Collusion Price Sustainability under Demand Uncertainty and Smooth Transition Market Share

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Abstract

We analyze the sustainability over time of collusion equilibrium in a two firms market with uncertain demand and risk neutrality, modeling uncertainty under several different distributional assumptions. Expected demand is assumed to be subject to inertia in that a difference between the two firms' prices results in a smooth variation of the market share instead of a discrete 0-1 outcome; demand is modeled as continuous in the price difference and secret price cuts result in the increase of the own market share and profit. We show that when secret price cuts cannot be observed directly and cheating may be inferred only on the ground of the own profit's level, the higher demand uncertainty, the more deviating from collusion equilibrium pays. Under the assumption of trigger strategies and firms employing a tail test based upon a threshold profit level to detect price cuttings, we find that strict detection rules result to be less effective than milder ones in order to avoid deviation.

Keywords: duopoly collusion, demand uncertainty, trigger strategies, secret price cuttings, tail test.

JEL Classification: D43, L11, L13

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1 Introduction

Theoretical work starting with Stigler (1964) suggests that tacit collusion sustainability in a repeated game environment is more difficult when firms face uncertain, unobservable demand. If firms maximize expected profits, unobservability of the state of demand and the only knowledge of the own performance makes it harder, to distinguish the rivals' cheating from bad performances due to demand fluctuations.

We show that in a duopoly where: *a)* firms are identical and split demand equally if they charge equal price, *b)* secret price cuts result in an increase of the low-price firm's market share, though this won't obtain *all* the demand, *c)* a tail test is used in order to detect deviations, and *d)* firms use trigger strategies after detecting deviations; then the threshold level of the discount factor over which collusion is sustainable depends critically on the level of the demand uncertainty and on the tail test employed. Consistently with intuition, we show that: *a)* the higher volatility, the more collusion is difficult until becoming impossible; *b)* under various circumstances, the higher demand uncertainty, the lower the benchmark for the tail test is to be in order for collusion to be sustainable.

We have chosen to model market share as sensitive to price cuts but with inertia. The specification we consider embeds anyway the standard possibility that the low price firm gets all demand. However, this is only a special case: modeling market share as being subject to a transition regime means considering the possibility that, for example, purchasers are unwilling to change seller even when identical, cheaper substitutes are available. Or we may interpret it as a case where information on the price is not available to all consumers but only to a part of them, with the proportion of "aware" consumers increasing with the price spread. Actually, as shown in Appendix I, our assumptions on market share embed the case of splitting consumers. We don't allow this inertia to be time varying for computational simplicity.

The environment we consider and the results we obtain are as follows. We consider that case of a two firms market without the possibility of new competitor's entrance and where firms maximize expected profits by risk neutrality, as in Green and Porter (1984). Firm's strategic behavior takes place in an infinitely repeated game context and it includes tacit collusion with firms maximizing the industry's profit, deviation from collusion where one firm maximizes its own profit given the other's collusive behavior, and competition. All maximization problems are w.r.t. the *price*. Given that both demand and the rival's profits are never observable, in every period each firm will employ a tail test based on comparing its previous profit with a threshold level; both the former and the discount factor are assumed to be fixed over time and deterministic, while demand is subject to uncertainty. This allows to avoid the dependence of collusive equilibrium price on the discount factor, which has been considered by Dal Bo' (2001). Demand uncertainty is modeled as a continuous distributions whose details are in Section 2.1. Firms will discount expected profits by the discount factor and taking into consideration the presence of the rival's tail test: there-

fore they will make decision on the *expected* sum of expected profits.
The main results derived are two. First, volatility is crucial in determining the set of equilibrium outcomes and hence the sustainability of tacit. We show that under various circumstances, the higher demand volatility, the higher the care of the future required in order for collusion to be possible. Moreover, there exists a threshold level for volatility over which collusion is never possible. Second, we show that for any degree of uncertainty there exists an "optimal" level of the benchmark that triggers the tail test, "optimal" meaning that *ceteris paribus* it minimizes the lower bound of the discount factor above which collusion is sustainable. Our conclusion is that - if some quite general condition that will be specified later on hold - the higher uncertainty, the lower the aforementioned optimal level is: in order to maintain collusion, the tail test is required to be milder as demand volatility increases.

