for $F = 2$ which are not included in Table 6. Its row $\text{RMS}^2/\text{St. Dev.}^2$ is included because although the traditional measure of risk is variance (the square of the standard deviation of the row labeled "Avg. annualized return over $2 *$ initial duration"), a better measure of risk might be the square of Table 6's 2.0 column's "RMS FE," the argument being that short-term fluctuations

[18]The sharp interest rate peak of 1932 in Figure 6 was due to a rise in long-term Treasury rates (FRED's "Long-term U.S. Government Securities (Discontinued Series)") of slightly more than 1% coupled with an increase in the spread between these two rates from approximately 2% in early 1929 to more than 7% in mid-1932.

[19]The direction of the yield path did play a role in Column G of Table 1, which showed that forecast error would be lowered by using larger F's when $\Delta Y > 0$ and smaller F's when $\Delta Y < 0$, but that was a consequence of using the geometric mean, which is inapplicable to the continuously-compounded yields and returns of this Section.

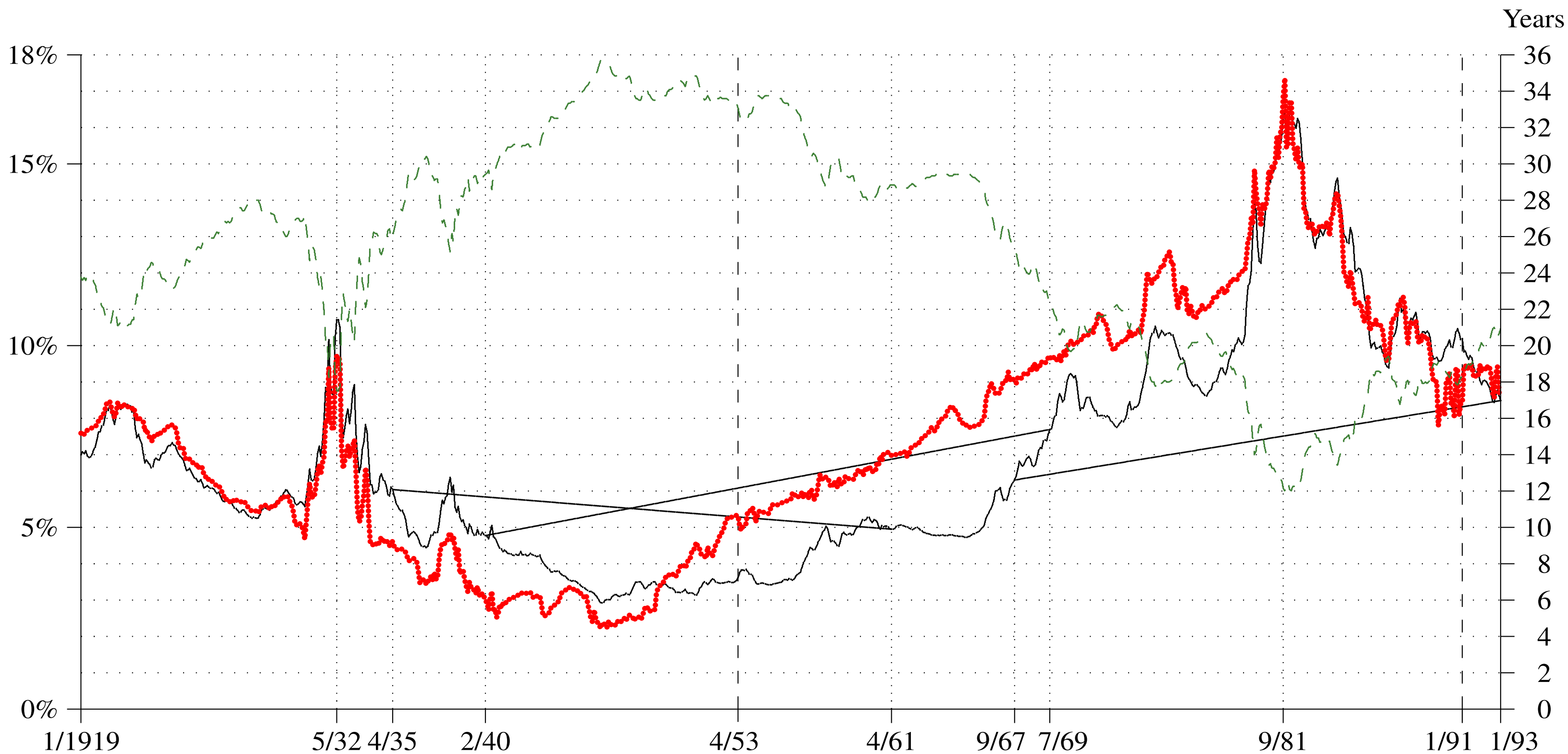

Figure 6. Results for $F = 2$ for the Long-term Corporate series. The vertical axis is continually-compounded annualized yield or return. The horizontal axis is the initial purchase date. The scale of both axes is the same as in Figure 4. The solid line is the yield and the dotted red line is the forward realized return over a period twice its initial duration. The right-hand axis is in years and the dashed green line, which is read on that axis, represents two times duration. The tick mark between 10/88 and 2/93 is 3/91, the last date of purchase. Using the code of Figure 5, the overall fit is BB-BB.

