RELATIVE PRICES IN THE CLASSICAL THEORY: FACTS AND FIGURES

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Abstract. This paper surveys, extends and illustrates recent results on the movement of relative prices of production when distribution changes.

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Contents

Whereas labor values are independent of distribution, prices of production vary if profits rise at the expense of wages or vice versa. The Classicals were aware of this fact and tried to develop theories which would explain these variations. Notable are here Ricardo's search for an "invariant commodity," produced by an industry which would use some average proportion of labor and capital, and the various "solutions" of the "transformation problem" in Marx's theory, which try to explain the sum of all profits by the difference between the value produced by labor and the wage.

Ricardo states repeatedly the idea sustaining his search for an invariant commodity: Due to increasing costs of capital, the price of a capital-intensive commodity, i.e. of one produced by an industry with a high rate of $\frac{\text{value of capital}}{\text{amount of wages}}$, should increase relatively to the price of the labor-intensive commodity as the profit rate increases. This law, which was accepted by all the Classicals, holds true for an economy with two industries and two goods. If p_1 is the price of the capital-intensive good, then the relative price $p_1(r)/p_2(r)$ increases with the profit rate r. But Sraffa (1960, §20) pointed out that such a rule does not extend to economies with three or more industries. A "capital-intensive" industry may use "labor-intensive" commodities as inputs, so that its costs do not vary as expected. For more than two industries, one cannot even uphold the notions "capital-intensive" and "labor-intensive," because the ranking of industries may depend on the profit rate.

Only very few economically intuitive regularities have been discovered that govern the movement of prices of production when distribution changes. The existence, uniqueness, and positivity of prices of production themselves is one of them, the fall of real wages, regardless of numeraire chosen, if the profit rate rises, is another. A more recent result, which is reviewed in this paper, is that prices move closer to the Perron-Frobenius price vector, the price vector which would prevail under zero wages, as wages fall. The concept of a distance to be used here is that of the "Hilbert distance."

The purpose of this study is to give a broad overview, using a variety of mathematical tools, about what can be said about relative prices of production if distribution changes. We are presenting result concerning the monotonicity of the movement, and its curvature, from some adequately defined point of view. We find that even though there are no obvious economic reasons for it, the trajectories of relative prices follow a strict geometric choreography. The "evidence" given by a series of computer-generated graphs will be our starting point to discuss the regularities in the movement of relative prices as distribution changes. A self-contained presentation of the mathematical proofs is given in the Appendix.

1. Curves of Prices of Production

1.1. **Representation of the Technology.** In our graphs we will display three-industry, three-commodity economies—but the intention is to exhibit general properties, and the proofs in the Mathematical Appendix hold for general n. All processes of production are of the single-production type, and there is no choice of techniques. The economies are basic in Sraffa's sense. The production technology can therefore be represented by a semipositive irreducible 3×3 matrix $\boldsymbol{A} \gneqq \boldsymbol{O}$ and a semipositive 3×1 column vector $\boldsymbol{\ell} \gneqq \boldsymbol{o}$, which collect the input and labor coefficients, respectively. We are using the convention that commodities, the vector of labor requirements, and price vectors are column vectors, while processes and baskets are row vectors (written as the transposes of column vectors). The transpose of a matrix $\boldsymbol{A}$ is $\boldsymbol{A}^\top$. A matrix or vector is called semipositive, notation $\boldsymbol{A} \gneqq \boldsymbol{O}$, if all elements are nonnegative and at least one element is positive. If all elements are positive we will write $\boldsymbol{H} \gg \boldsymbol{O}$.

Even though the numerical data $\boldsymbol{A}$ and $\boldsymbol{\ell}$ are necessary to define the technology, a technology must not be confused with this numerical representation, more precisely with any of its representations. Given a technology, the choice of physical units of measurement for the n goods determines a representation $(\boldsymbol{A}, \boldsymbol{\ell})$. If other units are chosen, the same technology is represented by other data $(\boldsymbol{A}', \boldsymbol{\ell}')$. Conversion rules are given in Fact 1 in the Appendix. Only those properties of $(\boldsymbol{A}, \boldsymbol{\ell})$ which do not depend on the choice of units are properties of the technology itself. For example, the diagonal elements a_{ii} of $\boldsymbol{A}$, which give the proportion of good i entering directly into its own production, are invariants of the technology; a technology is basic iff one/any of its representative matrices is irreducible; and a physical commodity bundle which is Sraffa's standard commodity in one representation is a standard commodity in all representations (up to a normalization factor). A change in measuring units also changes the price vectors; but the total price of a given physical commodity bundle is invariant, and also the capital/labor ratio of an industry is invariant. By contrast, the concept of Euclidean angles or distances between price vectors is problematic, since they depend on the representation of the technology.

1.2. **Definition of Prices of Production.** The vector $\boldsymbol{p}$ of prices of production and the wage w associated with an equalized profit rate r for technology $(\boldsymbol{A}, \boldsymbol{\ell})$ satisfy

$$(1 + r)\boldsymbol{A}\boldsymbol{p} + w\boldsymbol{\ell} = \boldsymbol{p}. \tag{1}$$

If $w = 0$, semipositive price vectors are possible only for one profit rate, the maximal profit rate r^*. The corresponding price vectors are eigenvectors of $\boldsymbol{A}$, namely, they are proportional to the (up to a factor) unique positive Perron-Frobenius eigenvector $\boldsymbol{p}^*$.

Although economists usually use zero as the lower bound of the profit rate, we will allow profit rates as low as -1. For all profit rates $-1 \leq r < r^*$, (1) can be solved

$$\boldsymbol{p} = w(\boldsymbol{I} - (1 + r)\boldsymbol{A})^{-1}\boldsymbol{\ell}, \tag{2}$$

If $-1 < r < r^*$, then $(\boldsymbol{I} - (1 + r)\boldsymbol{A})^{-1} \gg \boldsymbol{O}$.

Most contemporary economists start with equation (1) which is due to Sraffa, assuming that the wage is paid *post factum* (Sraffa, 1960, §9). An equation more faithful to the classical economists' idea of a wage advanced from capital is

$$(1 + r)(\boldsymbol{A}\boldsymbol{p} + w\boldsymbol{\ell}) = \boldsymbol{p}. \tag{3}$$

If $\boldsymbol{p}$ satisfies (3) for wage w, then it satisfies (1) for wage $(1+r)w$; one will therefore obtain the same relative prices with (1) and (3), and it suffices to study (1).

1.3. **Normalizations.** If $\boldsymbol{p}$ and w are solutions of (1) for a given r, then any scalar multiples $\alpha\boldsymbol{p}$ and αw are solutions as well. In order to remove this ambiguity, we will normalize prices and wages, using a fixed basket $\boldsymbol{g} \gg \boldsymbol{o}$ as numeraire. Normalized prices satisfy the condition $\boldsymbol{g}^\top \mathbf{p} = 1$, and the normalized wage satisfies (1) together with the normalized price vector. For every rate of profit there is one normalized price vector and one normalized wage.

If $\boldsymbol{p} = [p_1, \ldots, p_n]^\top$ is any given price vector, we will write $\mathbf{p} = [\mathrm{p}_1, \ldots, \mathrm{p}_n]^\top$ (roman font instead of italics) for its normalized version. Normalization conve-

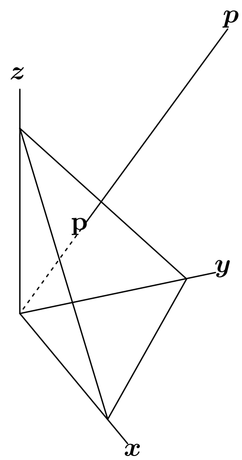

FIGURE 1. Unit simplex in $\mathbb{R}^3$

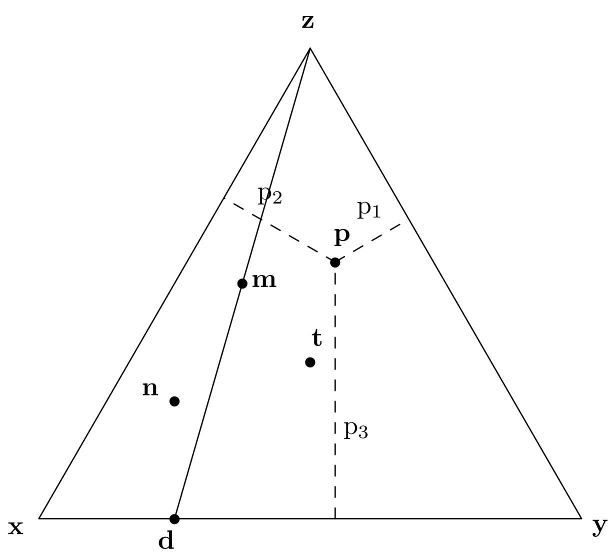

FIGURE 2. Equilateral map of unit simplex

niently reduces the dimension of the space of possible prices by one. If $n = 3$, all normalized prices are located on a simplex of $\mathbb{R}^3$, as shown in Figure 1.

The normalized wage depends negatively on the rate of profit, varying between $1/\boldsymbol{g}^\top\boldsymbol{\ell}$ and 0. Normalized prices form a continuous curve starting (for $r = -1$) at the normalized labor coefficient $\mathbf{l}$ and converging (for $r \to r^*$) towards the normalized Frobenius eigenvector $\mathbf{p}^*$. This end point of the price curve does not depend on $\boldsymbol{\ell}$. For $r = 0$, prices are proportional to labor values.

1.4. **Barycentric Coordinates.** Normalized prices in our three-commodity example economies will henceforth be represented as points in an equilateral plane triangle, as in Figure 2, which is a map of the simplex in Figure 1. Arbitrary irregularly shaped normalizations simplices can be mapped into the equilateral triangle using the *barycentric coordinates*. This is not an Euclidean mapping of the normalization simplex into the plane, i.e., it does not keep distances and angles intact. But this is not much of a loss since distances and angles depend on the representation of the technology. The barycentric coordinates of the normalized price vector, by contrast, depend on the technology and on the physical commodity bundle represented by the normalization vector, but not on the units in which the different commodities are measured.

Every point in the normalization simplex of Figure 1 is a weighted average of the corner points, and the barycentric coordinates of this point are these weights. Since the weights sum to 1, the three barycentric coordinates can be represented as points in an equilateral triangle. Assuming that the triangle in Figure 2 has height 1, the barycentric coordinates of the normalized price vector p_1, p_2, p_3 are the Euclidean distances of point $\mathbf{p}$ in Figure 2 to the sides of the triangle $\overline{\mathbf{yz}}$, $\overline{\mathbf{zx}}$, and $\overline{\mathbf{xy}}$.

From the star-like arrangement of the barycentric coordinates of every vector, as exemplified with vector $\mathbf{p}$, follows that

- point $\mathbf{d}$, for instance, on $\overline{\mathbf{xy}}$, has third coordinate equal to 0;
- point $\mathbf{n}$ close to $\mathbf{x}$ has small second and third coordinates relatively to its first;
- all points $\mathbf{m}$ on $\overline{\mathbf{zd}}$ have same ratio $\mathrm{m}_2/\mathrm{m}_1$ between their second and first coordinates.

If $\boldsymbol{g}$ is the normalization vector, then the ith barycentric coordinates of the price vector is the price of g_i units of commodity i. If the physical units of the economy are chosen such that the normalization vector is the vector $[1, 1, \dots, 1]$, then the components of the normalized price vector coincide with its barycentric coordinates. In this paper the normalization vector will usually be $[1, 1, \dots, 1]$, therefore the barycentric mapping of the normalization simplex will be an Euclidean one after all.

1.5. **Special Normalizations.** Certain normalizations have special properties. We will discuss here the normalization by the net product and the normalization by Sraffa's standard commodity.

Normalization by the Net Product: Let $\boldsymbol{x}^\top$ be the gross product and $\boldsymbol{y}^\top = \boldsymbol{x}^\top(\boldsymbol{I} - \boldsymbol{A})$ the net product of the entire economy. Furthermore define the vector of labor values by the equation $\boldsymbol{\lambda} = \boldsymbol{A}\boldsymbol{\lambda} + \boldsymbol{\ell}$. The normalization which sets the sum of prices of the net product equal to the sum of values of the net product, i.e., which imposes on every price vector $\boldsymbol{p}$ the identity $\boldsymbol{y}^\top\boldsymbol{p} = \boldsymbol{y}^\top\boldsymbol{\lambda}$, is the normalization recommended by the "new solution to the transformation problem" (Duménil, 1980; Foley, 1982), discussed in (Glick and Ehrbar, 1987). Since the value of the net product is equal to the total labor time performed, one can write this normalization also as $\boldsymbol{y}^\top\boldsymbol{p} = \boldsymbol{x}^\top\boldsymbol{\ell}$. One sees easily that in this normalization, the sum of profits is equal to the sum of surplus-values: $\boldsymbol{x}^\top\boldsymbol{p} - \boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{p} - \boldsymbol{x}^\top\boldsymbol{\ell}w = \boldsymbol{x}^\top\boldsymbol{\ell}(1-w)$. (Profit rates need not be equalized for this to hold.)

Normalization by Sraffa's Standard Commodity: If the basket used for normalization is a left Perron-Frobenius eigenvector of $\boldsymbol{A}$, call it $\boldsymbol{q}^*$, (i.e. a multiple of

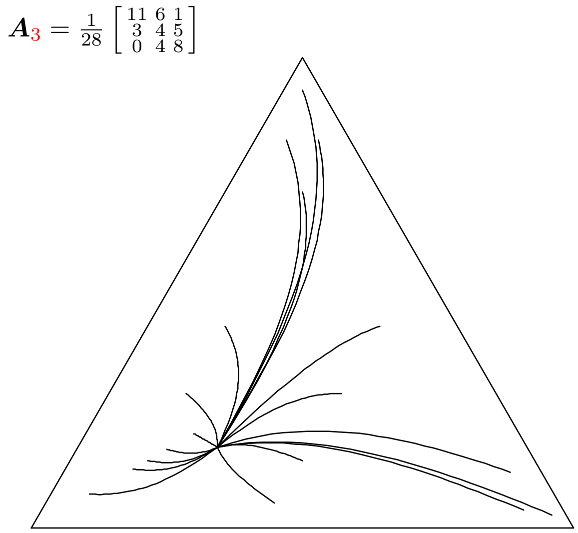

FIGURE 3. Price curves for different vectors $\boldsymbol{\ell}$ but same $\boldsymbol{A} = \boldsymbol{A}_3$

Sraffa's standard commodity), then the normalized wage is easy to compute:

$$w = \Big(1 - \frac{1+r}{1+r^*}\Big)\frac{1}{\boldsymbol{q}^{*\top}\boldsymbol{\ell}}. \tag{4}$$

Sraffa's normalization amounts to setting $\boldsymbol{q}^{*\top}\boldsymbol{\ell} = r^*/(1+r^*)$ so that (4) becomes $w = 1 - (r/r^*)$, but we will not impose this condition here. Normalized prices can be written in terms of the normalized labor coefficients $\mathbf{l} = \boldsymbol{\ell}/\boldsymbol{q}^{*\top}\boldsymbol{\ell}$ as follows:

$$\mathbf{p} - \mathbf{p}^* = \Big(\boldsymbol{I} - (1+r)\boldsymbol{A}\Big)^{-1}(\mathbf{l} - \mathbf{p}^*)\Big(1 - \frac{1+r}{1+r^*}\Big) \tag{5}$$

The vector in the normalization simplex going from $\mathbf{p}(r)$ to $\mathbf{p}^*$ is therefore a linear function of the vector going from $\mathbf{l}$ to $\mathbf{p}^*$. This accounts for the especially simple properties of price vectors in this normalization. (See Facts 3, 18, 27, 28, 33, 31 in the Appendix.) We will call this normalization the "standard normalization." Most of the Figures shown use the standard normalization, i.e., they describe a special situation. The general case differs from this special situation by a projective transformation (collineation).

2. Intersections of Price Curves

2.1. **Same Input Matrix: At Most One Intersection Point.** In order to study the dependency of the price curve on the labor vector, we will fix the input matrix $\boldsymbol{A}$ and allow the labor vector to vary. This corresponds to a variation of the intensity of labor with otherwise unchanged technology. But the purpose of this exercise goes beyond this special case. By separating the influence of $\boldsymbol{\ell}$ from that of $\boldsymbol{A}$, we gain valuable structural information about $\boldsymbol{A}$.

Figures 3 and 4 show a number of such price curves. To construct these two Figures, two different input coefficient matrix were chosen, and price curves were drawn with various starting points, corresponding to various vectors $\boldsymbol{\ell}$. As mentioned earlier, the normalization vector in all our examples is $[1,1,1]^\top$, so that

$$\boldsymbol{A}_4 = \frac{1}{12}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 4 \\ 5 & 1 & 2 \end{bmatrix}$$

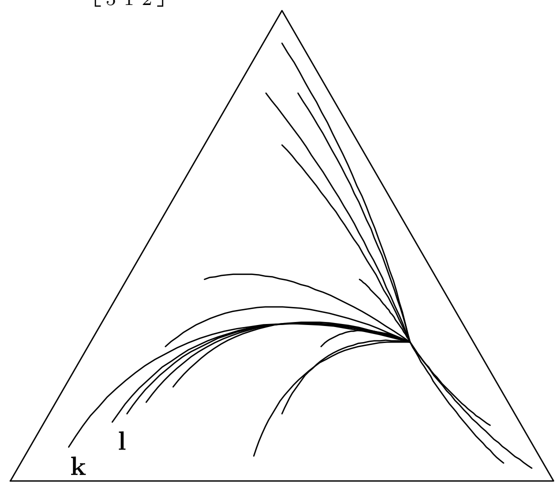

FIGURE 4. Same set of $\boldsymbol{\ell}$ vectors as in Figure 3 but a different matrix $\boldsymbol{A} = \boldsymbol{A}_4$

the equilateral triangle of barycentric coordinates is at the same time a Euclidean head-on view of the normalization simplex shown in Figure 1.

The most obvious feature of Figures 3 and 4 is the convergence of all price curves for the same input matrix toward the common endpoint $\boldsymbol{p}^*$. But two different price curves associated with a given input matrix may sometimes also have an additional intersection point other than this endpoint. This means that two economies with the same input matrix but different labor requirements may have the same structure of relative prices. In the Figures shown, different price curves for the same $\boldsymbol{A}$ have at most one additional intersection point, and the Appendix brings a proof (Fact 15 c) that there can never be more, even in higher dimensions.

2.2. **A Surprising Collineation.** If one examines the Figures closely, one discovers that the beginning points of two price curves are always collinear with their intersection point. This can be seen especially clearly in Figure 4, where the curves starting at $\mathbf{k}$ and $\mathbf{l}$ and the three other curves starting on the extension of the line $\overline{\mathbf{kl}}$ all intersect in the same point which again lies on that straight line (Fact 14).

