Review of Time Value of Money

- These are my lecture notes from FCS 3450 on Present Value and Future Values. In this class I assume you have already learned these concepts from a previous lower division class such as FCS3450 or FCS3500. Please review this material and make sure you feel comfortable with these concepts. If you need any help on these concepts please contact me.

Concept 8. Future Value (FV)

- What is future value?
  - Future Value is the accumulated amount of your investment fund.
- Notations related to future value calculations:
  - P = principle (original invested amount)
  - r = interest rate for a certain period
  - n = number of periods

Simple Interest vs. Compounded Interest

- Simple interest means you only earn interest on the original invested amount.
- Compounded interest rate assumes that interest earnings are automatically reinvested at the same interest rate as is paid on the original invested amount.
- Example: You save $100 in a savings account with an annual r=3%
  - If simple interest:
    - End of year 3 = $100+$3+$3+$3 = $109
  - If compounded annually:
    - End of year 1: $100 * (1+3%) = $103.00
    - End of year 2: $103 * (1+3%) = $106.09
    - End of year 3: $106.09 * (1+3%) = $109.27

Future Value for One-Time Investment

- To compute future value for one-time investments (as opposed to multiple investments), one uses Future Value Factor (FVF).
- What is Future Value Factor (FVF)?
  - $FVF= (1+r)^n$
- What does FVF mean?
  - FVF is how much one dollar will generate in the future given interest rate r and period n.
- How do you use FVF to figure out the future value of one-time investments?
  - $FV=P*FVF=P*(1+r)^n$

Example

- You put $10,000 in a CD account for 2 years. The account pays a 4% annual interest rate. How much money will you have at the end if annual compounding is used? How about monthly compounding? How about daily compounding?
  - Note: for annual compounding, use annual interest rate and number of years. For monthly compounding, use monthly interest rate (annual/12) and number of months. For daily compounding, use daily interest rate (annual/365) and number of days.

- Annual compounding
  - FV=10,000*(1+4%)^2=10,000*1.0816=$10,816.00
- Monthly compounding
  - Monthly interest rate: rm = 4%/12 = 0.3333%, n=2*12=24
  - FV=10,000*(1+0.3333%)^24=10,000*1.083134

    =$10,831.34
- Daily compounding
  - Daily interest rate: rd=4%/365=0.0110%, n=2*365=730
  - FV=10,000*(1+0.0110%)^730=10,000*1.083607

    = $10,836.07

Note: For all FV computations please keep the decimal point to 6 digits (4 digits when % sign is used). For money amount use two digits (to cents)

- You put $20,000 in a CD account for 10 years. The account pays a 6% annual interest rate. How much money will you have at the end if annual compounding is used? How about monthly compounding? How about daily compounding?

- Annual compounding
  - FV=20,000*(1+6%)^10=20,000*1.790848=$35816.95
- Monthly compounding
  - Monthly interest rate: rm = 6%/12 = 0.5%, n=10*12=120
  - FV=20,000*(1+0.5%)^120=20,000*1.819397=$36387.93
- Daily compounding
  - Daily interest rate: rd=6%/365=0.0164%, n=10*365=3650
  - FV=20,000*(1+0.0164%)^3650=20,000*1.822029= $36440.58

FV of Periodical Investments

- What is periodical investments?
  - Periodical investments are multiple investments that are made at certain time intervals.
- How to calculate the future value of periodical investments? It is probably best illustrated using an example.

Example

- Suppose you have decided to save some money to pay for a vacation. You can afford to save $100 a month. You put the money in a money market account which pays an 8% annual interest rate, compounded monthly. How much money will you have at the end of the 12th month?
  - Note: we can treat this as 12 separate $100 investments that are in the bank for different length of time.