	3 year Treasury	5 year Treasury	75 month duration Treasury	10 year Treasury	20 year Treasury	Long-Term Corporate
FULL PERIOD						
Last date of 2 ∗ initial duration	11/1/12	12/1/11	2/1/11	3/1/10	4/1/09	4/1/09
Avg. annualized return over 2 ∗ initial duration	6.0% ±2.7%	6.4% ±2.7%	6.7% ±2.9%	6.9% ±3.0%	7.9% ±2.4%	9.6% ±2.7%
$RMS^2/St. Dev.^2$	0.13	0.12	0.13	0.10	0.39	0.49
Correlation Coefficient	0.94	0.94	0.93	0.95	0.96	0.94
EARLY PERIOD						
Last date of 2 ∗ initial duration	5/1/80	7/1/79	7/1/78	7/1/77		
Avg. annualized return over 2 ∗ initial duration	4.4% ±1.5%	4.1% ±1.2%	3.4% ±0.8%	3.1% ±0.5%		
$RMS^2/St. Dev.^2$	0.20	0.19	0.31	0.65		
Correlation Coefficient	0.92	0.92	0.85	0.87		
LATE PERIOD						
Last date of 2 ∗ initial duration	11/1/12	12/1/11	2/1/11	3/1/10	4/1/09	4/1/09
Avg. annualized return over 2 ∗ initial duration	6.4% ±2.4%	7.2% ±2.1%	7.9% ±1.8%	8.7% ±1.9%	10.0% ±1.6%	11.5% ±2.2%
$RMS^2/St. Dev.^2$	0.18	0.11	0.17	0.08	0.10	0.13
Correlation Coefficient	0.93	0.97	0.97	0.98	0.98	0.95

Table 7: Results for $F = 2$. Numbers after ± are standard deviations.

are unimportant, and that the important source of risk for the long-term investor is having realized return turn out to be unequal to initial yield. This row gives the latter divided by the former.

Only as a cautionary tale does Table 7 report the "Correlation Coefficient" between initial yield and subsequent return, since if the data *exactly* followed $\overline{R}_{amFt} = aY_{mt} + b$ but a was not equal to one or b was not equal to zero, then the correlation coefficient would be a perfect 1.0, but the hypothesis proposed in this paper would have failed. The "full period" portion of Table 7 gives an example of how misleading this correlation coefficient can be: there the 20-year Treasury has the highest correlation coefficient at 0.96 but, as shown in Figure 5, one of the worst fits by all of the legitimate criteria.

Post-1953, Figure 5 shows that the only large and persistently positive or negative forecast errors occur for bonds influenced by the 1970's. Outside of those bonds, forecast errors tend to be less than ±1% per year, except for the 3-year series where errors remain larger but rarely reach ±2% per year, and forecast errors change sign at least twice a decade, often much more frequently.

The centered R^2 values in Table 6's 2.0 column are quite good, 0.82 or higher, except for the longest bonds of the full and early periods. For the full period the root mean square forecast error at $F = 2$ was around one percent per year for 10-year and shorter bonds, and between 1.5 and 1.9 percent for longer bonds. In the early period, initial yield is ±1% of realized return more than 91% of the time.

	Actual Path	Linear Path
Return Approximation	$R_1 \approx 1\% - (D-1)(+1\%)$ $R_2 \approx 3\% - (D-1)(+0\%)$ (a) $\overline{R}_a \approx 2.5\% - 0.005D$	$R_1 \approx 1\% - (D-1)(+\frac{1}{2}\%)$ $R_2 \approx 2\% - (D-1)(+\frac{1}{2}\%)$ (b) $\overline{R}_a \approx 1.5\% - 0.005D$
No Return Approximation	$R_1 = \ln\left[e^{1\%} - 1 + \frac{PV(3\%)}{PV(1\%)}\right]$ $R_2 = \ln\left[e^{3\%} - 1 + \frac{PV(3\%)}{PV(3\%)}\right]$ (c) $\overline{R}_a = 1.5\% + \frac{1}{2}\ln\left[e^{1\%} - 1 + \frac{PV(3\%)}{PV(1\%)}\right]$	$R_1 = \ln\left[e^{1\%} - 1 + \frac{PV(2\%)}{PV(1\%)}\right]$ $R_2 = \ln\left[e^{2\%} - 1 + \frac{PV(3\%)}{PV(2\%)}\right]$ (d) $\overline{R}_a =$ $\frac{1}{2}\ln\left\{\left[e^{1\%} - 1 + \frac{PV(2\%)}{PV(1\%)}\right] * \left[e^{2\%} - 1 + \frac{PV(3\%)}{PV(2\%)}\right]\right\}$

Table 8: Sources of error in continuously-compounded returns. Returns with no approximation use Section 1's (3).

In the late period, initial yield is $\pm 1\%$ of realized return more than 81% of the time except for the 3-year Treasury at 60%. Predictions for the full period are not as precise, but initial yield is $\pm 2\%$ of realized return more than 83% of the time except for the long-term corporate, when it is this accurate only 60% of the time. In summary, almost all the time, initial yield is within a percent or two of realized return with a horizon of twice initial duration.