The starting points of the price curves are the normalized prices associated with profit rate $r = -1$. The value $r = -1$ is not special here. The normalized prices associated with any other profit rate are collinear with the intersection point as well (Fact 15 b). Figure 5 gives an enlarged view of the situation in Figure 4, with the price curves dashed at equal profit rates.

If the intersection point moves towards $\mathbf{p}^*$, then the two price curves end up being tangent at $\mathbf{p}^*$. At the same time, the straight line connecting their beginning points moves towards $\mathbf{p}^*$. All normalized price curves which are tangent at $\mathbf{p}^*$ start therefore on the same ray emanating from $\mathbf{p}^*$. This is illustrated in the right half of Figure 5.

2.3. **Radial "Stretching" of the Price Curves.** Not only the direction in which the starting point lies relatively to the common endpoint, but the entire shape of

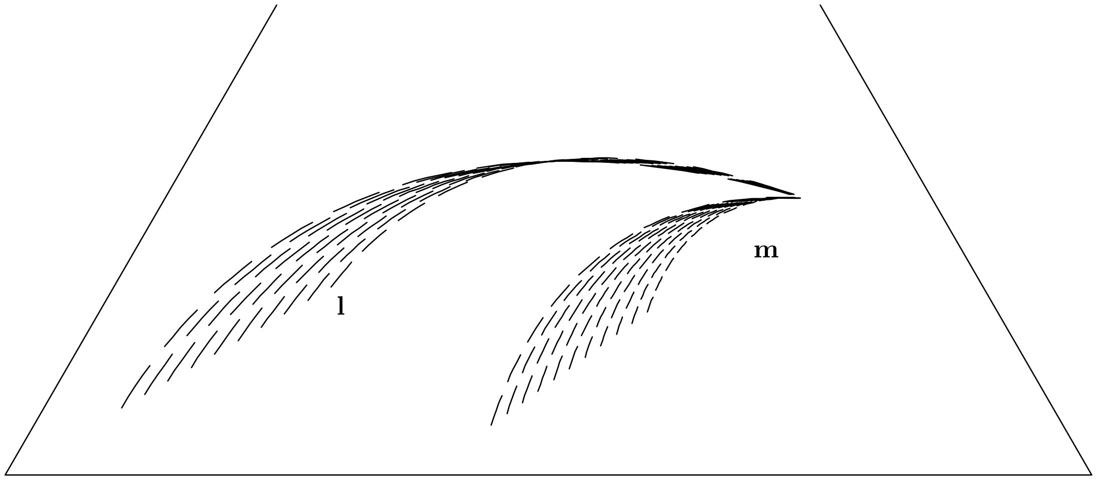

FIGURE 5. Intersecting and tangent price curves, gapped at equal profit rates.

the price curve is determined by the direction of the final tangent. In the *standard normalization*, which was chosen in these Figures (all example economies have constant column sums, i.e., the normalization vector $[1, 1, 1]$ is at the same time a left eigenvector), a movement of the labor vector in a straight line towards $\mathbf{p}^*$ or away from $\mathbf{p}^*$ induces an exact proportional scaling, a "shrinking" or "stretching," of the whole price curve. This follows from the linear relationship (5). Here we discovered a family of linear transformations which carry the price curves into each other. Other such linear transformations, called $\boldsymbol{A}$-endomorphisms, are described in the first paragraph of Section 4.1.

We will therefore not lose any information about the geometric shapes of the price curves if we draw in our Figures only the "long" price curves which start at the edge of the unit simplex. Any price curve starting at an interior point $\mathbf{l}$ is simply a scaled-down version of that "long" curve whose labor input vector is obtained by extending the line $\overline{\mathbf{p}^*\mathbf{l}}$ to the edge of the simplex.

2.4. **Nonintersection of "Long" Price Curves.** If one draws these long price curves (Figures 6–9), one discovers that they no longer intersect. Such a nonintersection property holds whenever the price curves start on the edge of a convex set which they will not leave again. Proof in Fact 16. The qualitative patterns shown in Figures 6–9 are representative for all technologies in the case $n = 3$, with 6 and 7 the generic case, and 8 and 9 the exceptional cases.

By the "trick" of stretching the price curves to the edge of the simplex, we have therefore generated families of nonintersecting price curves approaching their common endpoint from all directions. This is a good opportunity to visualize also the evolution of the profit rate along these curves. All curves (except for the straight ones) are dashed, and every single dash corresponds to a fixed profit rate differential—and, since we have standard normalization, also to an equal wage differential, see equation (4).

One discovers the following:

$$\boldsymbol{A}_3 = \frac{1}{28}\begin{bmatrix} 11 & 6 & 1 \\ 3 & 4 & 5 \\ 0 & 4 & 8 \end{bmatrix}$$

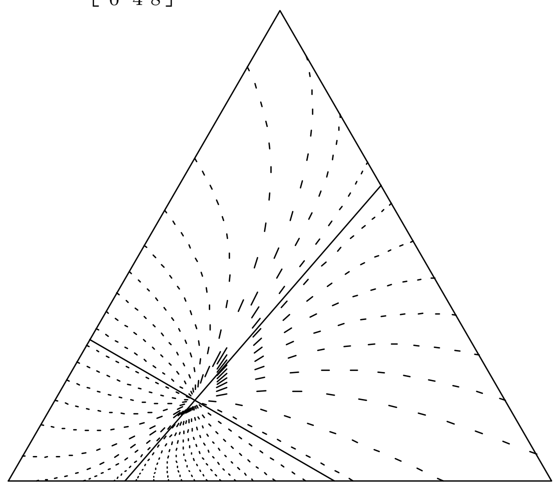

FIGURE 6. Representative Trajectories for $\boldsymbol{A}_3$ (same $\boldsymbol{A}$-Matrix as in Figure 3)

$$\boldsymbol{A}_4 = \frac{1}{12}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 4 \\ 5 & 1 & 2 \end{bmatrix}$$

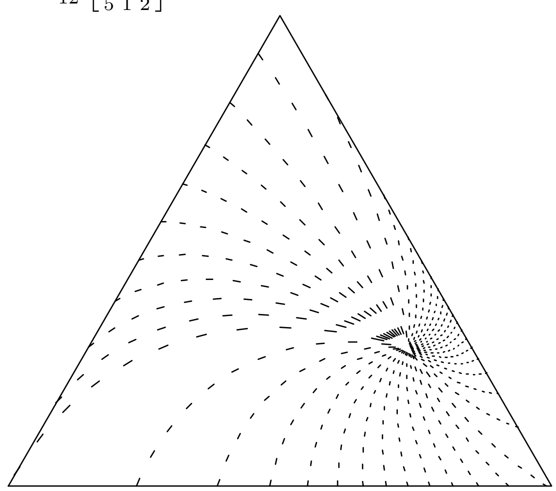

FIGURE 7. Representative Trajectories for $\boldsymbol{A}_4$

- For the curves starting at the edge of the unit simplex, the loci of equal profit rates are triangles with straight edges, we will call them "level triangles."
- Corresponding sides of different level triangles are "concurrent," i.e., they all meet in the same point. This common intersection point is at the same time the intersection of the "virtual" extensions of the two trajectories involved. This is shown in Figure 10 for the Northeast sides of the level triangles for $\boldsymbol{A}_8$. This is the same mathematical regularity which was already illustrated in Figure 5.

$\boldsymbol{A}_8 = \frac{1}{40}\begin{bmatrix} 12 & 7 & 1 \\ 3 & 8 & 9 \\ 5 & 5 & 10 \end{bmatrix}$

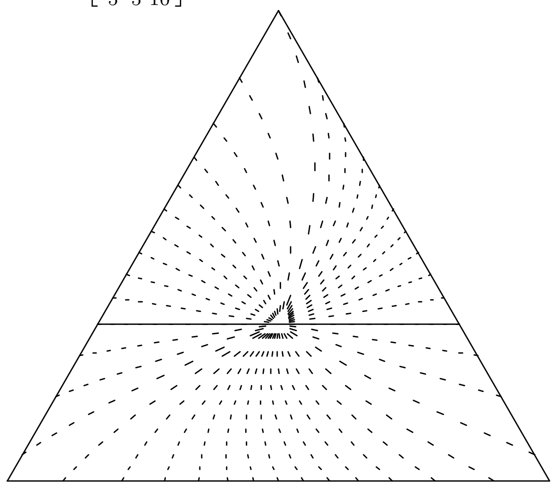

FIGURE 8. Limit Case $\boldsymbol{A}_8 = \frac{7}{10}\boldsymbol{A}_3 + \frac{3}{10}\boldsymbol{A}_4$

$\boldsymbol{A}_9 = \frac{1}{24}\begin{bmatrix} 0 & 4 & 4 \\ 7 & 3 & 7 \\ 5 & 5 & 1 \end{bmatrix}$

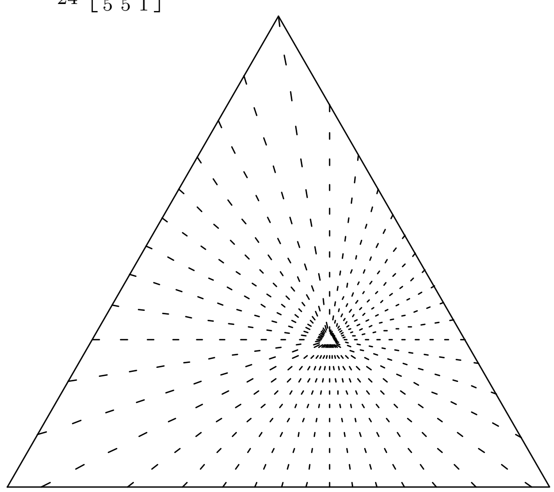

FIGURE 9. Second Limit Case: All Trajectories are Straight

- Every trajectory cuts each level triangle side with which it intersects in the same proportion. For example, that trajectory starting at the middle of the side of the simplex cuts each of the triangle sides in half, and similarly with trajectories starting at other proportions. This last property is an application of Fact 3; it only holds in the standard normalization.
- The "speed" along the price curves accelerates in some cases (Figures 6–8) and decelerates in others (Figure 9).

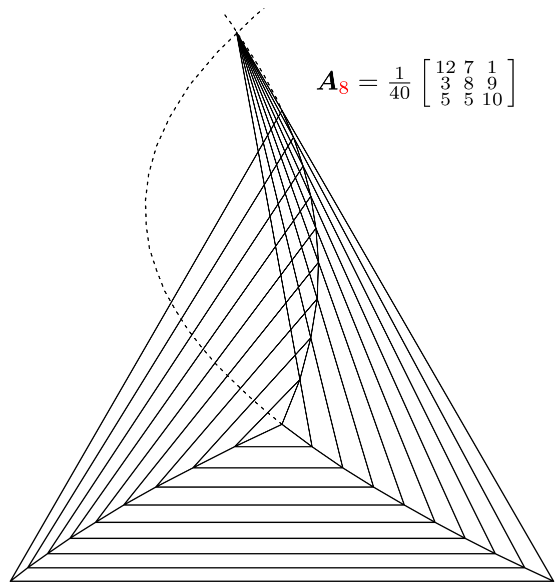

FIGURE 10. Corresponding sides of different "level triangles" are concurrent

3. The Shapes of the Price Curves

Figures 6–9 allow us to study the *shapes* of all possible price curves associated with a given input matrix.

3.1. **The Degree of Regularity of an Economy.** The given examples differ by the distribution of left-turning, right-turning, and straight curves over the triangle. Figure 6 has two directions in which the curves are straight; they divide the triangle into four sectors in which the curves turn alternatively right and left. In Figure 7, no curves are straight; all curves turn the same way (namely right) as they approach $\mathbf{p}^*$. In Figure 8, there is one direction of straight curves, which cuts the triangle into two regions. Both regions house curves turning the same way, namely right. The curves in the vicinity of the straight curves are still very straight. In Figure 9, all price curves are straight.

To understand the mathematics of straight price curves, it is useful to go back to unnormalized price vectors. When normalized prices move on the straight line $\overline{\mathbf{up}^*}$ in the normalization simplex, then the unnormalized price vectors remain in the plane defined by the unnormalized price vectors $\boldsymbol{u}$, $\boldsymbol{p}^*$, and the origin $\boldsymbol{o}$ (compare Figure 1). In other words, all price vectors are concentrated in a plane, they do not span the full $\mathbb{R}^3$. Economies whose price vectors do not span all of $\mathbb{R}^n$ are called *irregular*. The *degree of regularity* of an economy is the dimension of the price space of the economy.

The price space of an economy is defined as the vector space spanned by *all* price vectors. However this price space can also be defined from a single price vector $\boldsymbol{p} \neq \boldsymbol{p}^*$: it is the smallest $\boldsymbol{A}$-invariant subspace of $\mathbb{R}^n$ containing $\boldsymbol{p}$ (Fact 4 in the Appendix). A vector space $\mathcal{V} \subset \mathbb{R}^n$ is called "$\boldsymbol{A}$-invariant" if $\boldsymbol{p} \in \mathcal{V}$ implies $\boldsymbol{A}\boldsymbol{p} \in \mathcal{V}$.

This determination of the price space by one single price vector has an important implication for the intersection of price curves: if two different price curves associated with the same $\boldsymbol{A}$ pass through the same point, then they share the same price space. This is why one does not see any regular price curves crossing the straight lines in the examples. Regular and irregular price curves are confined to different areas of $\mathbb{R}^n$.

The concept of $\boldsymbol{A}$-invariant subspaces is a generalization of eigenvectors. Every subspace spanned by a set of right eigenvectors is $\boldsymbol{A}$-invariant, and so is every subspace orthogonal to a set of left eigenvectors. The real and imaginary parts of a complex eigenvector together span an invariant subspace. $\boldsymbol{A}$-invariance is responsible for the convenient properties of the standard normalization: it is the only normalization for which the normalization simplex is parallel to an $\boldsymbol{A}$-invariant hyperplane, namely, the hyperplane orthogonal to $\boldsymbol{q}^*$, which we will denote by $\mathcal{H}$.

To study invariant subspaces one needs the concept of a "complete chain." This is a strictly ascending sequence of invariant subspaces

$$\{\boldsymbol{o}\} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_r \subset \mathcal{V}_s \subset \mathcal{V}_t \subset \cdots \subset \mathcal{V}_{m-1} \subset \mathcal{V}_m = \mathbb{R}^n \tag{6}$$

which has maximal length, i.e., for which there are no invariant subspaces "between" $\mathcal{V}_r$ and $\mathcal{V}_s$. Every complete chain has the same length (Jordan-Hölder theorem, see Kirillov, 1976, p. 116), and subsequent links in the chain differ in dimension by either 1 or 2 (Fact 11).

The number of complete chains depends on $\boldsymbol{A}$. In one extreme case, there is only one complete chain consisting of subspaces which contain nonnegative vectors, in the other extreme, any subspace containing $\boldsymbol{p}^*$ is invariant.

Only those $\boldsymbol{A}$-invariant subspaces of $\mathbb{R}^n$ can be price spaces which contain some semipositive vector. Since such subspaces are uniquely defined by their intersections with $\mathcal{H}$ (Fact 12), this leads us again to consider the standard normalization, and look at $\boldsymbol{A}$-invariant subspaces of $\mathcal{H}$.

If $\boldsymbol{A}$ has a (real or complex) eigenvalue whose associated eigenspace has dimension 2 or larger, then *every* price vector is irregular (Fact 9). In the more usual situation that all eigenspaces are one-dimensional, irregular price spaces are the exception (Fact 8). If $\boldsymbol{A}$ has m real and q pairs of conjugate complex eigenvalues, each with a one-dimensional eigenspace, then all irregular prices are contained in m invariant hyperplanes (subspaces of $\mathbb{R}^n$ of dimension $n-1$) and q invariant "hyperlines" (subspaces of dimension $n-2$).

These invariant hyperplanes and hyperlines give important information about $\boldsymbol{A}$. The hyperplanes divide the normalization simplex into 2^k regions "capturing" the price curves: every price curve must remain in the region in which it was born. We will see below in Section 4.1 that these hyperplanes and hyperlines are also "attractive."

The determination of the entire price space by a single price vector $\boldsymbol{p}$, regardless of the profit rate attached to it, can also be interpreted to mean that a given price curve has the same degree of regularity along its entire length. This same concept is expressed in a different way by Fact 6: If an economy has degree of regularity s, then *any* s price vectors $\boldsymbol{p}(r_1), \cdots, \boldsymbol{p}(r_s)$ associated with s different profit rates $-1 \leq r_1 < \cdots < r_s \leq r^*$ are linearly independent. From this follows that any price curve which has three points on a straight line must entirely lie on this line; if any four points of a price curve lie in the same plane, or if any two of its tangents intersect, then the whole curve lies in that plane, etc.

3.2. **Orientation.** If $\boldsymbol{p}$ moves along a regular price curve, $\det(\boldsymbol{p}, \boldsymbol{A}\boldsymbol{p}, \boldsymbol{A}^2\boldsymbol{p}, \ldots, \boldsymbol{A}^{n-1}\boldsymbol{p})$ never vanishes. It has the same sign everywhere. How do price curves with positive determinant differ from price curves with negative determinant? Here we need to know that the sign of this determinant is the same as the sign of $\det(\boldsymbol{p}(r_1), \cdots, \boldsymbol{p}(r_n))$ for n ascending profit rates $-1 \leq r_1 < \cdots < r_n \leq r^*$ (Fact 7). If $n = 3$, the sign of the determinant of three successive points determines whether the price curve turns left or right. Also in higher dimensions this sign defines an *orientation.* A given curve has everywhere the same orientation, and curves with different orientation can never intersect. Certain areas of $\mathbb{R}^n$ only house left turning curves, and other areas only right turning curves. If the price curve is irregular, then determinants can be taken in its s-dimensional price space, and a similar result holds. One can therefore not only assign a degree of regularity but also an orientation to a price curve and to the region which it inhabits.

3.3. **Discussion of the Case $n = 3$.** In the case $n = 3$, things are especially simple since every $\boldsymbol{A}$-invariant hyperplane containing semipositive vectors has a basis consisting of eigenvectors ($\boldsymbol{p}^*$ and one other Non-Frobenius eigenvector), and is furthermore orthogonal to a left eigenvector (which cannot be $\boldsymbol{q}^*$). The straight price curves in our Figures are therefore the lines connecting $\mathbf{p}^*$ with the other eigenvectors. On the projective plane in which the normalization triangle is embedded these eigenvectors can be visualized to be located somewhere outside the normalization triangle. Equivalently, one can characterize the straight price curves as the lines orthogonal to the left Non-Frobenius eigenvectors.

- In Figure 6, matrix $\boldsymbol{A}$ admits two distinct real eigenvectors. The two (solid) straight price curves are segments of the line connecting $\boldsymbol{p}^*$ with one right eigenvector each, and at the same time they are the lines going through $\boldsymbol{p}^*$ orthogonal to the left Non-Frobenius eigenvectors.
- In Figure 7, matrix $\boldsymbol{A}$ admits no real eigenvectors (the eigenvalues are complex), hence no price curves are straight, they all turn the same way.