The rest of the paper is organized as follows. Below, some of the related literature on collusion is discussed. Section 2 provides the theoretical framework for demand uncertainty and market share. Section 3 considers the model, while Section 4 is aimed at presenting the main result arising from both the analytical and simulation viewpoint. Conclusions are in Section 5. We also sketched the simulation exercise on which part of the analysis is based in Appendix I, where we present some further structure necessary to fully understand our model for the market share.

1.1 Related literature

According to Dal Bo' (2001), the related literature falls into five categories: 1) studies of the effect of demand fluctuations on optimal tacit collusion; 2) studies of optimal punishment schemes under quantity competition; 3) repeated game with fixed discount factor; 4) empirical studies of collusive pricing; 5) studies of the role of oligopolies in macroeconomic fluctuations.
Our paper is mainly related to the first and the third fields of the analysis, even though some of the analysis can be referred to the second branch of the literature.
As to the literature dealing with the effect of demand variation on optimal tacit collusion, we follow the lines of the seminal work by Green and Porter (1984), modeling uncertainty in details in Section 2.1. Alike Green and Porter, however, our paper refers uncertainty to the *(total) quantity* demanded rather than directly to the *price*, still considering demand outcomes over time as independent of one another. Our aim, too, is different, too. While Green and Porter work showed the necessity for periodic price wars in order to maintain the incentives for collusion when firm cannot monitor the behavior of their rivals, we are more concerned with the *exogenous conditions* that make collusion possible. Literature has anyway also considered the case of demand correlation, within a Rotemberg and Saloner (1986) framework. Kandori (1991), Haltiwanger and Harrington (1991) and Bagwell and Staiger (1997) have alla considered tacit collusion under demand fluctuations; Dal Bo' (2001) has considered the case of

a stochastic discount factor - which may be related also to the third of the fields above - transferring the volatility from direct or inverse demand to this variable. As far as the third category is concerned, a standard result says that, for repeated games with fixed discount factor, the higher the discount factor, the bigger the set of equilibrium outcomes will be - see for example Abreu, Pierce and Stacchetti (1990). Dal Bo' (2001) has also show the striking result that under discount factor fluctuations it is not only the magnitude of the factor that counts in determining the set of equilibrium outcomes but also its volatility. Our paper remains in the classical field where the discount factor is not variable.

2 Modeling demand uncertainty and market share

This section is aimed at providing some considerations on two issues of basic importance for the analysis below: the demand volatility and the shape of the expected demand function.

2.1 Uncertainty

Demand uncertainty has been considered in several works and has been represented with various functional forms so far. Literature usually models the inverse demand function rather than the direct one, therefore assuming uncertainty to be relative to the price rather than to the quantity itself. The two main specifications that emerge from literature consider two kinds of uncertainty: the *additive* and the *multiplicative* one.
Additive uncertainty considers the variable affected by uncertainty, say $\tilde{k}$, to be given by a fixed, deterministic value, say k, plus a stochastic shock, say ϵ:

$$\tilde{k} = k + \epsilon$$

The functional form of the shock ϵ is what actually determines the uncertainty. So far literature has considered stochastic shocks with a bounded support. Porter (1986) has modeled price as the usual indirect demand plus a shock whose support is bounded so as to avoid negative prices. Reynolds and Wilson (1998) consider a demand uncertainty due to variability in the intercept of the demand function with a bounded support; the same assumption is made by Cason and Mason (1999), where the intercept of the inverse demand function is a discrete random variable with three possible outcomes (a mean, a high and a low value), and in Porter (1986), who considers a binary random price function.
Multiplicative uncertainty has been employed in order to model price variability, in the classical paper by Green and Porter (1984). Their model considers price at time t, $\widetilde{p}_t$, given by the multiplicative equation $\widetilde{p}_t = \theta_t p(q_t)$. Here $p(q_t)$ is the *expected* inverse demand at time t and the random variable θ_t is an *i.i.d.* random variable with expected value equal to one and continuous and integrable density function f. Two observations about this specification are possible. First,