5. Sources of Error

So far we have assumed linear yield paths and the Return Approximation, which for the example of Figure 1 gives rise to the $\overline{R}_a$ of Table 8's cell "(b)." The actual historical return comes about by dropping the linear path assumption and the Return Approximation, resulting for that example in the Table's "(c)." In this section we decompose the error, (b) minus (c), into two components:

1. a correction "*NL*" for the nonlinearity of the yield path (in the example above, "(a) minus (b)") ; and
2. a correction "*CRA*" for the Return Approximation (in the example above, "(d) minus (b)").

Note that *NL* in the example above, 1%, is also equal to the difference between the point $(2, 2\%)$ and the point $(2, 3\%)$ in Figure 1. (Since the actual path and our linear path share the same initial and final yields, the average capital gains along the two paths are the same, so the difference in return is just due to the difference in average income earned.) There will be a residual gap between "(b) plus *NL* plus *CRA*" and (c) for a few reasons: a theoretical reason ("(b) plus *NL* plus *CRA*" does not quite equal (c) due to interaction terms between the two approximations); and several empirical reasons (the data used in Sections 3 and 4 has nonconstant duration or, in the case of the constant-75-month-duration series, imperfect interpolation methods (and in the Long-term Corporate series, nonconstant duration and imprecisely known maturity); horizons rounded to an integer number of months; and possible imperfections in the Federal Reserve's method for calculating constant-maturity yields). However, this section will show that the residual gap was small, so *NL* and *CRA* do explain most of the error between (b) and (c) in our data.

To illustrate of how our definition for *NL* is calculated using actual data, for the 10-year Treasury between 2/76 and 4/90, first form a linear path between the actual 2/76 and 4/90 yields (the sloped dashed line drawn in Figure 4). Then calculate the (discrete-time analog of) the average area between the actual time path of yields and our chosen linear path, counting as negative the areas generated when actual yields are below the straight line. Formally, let $Y^{\ell}_{m\tau tF}$ be the value at date t of a straight line joining $Y_{m\tau}$ and $Y_{m(\tau+FD_\tau)}$. Then the measure of nonlinearity "*NL*" we use is[20]

$$NL_{m\tau} = \frac{1}{F \cdot D_\tau + 1} \sum_{t=\tau}^{\tau+FD_\tau} (Y_{mt} - Y^{\ell}_{m\tau tF}) \, .$$

By allowing positive and negative deviations from linearity to cancel, this measure ensures that "nonlinearity" will be furthest from zero when yields mostly deviate from linearity in a single direction. If the yield path is convex, the yield path will be below the linear path and $NL < 0$; if the yield path is concave, $NL > 0$. According to Proposition 4's corollary, if the yield path is convex and quadratic, forecast error is negative, and if the yield path is concave and quadratic, forecast error is positive, so if the actual yield paths are sufficiently close to being quadratic, *NL* will have the same sign as forecast error.

NL is the difference between the average yield along the realized path and the average yield along the linear path. To show this, denote the slope of the linear path $(Y_{m(\tau+FD_\tau)} - Y_{m\tau})/(F \cdot D_\tau)$ by s ; then since $Y^{\ell}_{m\tau tF} = Y_{m\tau} + s\,(t - \tau)$ for $\tau \le t \le \tau + FD_\tau$,

$$\begin{aligned} NL &= \frac{1}{F \cdot D_\tau + 1} \sum_{t=\tau}^{\tau+FD_\tau} (Y_{mt} - Y^{\ell}_{m\tau tF}) \\ &= \frac{1}{F \cdot D_\tau + 1} \sum_{t=\tau}^{\tau+FD_\tau} Y_{mt} - \frac{1}{F \cdot D_\tau + 1} \sum_{t=\tau}^{\tau+FD_\tau} Y_{m\tau} - \frac{s}{F \cdot D_\tau + 1} \sum_{t=\tau}^{\tau+FD_\tau} (t - \tau) \\ &= \frac{1}{F \cdot D_\tau + 1} \sum_{t=\tau}^{\tau+FD_\tau} Y_{mt} - Y_{m\tau} - \frac{s}{F \cdot D_\tau + 1} \frac{(F \cdot D_\tau)(F \cdot D_\tau + 1)}{2} \\ &= \frac{1}{F \cdot D_\tau + 1} \sum_{t=\tau}^{\tau+FD_\tau} Y_{mt} - \frac{Y_{m\tau} + Y_{m(\tau+FD_\tau)}}{2} \, . \end{aligned}$$

[20]This expression for *NL* has the same form as the measure of nonlinearity in equation (2) of Emancipator and Kroll (1993) except that they take the absolute value of the differences.

	3 year Treasury	5 year Treasury	75 month duration Treasury	10 year Treasury	20 year Treasury	Long-Term Corporate
FULL PERIOD						
Average *NL*	0.15%	0.12%	0.15%	0.21%	1.13%	1.31%
NL & Forec. Err. Corr. Coeff.	0.99	0.995	0.999	0.99	0.98	0.97
R^2NLFE	0.95	0.96	0.99	0.94	0.89	0.90
EARLY PERIOD						
Average *NL*	0.03%	0.02%	-0.04%	-0.20%		
NL & Forec. Err. Corr. Coeff.	0.995	0.995	0.998	0.99		
R^2NLFE	0.95	0.96	0.93	0.97		
LATE PERIOD						
Average *NL*	0.06%	-0.05%	-0.20%	-0.16%	-0.51%	-0.67%
NL & Forec. Err. Corr. Coeff.	0.996	0.996	0.999	0.98	0.93	0.87
R^2NLFE	0.96	0.95	0.98	0.94	0.34	0.55

Table 9: Results for $F = 2$. The abbreviation "R^2NLFE" stands for the centered R^2 of $\overline{R}_{amFt} - Y_{mt} = 1 * NL_{mFt} + 0$; the abbreviation "*NL* & Forec. Err. Corr. Coeff." stands for the correlation coefficient between *NL* and the forecast errors.