Figures 6 and 7 correspond to generic cases in dimension 3. Figures 8 and 9 illustrate the exceptional cases of double eigenvalues:

- In Figure 8, $\boldsymbol{A}$ has, besides the Perron-Frobenius eigenvalue, a double eigenvalue whose associated eigenspace is only one-dimensional. In this case there is one direction of straight price curves, on the line connecting $\boldsymbol{p}^*$ with the single right Non-Frobenius eigenvector, which is at the same time orthogonal to the single left Non-Frobenius eigenvector.
- In Figure 9, $\boldsymbol{A}$ has a double eigenvalue whose eigenspace has dimension two. In this case, there are infinitely many right hand eigenvectors; the economy $(\boldsymbol{A}, \boldsymbol{\ell})$ is irregular for every labor vector $\boldsymbol{\ell}$. Each of these infinitely many straight price curves points from $\boldsymbol{p}^*$ towards some right Non-Frobenius eigenvector, or equivalently it is the line orthogonal to some left Non-Frobenius eigenvector.

3.4. **Projective Geometry: Classification of all Collineations.** An alternative approach to the structure of $\boldsymbol{A}$-invariant subspaces uses the tools of *projective geometry.* The following mathematical apparatus is usually derived under the assumption that $\boldsymbol{A}$ is nonsingular, but those facts about collineations that are relevant for our purposes can also be proved under the weaker condition that the restriction of $\boldsymbol{A}$ to $\mathcal{H}$ is regular.

If price vectors are considered as homogeneous coordinates of the projective plane of *relative prices*, the transformation $\boldsymbol{p} \mapsto \boldsymbol{A}\boldsymbol{p}$ defines a collineation of the triangle of semipositive prices into itself, which can be extended to a collineation of the whole projective plane onto itself. Collineations are bijections which map collinear points into collinear points. Every collineation is induced by a matrix transformation of the homogeneous coordinates. The invariant subspaces of that matrix transformation correspond to the fixed points, fixed lines, or higher-dimensional fixed subspaces of the collineation. Irregular economies correspond to those fixed subspaces which are generated by a single semipositive vector. According to the Perron-Frobenius theorem, the collineation induced by $\boldsymbol{A}$ has exactly one fixed point inside the semipositive triangle, namely that generated by $\boldsymbol{p}^*$, and it has a fixed hyperplane which does not intersect the semipositive triangle, namely the hyperplane $\mathcal{H}$ consisting of all vectors $\boldsymbol{p}$ with ${\boldsymbol{q}^*}^\top \boldsymbol{p} = 0$. Any fixed subspace which intersects the semipositive triangle must therefore be spanned by $\boldsymbol{p}^*$ and a fixed subspace of that $\mathcal{H}$. And if this fixed subspace is generated by a single point, then its intersection with the hyperplane must also be generated by a single point. The study of critical subspaces of a technology is therefore that of the fixed subspaces generated by one point only of the collineation induced by $\boldsymbol{A}$ on $\mathcal{H}$.

The great advantage of this approach is that it reduces the dimensionality of the problem by 2. If $n = 3$, then the projective space of normalized prices is two-dimensional, and $\mathcal{H}$ is a projective line in this plane passing somewhere outside the semipositive triangle. (In the standard normalization, it is the line at infinity.) Restricted to straight lines, all collineations are "projectivities" (i.e., successions of "perspectivities"). Such projectivities can have no fixed points, one fixed point, two fixed points, or they leave the whole line pointwise fixed (Wylie, 1970, pp. 153, 156). (In this last case, the line itself is not generated by a single point.) This gives immediately the four cases discussed in Figures 7, 8, 6, 9.

3.5. **Two Types of Transition.** It has been said earlier that the case of a double eigenvalue is exceptional. It is however important as a limit case or transition when the data change continuously from 2 to 0 real eigenvalues. Let us, for instance, choose as the input matrix some linear combination $\alpha\boldsymbol{A}_3 + (1-\alpha)\boldsymbol{A}_4$ of the two matrices corresponding to Figures 3 and 4, and let us look at the variations of the critical lines and the curves themselves when α increases from 0 to 1. The characteristic polynomial $P_\alpha(\lambda)$ has one real root for $\alpha = 0$ and three for $\alpha = 1$. For the limit case $\boldsymbol{A}_8 = 0.7\boldsymbol{A}_3 + 0.3\boldsymbol{A}_4$, the two non-Frobenius eigenvalues coincide (Figure 11). In the unit simplex, the critical lines shown in Figure 6 move towards each other, coincide for $\alpha = 0.7$, and then vanish. The region occupied by left-turning curves, in our case, decreases. But for other data, the second type of transition represented in Figure 9 would be also possible, in which the critical lines do not coincide in the limiting case, but in which all price curves, whatever their starting points, become straight lines. Two qualitatively distinct types of transition are therefore possible between a set of curves as shown in Figure 6 and one as shown in Figure 7. These qualitative possibilities multiply in higher dimensions.

4. Comparative Statics

Assume the relative price vector $\mathbf{p}(r)$ is known at a given profit rate r, and the rate of profit changes (due to a change in distribution). To fix terminology, we will always assume the new profit rate to be *higher* than the old. What can we

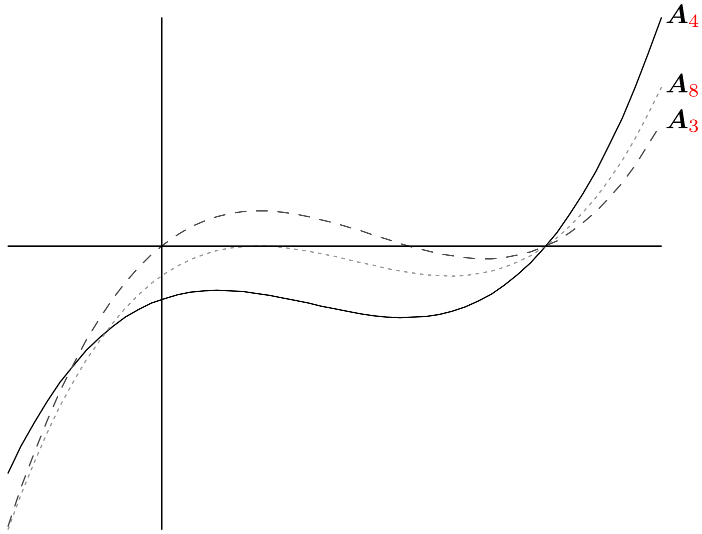

FIGURE 11. Characteristic Polynomials

say about the direction of the change of $\mathbf{p}$? Eventually, if the profit rate rises high enough, the price vector will end up at the Perron-Frobenius vector $\mathbf{p}^* = \mathbf{p}(r^*)$. Is there some sense in which this movement can be shown to be monotonic?

4.1. **Attractive Hyperplanes and Hyperellipses.** The decomposition of price curves formulated in Fact 19 and the characterization of indecomposable price curves in Fact 20 projects general price curves into one- or two-dimensional price curves, which are monotonic in certain ways. The general price curves inherit these monotonicity properties from these lower-dimensional price curves.

This decomposition also allows a better understanding of the difference between Figures 6 and 7. In 6, all price curves can be converted into each other by proportional scaling parallel to one of the critical lines, as illustrated in Figure 12. In Figure 7, they can be turned into each other by rotation along the ellipses, see Figure 13. One may say, Figure 6 contains two kinds of curves (those on the critical lines) and mixtures thereof, while Figure 7 contains only one kind of curve.

The comparative statics results formulated in Fact 20 can be seen in our examples. In Figure 12, the price curves confined to one of the critical lines approach $\mathbf{p}^*$ monotonically, the halfspaces bounded by hyperplanes orthogonal to the left eigenvector corresponding to this critical line are attractive halfspaces. In Figure 13, the ellipses are attractive, and all price curves cross the spokes of the ellipses in the same direction.

Here is the mathematical formula of the attractive ellipses. Assume $\boldsymbol{A}$ has a pair of conjugate complex eigenvalues λ_1 and λ_2, with associated left eigenvectors $\boldsymbol{q}_1 = \boldsymbol{s} + \boldsymbol{t}i$, $\boldsymbol{q}_2 = \boldsymbol{s} - \boldsymbol{t}i$. Then we know from Fact 20 that, in the standard normalization, the modulus of the complex inner product

$$k_i(r) = \left|\boldsymbol{q}_i^\top \mathbf{p}(r)\right| = \sqrt{\left(\boldsymbol{s}^\top \mathbf{p}(r)\right)^2 + \left(\boldsymbol{t}^\top \mathbf{p}(r)\right)^2} \qquad (i = 1, 2) \tag{7}$$

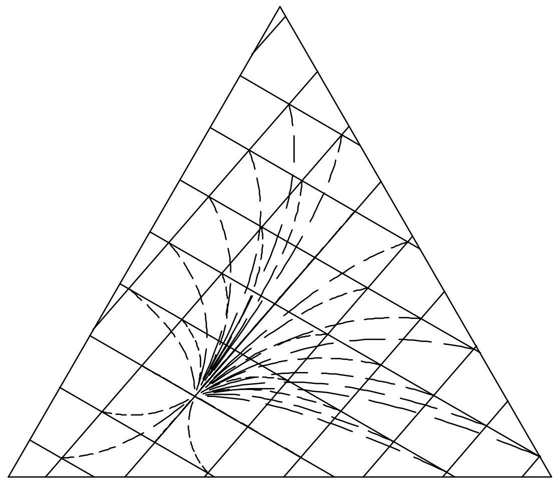

FIGURE 12. Attractive Halfspaces

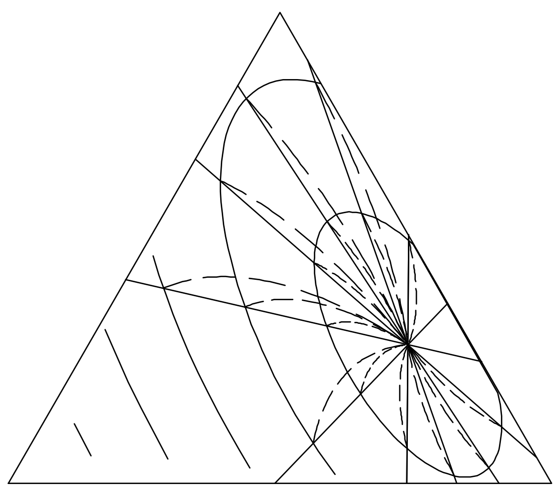

FIGURE 13. Attractive Ellipses

decreases as r increases. The loci of normalized price vectors $\mathbf{p}$ for which $\left|\boldsymbol{q}_i^\top \mathbf{p}(r)\right|$ is constant are elliptical hypercylinders (or, if the normalization simplex is two-dimensional, ellipses), which contain each other and shrink to a hyperline passing through the point $\mathbf{p}^*$. The fact that ratio (7) decreases as r increases means that the price curve successively enters into these ellipses, i.e., these ellipses are "attractive."

4.2. **Euclidean Angles Between Price Vectors.** Let us assume that matrix $\boldsymbol{A}$ is symmetric. This implies that there are $n-1$ real eigenvalues associated with $n-1$ critical hyperplanes, which are orthogonal to each other and to the normalization simplex for the standard normalization. We know by the previous section that the distance from the normalized $\mathbf{p}(r)$ to each of these critical hyperplanes decreases. By the theorem of Pythagoras, the distance between the normalized $\mathbf{p}(r)$ and $\boldsymbol{p}^*$ also decreases. Since the normalization vector is at the same time a right hand Frobenius eigenvector $\boldsymbol{p}^*$, the vector $\boldsymbol{p}^*$ is orthogonal to the simplex. Since the triangle $\overline{\boldsymbol{o}\boldsymbol{p}^*\boldsymbol{p}(r)}$ is rectangular at $\boldsymbol{p}^*$, the length of side $\overline{\boldsymbol{p}^*\boldsymbol{p}(r)}$ is a monotonic function of the angle between $\boldsymbol{p}^*$ and $\boldsymbol{p}(r)$. Therefore one can translate the above result into a form which is independent of the normalization chosen: If $\boldsymbol{A}$ is symmetric, the angle between $\boldsymbol{p}(r)$ and $\boldsymbol{p}^*$ decreases monotonically.

More generally, the angle between $\boldsymbol{p}(r)$ and $\boldsymbol{p}(s)$ is a monotonic function of r and s. All this is also true if $\boldsymbol{A}$ is normal, i.e., if $\boldsymbol{A}^\top\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^\top$, which is equivalent to complex but orthogonal (more precisely, unitary) eigenvectors.

Note that symmetry or normality of $\boldsymbol{A}$ is a property of the representation, not of the economy. Not every economy has a normal representation, and there is no economic reason why an economy should have one.

However, if matrix $\boldsymbol{A}$ is diagonalizable (which is the generic case, therefore a rather weak assumption) one can define a non-Euclidean angle such that the monotonicity property holds. Non-Euclidean angles correspond to attractive ellipses instead of attractive circles.

For a proof of all this, the reader is referred to (Bidard and Steedman, 1996) and (Bidard and Steedman, 2001).

4.3. **Arbitrary Normalization.** Most Figures have been drawn in the standard normalization, which makes the right hand non-Frobenius eigenvectors $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$ parallel to the normalization simplex (because ${\boldsymbol{q}^*}^\top\boldsymbol{p}_1 = {\boldsymbol{q}^*}^\top\boldsymbol{p}_2 = 0$). Under arbitrary normalization $\boldsymbol{g}^\top\mathbf{p}(r) = 1$, these non-Frobenius eigenvectors intersect the plane $\boldsymbol{g}^\top\mathbf{x} = 1$ in points P_1 and P_2 outside the simplex. In this case it is no longer true that the distance to the critical lines decreases monotonically, but the price curves pass all straight lines concurrent on P_1 and P_2 in the direction pointing towards the critical lines. In this case the Euclidean distance from a critical line may vary in a nonmonotonic way.

4.4. **The Inverse Problem: Determining Input Coefficients from Prices.** In order to apply the results obtained so far, $\boldsymbol{A}$ must be known. Before going on to Section 5, in which this information requirements will be relaxed, let us see how information about $\boldsymbol{A}$ can be built up from prices, or how price curves can be extrapolated without knowing $\boldsymbol{A}$. Relative price curves which have more than n points in common and which also have identical profit rates at these points, coincide (Fact 13). In our three-commodity economy, four points on any relative price curve, no three of which lie on a straight line, together with the associated profit rates, will identify this price curve along with the $\boldsymbol{A}$-matrix and (up to a factor) the $\boldsymbol{\ell}$-vector. If three points of a price curve lie on a straight line, then the price curve itself is straight (Fact 6). In this case, therefore, the price curve is identified by three instead of four points, but $\boldsymbol{A}$ is no longer determined uniquely.

If only prices are known but not the profit rates, then it will be shown in Section 6 that in a three-commodity economy, a price curve is determined if one knows five points on it.

5. Hilbert Circles

Results concerning the Hilbert metric have quite different information requirements than the results discussed until now. The Hilbert metric is therefore an alternative tool to gain insights about the behavior of price curves.

5.1. **Definition.** The Hilbert distance between two positive vectors $\boldsymbol{x} \gg \boldsymbol{o}$ and $\boldsymbol{y} \gg \boldsymbol{o}$ is defined as

$$d(\boldsymbol{x}, \boldsymbol{y}) = \log\Big(\max_i \frac{x_i}{y_i} \Big/ \min_i \frac{x_i}{y_i}\Big) \tag{8}$$

and has the following properties:

- $d(\lambda \boldsymbol{x}, \mu \boldsymbol{y}) = d(\boldsymbol{x}, \boldsymbol{y})$ for any $\lambda, \mu > 0$. The Hilbert distance only depends on the rays supported by $\boldsymbol{x}$ and $\boldsymbol{y}$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are price vectors, their Hilbert distance therefore only depends on relative prices.
- If the jth components of $\boldsymbol{x}$ and $\boldsymbol{y}$ are both multiplied by the same scalar, $d(\boldsymbol{x}, \boldsymbol{y})$ is unchanged. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are price vectors, $d(\boldsymbol{x}, \boldsymbol{y})$ is therefore independent of the choice of physical units for commodities. (This property is in sharp contrast with the Euclidean angle.)
- $d(\cdot, \cdot)$ satisfies the axioms of a distance between positive rays: it is symmetric, i.e., $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x})$; furthermore $d(\boldsymbol{x}, \boldsymbol{y}) = 0$ if and only if the relative prices are identical, and finally the triangle inequality holds $d(\boldsymbol{x}, \boldsymbol{z}) \leq d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z})$.
- A positive matrix induces contraction between rays: If $\boldsymbol{x} \gg \boldsymbol{o}$, $\boldsymbol{y} \gg \boldsymbol{o}$, $\boldsymbol{x} \neq \lambda \boldsymbol{y}$, and the matrix $\boldsymbol{H} \gg \boldsymbol{O}$, then
$$d(\boldsymbol{H}\boldsymbol{x}, \boldsymbol{H}\boldsymbol{y}) < d(\boldsymbol{x}, \boldsymbol{y}) \tag{9}$$
See (Seneta, 1981, Lemma 3.2 on p. 38) for a proof.

In order to give an intuition of the Hilbert metric, let us draw the "Hilbert circle" on the unit simplex with center $\mathbf{p}^*$ and going through $\mathbf{m}$, i.e., the set $\{\mathbf{x}\colon\ d(\mathbf{p}^*, \mathbf{x}) = d(\mathbf{p}^*, \mathbf{m})\}$. To simplify the exposition we will assume $\mathbf{p}^* = \mathbf{t}$, i.e., $\mathbf{p}^*$ lies in the center of the unit simplex, but the geometric construction of the Hilbert circle described here is valid in full generality. In our simplified situation, formula (8) shows that $d(\mathbf{t}, \mathbf{x})$ is nothing but the logarithm of the ratio between the extremal values of $\mathbf{x}$

$$d(\boldsymbol{t}, \boldsymbol{x}) = \log\Big(\max_i x_i \Big/ \min_i x_i\Big). \tag{10}$$

The geometric construction is illustrated in Figures 14 and 15. First draw lines $\overline{\mathbf{ata'}}$, $\overline{\mathbf{btb'}}$, and $\overline{\mathbf{ctc'}}$ which divide the triangle in six regions, each of which characterized by a specific ranking of the components of $\mathbf{x}$. E.g. point $\mathbf{m}$ is above line $\overline{\mathbf{aa'}}$, therefore $\mathrm{m}_2 < \mathrm{m}_3$; it is below line $\overline{\mathbf{bb'}}$, therefore $\mathrm{m}_1 < \mathrm{m}_3$; and it is left of line $\overline{\mathbf{cc'}}$, therefore $\mathrm{m}_1 > \mathrm{m}_2$. This established the order $\mathrm{m}_1 > \mathrm{m}_3 > \mathrm{m}_2$. (Figure 14). For all points $\mathbf{x}$ in that region, only x_1 and x_3 matter for the Hilbert distance from $\mathbf{t}$, and we know (see our above comments regarding Figure 2) that all points on $\overline{\mathbf{cm}}$ have the same ratio $\mathrm{x}_1/\mathrm{x}_3$. Therefore segment $\overline{\mathbf{di}}$ in Figure 14 is a part of the Hilbert circle we are drawing. At point $\mathbf{d}$, $\mathrm{x}_1 = \mathrm{x}_3$ and we enter the region where $\mathrm{x}_1 < \mathrm{x}_3$, i.e. where x_3 and x_2 become the extremal coordinates. There the points on the Hilbert circle are those belonging to segment $\overline{\mathbf{de}}$ in Figure 15 ($\mathbf{a}$, $\mathbf{d}$, and $\mathbf{e}$ are collinear), etc., and the Hilbert circle going through $\mathbf{m}$ is the hexagon $\overline{\mathbf{defghi}}$. The same geometric procedure will also produce the correct Hilbert circle if its center is not located at the center of the triangle.