- **Beginning of the month calculation (deposit money on the first day of every month):**
  - **Monthly interest rate rm= 8%/12=0.6667%**
  - **FV of $100 deposited on Jan. 1= $100 * $(1+0.6667\%)^{12}$ = $108.30**
  - **FV of $100 deposited on Feb. 1= $100 * $(1+0.6667\%)^{11}$ = $107.58**
  - **FV of $100 deposited on March 1=$100 * $(1+0.6667\%)^{10}$ = $106.87**
  - **FV of $100 deposited on April 1 = $100 * $(1+0.6667\%)^{9}$ = $106.16**
  - **FV of $100 deposited on May 1= $100 * $(1+0.6667\%)^{8}$ = $105.46**
  - **FV of $100 deposited on June 1= $100 * $(1+0.6667\%)^{7}$ = $104.76**
  - **FV of $100 deposited on July 1= $100 * $(1+0.6667\%)^{6}$ = $104.07**
  - **FV of $100 deposited on Aug. 1= $100 * $(1+0.6667\%)^{5}$ = $103.38**
  - **FV of $100 deposited on Sept. 1= $100 * $(1+0.6667\%)^{4}$ = $102.69**
  - **FV of $100 deposited on Oct. 1= $100 * $(1+0.6667\%)^{3}$ = $102.01**
  - **FV of $100 deposited on Nov. 1= $100 * $(1+0.6667\%)^{2}$ = $101.34**
  - **FV of $100 deposited on Dec. 1= $100 * $(1+0.6667\%)^{1}$ = $100.67**
- **Total FV = Sum of the FVs of the 12 periodical payments = $1253.29**

- **Note: With beginning of the month (BOM) calculation the last deposit, deposited on Dec. 1, earns one month of interest.**

- **End of the month calculation (deposit money on the last day of every month):**
  - **Monthly interest rate rm= 8%/12=0.6667%**
  - **FV of $100 deposited on Jan. 31= $100 * (1+0.6667%)^{11} = $107.58**
  - **FV of $100 deposited on Feb. 28= $100 * (1+0.6667%)^{10} = $106.87**
  - **FV of $100 deposited on March 31=$100 * (1+0.6667%)^{9} =$106.16**
  - **FV of $100 deposited on April 30 = $100 * (1+0.6667%)^{8} = $105.46**
  - **FV of $100 deposited on May 31= $100 * (1+0.6667%)^{7} = $104.76**
  - **FV of $100 deposited on June 30= $100 * (1+0.6667%)^{6} = $104.07**
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  - **FV of $100 deposited on Oct. 31= $100 * (1+0.6667%)^{2} = $101.34**
  - **FV of $100 deposited on Nov. 30= $100 * (1+0.6667%)^{1} = $100.67**
  - **FV of $100 deposited on Dec. 31= $100 * (1+0.6667%)^{0} = $100.00**
- **Total FV = Sum of the FV of the 12 periodical payments = $1244.99**

- **With end of the month calculation, the last deposit, which is deposited on Dec. 31, does not earn any interest. In fact, every deposit earns one month less of interest compared to the beginning of month situation.**

- Are there simpler ways of calculating FV for periodic investments?
  - If the monthly payments are equal, then we can simplify the problem by using Future Value Factor Sum (FVFS)

Future Value Factor Sum (FVFS)

- Beginning of the month (BOM) formula

$$FVFS = (1+r)^n + (1+r)^{n-1} + \ldots + (1+r)^1 = \frac{(1+r)^{n+1} - 1}{r} - 1$$

- End of the month (EOM) formula

$$FVFS = (1+r)^{n-1} + (1+r)^{n-2} \ldots + (1+r)^0 = \frac{(1+r)^n - 1}{r}$$

BOM or EOM

- The difference between Beginning of the Month (BOM) and End of the Month (EOM) is that with BOM the last investment earns interest for one period, while with EOM the last investment does not earn interest because it goes in and out of the account at the same time.
- Appendix FVFS Table shows EOM. Although, in most relevant cases of FVFS applications BOM is likely to be more appropriate.

FV for Periodical Payments

$FV = P_p * FVFS$

Where P_p=amount of periodical payments

- Now we use FVFS to solve the previous example problem.
  - Beginning of the month:

$$FV = P_p \times FVFS(r = 0.6667\%, n = 12, BOM)$$
$$= P_p \times (\frac{(1+r)^{n+1} - 1}{r} - 1)$$
$$= 100 \times (\frac{(1+0.6667\%)^{12+1} - 1}{0.6667\%} - 1)$$
$$= 100 \times 12.5330 = 1253.30$$

- End of the month:

$$FV = P_p \times FVFS(r = 0.6667\%, n = 12, EOM)$$

$$= P_p \times (\frac{(1+r)^n - 1}{r})$$

$$= 100 \times (\frac{(1+0.6667\%)^{12} - 1}{0.6667\%})$$

$$= 100 \times 12.4499 = 1244.99$$

Applications of FV

- Compute the FV of saving \$200 on the first day of each month for 6 months (withdraw at the end of the sixth month) at 12% annual interest rate, monthly compounding

- **n=6 months, monthly r=12%/12=1%=0.01, beginning of the month calculation**

$$FV = P_p \times FVFS(r = 1\%, n = 6, BOM)$$

$$= P_p \times \left(\frac{(1+r)^{n+1} - 1}{r} - 1\right)$$

$$= 200 \times \left(\frac{(1+1\%)^{6+1} - 1}{1\%} - 1\right)$$

$$= 200 \times 6.213535 = 1242.71$$

- Compute the FV of saving $200 on the last day of each month for 6 months (withdraw at the end of the sixth month) at 12% annual interest rate, monthly compounding.