The blue dash-dot lines in Figure 5 graph *NL*. To the extent that *NL* moves together with the forecast error, the former explains the latter. The graphs certainly suggest that forecast errors are largely due to nonlinearity as measured by *NL*.

One formal measure of how well *NL* fits forecast error is the correlation coefficient between them; another is the centered R^2 of $\overline{R}_{amFt} - Y_{mt} = 1 * NL_{mFt} + 0$. This last measure is reported below using the abbreviation "R^2NLFE."[21] Table 9 gives the correlation coefficients between the Nonlinearity Measure and the forecast error, and they are quite high, ranging from 0.97 to 0.999, except for the two longest bonds in the late period, which have 0.93 and 0.87. The R^2NLFE statistics are almost as good, except again for the two longest bonds in the later period.

For the Long-term Corporate series *NL* is graphed as the blue dash-dot line in Figure 7, together with the forecast error, which is the solid line. Their correlation coefficient is 0.93, their R^2NLFE is 0.75, and just as in Figure 5, in some periods it is difficult to distinguish by eye the forecast error from *NL*. Certainly nonlinearity is the source of a great deal of forecasting error. However, in Figure 7, nonlinearity does not explain forecasting error well circa 1932, 1981, and quite prominently in

[21] An alternative to *NL* as an explanation of forecast errors would be an estimate of the right-hand side of (17), where for a purchase date of t_1 one would set $z_1 = \arg\min \sum_{t=t_1}^{t_1+F*D_{t_1}} [Y(t_1) - z_1 t^2 - z_2 t - z_3]^2$ given the constraints $Y(t_1) = z_1 t_1^2 + z_2 t_1 + z_3$ and $Y(t_1 + F * D_{t_1}) = z_1(t_1 + F * D_{t_1})^2 + z_2(t_1 + F * D_{t_1}) + z_3$, which can be rewritten so that z_2 and z_3 drop out of the minimization problem, whose only unknown then is z_1. This measure captures only quadratic nonlinearity, not all nonlinearity, and using it on the long-term corporate data gave a predictor which was highly correlated with *NL* but had wider swings, which made it worse than *NL*. This measure is also much harder to calculate than *NL* because it requires solving an optimization problem for each purchase date, whereas *NL* just requires summing up differences.

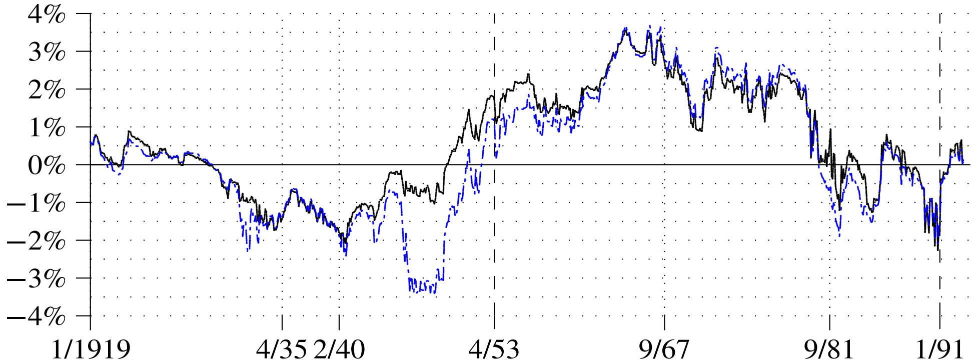

Figure 7. The dash-dot blue line is the nonlinearity measure *NL*, and the solid line is the forecast error, of Figure 6. The scale is the same as in Figure 5.

1945–1952, when forecast error deviated from *NL* to an extent probably unimaginable if the only the post-1953 data had been available. 1932 and 1981 are interest rate peaks and 1945–1952 is an interest rate trough, so they have large differences between their paths' initial and final yields, which Section 1's discussion of Proposition 3 warned us are just the circumstances—together with long duration—in which errors in the Return Approximation are most likely to be important.

To confirm this requires calculating our correction for the Return Approximation, *CRA*. The formal version of the definition we gave above is $CRA = \overline{R}^{\ell}_{amF\tau e} - \overline{R}^{\ell}_{amF\tau B}$ where the first is the arithmetic mean of exact return, starting at time τ and ending at time $\tau + F \cdot D_\tau$, of bonds following a linear yield path from $Y_{m\tau}$ to $Y_{m(\tau+FD_\tau)}$, and the second is the arithmetic mean of the Return Approximation of return over the same linear yield path. Because the formula for exact bond price changes is nonlinear, calculating the first quantity requires generating one counterfactual linear yield path between τ and $\tau + F \cdot D_\tau$ for each starting month (and each bond series and each F) in the data set, then calculating the exact price change over each month for each of the counterfactual paths.