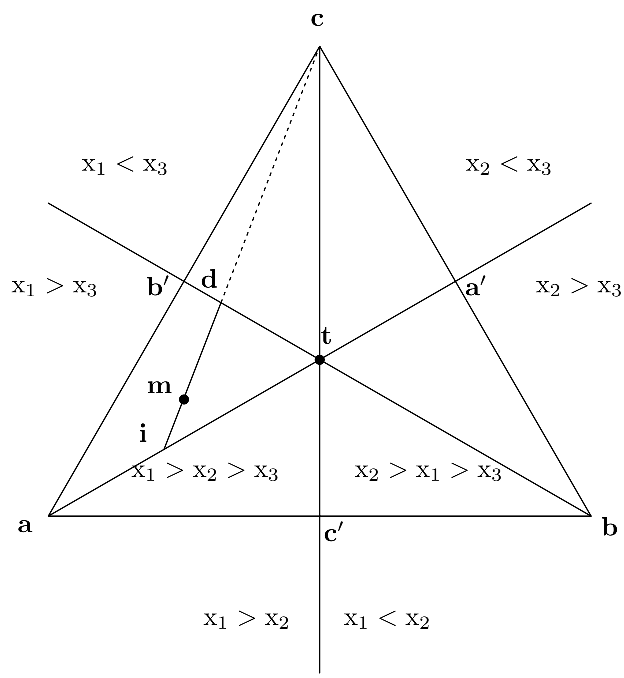

FIGURE 14. Preparation for Hilbert Circle through $\boldsymbol{m}$

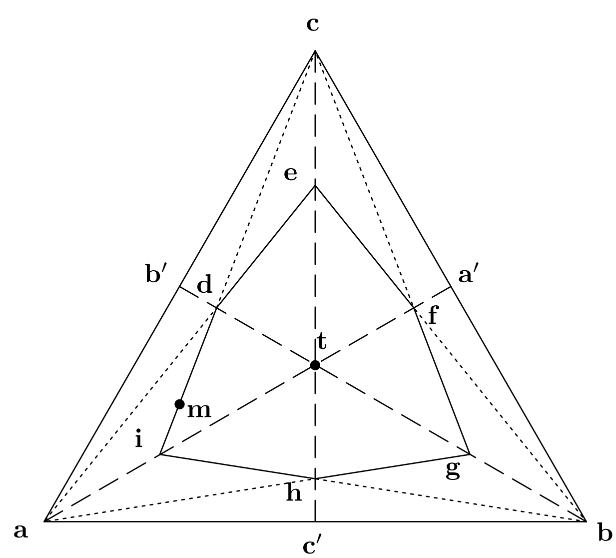

FIGURE 15. Construction of Hilbert Circle

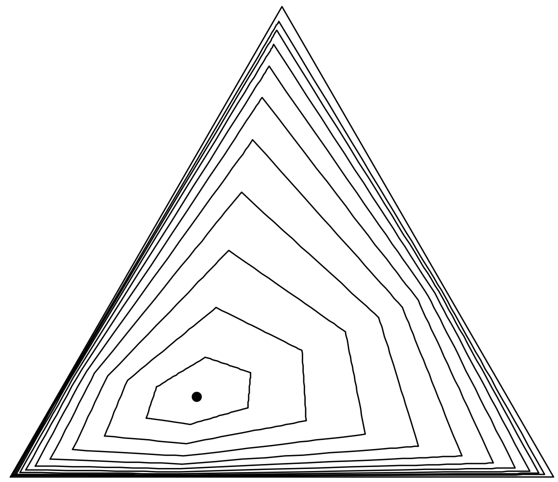

FIGURE 16. "Hilbert circles" with radius 0.5, 1, 1.5, 2, . . . around endpoint of curves in Figures 3 and 6

$\boldsymbol{A}_{17} = \frac{1}{1000}\begin{bmatrix} 498 & 1 & 1 \\ 498 & 1 & 1 \\ 1 & 1 & 498 \end{bmatrix}$ $\qquad$ $\boldsymbol{\ell}_{17} = \frac{1}{10}\begin{bmatrix} 1 \\ 10 \\ 10 \end{bmatrix}$

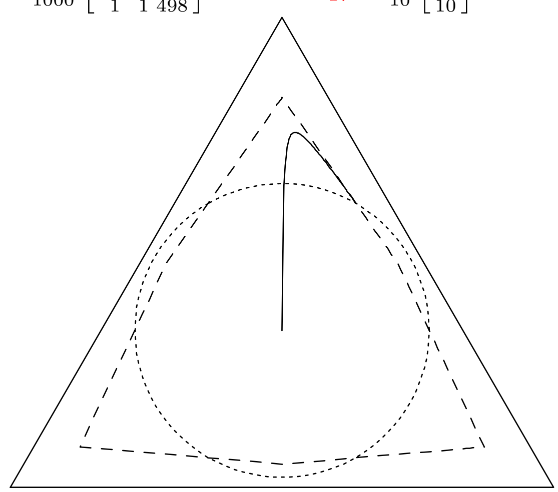

FIGURE 17. Euclidean Angle Counterexample

Figure 16 shows the deformation of the Hilbert circles if the endpoint is no longer in the center and if the radius increases. According to formula (10), the Hilbert distance to the center of the Hilbert disk grows to infinity as $\mathbf{m}$ approaches a side of the triangle.

5.2. **Hilbert Circles are Attractive.** The property of the Hilbert distance which makes it relevant for prices of production is a mathematical consequence of its contraction property: the Hilbert distance between $\boldsymbol{p}(r)$ and $\boldsymbol{p}^*$ is a decreasing function of the profit rate r, i.e., from $r < s < r^*$ follows $d(\boldsymbol{p}(s), \boldsymbol{p}^*) < d(\boldsymbol{p}(r), \boldsymbol{p}^*)$. Hilbert circles are therefore attractive for all profit rates. For a proof see Fact 23.

Figure 17 illustrates the economic law involving the Hilbert distance between $\mathbf{p}(r)$ and $\mathbf{p}^*$: the Hilbert circles are attractive, i.e. every price curve successively enters into them as the profit rate increases. This is even true for the curve drawn in Figure 17: Although the Euclidean angle increases locally, the Hilbert distance between $\boldsymbol{p}(r)$ and $\boldsymbol{p}^*$ is everywhere decreasing.

The Hilbert circle centered at $\mathbf{p}^*$ is a hexagon with straight sides. In exceptional cases, the price vector $\mathbf{p}(r)$ enters this hexagon at one of its corners, but usually it enters it on one of its sides. The local information provided by the Hilbert circle is therefore that the price curve crosses this side or, in the corner case, both sides involved, moving towards $\mathbf{p}^*$. There is also global information, since the remaining part of the curve for $s \geq r$ will remain within this Hilbert circle.

Here are some additional results regarding the Hilbert distance:

- Consider a Hilbert disk centered at $\boldsymbol{p}^*$, with radius equal to $d(\boldsymbol{p}^*, \boldsymbol{p}(t))$. The monotonicity property means that the part of the price curve corresponding to profit rates smaller than t is outside the disk. It can be shown (Fact 24) that the straight line segment joining any two outside points on the price curve also lies outside the Hilbert disk. A curve can therefore not make such a close turn around a corner of a Hilbert disk that a straight segment would cut into the Hilbert disk.
- Up to now we have only considered the distance between a price vector $\boldsymbol{p}(r)$ and the *end* $\boldsymbol{p}^*$ of the price curve. It can be shown (Fact 25) that the distance between $\boldsymbol{p}(r)$ and $\boldsymbol{p}(t)$ decreases as r increases: $d(\boldsymbol{p}(r), \boldsymbol{p}(t)) > d(\boldsymbol{p}(s), \boldsymbol{p}(t))$ whenever $r < s < t \leq r^*$. Therefore the Hilbert disks centered at $\boldsymbol{p}(t)$ are attractive for the part of the curve corresponding to profit rates *lower* than t. Curiously enough, if there are at least four commodities, then the expected *opposite* inequality does not hold: as the profit rate rises *above* t, the Hilbert circles centered on t will not always monotonically release the price curve again.

5.3. **Speed of Convergence.** Consider the Hilbert distance between $\boldsymbol{p}(r)$ and $\boldsymbol{p}^*$ as a function of r. It is a decreasing function. Its derivative at r^* indicates how fast the relative prices $\boldsymbol{p}(r)$ converge towards $\boldsymbol{p}^*$ in a neighborhood of r^*. And since the Hilbert distance between two close positive vectors is an approximation of the angle between them (more precisely: in a neighborhood of $\boldsymbol{p}^*$, the ratio between $d(\boldsymbol{p}^*, \boldsymbol{x})$ and the angle $(\boldsymbol{p}^*\boldsymbol{x})$ is lower and upper bounded by positive numbers, see Fact 26), such a derivative does give an idea of the speed of convergence according to common intuition.

It can be shown that the first and second derivatives exist. The first derivative (speed of convergence) depends on the relative position of $\boldsymbol{p}^*$ and the tangent to the curve at $\boldsymbol{p}^*$. See Fact 27 in the Appendix for the exact formula. Except in the case of uniform organic composition, the first derivative is finite and strictly negative, indicating that the convergence is neither extremely slow nor extremely fast. And the second derivative (acceleration of convergence) may be positive or negative, that is, the convergence may accelerate or decelerate according to the choice of the labor vector.

6. Outlook: Additional Results for $n = 3$

The following results hold for the case $n = 3$, but can be adequately generalized for higher dimensions.

- All price curves are conics. If there are two critical lines, the price curve starting at $\boldsymbol{\ell}$ also passes through the three right hand eigenvectors $\boldsymbol{p}^*$, $\boldsymbol{p}_1$, and $\boldsymbol{p}_2$ (Fact 29). More precisely, the price curve for $-1 \leq r \leq r^*$ is a part of this conic. The complementary part, which contains $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, is "virtual" and corresponds to profit rates outside this range. However, several properties mentioned earlier do hold for the virtual part. Compare for instance Figure 5 and Figure 10. In Figure 5, the price curves corresponding to different labor vectors $\boldsymbol{\ell}$ and $\boldsymbol{m}$ intersect at point $\boldsymbol{p}$ in the simplex and, more generally, $\boldsymbol{p}_{\boldsymbol{\ell}}(r)$, $\boldsymbol{p}_{\boldsymbol{m}}(r)$, and $\boldsymbol{p}$ are aligned for any r. In Figure 10, the price curves starting on the Northeast side of the simplex have no intersection in the simplex other than $\boldsymbol{p}^*$, see Section 2.4. But their virtual parts admit another intersection at $\hat{\boldsymbol{p}}$ outside the simplex, and for any r, the level lines are concurrent on $\hat{\boldsymbol{p}}$.
- Any conic is defined by four points and a tangent. Knowing $\boldsymbol{\ell}$, $\boldsymbol{p}^*$, $\boldsymbol{p}_1$, and $\boldsymbol{p}_2$, one additional piece of information is needed. Most conveniently it is the tangent in its beginning point $\boldsymbol{\ell}$ (Fact 30).
- Any conic is also defined by five points, no four of which are collinear. Therefore price curves belonging to possibly different $\boldsymbol{A}$ and $\boldsymbol{\ell}$ coincide if they have at least five intersection points, no four of which are collinear. Compare with Section 4.4; it was asserted there that four intersections at identical profit rates make the whole price curves coincide. Only one intersection point is gained if the price curves are not required to intersect at equal profit rates.
- Let there be only one critical line. There exists only one non-Frobenius eigenvector $\boldsymbol{p}_1$. In this case, the price curve passes through $\boldsymbol{\ell}$, $\boldsymbol{p}^*$, $\boldsymbol{p}_1$, and is tangent to the line passing through $\boldsymbol{p}_1$ which is orthogonal to the left hand Frobenius eigenvector $\boldsymbol{q}^*$. Again, an additional piece of information is needed, such as the tangent in $\boldsymbol{\ell}$, to define this price curve uniquely.
- In the standard normalization, the conics are hyperbolas if there are two critical lines, ellipses if there are no critical lines, and parabolas if there is one critical line (Fact 31).
- In the standard normalization, let $\boldsymbol{\ell}$ move parallel to a critical line. Then the tangents to the different price curves in $\boldsymbol{\ell}$ are concurrent on a point which belongs to the other critical line (if there are two critical lines) or belongs to the same critical line, if it is the only one (Fact 33).

7. Conclusion

We have studied the behavior of relative prices when the rate of profit varies. Several pieces of information have been assembled. We first noticed qualitative properties of the curve, e.g., the fact that they turn 'left' or 'right' according to the region to which the labor vector belongs, and within which the whole curve remains. These regions are delimited by critical lines or critical subspaces in higher dimensions, which correspond to invariant subspaces with respect to the input matrix. Quantitative properties have also been found. Our endeavor was to give content to the assertion that relative prices vary monotonically, which requires one to adopt an adequate measure of the distance between price vectors. We have thus been able to define attractive lines, attractive ellipses, and attractive Hilbert circles—certainly the most promising route of research.

These results clarify the question of relative prices. Even if their movement is undoubtedly complex, it is not chaotic, and follows some definite laws.

Appendix A. Mathematical Proofs

The Appendix gives references to justify the statements made in the body of the paper, and provides proofs of most results, including many new proofs. $(\boldsymbol{A}, \boldsymbol{\ell})$ represents a basic $n \times n$ economy without necessarily assuming $n = 3$.

We will call the vector $\boldsymbol{p} \neq \boldsymbol{o}$ a "virtual price vector for technology $(\boldsymbol{A}, \boldsymbol{\ell})$ associated with profit rate r," if a scalar w exists so that

$$(1+r)\boldsymbol{A}\boldsymbol{p} + w\boldsymbol{\ell} = \boldsymbol{p}. \tag{11}$$

If moreover $-1 \leq r < r^*$ (and therefore $w > 0$), then $\boldsymbol{p}$ is called a price of production vector, not qualified as "virtual."

Representations of the Technology.

Fact 1. *Here is a prescription how to transform $\boldsymbol{A}$, $\boldsymbol{\ell}$, and the price vectors $\boldsymbol{p}$ if physical units are changed. Assume 1 new unit of good i is equal to b_i old units of i, and 1 new unit of labor is equal to d old units of labor. Define the vector $\boldsymbol{b}$ with elements b_i, and the vector $\boldsymbol{c}$ with elements $c_i = 1/b_i$, i.e., 1 old unit of good i is equal to c_i new units of good i. Then one has to multiply all old price vectors element by element by the vector $\boldsymbol{b}$, multiply the matrix $\boldsymbol{A}$ element by element by the matrix $\boldsymbol{b}\boldsymbol{c}^\top$, and multiply the old labor vector element by element by the vector $\frac{1}{d}\boldsymbol{b}$*

Normalizations.

Fact 2. *The barycentric coordinates of normalized price vectors are independent of the representation of the economy.*

Proof. Assume $\boldsymbol{p}$ is normalized such that $\boldsymbol{g}^\top\boldsymbol{p} = 1$, i.e. $g_1p_1 + \cdots + g_np_n = 1$. Using the unit vectors $\boldsymbol{e}_i$ one can write

$$\boldsymbol{p} = p_1\boldsymbol{e}_1 + \cdots + p_n\boldsymbol{e}_n = g_1p_1\frac{\boldsymbol{e}_1}{g_1} + \cdots + g_np_n\frac{\boldsymbol{e}_n}{g_n}. \tag{12}$$

This shows that $\boldsymbol{p}$ is the weighted average of the scaled unit vectors $\frac{\boldsymbol{e}_i}{g_i}$, the corners of the simplex. The barycentric coordinates are therefore simply the g_ip_i. These are invariant under changes in representation, since a change in units of the ith commodity affects g_i and p_i in opposite ways and therefore leaves their product unchanged. □

Fact 3. *In the standard normalization, normalized prices for a given profit rate r are linear functions of the normalized labor vectors, with the origin in the space of normalized prices located at $\mathbf{p}^*$. The formulas for normalized wage and price are equations (4) and (5).*

Proof. For $r < r^*$, one obtains the normalized wage by combining equation (2) with $\boldsymbol{g}^\top \boldsymbol{p} = 1$:

$$w = \frac{1}{\boldsymbol{g}^\top (\boldsymbol{I} - (1+r)\boldsymbol{A})^{-1} \boldsymbol{\ell}}. \tag{13}$$

Seting $\boldsymbol{g} = \boldsymbol{q}^*$ in (13) gives (4).