- **n=6 months, monthly r=12%/12=1%=0.01, end of the month calculation**

$$FV = P_p \times FVFS(r = 0.6667\%, n = 12, EOM)$$

$$= P_p \times (\frac{(1+r)^n - 1}{r})$$

$$= 200 \times (\frac{(1+1\%)^6 - 1}{1\%})$$

$$= 200 \times 6.152015 = 1230.40$$

- Suppose you save $500 on the first day of each month for 6 months at 12% annual interest rate, compounded monthly, and then only put $200 on the first day of each month starting from the 7th month for another 6 months at the same interest rate. How much money will you have at the end of the 12th month?

- This is a more complicated scenario. You have to treat this like two investments. Investment one is a periodical investment of $500 per month for 6 month. After 6 months whatever amount there is will be treated for a one-time investment for another six months. Investment two is a periodical investment of $200 each month for 6 months. The total is just the sum of these two investments.
- Investment 1:

$$
\begin{aligned}
FV_1 &= P_p \times FVFS(r = 1\%, n = 6, BOM) \times (1+r)^n \\
&= P_p \times \left(\frac{(1+r)^{n+1} - 1}{r} - 1\right) \times (1+r)^n \\
&= 500 \times \left(\frac{(1+1\%)^{6+1} - 1}{1\%} - 1\right) \times (1+1\%)^6 \\
&= 500 \times 6.213535 \times 1.061520 = 3297.90
\end{aligned}
$$

- **Investment 2**
  - **Next 6 months: Pp=200, monthly r=12%/12=1%=0.01, n=6**

$$FV_2 = P_p \times FVFS(r = 1\%, n = 6, BOM)$$

$$= P_p \times (\frac{(1+r)^{n+1} - 1}{r} - 1)$$

$$= 200 \times (\frac{(1+1\%)^{6+1} - 1}{1\%} - 1)$$

$$= 200 \times 6.213535 = 1242.71$$

– **Total FV**
  - **FV=FV1+FV2=3297.90+1242.71=4540.61**

- Saving for vacation
- Mary is planning on saving some money for her next vacation. Her goal is to have $2000 saved after one year. If she decides to put an equal amount of money in a bank savings account every month on the first day, and the interest rate is 6% annually, how much should she save every month, if interest is compounded monthly?

- This is an application of FVFS because this is related to periodical payments
- Denote the monthly saving amount as M
  - Monthly interest rate r = 6%/12 = 0.5%

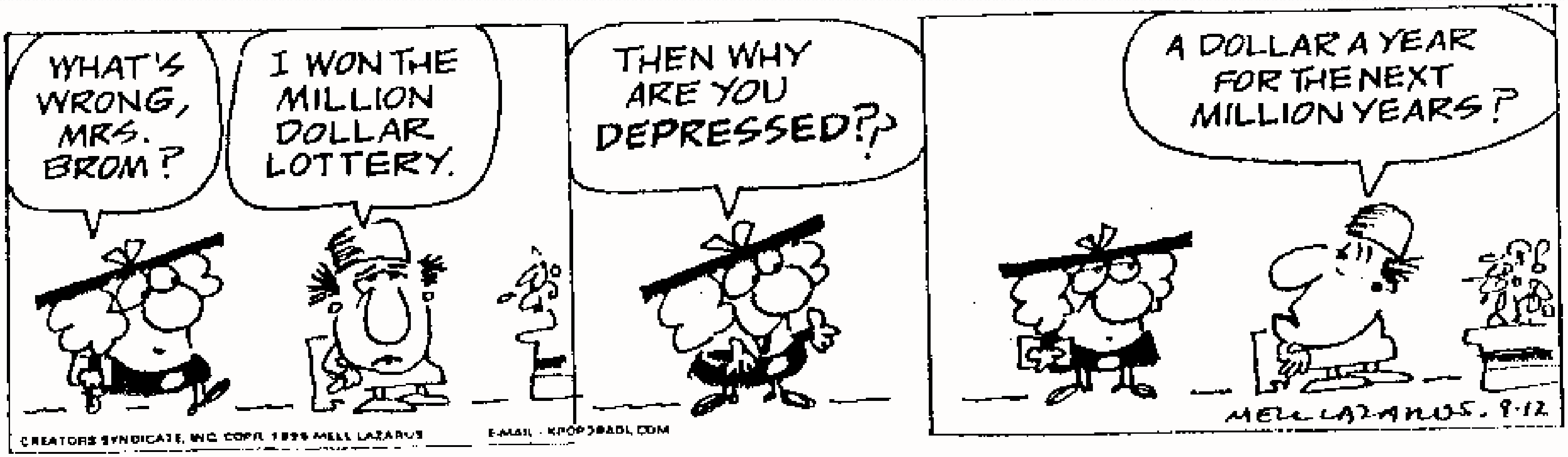
$$M \times FVFS(r = 0.5\%, n = 12, BOM) = 2{,}000,$$