notice that this formulation doesn't involve any constraint to the random component's support, which wouldn't prevent price from becoming lower than zero. This problem is actually negligible if the probability of negative prices is small enough, i.e. if $\int_{-\infty}^{0} p(q_t) f(x) dx$ is "small". Second, the *i.i.d.* assumption on θ_t allows for a kind of heteroscedasticity in the price generation process. Depending on the strategy chosen by the firms, i.e. on the total output level q_t, the expected price level in fact is allowed to be changing with the strategies, which is reflected in a different variance for the random variable $\theta_t p(q_t)$.

2.2 The expected demand function

The expected demand function (from now on: EDF) we consider is the direct one at each t, q_t. As the parametric specification we consider is assumed to be stable across time, we will drop the subscript t whenever unnecessary.

Consumers are considered to be in an environment where information on the two firms' prices is not symmetric and fully available. Let n be the total number of consumers: we assume they will be divided into two groups: the first group will buy from firm 1 and will consist of n_1 individuals, and the second, which buys from firm 2, is of $n_2 = n - n_1$. It is known that when the two prices are equal, the market will be symmetrically split between the two firms. If a price difference occurs, for example if $p_1 < p_2$, we assume that only a few of the n_2 consumers will start buying from firm 1, migrating o group 1. Firm 1 therefore will increase its market share, but it won't capture the whole market. Such an EDF will be defined a *smooth transition market share*, and the assumption on the smooth increase of the firm's market share as they cut their price will be referred to as the *smooth transition hypothesis*, in that we assume that the consumers' switching from one firm to the other follows a smooth, rather that discrete patterns such as the Bertrand 0, 1 one or the splitting consumer one.

This assumption may be interpreted and formalized as follows. Let: $q^{(ji)}$ be the quantity bought by the i-th consumer from the j-th firm, and p_j the j-th firm's price. Then the demand function for each consumer will be

$$q^{(1i)} = a^{(1)} - b^{(1)} p_1$$

$$q^{(2i)} = a^{(2)} - b^{(2)} p_2$$

The total demand is

$$q = \sum_{i=1}^{n_1} q^{1i} + \sum_{i=1}^{n_2} q^{2i} = n_1 q^{1i} + n_2 q^{2i} \tag{1}$$

Let now $\alpha = n_1/n$ be the market share for firm 1. If $a^{(1)} = a^{(2)} = a/n$ and $b^{(1)} = b^{(2)} = b/n$, then, after some manipulating, (1) turns out to be

$$q = a - b[\alpha p_1 + (1 - \alpha) p_2] \tag{2}$$

For (2) to be a smooth transition EDF some more structure is required. In particular, α is to be specified so that it can vary smoothly with the price difference and some other variables, according to the discussion above. This will be derived in the next Section.

3 The model: assumptions and analytic results

This section is divided into two subsections. The first presents the hypotheses on which the model we consider is based; the second contains the analytic results obtained.

3.1 Assumptions

The hypotheses on which the following analysis and considerations lie fall into four groups. Specifically, assumptions are relative to:

1. the *demand* the firms face;
2. the *firms' characteristics and behavior* and the *market structure*;
3. the *strategies* firm implement for each of the possible situations and the properties of the demand function required for the optimization problems involved;
4. the *distributional assumptions* of the demand function, which will characterize its uncertainty.

Before enumerating the assumptions, some preliminary notation - that will be further developed below - is required. We will refer to the i-th firm's price and quantity sold at time t as p_{it} and q_{it} respectively, q_t being the total demand $q_t = q_{1t} + q_{2t}$. Firm i's expected profit at time t will be referred to as Π_{it}, while the discounted sum of expected profits will be denoted as Π_i. Any random variable X will be indicated as $\widetilde{X}$ and its expected value as X.

1. *Demand:*

   As stated in the previous Section, total demand will be modeled according to a smooth transition specification.