Equations (1) and (4) give the following defining equation for normalized prices:

$$\Big(\boldsymbol{I} - (1+r)\boldsymbol{A}\Big)\mathbf{p} = \frac{\boldsymbol{\ell}}{\boldsymbol{q}^{*\top}\boldsymbol{\ell}}\Big(1 - \frac{1+r}{1+r^*}\Big) \tag{14}$$

This is a linear function in the normalized labor coefficient $\mathbf{l} = \boldsymbol{\ell}/\boldsymbol{q}^{*\top}\boldsymbol{\ell}$, but normalized labor coefficients do not form a linear space. Therefore we subtract the following identity involving the normalized Perron-Frobenius eigenvector $\mathbf{p}^*$:

$$\Big(\boldsymbol{I} - (1+r)\boldsymbol{A}\Big)\mathbf{p}^* = \mathbf{p}^*\Big(1 - \frac{1+r}{1+r^*}\Big) \tag{15}$$

This gives

$$\Big(\boldsymbol{I} - (1+r)\boldsymbol{A}\Big)(\mathbf{p} - \mathbf{p}^*) = (\mathbf{l} - \mathbf{p}^*)\Big(1 - \frac{1+r}{1+r^*}\Big) \tag{16}$$

and therefore (5). □

If $r = r^*$, then $\boldsymbol{I} - (1+r)\boldsymbol{A}$ is singular, but on $\mathcal{H}$ it remains regular. To prove this, take any vector $\boldsymbol{t}$ for which the value of $\boldsymbol{q}^{*\top}\boldsymbol{t}$ lies outside the interval $[0,1]$ (for instance $\boldsymbol{t} = -\mathbf{p}^*$ or $\boldsymbol{t} = -\boldsymbol{\iota}$ will do), and form the matrix $\boldsymbol{I} - \boldsymbol{t}\boldsymbol{q}^{*\top} - (1+r)\boldsymbol{A}$. We will show that this matrix, which coincides on $\mathcal{H}$ with $\boldsymbol{I} - (1+r)\boldsymbol{A}$, is nonsingular on all of $\mathbb{R}^n$ if r remains in the closed interval $-1 \le r \le r^*$. Take any $\boldsymbol{p}$ which satisfies

$$\Big(\boldsymbol{I} - \boldsymbol{t}\boldsymbol{q}^{*\top} - (1+r)\boldsymbol{A}\Big)\boldsymbol{p} = \boldsymbol{o}. \tag{17}$$

If one premultiplies (17) with $\boldsymbol{q}^{*\top}$ it follows

$$\Big(1 - \boldsymbol{q}^{*\top}\boldsymbol{t} - \frac{r+1}{r^*+1}\Big)\boldsymbol{q}^{*\top}\boldsymbol{p} = 0, \tag{18}$$

and therefore $\boldsymbol{q}^{*\top}\boldsymbol{p} = 0$. Therefore (17) becomes $\boldsymbol{p} = (1+r)\boldsymbol{A}\boldsymbol{p}$. If $-1 \le r < r^*$, then $\boldsymbol{I} - (1+r)\boldsymbol{A}$ is nonsingular and it follows $\boldsymbol{p} = \boldsymbol{o}$. Remains only to investigate the case $r = r^*$. Since there is, up to a scalar factor, only one eigenvector associated with the Perron-Frobenius eigenvalue, $\boldsymbol{p} = (1+r^*)\boldsymbol{A}\boldsymbol{p}$ implies $\boldsymbol{p} = \alpha\mathbf{p}^*$. Applying $\boldsymbol{q}^{*\top}\boldsymbol{p} = 0$ to this gives $\alpha = 0$, i.e., again $\boldsymbol{p} = \boldsymbol{o}$. This concludes the proof of nonsingularity. Instead of (5) one can therefore also write

$$\mathbf{p} - \mathbf{p}^* = \frac{r^* - r}{1+r^*}\Big(\boldsymbol{I} - \boldsymbol{t}\boldsymbol{q}^{*\top} - (1+r)\boldsymbol{A}\Big)^{-1}(\mathbf{l} - \mathbf{p}^*) \tag{19}$$

and (19) has the advantage over (5) that it is defined on the entire closed interval $-1 \le r \le r^*$. This may be helpful for computer simulations, and it establishes the continuity and differentiability of the price curves in their common endpoint.

The Price Space of an Economy. Given a technology $(\boldsymbol{A}, \boldsymbol{\ell})$, we define its price space $\mathcal{P}$ as the linear subspace of $\mathbb{R}^n$ spanned by all virtual price vectors $\boldsymbol{p}(r)$ where $r \neq -1$ and $1/(1+r)$ is not an eigenvector. The dimension of this price space, $s = \dim(\mathcal{P})$, is called the "degree of regularity" of the economy, notation $s = \deg(\boldsymbol{A}, \boldsymbol{\ell})$. The economy is called regular (Schefold, 1976) if $s = n$, and irregular if $s < n$.

It is easy to see that $\mathcal{P}$ is $\boldsymbol{A}$-invariant, because $\boldsymbol{\ell} \in \mathcal{P}$ by continuity, and $\boldsymbol{A}\boldsymbol{p}(r) = (\boldsymbol{p}(r) - w\boldsymbol{\ell})/(1+r) \in \mathcal{P}$. One can say even more:

Fact 4. *Given a profit rate r which is either $r = -1$ or, if it is not, satisfies the condition that $1/(1+r)$ is not an eigenvalue of $\boldsymbol{A}$. Let $\boldsymbol{p}(r)$ be the price vector associated with r. Then the price space of the technology $(\boldsymbol{A}, \boldsymbol{\ell})$ is the smallest $\boldsymbol{A}$-invariant space containing $\boldsymbol{p}(r)$.*

Proof. Take any $\boldsymbol{A}$-invariant $\mathcal{V}$ which contains $\boldsymbol{p}(r)$. Then $\mathcal{V}$ also contains $\boldsymbol{A}\boldsymbol{p}(r)$, hence $\mathcal{V} \ni -(1+r)\boldsymbol{A}\boldsymbol{p}(r) + \boldsymbol{p}(r) = w\boldsymbol{\ell}$. Since $1/(1+r)$ is not an eigenvalue, w must be nonzero, therefore $\mathcal{V} \ni \boldsymbol{\ell}$. Now take any other r such that $1/(1+r)$ is not an eigenvalue of $\boldsymbol{A}$, and any other $w > 0$. The mapping $\boldsymbol{p} \mapsto (\boldsymbol{I} - (1+r)\boldsymbol{A})\boldsymbol{p}$ is a monomorphism $\mathcal{V} \to \mathcal{V}$, therefore also an epimorphism. Since $\boldsymbol{\ell} \in \mathcal{V}$, the $\boldsymbol{p}$ satisfying the equation $(1+r)\boldsymbol{A}\boldsymbol{p} + w\boldsymbol{\ell} = \boldsymbol{p}$ is also in $\mathcal{V}$. Hence the price space $\mathcal{P}$ of the technology $(\boldsymbol{A}, \boldsymbol{\ell})$ is contained in all $\boldsymbol{A}$-invariant subspaces containing the given $\boldsymbol{p}(r)$. Since $\mathcal{P}$ itself is $\boldsymbol{A}$-invariant, the statement follows. □

Fact 5. *Assume the economy has degree of regularity s. Then s is the greatest integer such that the vectors $\boldsymbol{\ell}, \boldsymbol{A}\boldsymbol{\ell}, \boldsymbol{A}^2\boldsymbol{\ell}, \dots, \boldsymbol{A}^{s-1}\boldsymbol{\ell}$ are independent, and the price space can be written as $\mathcal{P} = \{\boldsymbol{\ell}, \boldsymbol{A}\boldsymbol{\ell}, \boldsymbol{A}^2\boldsymbol{\ell}, \dots, \boldsymbol{A}^{s-1}\boldsymbol{\ell}\}$.*

Proof. Follows immediately from Fact 4. □

Fact 6. *Assume the economy has degree of regularity s. For s distinct profit rates $r_1, \dots, r_s$ the vectors $\boldsymbol{p}(r_1), \dots, \boldsymbol{p}(r_s)$ span the price space $\mathcal{P}$.*

Proof. Fact 6 was first proved by Schefold (1976, Theorem 1.1) in the regular case, and extended by Raneda and Reus (1985) to $s < n$. The following simple proof by induction follows (Bidard and Salvadori, 1998, Theorem 1). Given s virtual price vectors $\boldsymbol{p}(r_i)$ associated with different profit rates. If $\boldsymbol{\ell}$ is one of the price vectors, then these price vectors must be rearranged in the form $\boldsymbol{\ell}, \boldsymbol{A}\boldsymbol{\ell}, \boldsymbol{A}^2\boldsymbol{\ell}, \dots, \boldsymbol{A}^{t-1}\boldsymbol{\ell}, \boldsymbol{p}(r_{t+1}), \dots, \boldsymbol{p}(r_s)$ where $1 \leq t \leq s$ and $\boldsymbol{A}^t\boldsymbol{\ell}$ is not one of the $\boldsymbol{p}(r_i)$. If $\boldsymbol{\ell}$ is not among the original price vectors, then set $t = 0$ in what follows. By construction, $r_i \neq -1$ for $t+1 \leq i \leq s$. Now assume there is a linear dependence relationship between those s vectors. Take the image of the relationship under $\boldsymbol{A}$, and substitute $(1+r_i)^{-1}(\boldsymbol{p}(r_i) - w_i\boldsymbol{\ell})$ for $\boldsymbol{A}\boldsymbol{p}(r_i)$ for $t+1 \leq i \leq s$. This generates a second linear dependence relationship among $\boldsymbol{\ell}, \boldsymbol{A}\boldsymbol{\ell}, \boldsymbol{A}^2\boldsymbol{\ell}, \dots, \boldsymbol{A}^t\boldsymbol{\ell}, \boldsymbol{p}(r_{t+1}), \dots, \boldsymbol{p}(r_s)$. After elimination of $\boldsymbol{p}(r_{t+1})$ between the first and second relationship, a new relationship is obtained among $\boldsymbol{\ell}, \boldsymbol{A}\boldsymbol{\ell}, \boldsymbol{A}^2\boldsymbol{\ell}, \dots, \boldsymbol{A}^t\boldsymbol{\ell}, \boldsymbol{p}(r_{t+2}), \dots, \boldsymbol{p}(r_s)$. Repeating this procedure, all original price vectors disappear and a contradiction is obtained since, by Fact 5, $\boldsymbol{\ell}, \boldsymbol{A}\boldsymbol{\ell}, \boldsymbol{A}^2\boldsymbol{\ell}, \dots, \boldsymbol{A}^{s-1}\boldsymbol{\ell}$ are independent. □

Fact 6 allows us to define the price space as the space spanned by all virtual price vectors (no restriction on the profit rates); but the same space is also spanned by all price vectors for profit rates r_i in the interval $-1 \leq r_i < r^*$ or $-1 \leq r_i \leq r^*$ etc.

Fact 7. *If the economy is regular, then for any $-1 \le r_1 < \cdots < r_n \le r^*$, and for any r, $-1 \le r < r^*$, the three determinants $\det\big(\boldsymbol{p}(r_1), \dots, \boldsymbol{p}(r_n)\big)$, $\det\big(\boldsymbol{p}(r), \boldsymbol{A}\boldsymbol{p}(r), \dots, \boldsymbol{A}^{n-1}\boldsymbol{p}(r)\big)$, and $\det(\boldsymbol{\ell}, \boldsymbol{A}\boldsymbol{\ell}, \dots, \boldsymbol{A}^{n-1}\boldsymbol{\ell})$ have the same sign.*

Proof. This was proved in (Bidard, 1991, p. 56). The determinant $\det\big(\boldsymbol{p}(r_1), \dots, \boldsymbol{p}(r_n)\big)$ does not vanish due to Fact 6, and varies continuously with $r_1 < \cdots < r_n$, therefore its sign is constant. For $r_k = -1 + \varepsilon k$ ($k = 1, \dots, n$; $\varepsilon > 0$) and using a Taylor expansion of $\boldsymbol{p}(r)$ up to order $n-1$ it is easily checked that the sign is that of $\det(\boldsymbol{\ell}, \boldsymbol{A}\boldsymbol{\ell}, \dots, \boldsymbol{A}^{n-1}\boldsymbol{\ell})$. Similarly, the determinant $\det(\boldsymbol{p}(r), \boldsymbol{A}\boldsymbol{p}(r), \dots, \boldsymbol{A}^{n-1}\boldsymbol{p}(r))$ does not vanish for $-1 \le r < r^*$, and by continuity, when r moves to -1, its sign is that of $\det(\boldsymbol{\ell}, \boldsymbol{A}\boldsymbol{\ell}, \dots, \boldsymbol{A}^{n-1}\boldsymbol{\ell})$. □

If the economy is irregular, prices vary in a s-dimensional subspace $\mathcal{P}$, and one can define a determinant within this subspace. For the same reason as above, it does not vanish and has a constant sign.

Fact 8. *An economy is irregular if and only if a (real or complex) left hand eigenvector $\boldsymbol{q}_i$ of $\boldsymbol{A}$ exists so that $\boldsymbol{q}_i^\top \boldsymbol{\ell} = 0$.*

Proof. If the vectors $\boldsymbol{\ell}, \boldsymbol{A}\boldsymbol{\ell}, \boldsymbol{A}^2\boldsymbol{\ell}, \dots, \boldsymbol{A}^{n-1}\boldsymbol{\ell}$ are dependent, the subspace $\mathcal{G} = \{\boldsymbol{y} : \boldsymbol{y}^\top\boldsymbol{\ell} = \boldsymbol{y}^\top\boldsymbol{A}\boldsymbol{\ell} = \boldsymbol{y}^\top\boldsymbol{A}^2\boldsymbol{\ell} = \boldsymbol{y}^\top\boldsymbol{A}^{n-1}\boldsymbol{\ell} = 0\}$ is not the null space. As $\mathcal{G}$ is invariant under $\boldsymbol{A}^\top$, it contains an eigenvector of $\boldsymbol{A}^\top$, i.e., a left hand eigenvector $\boldsymbol{q}_i$ of $\boldsymbol{A}$. Conversely, if $\boldsymbol{q}_i^\top\boldsymbol{\ell} = 0$ and $\boldsymbol{q}_i$ is a left hand eigenvector, all vectors $\boldsymbol{A}^k\boldsymbol{\ell}$ are orthogonal to $\boldsymbol{q}_i$, hence the n vectors $\boldsymbol{\ell}, \boldsymbol{A}\boldsymbol{\ell}, \boldsymbol{A}^2\boldsymbol{\ell}, \dots, \boldsymbol{A}^{n-1}\boldsymbol{\ell}$ are dependent. □

Fact 9. *If $\boldsymbol{A}$ has a (real or complex) eigenvalue with eigenspace of dimension greater than one, then every economy is irregular.*

Proof. Assume λ_i has two non-proportional left eigenvectors associated with it, call them $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. Given any $\boldsymbol{\ell}$, the vector $(\boldsymbol{q}_1^\top\boldsymbol{\ell})\boldsymbol{q}_2 - (\boldsymbol{q}_2^\top\boldsymbol{\ell})\boldsymbol{q}_1$ is a left eigenvector orthogonal to $\boldsymbol{\ell}$, therefore $(\boldsymbol{A}, \boldsymbol{\ell})$ is irregular. □

The literature related to Facts 8 and 9 is discussed in (Bidard and Salvadori, 1995, p. 389).

Fact 10. *Every $\boldsymbol{A}$-invariant subspace of $\mathcal{V} \subset \mathbb{R}^n$ contains an $\boldsymbol{A}$-invariant subspace which is either one- or two-dimensional, and an $\boldsymbol{A}$-invariant subspace whose dimension is by either one or two smaller than the dimension of $\mathcal{V}$.*

Proof. Whatever $\mathcal{V}$, we can always choose a basis of it and talk in terms of matrices. I.e., we can assume $\mathcal{V} = \mathbb{R}^m$ and $\boldsymbol{A}$ is a $m \times m$ matrix. The characteristic equation $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$ has at least one real or one pair of conjugate complex roots. A real eigenvector defines a one-dimensional $\boldsymbol{A}$-invariant real subspace. Now assume the eigenvectors are complex: call them $\boldsymbol{u} + i\boldsymbol{v}$, with eigenvalue $a + ib$, with $\boldsymbol{u}$, $\boldsymbol{v}$, a, and b real. Then the complex identity $\boldsymbol{A}(\boldsymbol{u} + \boldsymbol{v}i) = (\boldsymbol{u} + \boldsymbol{v}i)(a + bi)$ is equivalent to the identity between real partitioned matrices

$$\boldsymbol{A}\begin{bmatrix}\boldsymbol{u} & \boldsymbol{v}\end{bmatrix} = \begin{bmatrix}\boldsymbol{u} & \boldsymbol{v}\end{bmatrix}\begin{bmatrix} a & b \\ -b & a\end{bmatrix} \tag{20}$$

Since each column of the matrix on the righthand side is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, it follows that $\boldsymbol{u}$ and $\boldsymbol{v}$ span a 2-dimensional $\boldsymbol{A}$-invariant subspace.

$\mathbb{R}^m$ also has a one- or two-dimensional $\boldsymbol{A}^\top$-invariant subspace, call it $\mathcal{W}$, and the space orthogonal to $\mathcal{W}$ is $\boldsymbol{A}$-invariant and has dimension $m-1$ or $m-2$.

(But there is no guarantee that this orthogonal space has a basis consisting of eigenvectors.) □

Successive application of Fact 10 gives:

Fact 11. *There exists a complete chain of $\boldsymbol{A}$-invariant subspaces*

$$\{\boldsymbol{p}^*\} \subset \cdots \subset \mathcal{E}_r \subset \mathcal{E}_s \subset \mathcal{E}_t \subset \cdots \subset \mathbb{R}^n \tag{21}$$

such that the dimensions of consecutive subspaces differ by 1 or at most 2.

Fact 12. *Let $\mathcal{H}$ be the hyperplane orthogonal to $\boldsymbol{q}^*$. There exists a one-to-one correspondence between $\boldsymbol{A}$-invariant subspaces of $\mathbb{R}^n$ which contain some semipositive vector and $\boldsymbol{A}$-invariant subspaces of $\mathcal{H}$.*

Proof. The condition "containing some semipositive vector" may be replaced by "containing $\boldsymbol{p}^*$," because if an $\boldsymbol{A}$-invariant subspace contains a semipositive vector $\boldsymbol{\ell} \gneqq \boldsymbol{o}$, it must contain the end point $\boldsymbol{p}^*$ of the price curve starting at $\boldsymbol{\ell}$.

An invariant subspace $\mathcal{E}$ containing $\boldsymbol{p}^*$ can be decomposed as the direct sum $\mathcal{E} = \{\boldsymbol{p}^*\} \oplus \mathcal{S}$, where $\mathcal{S} = \mathcal{E} \cap \mathcal{H}$ and $\{\boldsymbol{p}^*\}$ is, of course, the space spanned by $\boldsymbol{p}^*$. Since both $\mathcal{H}$ and $\mathcal{E}$ are $\boldsymbol{A}$-invariant, so is $\mathcal{S}$. Therefore the assignment $\mathcal{E} \mapsto \mathcal{S} = \mathcal{E} \cap \mathcal{H}$ defines a one-to-one correspondence between $\boldsymbol{A}$-invariant subspaces $\mathcal{E}$ of $\mathbb{R}^n$ which contain some semipositive vector and all $\boldsymbol{A}$-subspaces $\mathcal{S} \subset \mathcal{H}$. The inverse mapping is $\mathcal{S} \mapsto \mathcal{E} = \{\boldsymbol{p}^*\} \oplus \mathcal{S}$. □

Identification of Price Curves and of Technical Data. If enough price vectors associated with some technique are known, then Fact 6 allows to identify the price space $\mathcal{P}$ and the degree s as $s = \dim \mathcal{P}$. If the profit rates associated with these prices are also known, then the following can be said about the associated techniques:

Fact 13. *(Compare Schefold, 1976). Let $(\boldsymbol{A}, \boldsymbol{\ell})$ be an economy of degree $s = \deg(\boldsymbol{A}, \boldsymbol{\ell})$ with price space $\mathcal{P}$. Consider a second economy $(\boldsymbol{B}, \boldsymbol{m})$. Then the following three properties are equivalent:*

(i) The virtual relative prices associated with $(\boldsymbol{A}, \boldsymbol{\ell})$ and $(\boldsymbol{B}, \boldsymbol{m})$ coincide for $s+1$ different profit rates.