$$M = 2{,}000 \Big/ FVFS(r = 0.5\%, n = 12, BOM)$$

$$= 2{,}000 \Big/ \left(\frac{(1+0.5\%)^{12+1} - 1}{0.5\%} - 1 \right)$$

$$= 2{,}000 \Big/ 12.397240 = 161.33$$

Concept 9: Present Value



- Is the value of a dollar received today the same as received a year from today?
  - A dollar today is worth more than a dollar tomorrow because of inflation, opportunity cost, and risk
- Bring the future value of money back to the present is called finding the Present Value (PV) of a future dollar

Discount Rate

- To find the present value of future dollars, one way is to see what amount of money, if invested today until the future date, will yield that sum of future money
- The interest rate used to find the present value = discount rate
- There are individual differences in discount rates
  - Those who are present-oriented (high rate of time preference) have higher discount rates
  - Those who are future-oriented (low rate of time preference) have lower discount rates.
- We use the notation r to denote discount rate in Present Value computations because interest rate is often used to approximate discount rate. The issue of compounding also applies to Present Value computations.

Present Value Factor

- To bring one dollar in the future back to present, one uses the Present Value Factor (PVF):

$$PVF = \frac{1}{(1+r)^n}$$

Present Value (PV) of Lump Sum Money

- For lump sum payments, Present Value (PV) is the amount of money (denoted as P) times PVF Factor (PVF)

$$PV = P \times PVF = P \times \frac{1}{(1+r)^n}$$

An Example Using Annual Compounding

- Suppose you are promised a payment of $100,000 after 10 years from a legal settlement. If your discount rate is 6%, what is the present value of this settlement?

$$PV = P \times PVF = 100{,}000 \times \frac{1}{(1+6\%)^{10}} = 55{,}839.48$$

An Example Using Monthly Compounding

- You are promised to be paid $10,000 in 10 years. If you have a discount rate of 12%, using monthly compounding, what is the present value of this $10,000?
- First compute monthly discount rate

  Monthly r = 12%/12=1%, n=120 months

$$PV = P \times PVF = 10{,}000 \times \frac{1}{(1+1\%)^{120}} = 10{,}000 * 0.302995 = \$3{,}029.95$$

An Example Comparing Two Options

- Suppose you have won lottery. You are faced with two options in terms of receiving the money you have won: (1) $10,000 paid now; (2) $15,000 paid five years later. Which one would you take? Use annual compounding and a discount rate of 10% first and an discount rate of 5% next.

- Your answer will depend on your discount rate:
  - Discount rate r=10% annually, annual compounding
    - Option (1): PV=10,000 (note there is no need to convert this number as it is already a present value you receive right now).
    - Option (2): PV = 15,000 *(1/ (1+10%)^5) = $9,313.82
    - Option (1) is better
  - Discount rate r= 5% annually, annual compounding
    - Option (1): PV=10,000
    - Option (2): PV = 15,000*(1/ (1+5%)^5) = $11,752,89
    - Option (2) is better

Present Value (PV) of Periodical Payments

- For the lottery example, what if the options are (1) $10,000 now; (2) $2,500 every year for 5 years, starting from a year from now; (3) $2,380 every year for 5 years, starting from now?
  - The answer to this question is quite a bit more complicated because it involves multiple payment for two of the three options. First, let's again assume annual compounding with a 10% discount rate.