Performing the required calculations and then adding *CRA* to *NL* should provide a close match to forecast error. Figure 8 tests this; it is like Figure 7, but its blue dash-dot line is $NL + CRA$ instead of just *NL*. The fit is much improved; incorporating *CRA* eliminates the unexplained forecast error around the interest rate peaks and trough.

We calculated *CRA* for all the other bond series, and choose in Table 10 to summarize the results by reporting only the size of root mean square of "forecast error minus *NL* minus *CRA*," although the other descriptors used in Table 6 could be displayed as well. The "RMS FE − *NL* − *CRA*" numbers in the "2" column are remarkably small, and, if not the smallest in their row, are only one basis point more than the smallest in their row. If one knew in advance how to correct for nonlinearity and the Return Approximation, then $F = 2$ gives the best match between initial yield and return, as suggested by theory. It is not possible to know in

***F*:**	0.75	1	1.25	1.5	1.6	1.7	1.75	1.8	1.9	**2**	2.1	2.2	2.3	2.4	2.5
3 Year, Full Period															
RMS FE	1.74%	1.30%	1.05%	0.92%	0.90%	0.92%	0.91%	0.92%	0.93%	**0.96%**	0.98%	0.99%	1.01%	1.02%	1.05%
RMS FE − *NL*	1.61%	1.07%	0.69%	0.42%	0.33%	0.25%	0.21%	0.18%	0.13%	**0.12%**	0.13%	0.17%	0.22%	0.27%	0.32%
RMS FE − *NL* − *CRA*	1.13%	0.70%	0.43%	0.26%	0.21%	0.17%	0.15%	0.14%	0.11%	**0.10%**	0.10%	0.11%	0.13%	0.14%	0.16%
5 Year, Full Period															
RMS FE	1.82%	1.31%	1.07%	0.98%	0.96%	0.93%	0.94%	0.93%	0.93%	**0.94%**	0.97%	0.96%	0.97%	0.98%	1.00%
RMS FE − *NL*	1.68%	1.11%	0.74%	0.46%	0.35%	0.25%	0.21%	0.18%	**0.12%**	0.11%	0.15%	0.20%	0.25%	0.31%	0.37%
RMS FE − *NL* − *CRA*	1.08%	0.66%	0.41%	0.25%	0.20%	0.16%	0.14%	0.12%	0.10%	**0.09%**	0.08%	0.09%	0.11%	0.12%	0.14%
75 Month, Full Period															
RMS FE	2.13%	1.60%	1.27%	1.13%	1.10%	1.06%	1.05%	1.03%	1.02%	**1.03%**	1.05%	1.07%	1.12%	1.17%	1.20%
RMS FE − *NL*	2.00%	1.33%	0.90%	0.54%	0.43%	0.32%	0.28%	0.22%	0.12%	**0.04%**	0.08%	0.17%	0.26%	0.34%	0.43%
RMS FE − *NL* − *CRA*	0.95%	0.63%	0.39%	0.24%	0.19%	0.15%	0.13%	0.11%	0.10%	**0.09%**	0.08%	0.08%	0.09%	0.10%	0.12%
10 Year, Full Period															
RMS FE	1.99%	1.73%	1.28%	1.00%	1.07%	1.02%	0.99%	0.93%	0.92%	**0.91%**	0.92%	0.98%	1.05%	1.11%	1.20%
RMS FE − *NL*	1.90%	1.42%	0.88%	0.57%	0.45%	0.32%	0.27%	0.23%	0.21%	**0.22%**	0.27%	0.35%	0.44%	0.50%	0.59%
RMS FE − *NL* − *CRA*	0.97%	0.60%	0.37%	0.22%	0.18%	0.14%	0.12%	0.11%	0.09%	**0.07%**	0.07%	0.08%	0.09%	0.10%	0.12%
20 Year, Full Period															
RMS FE	2.88%	1.80%	1.43%	1.20%	1.11%	1.04%	1.01%	0.95%	0.91%	**0.96%**	1.05%	1.08%	1.16%	1.23%	1.35%
RMS FE − *NL*	2.49%	1.51%	1.02%	0.62%	0.55%	0.52%	0.50%	0.43%	0.45%	**0.32%**	0.31%	0.29%	0.32%	0.30%	0.28%
RMS FE − *NL* − *CRA*	0.91%	0.56%	0.34%	0.20%	0.16%	0.12%	0.10%	0.08%	0.05%	**0.03%**	0.03%	0.05%	0.07%	0.09%	0.10%
LTCorp Full Period															
RMS FE	3.04%	1.99%	1.39%	1.29%	1.23%	0.99%	1.00%	0.95%	1.03%	**1.24%**	1.34%	1.45%	1.59%	1.66%	1.65%
RMS FE − *NL*	2.63%	1.66%	1.10%	0.65%	0.66%	0.59%	0.57%	0.52%	0.45%	**0.37%**	0.34%	0.33%	0.29%	0.29%	0.32%
RMS FE − *NL* − *CRA*	0.91%	0.55%	0.34%	0.20%	0.16%	0.12%	0.10%	0.08%	0.05%	**0.03%**	0.03%	0.05%	0.07%	0.09%	0.10%
LTCorp, 1/1919–4/2014															
RMS FE	2.59%	1.76%	1.30%	1.19%	1.18%	1.10%	1.13%	1.17%	1.28%	**1.40%**	1.50%	1.59%	1.71%	1.78%	1.83%
RMS FE − *NL*	2.33%	1.53%	1.02%	0.60%	0.56%	0.51%	0.53%	0.53%	0.53%	**0.62%**	0.55%	0.57%	0.56%	0.59%	0.64%
RMS FE − *NL* − *CRA*	0.84%	0.51%	0.32%	0.19%	0.15%	0.12%	0.11%	0.09%	0.07%	**0.06%**	0.06%	0.06%	0.07%	0.09%	0.10%

Table 10: RMS forecast error, forecast error minus *NL*, and forecast error minus *NL* minus *CRA*, for various values of F.