(ii) The curves of virtual relative prices associated with $(\boldsymbol{A}, \boldsymbol{\ell})$ and $(\boldsymbol{B}, \boldsymbol{m})$ coincide for any profit rate.

(iii) $\boldsymbol{m} = \mu \boldsymbol{\ell}$ for some $\mu \neq 0$, and $\boldsymbol{B} = \boldsymbol{A} + \boldsymbol{\ell} \boldsymbol{d}^\top + \boldsymbol{C}$ where $\boldsymbol{d}^\top$ is any row vector and the matrix $\boldsymbol{C}$ satisfies $\boldsymbol{C} \boldsymbol{f} = \boldsymbol{o}$ for all $\boldsymbol{f} \in \mathcal{P}$.

Proof of (i)⇒(iii). The prices and profit rates which coincide will be called $\boldsymbol{p}_i$ and r_i $(i = 1, \ldots, s+1)$. First look at the case that all $r_i \neq -1$. Define $\lambda_i = (1+r_i)^{-1}$ and write $\boldsymbol{A}\boldsymbol{p}_i = \lambda_i(\boldsymbol{p}_i - w_i \boldsymbol{\ell})$ and $\boldsymbol{B}\boldsymbol{p}_i = \lambda_i(\boldsymbol{p}_i - v_i \boldsymbol{m})$ for $i = 1, \ldots, s+1$ and appropriate v_i and w_i. Due to Fact 6, there exists a relationship $\alpha_1 \boldsymbol{p}_1 + \cdots + \alpha_{s+1} \boldsymbol{p}_{s+1} = \boldsymbol{o}$ which is up to a factor the only linear combination which annuls these vectors. Premultiplication of this relationship by $\boldsymbol{A}$ and by $\boldsymbol{B}$ gives

$$\sum_{i=1}^{s+1} \alpha_i \lambda_i \boldsymbol{p}_i = \Big(\sum_{i=1}^{s+1} \alpha_i \lambda_i w_i \Big) \boldsymbol{\ell} = \Big(\sum_{i=1}^{s+1} \alpha_i \lambda_i v_i \Big) \boldsymbol{m}. \tag{22}$$

Since all λ_i are different, the leftmost sum cannot be zero. Therefore $\boldsymbol{m}$ is proportional to $\boldsymbol{\ell}$, say $\boldsymbol{m} = \mu \boldsymbol{\ell}$ with $\mu \neq 0$, and therefore $(\boldsymbol{B} - \boldsymbol{A})\boldsymbol{p}_i = \lambda_i(\mu v_i - w_i)\boldsymbol{\ell}$ for all i (i.e., one has $s+1$ such identities). Now look at the case that there is a j with

$r_j = -1$; then identity $\boldsymbol{m} = \mu\boldsymbol{\ell}$ with $\mu \neq 0$ follows from the equality of the jth relative prices, and $(\boldsymbol{B}-\boldsymbol{A})\boldsymbol{p}_i = \lambda_i(\mu v_i - w_i)\boldsymbol{\ell}$ holds for all $i \neq j$, i.e., still for the prices associated with s different profit rates.

Since by Fact 6 the prices associated with s (and a fortiori those with $s+1$) different profit rates span $\mathcal{P}$, it follows in either case that $(\boldsymbol{B}-\boldsymbol{A})\mathcal{P}$ is the line spanned by $\boldsymbol{\ell}$. In other words, a $\boldsymbol{d}^\top$ exists so that for all $\boldsymbol{x} \in \mathcal{P}$, $(\boldsymbol{B}-\boldsymbol{A})\boldsymbol{x} = \boldsymbol{\ell}\boldsymbol{d}^\top\boldsymbol{x}$. Therefore $\boldsymbol{B}-\boldsymbol{A} = \boldsymbol{\ell}\boldsymbol{d}^\top + \boldsymbol{C}$ where $\boldsymbol{C}\mathcal{P} = \{\boldsymbol{o}\}$. □

Proof of (iii)⇒(ii). Take any $\boldsymbol{p}$ with $(1+r)\boldsymbol{A}\boldsymbol{p} = \boldsymbol{p} - w\boldsymbol{\ell}$ for some w, and define $v = \big(w - (1+r)\boldsymbol{d}^\top\boldsymbol{p}\big)/\mu$. Then $\boldsymbol{p}$ satisfies $(1+r)\boldsymbol{B}\boldsymbol{p} = \boldsymbol{p} - v\boldsymbol{m}$. Therefore any relative price vector associated with $(\boldsymbol{A},\boldsymbol{\ell})$ is a relative price vector associated $(\boldsymbol{B},\boldsymbol{m})$, and vice versa. Hence the two curves coincide. □

Since (ii) clearly implies (i), this concludes the proof of Fact 13.

The correspondence between $(\boldsymbol{A},\boldsymbol{\ell})$ and $(\boldsymbol{B},\boldsymbol{m})$ is easily understood:

- The transformation $\boldsymbol{\ell} \mapsto \boldsymbol{m} = \mu\boldsymbol{\ell}$ reduces prices in terms of wage by a factor μ, but does not affect relative prices.
- If $(\boldsymbol{A},\boldsymbol{\ell})$ is irregular, the price curve lies in subspace $\mathcal{P}$ of dimension s and, even if the price-and-wage vectors were known (Schefold, 1976), the observation of the curve leaves $n(n-s)$ degrees of freedom on the input matrix, which is only determined up to a matrix $\boldsymbol{C}$ satisfying $\boldsymbol{C}\mathcal{P} = \{\boldsymbol{o}\}$.
- For the sake of simplicity, assume $s = n$, hence $\boldsymbol{C} = \boldsymbol{O}$, and $\boldsymbol{d} > \boldsymbol{o}$. Matrix $\boldsymbol{B} = \boldsymbol{A} + \boldsymbol{d}^\top\boldsymbol{\ell}$ can then be interpreted as a "socio-technical" matrix, obtained by incorporating a minimal wage basket $\boldsymbol{d}^\top$ into the input coefficient matrix. At any profit rate, the nominal wages paid by the two techniques $(\boldsymbol{A},\boldsymbol{\ell})$ and $(\boldsymbol{B},\boldsymbol{m})$ will then differ by the value of basket $\boldsymbol{d}^\top$ and, since this is the only difference, the relative prices are identical.

Intersection of Price Curves.

Fact 14. *If the normalized price curves starting in* $\mathbf{l}$ *and* $\mathbf{m}$ *intersect in* $\mathbf{p}$*, then* $\mathbf{l}$*,* $\mathbf{m}$*, and* $\mathbf{p}$ *are collinear.*

Proof. This is a special case of Fact 15 b for $t = -1$. □

Fact 15. *Assume the economies* $(\boldsymbol{A},\boldsymbol{\ell})$ *and* $(\boldsymbol{A},\boldsymbol{m})$ *have the same input matrix, but their labor vectors are not proportional to each other. If their virtual relative price curves intersect at a point* $\boldsymbol{p}$ *which is not an eigenvector of* $\boldsymbol{A}$*, then the following holds:*

(a) The profit rates which $(\boldsymbol{A},\boldsymbol{\ell})$ *and* $(\boldsymbol{A},\boldsymbol{m})$ *associate with* $\boldsymbol{p}$ *are different.*

(b) Given any t *such that* $1/(1+t)$ *is not an eigenvalue of* $\boldsymbol{A}$*. Let* $\mathbf{p}_{\boldsymbol{\ell}}$ *and* $\mathbf{p}_{\boldsymbol{m}}$*, respectively, be the price vectors in economy* $(\boldsymbol{A},\boldsymbol{\ell})$ *and* $(\boldsymbol{A},\boldsymbol{m})$ *for profit rate* t*. Then* $\mathbf{p}_{\boldsymbol{\ell}}$*,* $\mathbf{p}_{\boldsymbol{m}}$*, and* $\boldsymbol{p}$ *are linearly dependent, i.e., the normalized prices are collinear.*

(c) If the normalized virtual price curves of $(\boldsymbol{A},\boldsymbol{\ell})$ *and* $(\boldsymbol{A},\boldsymbol{m})$ *are not both straight lines, then* $\boldsymbol{p}$ *is their only intersection point which is not an eigenvector of* $\boldsymbol{A}$*.*

Proof. The vector $\boldsymbol{p} \neq \boldsymbol{o}$ is an intersection point of the two virtual relative price curves if and only if

$$\boldsymbol{p} = (1+r)\boldsymbol{A}\boldsymbol{p} + w\boldsymbol{\ell} \tag{23}$$

$$\boldsymbol{p} = (1+s)\boldsymbol{A}\boldsymbol{p} + v\boldsymbol{m} \tag{24}$$

for some r, s, w and v. Both wages w and v are nonzero because $\boldsymbol{p}$ is not an eigenvector. If $r = s$ then it follows $w\boldsymbol{\ell} = v\boldsymbol{m}$, which was ruled out by assumption. This proves part (a).

Now assume $\boldsymbol{p_\ell}$ is the price vector for profit rate t under technology $(\boldsymbol{A}, \boldsymbol{\ell})$, and $\boldsymbol{p_m}$ is price vector for the same profit rate t under technology $(\boldsymbol{A}, \boldsymbol{m})$. From the assumption that $1/(1+t)$ is not an eigenvalue of $\boldsymbol{A}$ follows that the wages associated with $\boldsymbol{p_\ell}$ and $\boldsymbol{p_m}$ are both nonzero. Then we can assume, without loss of generality, that these wages are again w and v; if they are not, a simple rescaling of $\boldsymbol{p_\ell}$ and $\boldsymbol{p_m}$ will achieve this. In other words, from now on $\boldsymbol{p_\ell}$ and $\boldsymbol{p_m}$ are such that

$$\boldsymbol{p_\ell} = (1+t)\boldsymbol{A}\boldsymbol{p_\ell} + w\boldsymbol{\ell} \tag{25}$$

$$\boldsymbol{p_m} = (1+t)\boldsymbol{A}\boldsymbol{p_m} + v\boldsymbol{m} \tag{26}$$

Subtract (23) from (25) and rearrange to get

$$\big(\boldsymbol{I} - (1+t)\boldsymbol{A}\big)(\boldsymbol{p_\ell} - \boldsymbol{p}) = (r-t)\boldsymbol{A}\boldsymbol{p}, \tag{27}$$

and since $1/(1+t)$ is not an eigenvalue of $\boldsymbol{A}$, one can solve

$$\boldsymbol{p_\ell} - \boldsymbol{p} = (r-t)\big(\boldsymbol{I} - (1+t)\boldsymbol{A}\big)^{-1}\boldsymbol{A}\boldsymbol{p}. \tag{28}$$

If one subtracts (24) from (26), one gets by the same manipulations

$$\boldsymbol{p_m} - \boldsymbol{p} = (s-t)\big(\boldsymbol{I} - (1+t)\boldsymbol{A}\big)^{-1}\boldsymbol{A}\boldsymbol{p}. \tag{29}$$

(28) and (29) say that $\boldsymbol{p_\ell} - \boldsymbol{p}$ is a scalar multiple of $\boldsymbol{p_m} - \boldsymbol{p}$, therefore the three vectors are linearly dependent.

To show uniqueness of $\boldsymbol{p}$, i.e., to prove (c), let us work here with the normalized versions of the price curves. Applied to the normalized vectors, part (b) means that $\boldsymbol{p}$ lies on the straight line through $\boldsymbol{\ell}$ and $\boldsymbol{m}$. Since the curves themselves are not straight, there must be a profit rate s such that $1/(1+s)$ is not an eigenvalue and at least one of the prices for profit rate s, which we call again $\boldsymbol{p_\ell}$ and $\boldsymbol{p_m}$, are not on this straight line—hence the line through $\boldsymbol{p_\ell}$ and $\boldsymbol{p_m}$ does not coincide with the line through $\boldsymbol{\ell}$ and $\boldsymbol{m}$. Therefore, by part (b) again, $\boldsymbol{p}$ lies on the intersection of these two different lines, which makes it unique. This completes the proof of Fact 15. ☐

Next we give a method to construct families of price curves which have no intersection points at all.

Fact 16. *All normalized price curves which start at the edge of some convex set $\mathcal{R}$ and stay entirely inside $\mathcal{R}$ have no intersections other than their common endpoint.*

Proof. Take two trajectory segments, starting at points $\boldsymbol{\ell}$ and $\boldsymbol{m}$, respectively, on the edge of the region $\mathcal{R}$, and assume they intersect at point $\boldsymbol{p}$, which is not their common endpoint. At $\boldsymbol{p}$, the first trajectory has profit rate r, and the second profit rate s. By Fact 15 (b), $\boldsymbol{p}$ lies on the straight line going through $\boldsymbol{\ell}$ and $\boldsymbol{m}$, and since it must be inside $\mathcal{R}$, it must be between $\boldsymbol{\ell}$ and $\boldsymbol{m}$. This contradicts equations (28) and (29) which imply in our case that

$$\boldsymbol{p} = \frac{s+1}{s-r}w\boldsymbol{\ell} - \frac{r+1}{s-r}v\boldsymbol{m}. \tag{30}$$

Since the coefficients of $\boldsymbol{\ell}$ and $\boldsymbol{m}$ have opposite signs, $\boldsymbol{p}$ cannot lie between $\boldsymbol{\ell}$ and $\boldsymbol{m}$. ☐

Examples of convex sets which the price curves never leave again are any Hilbert disk around $\mathbf{p}^*$, and the unit simplex itself.

Tangents. If organic composition is not uniform, the unnormalized price curve belongs to a cone of dimension 2, whatever the number of commodities. At any point $\boldsymbol{p}(r)$, this cone admits a tangent plane (of dimension 2), which is spanned by vector $\boldsymbol{p}(r)$ and another vector which we now describe.

- at any profit rate r, $-1 \leq r < r^*$, vector $(\boldsymbol{I} - (1+r)\boldsymbol{A})^{-1}\boldsymbol{A}\boldsymbol{p}$ is tangent. (This results from the expression of the derivative of vector $\boldsymbol{p}(r)$ with respect to r.) In particular, vector $\boldsymbol{A}\boldsymbol{\ell}$ is tangent at the beginning of the curve.
- At r^*, a tangent vector also exists. Since $\boldsymbol{I} - (1+r^*)\boldsymbol{A}$ is singular, its description first requires to isolate this singularity (Fact 17), before the tangent vector can be characterized in Fact 18.

Fact 17. *Let $\mathcal{H}$ be the hyperplane defined by $\mathcal{H} = \{\boldsymbol{x} :\ {\boldsymbol{q}^*}^\top \boldsymbol{x} = 0\}$. For any non-eigenvalue λ, let $\boldsymbol{G}_\lambda$ be the matrix representative of the endomorphism g_λ defined by the two conditions: $g_\lambda(\boldsymbol{p}^*) = \boldsymbol{o}$ and g_λ coincides on $\mathcal{H}$ with $(\lambda\boldsymbol{I} - \boldsymbol{A})^{-1}$. The following matrix identity holds:*

$$(\lambda\boldsymbol{I} - \boldsymbol{A})^{-1} = (\lambda - \lambda^*)^{-1}\boldsymbol{p}^*{\boldsymbol{q}^*}^\top/({\boldsymbol{q}^*}^\top\boldsymbol{p}^*) + \boldsymbol{G}_\lambda \tag{31}$$

Proof. Since $\boldsymbol{p}^* \notin \mathcal{H}$, the whole space is decomposed as a direct sum $\{\boldsymbol{p}^*\} \oplus \mathcal{H}$. It is immediately checked that the two members of the above identity coincide on $\{\boldsymbol{p}^*\}$ and on $\mathcal{H}$. □

Fact 18. *Let $\boldsymbol{q}^*$ and $\boldsymbol{p}^*$ be normalized by setting ${\boldsymbol{q}^*}^\top\boldsymbol{\ell} = {\boldsymbol{q}^*}^\top\boldsymbol{p}^* = 1$. There exists a unique vector $\boldsymbol{c}$, solution to*

$$\begin{aligned} (\lambda^*\boldsymbol{I} - \boldsymbol{A})\boldsymbol{c} &= \boldsymbol{\ell} - \boldsymbol{p}^* && (32)\\ {\boldsymbol{q}^*}^\top\boldsymbol{c} &= 0 && (33)\end{aligned}$$

This vector lies in the tangent plane. It is the tangent to the normalized price curve in the normalization by the standard commodity.

Proof. $\mathcal{H}$ is, as always, the hyperplane orthogonal to $\boldsymbol{q}^*$. The endomorphism represented by $\lambda^*\boldsymbol{I} - \boldsymbol{A}$ sends $\mathcal{H}$ into $\mathcal{H}$ and, since $\boldsymbol{p}^* \notin \mathcal{H}$, its restriction to $\mathcal{H}$ is an isomorphism of $\mathcal{H}$. Let its inverse be denoted by $f : \mathcal{H} \to \mathcal{H}$. Vector $\boldsymbol{\ell} - \boldsymbol{p}^* \in \mathcal{H}$, and the above equations define $\boldsymbol{c}$ as $\boldsymbol{c} = f(\boldsymbol{\ell} - \boldsymbol{p}^*)$. When identity (31) is applied to vector $\boldsymbol{\ell}$, it appears that the normalized price vector $\boldsymbol{p}(\lambda)$ is equal to $\boldsymbol{c}_\lambda = \boldsymbol{G}_\lambda(\boldsymbol{\ell})$, whose limit as $\lambda \to \lambda^*$ is $\boldsymbol{c}$.