- **Annual discount rate r= 10%, annual compounding**
  - **Option (1): PV=10,000**
  - **Option (2):**
    - **PV of money paid in 1 year = 2500*[1/(1+10%)^1] = 2272.73**
    - **PV of money paid in 2 years = 2500*[1/(1+10%)^2] = 2066.12**
    - **PV of money paid in 3 years = 2500*[1/(1+10%)^3] = 1878.29**
    - **PV of money paid in 4 years = 2500*[1/(1+10%)^4] = 1707.53**
    - **PV of money paid in 5 years = 2500*[1/(1+10%)^5] = 1552.30**
    - **Total PV = Sum of the above 5 PVs = 9,476.97**
  - **Option (3):**
    - **PV of money paid now (year 0) = 2380 (no discounting needed)**
    - **PV of money paid in 1 year = 2380*[1/(1+10%) ^{1}] = 2163.64**
    - **PV of money paid in 2 years = 2380*[1/(1+10%)^2] = 1966.94**
    - **PV of money paid in 3 years = 2380*[1/(1+10%)^3] = 1788.13**
    - **PV of money paid in 4 years = 2380*[1/(1+10%)^4] = 1625.57**
    - **Total PV = Sum of the above 5 PVs = 9,924.28**
- **Option (1) is the best, option (3) is the second, and option (2) is the worst.**

- Are there simpler ways to compute present value for periodical payments?
  - Just as in Future Value computations, if the periodic payments are equal value payments, then Present Value Factor Sum (PVFS) can be used.
- Present Value (PV) is the periodical payment times Present Value Factor Sum (PVFS). In the formula below P_p denotes the periodical payment:
  - $PV=P_p*PVFS$

Present Value Factor Sum (PVFS)

- If the first payment is paid right now (so the first payment does not need to be discounted), it is called the Beginning of the month (BOM):

$$PVFS = \frac{1}{(1+r)^0} + \frac{1}{(1+r)^1} + \ldots + \frac{1}{(1+r)^{n-1}}$$

$$= 1 + \frac{1 - \frac{1}{(1+r)^{n-1}}}{r}$$

- If the first payment is paid a period away from now (like end of this month, beginning of next month, or end of this year, etc.), then the first payment needs to be discounted for one period. In this case, the end of the month (EOM) formula applies:

$$PVFS = \frac{1}{(1+r)^1} + \ldots + \frac{1}{(1+r)^n}$$

$$= \frac{1 - \frac{1}{(1+r)^n}}{r}$$

BOM or EOM

- In most cases End of the Month (EOM) is used in PVFS computation. So use EOM as the default unless the situation clearly calls for Beginning of the Month (BOM) calculation.
- Appendix PVFS Table uses EOM.

- Use PVFS to solve the example problem but use a 5% discount rate:
  - discount rate r=5%
    - Option (1): PV = 10,000
    - Option (2):

$$PV = 2500 \times PVFS(r = 5\%, n = 5, EOM)$$

$$= 2500 \times \frac{1 - \frac{1}{(1+5\%)^5}}{5\%} = 2500 \times 4.329477 = 10,823.69$$

- Option (3):

$$PV = 2380 \times PVFS(r = 5\%, n = 5, BOM)$$

$$= 2380 \times (1 + \frac{1 - \frac{1}{(1+5\%)^{5-1}}}{5\%}) = 2380 \times 4.545951 = 10,819.36$$

Option (2) is the best.

Applications of Present Value: Computing Installment Payments

- You buy a computer. Price=\$3,000. No down payment. r=18% with monthly compounding, n=36 months. What is your monthly installment payment M?
  - The basic idea here is that the present value of all future payments you pay should equal to the computer price.

- Answer:
  - Apply PVFS, n=36, monthly r=18%/12=1.5%, end of the month because the first payment usually does not start until next month (or else it would be considered a down payment)

$$3000 = M \times PVFS(r = 1.5\%, n = 36, EOM),$$

$$M = \frac{3000}{PVFS(r = 1.5\%, n = 36, EOM)}$$

$$= 3000 \Big/ \frac{1 - \frac{1}{(1+1.5\%)^{36}}}{1.5\%}$$

$$= 3000 / 27.660684 = 108.46$$

Application of Present Value: Rebate vs. Low Interest Rate

- Suppose you are buying a new car. You negotiate a price of $12,000 with the salesman, and you want to make a 30% down payment. He then offers you two options in terms of dealer financing: (1) You pay a 6% annual interest rate for a four-year loan, and get $600 rebate right now; or (2) You get a 3% annual interest rate on a four-year loan without any rebate. Which one of the options is a better deal for you, and why? What if you only put 5% down instead of 30% down (Use monthly compounding)
  - In this case because your down payment is the same for these two options, and both loans are of four years, comparing monthly payments is sufficient.