   (a) the expected total demand at time t follows the specification of equation (2) and the smooth transition requirement for α. Notice that, by definition, $q_{1t} = \alpha_t q_t$, $q_{2t} = (1 - \alpha_t)q_t$, the same holding for the random variables;

   (b) the firm 1's market share α_t is a function of the difference between the two firms' prices and of the rival's price level:

   $$\alpha_t = \alpha(x_t, p_{2t})$$

   with $x_t = p_{1t} - p_{2t}$. The following further conditions are required:

   i. $\alpha_t(0, p_{2t}) = \frac{1}{2}$ for any p_{2t};

ii. $\frac{\partial}{\partial \mathrm{x_t}}\alpha_t \leq 0$ for any x_t, p_{2t};

iii. $\frac{\partial^2}{\partial^2 \mathrm{x_t}}\alpha_t = \begin{cases} > 0 & \text{for } x_t > 0 \\ < 0 & \text{for } x_t < 0 \end{cases}$;

iv. $\lim_{x_t \to +\infty} \alpha_t(x_t, p_{2t}) = 0$

$\lim_{x_t \to -\infty} \alpha_t(x_t, p_{2t}) = 1$

v. $\frac{\partial}{\partial \mathrm{p_{2t}}}\alpha_t = \begin{cases} > 0 & \text{for } x_t > 0 \\ < 0 & \text{for } x_t < 0 \end{cases}$;

vi. $\lim_{p_{2t} \to c^+} \alpha_t(x_t, p_{2t}) = \mathrm{I}[x_t < 0]$

with c being the marginal cost for each firm (see below for further details) and $\mathrm{I}[\cdot]$ an index function assuming value 1 if the condition in braces holds and 0 otherwise. Notice that clearly $p_{1t}, p_{2t} \in (c, +\infty)$.

The hypotheses above include some of the functional requirements for the market share; other requirements, that enter the firms' optimization problems, will be reported below. A brief comment on them is as follows: assumption *i* states that when there is no difference between the firm prices, the market demand will be split between the two firms, *regardless of the price level*, which affects anyway the total demand level; assumption *iii* is a symmetry condition around $x_t = 0$ and states that the marginal effect of a price cutting is decreasing (though always positive); assumptions *v* and *vi* are required to obtain certain results for the competition equilibrium and they will be made clearer in the third subsection. In Appendix I we will report some further results from simulation on the market share α.

2. *Firms' behavior and market structure:*

This subsection is aimed at modeling the firms' behavior and the environment where decisions are made.

(a) firms are assumed to have the same, linear cost structure. If C_i is the total cost of firm i, then $C_{it} = cq_{it}$, c being the marginal (and average) cost;

(b) firms are risk neutral and they maximize their expected profit, independently of the level of the uncertainty;

(c) Π_{1t} and Π_{2t} are given respectively by:

$$\Pi_{1t} = \alpha_t(p_{1t} - c)q_t \tag{3}$$

$$\Pi_{2t} = (1 - \alpha_t)(p_{2t} - c)q_t \tag{4}$$

(d) each firm knows its own expected outcomes for any strategy.

Assumption *b)* is a common representation of risk neutrality - see for example Green and Porter (1984); assumption *d)* means that each firm knows its expected profit under collusion, competition and in the cases of its own or the rival's deviation. What firms do *not* know is the strategy currently implemented by their competitor.

3. *Firms' strategies:*

   In this subsection, we specify the single-period component game G for our discounted duopoly supergame. In particular, we formalize the firms' strategies under various circumstances and the rules according to which each firm reverts from one strategy to another. In the following lines, and in the rest of the paper as well, we will denote at some times the density function for a generic random variable $\widetilde{X}$ as $\widetilde{X}$ itself, whenever this will not be ambiguous.
   The possible strategies firm 1 (and, by symmetry, firm 2 as well) may implement are: collusion, deviation while the other firm is still playing according to the collusion rule and competition. The following assumption describe a situation where a firm keeps respecting the collusion price until it doesn't notice a (possible) deviation. Firms cannot observe the quantity demanded at each period. This results in employing a tail test.