Here is a more explicit computation of the derivatives of the price curves in the standard normalization: Differentiation of (16) gives

$$\big(\boldsymbol{I} + (1+r)\boldsymbol{A}\big)\dot{\mathbf{p}}(r) = \boldsymbol{A}\big(\mathbf{p}(r) - \mathbf{p}^*\big) - \frac{1}{1+r^*}(\mathbf{l} - \mathbf{p}^*). \tag{34}$$

With the help of (5) this gives, for the case $-1 \leq r < r^*$

$$\dot{\mathbf{p}}(r) = \Big(\big(\boldsymbol{I} - (1+r)\boldsymbol{A}\big)^{-1}\boldsymbol{A} - \frac{1}{r^* - r}\boldsymbol{I}\Big)\big(\mathbf{p}(r) - \mathbf{p}^*\big). \tag{35}$$

For $r = -1$ this simplifies to

$$\dot{\mathbf{p}}(-1) = \Big(\boldsymbol{A} - \frac{1}{1+r^*}\boldsymbol{I}\Big)(\mathbf{l} - \mathbf{p}^*). \tag{36}$$

For $r = r^*$, (34) gives

$$\big(\boldsymbol{I} - (1+r^*)\boldsymbol{A}\big)\dot{\mathbf{p}}(r^*) = -\frac{1}{1+r^*}(\mathbf{l} - \mathbf{p}^*). \tag{37}$$

□

Since $\boldsymbol{q}^{*\top}\dot{\mathbf{p}}(r) = 0$, the following equation is also valid, for any vector $\boldsymbol{t}$:

$$\Big(\boldsymbol{I} - \boldsymbol{t}\boldsymbol{p}^{*\top} - (1+r^*)\boldsymbol{A}\Big)\dot{\mathbf{p}}(r) = -\frac{1}{1+r^*}(\mathbf{l} - \mathbf{p}^*), \tag{38}$$

If $\boldsymbol{t}$ is chosen as in Fact 3, one can take the inverse to get

$$\dot{\mathbf{p}}(r^*) = -\frac{1}{1+r^*}\Big(\boldsymbol{I} - \boldsymbol{t}\boldsymbol{p}^{*\top} - (1+r^*)\boldsymbol{A}\Big)^{-1}(\mathbf{l} - \mathbf{p}^*). \tag{39}$$

Decomposition of Price Curves. In Fact 19 we assume that $\boldsymbol{A}$ is diagonalizable, i.e., its eigenvectors span all of $\mathbb{R}^n$. This is the generic situation; those $\boldsymbol{A}$ which are not diagonalizable are exceptional. If $\boldsymbol{A}$ is diagonalizable, then all price curves of economies with higher degree of regularity (higher dimensions) can be constructed as the sums of price curves located in one- or two-dimensional $\boldsymbol{A}$-invariant subspaces:

Fact 19. *Assume $\boldsymbol{A}$ is diagonalizable and has m different real and q different pairs of conjugate complex eigenvalues. (Each eigenvalue is counted here only once, even if its associated eigenspace has higher dimension). In the standard normalization, with the origin of the coordinate system moved to $\mathbf{p}^*$, any price curve can be decomposed as the sum of $m' + q'$ price curves, where $m' \le m$ and $q' \le q$, with m' price curves confined to one-dimensional lines and q' confined to two-dimensional planes.*

Proof. To prove this, we will construct $m' + q'$ idempotent matrices $\boldsymbol{P}_i$ ($i = 1, \dots, m' + q'$), where $m' \le m$ and $q' \le q$. They are "idempotent" in the sense that $\boldsymbol{P}_i\boldsymbol{P}_i = \boldsymbol{P}_i$, but $\boldsymbol{P}_i$ are not necessarily symmetric. In other words, they are projection matrices, but not necessarily orthogonal projection matrices. They have the following properties. For $1 \le i \le m'$, the $\boldsymbol{P}_i$ project on 1-dimensional subspaces, and for $m' + 1 \le i \le m' + q'$, on 2-dimensional subspaces of $\mathbb{R}^n$. Furthermore, each $\boldsymbol{P}_i$ satisfies

$$\boldsymbol{P}_i\boldsymbol{A} = \boldsymbol{A}\boldsymbol{P}_i \tag{40}$$

and finally, the following holds:

$$\mathbf{l} - \mathbf{p}^* = \sum_{i=1}^{m'+q'} \boldsymbol{P}_i(\mathbf{l} - \mathbf{p}^*) \tag{41}$$

Remember that $\mathbf{l}$ is the normalized vector that is a scalar multiple of $\boldsymbol{\ell}$, and we are using the standard normalization.

Before constructing the $\boldsymbol{P}_i$, we will verify that the application of projection matrices $\boldsymbol{P}_i$ with the above properties to the price curve starting at $\mathbf{l}$ gives the decomposition specified in Fact 19. Premultiplication of equation (16) by $\boldsymbol{P}_i$ gives, using (40):

$$\big(\boldsymbol{I} - (1+r)\boldsymbol{A}\big)\boldsymbol{P}_i\big(\mathbf{p}(r) - \mathbf{p}^*\big) = \boldsymbol{P}_i(\mathbf{l} - \mathbf{p}^*)\Big(1 - \frac{1+r}{1+r^*}\Big) \tag{42}$$

Now call $\mathbf{x}_i(r)$ the normalized price vector for profit rate r with normalized labor vector $\boldsymbol{P}_i\mathbf{l} + (\boldsymbol{I} - \boldsymbol{P}_i)\mathbf{p}^*$. Its defining equation is

$$\big(\boldsymbol{I} - (1+r)\boldsymbol{A}\big)\big(\mathbf{x}_i(r) - \mathbf{p}^*\big) = \Big(\boldsymbol{P}_i\mathbf{l} + (\boldsymbol{I} - \boldsymbol{P}_i)\mathbf{p}^* - \mathbf{p}^*\Big)\Big(1 - \frac{1+r}{1+r^*}\Big) \tag{43}$$

Since the righthand sides of (42) and (43) are equal, it follows $\mathbf{x}_i(r) - \mathbf{p}^* = \boldsymbol{P}_i(\mathbf{p}(r) - \mathbf{p}^*)$, i.e., $\mathbf{x}_i(r) - \mathbf{p}^*$ remains in the subspace on which $\boldsymbol{P}_i$ projects.

Now sum (42) over all i to get, using (41):

$$\big(\boldsymbol{I} - (1+r)\boldsymbol{A}\big)\sum_{i=1}^{m+q}\big(\mathbf{x}_i(r) - \mathbf{p}^*\big) = (\mathbf{l} - \mathbf{p}^*)\Big(1 - \frac{1+r}{1+r^*}\Big) \tag{44}$$

From this follows

$$\sum_{i=1}^{m+q}(\mathbf{x}_i(r) - \mathbf{p}^*) = \mathbf{p}(r) - \mathbf{p}^*, \tag{45}$$

i.e., the $\mathbf{x}_i(r) - \mathbf{p}^*$ form indeed a decomposition of $\mathbf{p}(r) - \mathbf{p}^*$.

To conclude the proof, we have to construct the $\boldsymbol{P}_i$ with the required properties. Since $\boldsymbol{A}$ is diagonalizable, one can find n generally complex linear independent left eigenvectors $\boldsymbol{q}_1, \dots, \boldsymbol{q}_n$, associated with (not necessarily distinct) eigenvalues $\lambda_1, \dots, \lambda_n$. In what follows, the subscripts j and k go from 1 to n, while the subscript i goes from 1 to $m' + q'$. λ_j or λ_k refers to a different indexing than λ_i: in the former, the same eigenvalue can appear more than once, in the latter it cannot; and if the eigenvalue is complex, then the latter indexing refers to pairs of conjugate complex eigenvalues, while the former indexing lists each eigenvalue of this pair separately.

Our complex basis of the n-dimensional complex space can be constructed in such a way that the following holds:

- If λ_k is real, then $\boldsymbol{q}_k$ is also real.
- If λ_k is complex, then both $\boldsymbol{q}_k$ and its conjugate complex $\overline{\boldsymbol{q}_k}$ is in this basis.
- If $\lambda_k = \lambda_j$ and $\boldsymbol{q}_k^\top(\mathbf{l} - \mathbf{p}^*) \neq 0$, then $\boldsymbol{q}_j^\top(\mathbf{l} - \mathbf{p}^*) = 0$. I.e., for each eigenvalue there is at most one eigenvector in this basis which satisfies $\boldsymbol{q}_k^\top(\mathbf{l} - \mathbf{p}^*) \neq 0$. This can be easily achieved; simply replace $\boldsymbol{q}_j$ by $\boldsymbol{q}_j - \frac{\boldsymbol{q}_j^\top(\mathbf{l}-\mathbf{p}^*)}{\boldsymbol{q}_k^\top(\mathbf{l}-\mathbf{p}^*)}\boldsymbol{q}_k$.

If $\boldsymbol{L}$ is the matrix which has these left eigenvectors as rows, then the matrix inverse $\boldsymbol{R} = \boldsymbol{L}^{-1}$ has right eigenvectors as its columns. The column vectors of $\boldsymbol{R}$ will be called $\boldsymbol{p}_1, \dots, \boldsymbol{p}_n$. Since $\boldsymbol{R}\boldsymbol{L}(\mathbf{l} - \mathbf{p}^*) = \mathbf{l} - \mathbf{p}^*$ and $\boldsymbol{R}\boldsymbol{L} = \sum_k \boldsymbol{p}_k\boldsymbol{q}_k^\top$, it follows that also

$$\sum_{k:\, \boldsymbol{q}_k^\top(\mathbf{l}-\mathbf{p}^*)\neq 0} \boldsymbol{p}_k\boldsymbol{q}_k^\top(\mathbf{l} - \mathbf{p}^*) = \mathbf{l} - \mathbf{p}^* \tag{46}$$

Now let us go over to our subscripts i. For every real eigenvalue λ_i for which there is a left eigenvector $\boldsymbol{q}_k$ such that $\boldsymbol{q}_k^\top(\mathbf{l} - \mathbf{p}^*) \neq 0$ (there can be at most one k for every λ_i), we define $\boldsymbol{P}_i = \boldsymbol{p}_k\boldsymbol{q}_k^\top$. For every pair of conjugate complex eigenvectors $(\lambda_i, \overline{\lambda_i}$ for which there is an associated eigenvector $\boldsymbol{q}_k$ such that $\boldsymbol{q}_k^\top(\mathbf{l} - \mathbf{p}^*) \neq 0$ we define $\boldsymbol{P}_i = \boldsymbol{p}_k\boldsymbol{q}_k^\top + \overline{\boldsymbol{p}_k\boldsymbol{q}_k}^\top$. If $\boldsymbol{q}_k = \boldsymbol{s} + i\boldsymbol{t}$ and $\boldsymbol{p}_k = \boldsymbol{u} + i\boldsymbol{v}$, then one can also write it as $\boldsymbol{P}_i = 2\begin{bmatrix}\boldsymbol{u} & \boldsymbol{v}\end{bmatrix}\begin{bmatrix}\boldsymbol{s}^\top \\ \boldsymbol{t}^\top\end{bmatrix}$. These projection matrices have all the required properties. $\square$

Degrees of Regularity 2 and 3. Due to the decomposition discussed in Fact 19, a characterization of price curves in one- and two-dimensional subspaces of the normalization hyperplane (which have degrees of regularity 2 and 3 respectively) will also give information about higher-dimensional price curves. Fact 20 gives a complete characterization of price curves of degrees 2 or 3, and a partial characterization of price curves of higher degrees. The formulas for degrees 2 and 3 look identical, but in the case of degree 2 they are to be interpreted in the field of real numbers, and in the case of degree 3 in the complex numbers.

Fact 20. *Let $\boldsymbol{q}_j$ be a (possibly complex) left eigenvector of $\boldsymbol{A}$ with eigenvalue λ_j, and let $\mathbf{p}(r)$ be the normalized price vector associated with the normalized labor vector $\mathbf{l}$ (using the standard normalization). Then there is a function $k_j(r)$, which depends on $\boldsymbol{A}$ only through r^* and λ_j, with the following defining property:*

$$\boldsymbol{q}_j^\top \mathbf{p}(r) = k_j(r)\boldsymbol{q}_j^\top \mathbf{l} \tag{47}$$

The modulus of this function, $|k_j(r)|$, declines monotonically as r rises from -1 to r^*. If $k_j(r)$ is complex, then the derivative of its argument $\arg k_j(r)$ does not change sign, i.e., $k_j(r)$ moves monotonically either clockwise or counterclockwise across the complex plane.

Proof. The decline of the modulus $|k_j(r)|$ was first established in (Bidard and Steedman, 1996, Theorem 1). To prove Fact 20, write $\lambda_j = a + bi$ and $\lambda^* = (1 + r^*)^{-1}$ for the eigenvalues:

$$\boldsymbol{q}_j^\top \mathbf{p}(r) = \frac{\boldsymbol{q}_j^\top \boldsymbol{p}(r)}{\boldsymbol{q}^{*\top} \boldsymbol{p}(r)} = \frac{\boldsymbol{q}_j^\top (\boldsymbol{I} - (1+r)\boldsymbol{A})^{-1}\boldsymbol{\ell}}{\boldsymbol{q}^{*\top} (\boldsymbol{I} - (1+r)\boldsymbol{A})^{-1}\boldsymbol{\ell}} = \frac{(1-(1+r)\lambda_j)^{-1}}{(1-(1+r)\lambda^*)^{-1}} \frac{\boldsymbol{q}_j^\top \boldsymbol{\ell}}{\boldsymbol{q}^{*\top}\boldsymbol{\ell}} = \frac{1-(1+r)\lambda^*}{1-(1+r)\lambda_j} \boldsymbol{q}_j^\top \mathbf{l} \tag{48}$$

Therefore

$$k_j(r) = \frac{1-(1+r)\lambda^*}{1-(1+r)\lambda_j} = \frac{\lambda - \lambda^*}{\lambda - \lambda_j} \qquad \text{where } \lambda = (1+r)^{-1} \tag{49}$$

We have $a < \lambda^* < \lambda$, because λ^* is the dominant eigenvalue. Now take the logarithmic derivative with respect to λ:

$$\frac{d}{d\lambda} \log \frac{\lambda - \lambda^*}{\lambda - \lambda_j} = \frac{1}{\lambda - \lambda^*} - \frac{1}{\lambda - \lambda_j} = \frac{\lambda^* - \lambda_j}{(\lambda - \lambda^*)(\lambda - \lambda_j)} = \frac{(\lambda^* - \lambda_j)(\lambda - \bar{\lambda}_j)}{(\lambda - \lambda^*)|\lambda - \lambda_j|^2} \tag{50}$$

The denominator is always positive, and the numerator has real part $(\lambda - a)(\lambda^* - a) + b^2 > 0$ and imaginary part $b(\lambda^* - \lambda)$ (which has the same sign as b). Since, for a complex number x, $\log x = \log |x| + i \arg x$, the statement follows. □

Euclidean Angle Between Price Vectors.

Fact 21. *If the input matrix is normal, the Euclidean angle between $\boldsymbol{p}(r)$ and $\boldsymbol{p}^*$ is a decreasing function of the profit rate. (Section 4.2).*

This property is established in (Bidard and Steedman, 1996). If the matrix is normal, it admits orthogonal critical lines, and the result follows from Pythagoras.

Hilbert Circles. The lemmas collected in Fact 22 will be used in the proofs below:

Fact 22. *(a) The Hilbert distance only depends on relative prices.*
(b) $d(\boldsymbol{x}, \alpha\boldsymbol{x} + \beta\boldsymbol{y}) \leq d(\boldsymbol{x}, \boldsymbol{y})$ for any $\alpha, \beta \geq 0$.
(c) $d(\boldsymbol{H}\boldsymbol{x}, \boldsymbol{H}\boldsymbol{y}) \leq d(\boldsymbol{x}, \boldsymbol{y})$ for any positive matrix $\boldsymbol{H}$.

Now define the matrix

$$\boldsymbol{H}(r,s) = \big(\boldsymbol{I}-(1+s)\boldsymbol{A}\big)^{-1}\big(\boldsymbol{I}-(1+r)\boldsymbol{A}\big) = \Big(\boldsymbol{I}-(s-r)\big(\boldsymbol{I}-(1+r)\boldsymbol{A}\big)^{-1}\Big)^{-1}. \quad (51)$$

Then for $-1 \le r < s < t \le r^*$, $\boldsymbol{H} \gg \boldsymbol{O}$ *and*

(d) $\boldsymbol{H}(s,r)\boldsymbol{p}(r) = \boldsymbol{p}(s)$ *and* $\boldsymbol{H}(t,r)\boldsymbol{p}(r) = \boldsymbol{p}(t)$.

(e) $(t-r)\boldsymbol{H}(t,r) = \boldsymbol{H}(s,r)\Big((s-r)\boldsymbol{I} + (t-s)\boldsymbol{H}(t,r)\Big)$.

(f) $\boldsymbol{H}(s,r)\boldsymbol{p}^* \propto \boldsymbol{p}^*$.

Fact 23. *The successive Hilbert circles centered at* $\boldsymbol{p}^*$ *are attractive: If* $r < s < r^*$ *then* $d(\boldsymbol{p}(s),\boldsymbol{p}^*) \le d(\boldsymbol{p}(r),\boldsymbol{p}^*)$.

Proof. This is a special case of Fact 25, but here is a direct proof: Using properties (d), (f), (a), and (c) in Fact 22 successively, one obtains $d(\boldsymbol{p}(s),\boldsymbol{p}^*) = d(\boldsymbol{H}(s,r)\boldsymbol{p}(r),\boldsymbol{H}(s,r)\boldsymbol{p}^*) \le d(\boldsymbol{p}(r),\boldsymbol{p}^*)$. □

Fact 24. *Let* $r < t \le r^*$. *No point on the straight line segment* $\overline{\boldsymbol{p}(r)\boldsymbol{p}(t)}$ *has a smaller Hilbert distance from* $\boldsymbol{p}^*$ *than* $\boldsymbol{p}(t)$.

Proof. Choose any s with $r < s < t$. Using properties (d), (a), (f), (e), (c), and (d) successively, one obtains

$$\begin{aligned}
d(\boldsymbol{p}(t),\boldsymbol{p}^*) &= d(\boldsymbol{H}(t,r)\boldsymbol{p}(r),\boldsymbol{H}(s,r)\boldsymbol{p}^*) && (52)\\
&= d(\boldsymbol{H}(s,r)\Big(\frac{s-r}{t-r}\boldsymbol{I} + \frac{t-s}{t-r}\boldsymbol{H}(t,r)\Big)\boldsymbol{p}(r),\boldsymbol{H}(s,r)\boldsymbol{p}^*) && (53)\\
&\le d(\frac{s-r}{t-r}\boldsymbol{p}(r) + \frac{t-s}{t-r}\boldsymbol{H}(t,r)\boldsymbol{p}(r),\boldsymbol{p}^*) && (54)\\
&= d(\frac{s-r}{t-r}\boldsymbol{p}(r) + \frac{t-s}{t-r}\boldsymbol{p}(t),\boldsymbol{p}^*) && (55)
\end{aligned}$$

By an appropriate choice of s, the first argument in the last expression can be made to represent any point on the straight line segment $\overline{\boldsymbol{p}(r)\boldsymbol{p}(t)}$. This segment is therefore outside or at most at the edge of the Hilbert circle passing through $\boldsymbol{p}(t)$. □

Fact 25. *The successive Hilbert circles centered at* $\boldsymbol{p}(t)$ *are attractive: whenever* $r < s < t \le r^*$, *then* $d(\boldsymbol{p}(s),\boldsymbol{p}(t)) \le d(\boldsymbol{p}(r),\boldsymbol{p}(t))$.