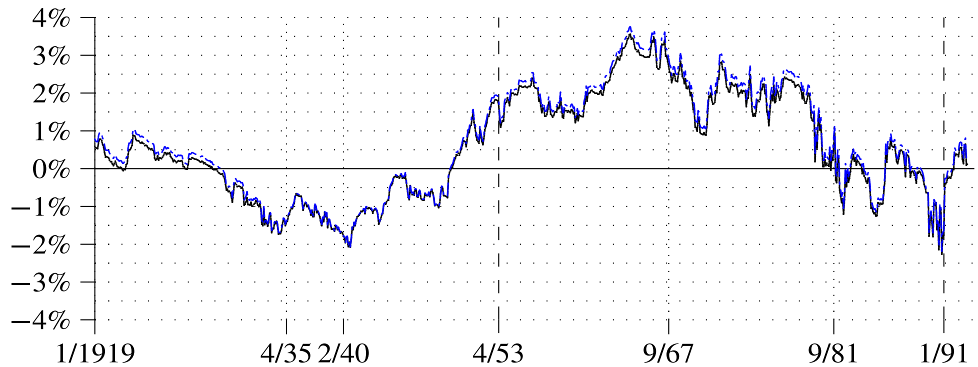

Figure 8. The dash-dot blue line is the nonlinearity measure *NL* added to the correction for the Return Approximation *CRA*, and the solid line is the forecast error, of Figure 6. The scale is the same as in Figure 5.

advance how to correct for nonlinearity and the Return Approximation, but to the extent they are as likely to increase return as to decrease it, $F \approx 2$ remains the best choice for ex ante prediction of the future. Ex post, $F \approx 2$ will not have been the best choice for any era, due to unpredictable idiosyncracies.

Finally, for $F = 2$ predicted return (initial yield plus *NL* plus *CRA*) and actual return is graphed in Figure 9. It reinforces the message of Figure 8 that "initial yield plus *NL* plus *CRA*" is quite close to actual return—so close that in Figure 9, for 75-month the constant-duration series there is hardly any difference between them at this scale. (The vertical scale is the same one used in all the previous yield graphs, namely Figures 4, 5, 6, 7, and 8.)

A closer examination of 9's residuals is permitted by magnifying the vertical scale ten times, as in Figure 10. The residuals in Figure 10 reflect the interaction terms between *NL* and *CRA*, and approximations such as our horizons being rounded to an integer number of months. The cycles of the 75-month constant-duration series might reflect slow changes in the nonlinearity of the yield curve (that is, the instantaneous graph of maturity versus yield; we treated the yield curve as linear when we used linear interpolation between yields to construct the constant-duration yield series). Figure 10's graphs of the constant-maturity series are never positive and are quite smooth. The artificial smoothness may come about because the constant-maturity series have an unchanging "maturity" argument in their bond pricing formulas, while that argument changes every month for the constant-duration series. Figures 9 and 10 suggest that the model of Sections 1 and 2 does apply best to constant-duration, rather than constant-maturity, portfolios.

Overall, the three factors of initial yield, *NL*, and *CRA* did capture all important aspects of the 75-month constant-duration rolling-bond portfolio returns over approximately $2D-1$ periods. Even for the constant-maturity portfolios, the residuals are small enough to be of little significance. Of the three factors, initial yield