   (a) under *collusion* the two firms will maximize, at any t, the joint profit of the industry w.r.t. their prices which, by symmetry, will be equal:

   $$\begin{array}{cc} \max\limits_{p_{1t}, p_{2t}} & (\Pi_{1t} + \Pi_{2t}) \\ s.t. & p_{1t} = p_{2t} \end{array} \tag{5}$$

   The solution to this maximization problem will be referred to as p^c and Π^c. Solutions are of course the same at each t, which justifies our dropping the subscript t;

   (b) in case of *deviation*, the price cutting firm (say 1), will maximize its own profit w.r.t. its own price assuming the other firm (say 2) will maintain its price to the optimal collusion level p^c. 1's optimization problem is hence as follows:

   $$\max_{p_{1t}} \quad \Pi_{1t}(p_{1t}, p^c) \tag{6}$$

   The solution to this problem will be referred to as p_1^d and Π_i^d. Here too the optimum remains stable across time;

   (c) when there is *competition* between the two firms, each will employ the well-known reaction function that takes into account of the other firm's price choice. The two maximization problems whose solutions are respectively 1's and 2's reaction function are as follows:

$$\max_{p_{1t}} \quad \Pi_{1t}(p_{1t}, p_{2t}) \tag{7}$$

$$\max_{p_{2t}} \quad \Pi_{2t}(p_{1t}, p_{2t}) \tag{8}$$

Solutions will be symmetric - i.e. the same for either firm - and denoted as p^{co} and Π^{co}.

(d) at each t, both firms will try and detect whether there has been a secret price cut in the previous period by considering their own profit in $t-1$. The critical set of *own* profits that trigger the reversion to competition is $[0, \widehat{\Pi}]$ - i.e. firm i will react to deviation if $\Pi_{it-1} \in [0, \widehat{\Pi}]$. We will refer to the set upper bound $\widehat{\Pi}$ as the *threshold* profit. Being it impossible to observe either the rival's behavior or q_{t-1}, each firm employs a tail test - see Porter (1983) and Porter (1986). It is in fact impossible to observe the rival's behavior or the total demand level at $t-1$. The tail test triggers competition regardless of whether profit decreasing is due to the other firm's cheating or to a demand shock, as in Green and Porter (1984).

(e) firms do not consider review strategies: the threshold profit at time t is in fact independent of profits prior to $t-1$.

The last assumption is worth commenting further. The cheating detection tail test suggested above consists of choosing, at each t, whether to keep fixing the collusion price p^c and producing the collusion output, or to revert to competition on the ground of a threshold level of the own profit. Such a strategy is similar to the one analyzed by Abreu, Pearce and Stacchetti (1986), where the control variable is the firm's own price.

4. *Distributional assumptions:*

   (a) the total demand $\widetilde{q_t}$ is assumed to be a normally distributed random variable, with expected value q_t and variance σ^2. We will denote this as $\widetilde{q_t} \sim N[q_t, \sigma^2]$;

Notice that the distributional assumption, under no restrictions such as the *i.i.d.* requirement, allows for heteroscedasticity. Actually, assuming $\widetilde{q_t}$ as a normally distributed variable allows for both the additive and multiplicative representation of the uncertainty.
We will say that uncertainty is *additive* if one considers the demand $\widetilde{q_t}$ expressed as

$$\widetilde{q_t} = q_t + \epsilon_t \tag{9}$$

where $\epsilon_t \sim N[0, \sigma^2]$.
Multiplicative uncertainty can be represented considering the random variable $\widetilde{\theta} \sim N(1, \sigma^2)$ and, for any t, the relationship