Proof. Using properties (d), (e), (a), (c), (b), and (d) in Fact 22 successively, one obtains:

$$\begin{aligned}
d(\boldsymbol{p}(s),\boldsymbol{p}(t)) &= d(\boldsymbol{H}(s,r)\boldsymbol{p}(r),\boldsymbol{H}(t,r)\boldsymbol{p}(r)) && (56)\\
&= d\Big(\boldsymbol{H}(s,r)\boldsymbol{p}(r),\boldsymbol{H}(s,r)((s-r)\boldsymbol{I}+(t-s)\boldsymbol{H}(t,r))\boldsymbol{p}(r)\Big) && (57)\\
&\le d\Big(\boldsymbol{p}(r),(s-r)\boldsymbol{p}(r)+(t-s)\boldsymbol{H}(t,r)\boldsymbol{p}(r)\Big) && (58)\\
&\le d\Big(\boldsymbol{p}(r),\boldsymbol{H}(t,r)\boldsymbol{p}(r)\Big) = d(\boldsymbol{p}(r),\boldsymbol{p}(t)). && (59)
\end{aligned}$$

□

The symmetric property $d(\boldsymbol{p}(r),\boldsymbol{p}(s)) < d(\boldsymbol{p}(r),\boldsymbol{p}(t))$ holds if $n=3$ but a counterexample for $n=4$ is, according to (Bidard and Krause, 1996):

$$\boldsymbol{A} = \frac{1}{540}\begin{bmatrix} 423 & 360 & 0 & 0\\ 83 & 0 & 10 & 0\\ 0 & 2629 & 0 & 1332\\ 0 & 18 & 0 & 477 \end{bmatrix} \qquad \boldsymbol{\ell} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad \boldsymbol{p}(0) = \begin{bmatrix}20\\5\\50\\10\end{bmatrix} \qquad \boldsymbol{p}^* = \begin{bmatrix}5\\1\\8\\1\end{bmatrix} \quad (60)$$

$r^* = 1/11$; $d(\boldsymbol{\ell}, \boldsymbol{p}(0)) = \log 10 > \log 8 = d(\boldsymbol{\ell}, \boldsymbol{p}^*)$. However computer simulations (made by L. Pierre) show that such counterexamples are rare.

Fact 26. *In a neighborhood of any positive vector $\boldsymbol{x} \gg \boldsymbol{o}$, the ratio between the Euclidean angle $(\boldsymbol{x}, \boldsymbol{y})$ and the Hilbert distance $d(\boldsymbol{x}, \boldsymbol{y})$ is lower and upper bounded by positive scalars.*

Proof. Let us write

$$y_i = x_i + \varepsilon_i x_i \tag{61}$$

with ε_i small. The cosine of the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$ is

$$\cos(\boldsymbol{x}, \boldsymbol{y}) = \Bigl(n + \sum_i \varepsilon_i\Bigr)\Bigl(n \sum_i (1 + \varepsilon_i)^2\Bigr)^{-1/2} \tag{62}$$

Using a Taylor expansion, one obtains

$$\begin{aligned} \cos(\boldsymbol{x}, \boldsymbol{y}) &= 1 - \frac{1}{2}\left(\frac{\sum_i \varepsilon_i^2}{n} - \Bigl(\frac{\sum_i \varepsilon_i}{n}\Bigr)^2\right) + o(\varepsilon^2) && (63) \\ &= 1 - \frac{1}{2}\operatorname{var}(\varepsilon) + o(\varepsilon^2). && (64) \end{aligned}$$

Hence the Euclidean angle between $\boldsymbol{x}$ and $\boldsymbol{y}$ is $(\boldsymbol{x}, \boldsymbol{y}) = \sigma(\varepsilon) + o(\varepsilon)$ where σ is the standard deviation of the ε_i.

A similar expansion of the Hilbert distance shows that

$$d(\boldsymbol{x}, \boldsymbol{y}) = \log\Bigl(\max_i(1 + \varepsilon_i)\Big/\min_i(1 + \varepsilon_i)\Bigr) = e(\varepsilon) + o(\varepsilon) \tag{65}$$

where $e(\varepsilon) = \max_i \varepsilon_i - \min_i \varepsilon_i$ is the range of the ε_i. The result then follows from the inequality

$$2 \le e/\sigma \le \sqrt{2n}. \tag{66}$$

□

By means of compactness arguments, this local relationship can be extended to any compact set in the interior of the simplex.

Fact 27. *Let $\boldsymbol{p}^*$ be normalized by setting ${\boldsymbol{q}^*}^\top \boldsymbol{p}^* = {\boldsymbol{q}^*}^\top \boldsymbol{\ell}$. Let $\boldsymbol{c}$ be the vector defined in Fact 18. The first derivative d_1 of $d(\boldsymbol{p}(r), \boldsymbol{p}^*)$ with respect to $\lambda = 1/(1+r)$ at $\lambda = \lambda^*$ is given by*

$$d_1 = c_i/p_i^* - c_j/p_j^* \tag{67}$$

where $c_i/p_i^* = \max_k c_k/p_k^*$ and $c_j/p_j^* = \min_k c_k/p_k^*$.

Proof. Let $\boldsymbol{p}(r) = (\lambda\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{\ell}$, with a slight misuse of notation. Writing $\boldsymbol{c}_\lambda = \boldsymbol{G}_\lambda\boldsymbol{\ell}$ it follows from Fact 17 that $\boldsymbol{c}_\lambda = \boldsymbol{c} + o(1)$ when $\lambda \to \lambda^*$.

$$d((\lambda\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{\ell}, \boldsymbol{p}^*) = d((\lambda - \lambda^*)^{-1}\boldsymbol{p}^* + \boldsymbol{G}_\lambda\boldsymbol{\ell}, \boldsymbol{p}^*) = d((\lambda - \lambda^*)^{-1}\boldsymbol{p}^* + \boldsymbol{c}_\lambda, \boldsymbol{p}^*) \tag{68}$$

For every component k, we have

$$\begin{aligned} \Bigl((\lambda - \lambda^*)^{-1}p_k^* + c_{\lambda k}\Bigr)\Big/p_k^* &= \Bigl((\lambda - \lambda^*)^{-1}p_k^* + c_k + o(1)\Bigr)\Big/p_k^* \\ &= (\lambda - \lambda^*)^{-1}\Bigl(1 + (\lambda - \lambda^*)c_k/p_k^* + o(\lambda - \lambda^*)\Bigr), && (69) \end{aligned}$$

hence

$$\log\Bigl\{\Bigl((\lambda - \lambda^*)^{-1}p_k^* + c_{\lambda k}\Bigr)\Big/p_k^*\Bigr\} = \log(\lambda - \lambda^*)^{-1} + (\lambda - \lambda^*)c_k/p_k^* + o(\lambda - \lambda^*). \tag{70}$$

Therefore

$$d((\lambda \boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{\ell}, \boldsymbol{p}^*) = (\lambda - \lambda^*)(\max_k c_k/p_k^* - \min_k c_k/p_k^*) + o(\lambda - \lambda^*). \tag{71}$$

Hence the conclusion. □

Note that d_1 is strictly positive, because vectors $\boldsymbol{c}$ and $\boldsymbol{p}^*$ are not proportional (except if $\boldsymbol{c} = \boldsymbol{o}$, i.e., if $\boldsymbol{\ell} \propto \boldsymbol{p}^*$). The derivative of $d(\boldsymbol{p}(r), \boldsymbol{p}(r^*))$ with respect to r follows from the calculation of d_1 and is strictly negative.

Fact 28. *There exists a unique vector $\boldsymbol{e}$, solution to*

$$(\lambda^* \boldsymbol{I} - \boldsymbol{A})\boldsymbol{e} = \boldsymbol{c} \tag{72}$$

$$\boldsymbol{q}^{*\top}\boldsymbol{e} = 0 \tag{73}$$

Let d_2 be the second derivative of $d(\boldsymbol{p}(r), \boldsymbol{p}^*)$ with respect to λ, at $\lambda = \lambda^*$. Then

$$d_2 = -(c_i/p_i^*)^2 - 2e_i/p_i^* + (c_j/p_j^*)^2 + 2e_j/p_j^* \tag{74}$$

where i and j are defined as in Fact 27.

Proof. Since $\boldsymbol{c} \in \mathcal{H}$ and $\lambda^*\boldsymbol{I} - \boldsymbol{A}$ is an isomorphism in $\mathcal{H}$, vector $\boldsymbol{e}$ is uniquely defined. The calculation of the second derivative requires an expansion with one more term than in the previous proof. According to equation (31) applied to vector $\boldsymbol{\ell}$, we have

$$\begin{aligned}\boldsymbol{p}(\lambda) &= (\lambda - \lambda^*)^{-1}\boldsymbol{p}^* + \boldsymbol{G}_\lambda(\boldsymbol{\ell} - \boldsymbol{p}^*)\\ &= (\lambda - \lambda^*)^{-1}\boldsymbol{p}^* + \boldsymbol{G}(\boldsymbol{\ell} - \boldsymbol{p}^*) - (\lambda - \lambda^*)(\boldsymbol{G}^2(\boldsymbol{\ell} - \boldsymbol{p}^*) + o(\lambda - \lambda^*)\\ &= (\lambda - \lambda^*)^{-1}\boldsymbol{p}^* + \boldsymbol{c} - (\lambda - \lambda^*)\boldsymbol{e} + o(\lambda - \lambda^*);\end{aligned} \tag{75}$$

hence

$$\begin{aligned}\log(\lambda - \lambda^*)p_k/p_k^* &= \log\Big(1 + (\lambda - \lambda^*)c_k/p_k^* - (\lambda - \lambda^*)^2 e_k/p_k^* + o(\lambda - \lambda^*)^2\Big)\\ &= (\lambda - \lambda^*)c_k/p_k^* - (\lambda - \lambda^*)^2\Big((c_k/p_k^*)^2/2 + e_k/p_k^*\Big) + o(\lambda - \lambda^*)^2.\end{aligned} \tag{76}$$

Hence the conclusion. □

Conics.

Fact 29. *(in $\mathbb{R}^3$:) Normalized prices $\boldsymbol{p}(r)$ are located on a conic. This conic goes through the normalized labor vector $\boldsymbol{\ell}$ and through any (real or complex) eigenvectors of $\boldsymbol{A}$.*

Proof. This was first proved in (Steedman, 1996). Here is a shorter proof: By definition of the prices of production, the three vectors $\boldsymbol{Ap}$, $\boldsymbol{p}$, and $\boldsymbol{\ell}$ are dependent. Hence the unnormalized prices satisfy $F(\boldsymbol{p}) := \det(\boldsymbol{Ap}, \boldsymbol{p}, \boldsymbol{\ell}) = 0$, which is the equation of a quadratic cone. The normalized prices are located at the intersection of this cone with a plane, which is a conic. Clearly, the equation of the cone is satisfied by the vector $\boldsymbol{\ell}$ and any eigenvector of $\boldsymbol{A}$. □

One also already knows when this conic is degenerate: this happens if and only if $\boldsymbol{\ell}$ belongs to a critical line. In the following, we will leave this case apart.

Fact 30. *The plane tangent in $\boldsymbol{\ell}$ to the quadratic cone of the unnormalized prices is the one containing $\boldsymbol{\ell}$ and $\boldsymbol{A\ell}$.*

Proof. The derivative of $\boldsymbol{p}(r,1)$ at $r=-1$ is $\boldsymbol{A\ell}$. It is the tangent of the price curve starting at $\boldsymbol{p}$, if one normalizes the price curve not by some numeraire but if one keeps w constant. $\square$

Fact 31. *(in $\mathbb{R}^3$:) In the standard normalization, the conic defined in Fact 29 is an ellipse, parabola, or hyperbola, according to the number 0, 1, or 2 of real Non-Frobenius eigenvalues of $\boldsymbol{A}$. In the case of a parabola, the line connecting $\mathbf{p}^*$ with the unique other eigenvector is parallel to the axis of the parabola, and in the case of a hyperbola, the two lines connecting $\mathbf{p}^*$ with the two other real eigenvectors are parallel to the asymptotes of the hyperbola.*

Proof. Consider the intersection of the quadratic cone with the plane $\{\boldsymbol{x} : \boldsymbol{q}^{*\top}\boldsymbol{x} = 1\}$. In order to recognize the type of the conic consider the values of function F on a line of this plane going through $\boldsymbol{p}^*$. I.e., we calculate $F(\boldsymbol{p}^* + t\boldsymbol{u})$ for a vector $\boldsymbol{u}$ such that $\boldsymbol{q}^{*\top}\boldsymbol{u} = 0$, and t varying. One obtains $F(\boldsymbol{p}^* + t\boldsymbol{u}) = t^2 \det(\boldsymbol{Au}, \boldsymbol{u}, \boldsymbol{\ell}) + t\det(\boldsymbol{Au} - \lambda^*\boldsymbol{u}, \boldsymbol{p}^*, \boldsymbol{\ell})$ whose sign for t great enough is that of $\det(\boldsymbol{Au}, \boldsymbol{u}, \boldsymbol{\ell})$. Hence

- If $\det(\boldsymbol{Au}, \boldsymbol{u}, \boldsymbol{\ell})$ has the same sign whatever $\boldsymbol{u}$, the conic is an ellipse.
- If $\det(\boldsymbol{Au}, \boldsymbol{u}, \boldsymbol{\ell})$ is positive for some values of $\boldsymbol{u}$ and negative for other values, the conic is a hyperbola, and the asymptotic directions are those of vectors $\boldsymbol{u}$ such that $\det(\boldsymbol{Au}, \boldsymbol{u}, \boldsymbol{\ell}) = 0$ and $\boldsymbol{q}^{*\top}\boldsymbol{u} = 0$, i.e., they are the two real distinct non-Frobenius eigenvectors of $\boldsymbol{A}$.
- In the limit case where $\det(\boldsymbol{Au}, \boldsymbol{u}, \boldsymbol{\ell})$ vanishes in one direction, the conic is a parabola.

$\square$

Fact 32. *If the conic is a hyperbola or ellipse, then its center of symmetry is a vector $\mathbf{c}$ proportional to*

$$\mathbf{c} \propto (\lambda^* - \lambda_{\boldsymbol{u}})\det(\mathbf{p}^*, \boldsymbol{u}, \boldsymbol{\ell})\boldsymbol{v} + (\lambda_{\boldsymbol{u}} - \lambda_{\boldsymbol{v}})\det(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\ell})\mathbf{p}^* + (\lambda_{\boldsymbol{v}} - \lambda^*)\det(\boldsymbol{v}, \mathbf{p}^*, \boldsymbol{\ell})\boldsymbol{u} \tag{77}$$

Proof. The center of symmetry of the conic is a point $\mathbf{c} = \mathbf{p}^* + \alpha\boldsymbol{u} + \beta\boldsymbol{v}$, where $\mathbf{p}^*$, $\boldsymbol{u}$, and $\boldsymbol{v}$ are eigenvectors, such that if $\mathbf{c} + \boldsymbol{x}$ belongs to the conic, so does $\mathbf{c} - \boldsymbol{x}$ for any $\boldsymbol{x} \in \mathcal{H} = \{\boldsymbol{x} : \mathbf{q}^{*\top}\boldsymbol{x} = 0\}$.

$$\begin{aligned} 0 &= \det\big(\boldsymbol{A}(\mathbf{c} + \boldsymbol{x}), (\mathbf{c} + \boldsymbol{x}), \boldsymbol{\ell}\big) \\ &= \det(\boldsymbol{A}\mathbf{c}, \mathbf{c}, \boldsymbol{\ell}) + \det(\boldsymbol{Ax}, \boldsymbol{x}, \boldsymbol{\ell}) + \Big(\det(\boldsymbol{A}\mathbf{c}, \boldsymbol{x}, \boldsymbol{\ell}) + \det(\boldsymbol{Ax}, \mathbf{c}, \boldsymbol{\ell})\Big) \end{aligned} \tag{78}$$

A sufficient condition is that the large parenthesis is identically 0 for all $\boldsymbol{x} \in \mathcal{H}$, which is the case if it vanishes for $\boldsymbol{x} = \boldsymbol{u}$ and $\boldsymbol{x} = \boldsymbol{v}$. For $\boldsymbol{x} = \boldsymbol{u}$ one obtains

$$0 = \det(\boldsymbol{A}\mathbf{p}^* + \alpha\boldsymbol{Au} + \beta\boldsymbol{Av}, \boldsymbol{u}, \boldsymbol{\ell}) + \det(\boldsymbol{Au}, \mathbf{p}^* + \alpha\boldsymbol{u} + \beta\boldsymbol{v}, \boldsymbol{\ell}) \tag{79}$$

hence, after simplifications

$$\beta = (\lambda^* - \lambda_{\boldsymbol{u}})\det(\mathbf{p}^*, \boldsymbol{u}, \boldsymbol{\ell})\Big/(\lambda_{\boldsymbol{v}} - \lambda_{\boldsymbol{u}})\det(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\ell}). \tag{80}$$

Similarly

$$\alpha = (\lambda^* - \lambda_{\boldsymbol{v}})\det(\mathbf{p}^*, \boldsymbol{v}, \boldsymbol{\ell})\Big/(\lambda_{\boldsymbol{u}} - \lambda_{\boldsymbol{v}})\det(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\ell}). \tag{81}$$

These formulas are independent of any normalisation of $\boldsymbol{u}$ and $\boldsymbol{v}$. Formula (77) is a symmetric version of this. $\square$

Fact 33. *Assume the economy has two critical lines, call them i and j. Consider the tangents to the price curves at point $\boldsymbol{\ell}$ when $\boldsymbol{\ell}$ moves, in the standard normalization, parallel to critical line i. These tangents are concurrent on a point which belongs to the other critical line j.*

Proof. According to Fact 30, vector $\boldsymbol{x} = (\boldsymbol{A\ell} - \lambda_i \boldsymbol{\ell})/(\lambda^* - \lambda_i)$ belongs to the tangent plane. Consider vector $\boldsymbol{p}^* + \boldsymbol{x}$: since ${\boldsymbol{q}^*}^\top \boldsymbol{x} = 1$, the tangent in $\boldsymbol{\ell}$ to the normalized prices is the straight line passing through the extremities of $\boldsymbol{\ell}$ and $\boldsymbol{x}$. Since ${\boldsymbol{q}_j}^\top(\boldsymbol{x} - \boldsymbol{p}^*) = 0$, the extremity of $\boldsymbol{x}$ belongs to the critical line j. Moreover, $\boldsymbol{x}$ is invariant when $\boldsymbol{\ell}$ is replaced by $\boldsymbol{\ell} + t\boldsymbol{p}_i$ for any t. □

These properties can be extended to higher dimensions. Let us leave apart the case of uniform organic composition.

- For a price of production vector $\boldsymbol{p}$ in $(\boldsymbol{A}, \boldsymbol{\ell})$, matrix $[\boldsymbol{Ap}, \boldsymbol{p}, \boldsymbol{\ell}]$ has rank 2. Hence any 3 determinant extracted from it is zero. There are $n-2$ independent determinants of this type, and each of them provides the equation of a quadratic cone of dimension $n-1$. The price curve is therefore located at the intersection of $n-2$ quadratic cones, plus the affine hyperplane used for the normalization.
- The price curve may be parametrized. The simplest way to proceed is to write down explicitly equation (2). Using Cramer's rule and setting $w = \det(\boldsymbol{I} - (1+r)\boldsymbol{A})$, one obtains $p_i = p_i(r)$, where $p_i(r)$ is a polynomial of degree $n-1$ in r.

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