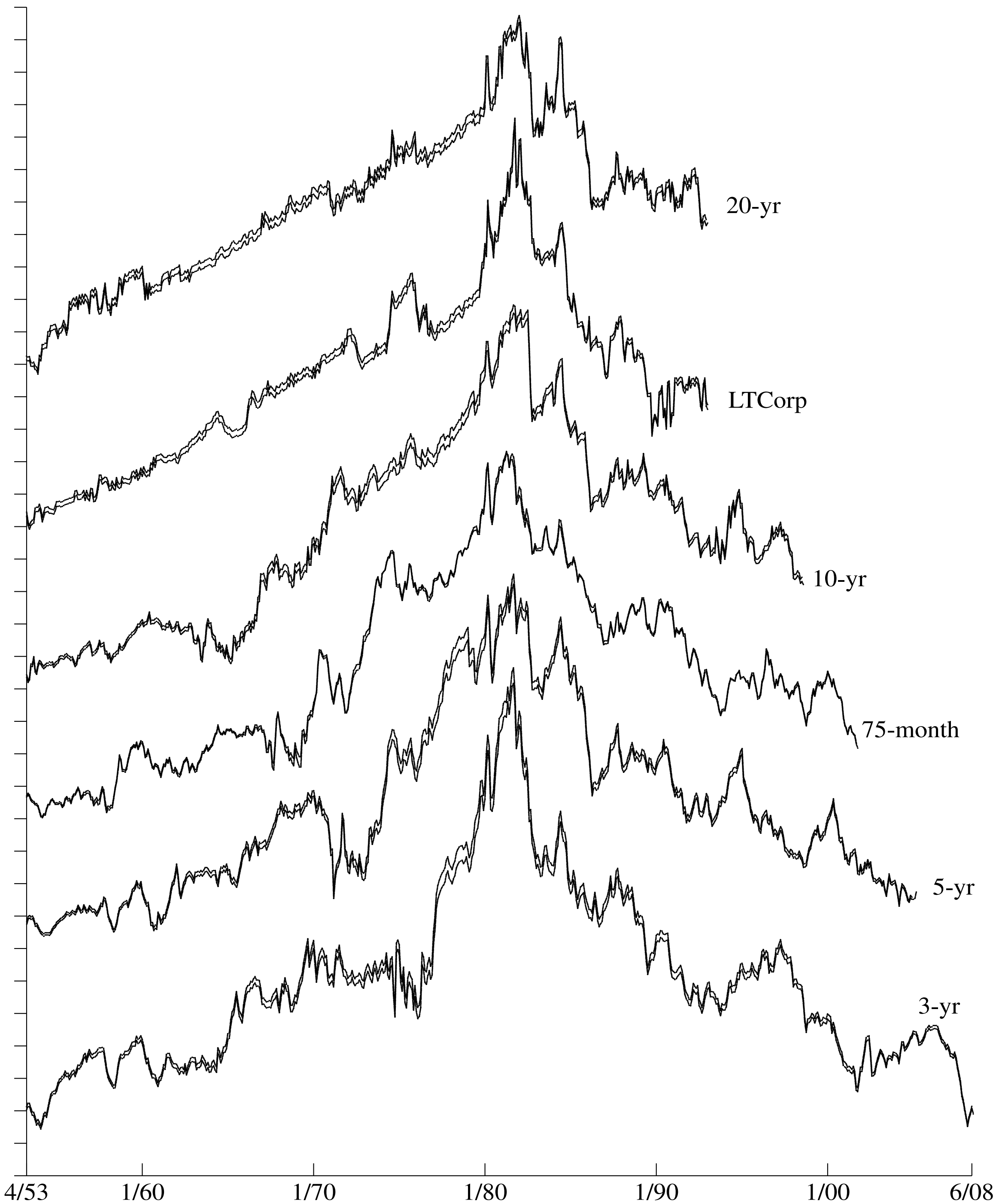

Figure 9. Actual forward return and "initial yield plus *NL* plus *CRA*," over a period twice initial duration, for each bond series. For legibility the graphs have been separated, so absolute yields are not shown, but 1% tick marks appear on the vertical axis to show the relative scale for all.

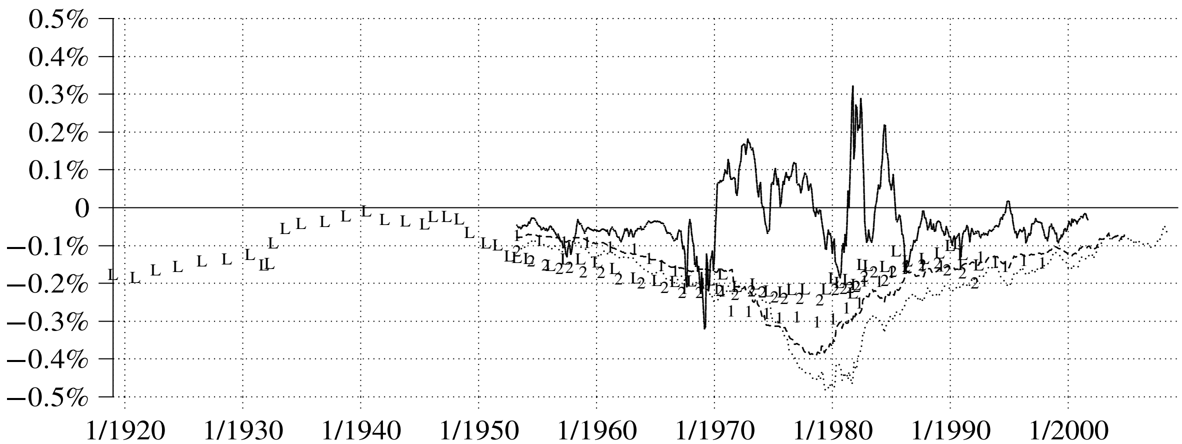

Figure 10. For $F = 2$, forecast error minus *NL* minus *CRA* for 75-month constant duration (solid line), 3 year (dotted line), 5 year (dashed line), 10 year (symbol "1"), 20 year (symbol "2"), and Long-term Corporate (symbol "L"). The vertical scale's tick marks appear at every 0.1%, in contrast with the 1% spacing of the previous graphs.

is the only one knowable in advance, confirming its significance as a tool for forecasting returns.