30% down situation

- **Option 1. Amount borrowed is 12,000*(1-30%) – 600 =7,800**
  - **Monthly r=6%/12=0.5%, n=48 months**

$$M = \frac{7800}{PVFS(r = 0.5\%, n = 48, EOM)}$$

$$= 7800 \Big/ \frac{1 - \frac{1}{(1+0.5\%)^{48}}}{0.5\%}$$

$$= 7800 / 42.580318 = 183.18$$

- **Option 2. The amount borrowed: 12,000*(1-30%)=8,400**
  - **Monthly r=3%/12=0.25%, n=48 months**

$$M = \frac{8400}{PVFS(r = 0.25\%, n = 48, EOM)}$$

$$= 8400 \Big/ \frac{1 - \frac{1}{(1+0.25\%)^{48}}}{0.25\%}$$

$$= 8400 / 45.178695 = 185.93$$

Option 1 is better because it has a lower monthly payment

5% down situation

- **Option 1. Amount borrowed is 12,000*(1-5%) – 600 =10,800**
  - **Monthly r=6%/12=0.5%, n=48 months**

$$M = \frac{10{,}800}{PVFS(r = 0.5\%, n = 48, EOM)}$$

$$= 10{,}800 \Big/ \frac{1 - \frac{1}{(1+0.5\%)^{48}}}{0.5\%}$$

$$= 10{,}800 \big/ 42.580318 = 253.64$$

- **Option 2. The amount borrowed: 12,000*(1-5%)=11,400**
  - **Monthly r=3%/12=0.25%, n=48 months**

$$M = \frac{11{,}400}{PVFS(r = 0.25\%, n = 48, EOM)}$$

$$= 11{,}400 \Big/ \frac{1 - \frac{1}{(1+0.25\%)^{48}}}{0.25\%}$$

$$= 11{,}400 \big/ 45.178695 = 252.33$$

Option 2 is better now because it has a lower monthly payment

Application of Present Value: Annuity

- Annuity is defined as equal periodic payments which a sum of money will produce for a specific number of years, when invested at a given interest rate.
- Example: You have built up a nest egg of $100,000 which you plan to spend over 10 years. How much can you spend each year assuming you buy an annuity at 7% annual interest rate, compounded annually ?

- Annuity calculation is an application PVFS because the present value of all future annuity payments should equal to the nestegg one has built up.

$$100{,}000 = M \times PVFS(r = 7\%, n = 10, EOM),$$

$$M = \frac{100{,}000}{PVFS(r = 7\%, n = 10, EOM)}$$

$$= 100{,}000 \Big/ \frac{1 - \frac{1}{(1+7\%)^{10}}}{7\%}$$

$$= 100{,}000 \Big/ 7.023582 = \$14{,}237.75$$

- If you know how much money you want to have every year, given the interest rate and the initial amount of money, you can compute how long the annuity will last. Say you have $10,000 now, you want to get $2,000 a year. The annual interest rate is 7% with annual compounding (EOM)

- Approximate solution:
  - Step 1: \$10,000/\$2,000 = 5
  - Step 2: Find a PVFS that is the closest possible to 5
    - PVFS(r=7%, n=5, EOM) = 4.100197
    - PVFS(r=7%, n=6, EOM) = 4.76654 close to 5
    - PVFS(r=7%, n=7, EOM) = 5.389289 close to 5
  - Because 5 is in-between PVFS(n=6) and PVFS(n=7), this annuity is going to last between 6 and 7 years
- Exact solution:
  - \$10,000/\$2,000=5
  - 5=PVFS (r=7%, n=?, EOM) => 5=[1- 1/(1+7%)^n]/7%
  - 0.35=1-1/(1.07)^n
  - 0.65=1/(1.07)^n
  - 1/0.65=(1.07)^n
  - Log(1/0.65)=n log(1.07)
  - n=log(1/0.65)/log(1.07)=6.37 years
- Note: Homework, Quiz and Exam questions will ask for approximate solution, not the exact solution, although for those who understand the exact solution the computation can be easier.

Appendix: An Step-by-Step Example for PVFS Computation

$$PVFS(n=5, r=7\%, EOM) = \frac{1-\frac{1}{(1+7\%)^5}}{7\%} = \frac{1-\frac{1}{(1+0.07)^5}}{0.07} = \frac{1-\frac{1}{1.07^5}}{0.07}$$

$$= \frac{1-\frac{1}{1.402552}}{0.07} = \frac{1-0.712986}{0.07} = \frac{0.287014}{0.07} = 4.100197$$