Conclusion

Assuming linear yield paths, a linear approximation of the relationship between bond price and yield change, and using an approximation that after one month the duration of a bond shrinks by one month, the initial continuously-compounded yield of a constant-duration bond portfolio equals its arithmetic mean continuously-compounded return over a period slightly less than twice duration. Twice duration gave low errors in most of our samples of historical yield paths, despite all but one of them being constant-maturity instead of constant-duration. As anticipated in Section 2, in those historical samples the forecast errors were mostly explained by the convexity of the yield path, with negative forecast errors occurring when the yield path was generally convex and positive forecast errors occurring when the yield path was generally concave. The Return Approximation explained most of the remaining forecast errors.

The extent of bond returns' predictability demonstrated in this paper may make investment-grade bonds appear less risky to long-term investors than the bonds' variance has in the past made them appear. On the other hand, with long-term bonds, experiencing a fixed annual forecast error of a percent or two for a period of almost thirty years will lead to a substantial difference in final value. Furthermore, what predictability we have discovered only applies to nominal returns, while what most investors care about are bonds' real returns, which can be quite hard to predict. For example, long-term corporate bonds, which had the highest real return over the full period (Table 3), during the early period turned an initial $10,000 investment into $5,597 in constant dollars. Theory suggests the initial yield on inflation-indexed bonds predicts their real return over twice their duration minus one rollover period; once more data becomes available it will be interesting to see how close that fit is.

BIBLIOGRAPHY

Bova, Anthony (2013), "Hypothetical Equity Returns Models under Rising Rates.." Society of Actuaries Investment Symposium, March 14–15, Session R5, "A Fresh Look at the Equity Risk Premium." At `http://www.soa.org/Files/Pd/2013/investment-sym/2013-ny-invest-sym-r5.pdf`.

Eisenhauer, Joseph G. (2003), "Regression Through the Origin," *Teaching Statistics* **25/3**: 76—80.

Emancipator, Kenneth, and Martin H. Kroll (1993), "A Quantitative Measure of Nonlinearity," *Clinical Chemistry* **39/5**: 766—772.

Fridson, Martin S., and Xiaoyi Xu (2014), "Duration Targeting: No Magic for High-Yield Investors," *Financial Analysts Journal* **70/3**: 28—33.

Gay, Chris (March 3, 2014), "How to Predict the Next Decade's Bond Returns; Current Yields Are a Good Indicator of What You Will Earn Over Time." *Wall Street Journal*.

Gleich, David (2005), "Finite Calculus: A Tutorial for Solving Nasty Sums." At `https://www.cs.purdue.edu/homes/dgleich/publications/finite-calculus.pdf`.

Hall, Robert E. (2001), "The Stock Market and Capital Accumulation," *American Economic Review* **91/5**: 1185—1202.

Ivaschenko, Iryna (January 2003), "How Much Leverage is Too Much, or Does Corporate Risk Determine the Severity of a Recession? (IMF Working Paper)." At `www.imf.org/external/pubs/ft/wp/2003/wp0303.pdf`.

Langeteig, Terence C., Martin L. Leibowitz and Stanley Kogelman (1990), "Duration Targeting and the Management of Multiperiod Returns," *Financial Analysts Journal* **46/5**: 35—45.

Leibowitz, Martin, and Anthony Bova (2013), "Duration Targeting and Index Convergence," *Investment Management Journal (Morgan Stanley)* **3/1**: 73—80.

Leibowitz, Martin L., Anthony Bova, and Stanley Kogelman (2014), "Long-Term Bond Returns under Duration Targeting," *Financial Analysts Journal* **70/1**: 31—51.

Leibowitz, Martin L., Anthony Bova, Stanley Kogelman, and Sidney Homer (2013), *Inside the Yield Book: The Classic That Created the Science of Bond Analysis (3rd Edition).* Somerset, New Jersey: John Wiley & Sons..

Leibowitz, Martin, and Anthony Bova (December 18, 2012), "Duration Targeting: A New Look at Bond Portfolios (Morgan Stanley Research)." At `www.q-group.`

org/wp-content/uploads/2014/01/2013sp-Leibowitz-Bova-Duration-Targeting-121812.pdf.
Mindlin, Dimitry, (August 14, 2011), "On the Relationship between Arithmetic and Geometric Returns." At http://ssrn.com/abstract=2083915 or http://dx.doi.org/10.2139/ssrn.2083915.
McNabb, William (August 5, 2014), "Taking Stock: A Conversation with Vanguard CEO Bill McNabb." At https://institutional.vanguard.com/VGApp/iip/site/institutional/researchcommentary/article/InvComTakingStock.
Naik, Vipaul (no date), "Summation Notation, Series, and Convergence Basics." At files.vipulnaik.com/math-153-sequence/seriesandconvergence.pdf
Potts, Tom, and William Reichenstein (November 2004), "Predictability of Fixed-Income Fund Returns," *Journal of Financial Planning* **17/11**:62—71.
Wooldridge, Jeffrey (2012), *Introductory Econometrics: A Modern Approach (5th edition)*. Mason, Ohio: South-Western Cengage Learning.
Yogaranpan, Yanagan (July 2005), "Expanding the Geometric Mean Return." Research Paper No. 2005/04, Division of Economic and Financial Studies, Macquarie University. At www.businessandeconomics.mq.edu.au/our_departments/Applied_Finance_and_Actuarial_Studies/acst_docs/research_papers/2005/200504_Geometric_Mean_Return.pdf.