$$\widetilde{q_t} = \theta q_t \tag{10}$$

This relationship will allow for heteroscedasticity due to different strategies, even though no variations across time are allowed, due to the *i.i.d.* requirement. According to the firms' behavior, in fact, total quantity will assume different values which will cause the variance to change. For simplicity, we will consider both ϵ_t and $\widetilde{\theta}$ as *i.i.d.* variables.
Notice that in the additive case, demand uncertainty is assumed to be homoscedastic not only across time, but also with strategies. Regardless of the firms' behavior, in fact, the variance of the total demand will be a constant, equal to σ^2. Not withstanding this, the profit is still heteroscedastic across time, but homoscedastic w.r.t. the strategies. Due to this, the apparent drawback of additive representation is overcome. Hence, in the rest of the paper we will employ the additive form, which is slightly easier from the algebraic viewpoint. A subsection will be dedicated to the multiplicative case, mostly in order to show that employing the additive specification instead of the multiplicative one may be done almost w.l.o.g. Last, notice that making variance - and therefore uncertainty - vary across time is possible by removing the *i.i.d.* requirement for the random variables ϵ_t or $\widetilde{\theta}$. One might want to model an uncertainty which is sensitive to its own past values. The *i.i.d.* hypothesis seems to be quite strict: demand variability could follow for instance a more complex process, such as a martingale, as suggested in a note in Green and Porter's article. We will limit our analysis to the *i.i.d.* case.

3.2 The model

This subsection is organized as follows. First, we present some theorems and lemmata in order to give the solutions for the model at time t, i.e. for the single period component game G. Second, we model the infinitely repeated game $G(\delta)$, defined by the component game G and the *discount factor* $\delta \in (0, 1)$. Third, we analyze the circumstances under which collusive solution is the equilibrium one. We remind that deviation from collusion will be studied w.r.t. firm 1, being the opposite case (1 colluding and 2 deviating) exactly symmetric.

3.2.1 Theorems and other results for the static game

THEOREM 3.1

Let q^c be the total quantity under collusion. The solution of problem (5) is

$$q^c = \frac{\mathrm{a} - \mathrm{bc}}{2}$$

$$\begin{cases} p^c = \frac{\mathrm{a+bc}}{\mathrm{2b}} \\ \alpha^c = \frac{1}{2} \\ \Pi^c = \frac{(\mathrm{a-bc})^2}{\mathrm{8b}} \end{cases} \tag{11}$$

and it holds that

$$\widetilde{\Pi}^c \sim N[\Pi^c, \frac{1}{4}(\frac{\mathrm{a-bc}}{\mathrm{2b}})^2\sigma^2] \tag{12}$$

THEOREM 3.2

Let α^d be firm 1's market share and q^d the total quantity under deviation. If the following optimum uniqueness condition holds

$$\begin{aligned} \frac{\partial}{\partial \mathrm{p}_1}\Pi_1 = \quad & \alpha'(p_1 - c)\{a - b[\alpha p_1 + (1-\alpha)p^c]\} + \alpha\{a - b[\alpha p_1 + (1-\alpha)p^c]\} + \\ & + \alpha(p_1 - c)(-b\alpha - b\alpha' p_1 + b\alpha' p^c) \neq 0 \text{ for any } p_1 \neq p_1^d \end{aligned} \tag{13}$$

and if

$$\alpha'(0) < -\frac{\mathrm{b}}{2(\mathrm{a-bc})} \tag{14}$$

then the unique solution for the optimization problem p_1^d is always a maximum and the following relationships hold:

$$q^d = a - b[\alpha^d p_1^d + (1-\alpha^d)p^c]$$

$$\begin{cases} p_1^d < p^c \\ \Pi_1^d = \alpha^d(p_1^d - c)q^d > \Pi^c \\ \alpha^d > \frac{1}{2} \end{cases} \tag{15}$$

$$\Pi_2^d = (1-\alpha^d)(\frac{\mathrm{a-bc}}{\mathrm{2b}})q^d < \Pi^c \tag{16}$$

and besides

$$\widetilde{\Pi}_2^d \sim N[\Pi_2^d, (1-\alpha^d)^2(\frac{\mathrm{a-bc}}{\mathrm{2b}})^2\sigma^2] \tag{17}$$

THEOREM 3.3

If

$$\alpha'(0, p_2) < -\frac{[\frac{1}{2}(\mathrm{a-bp_2}) - \frac{1}{4}\mathrm{b(p_2-c)}]}{(\mathrm{p_2-c})(\mathrm{a-bp_2})} \text{ for any } p_2 > c \tag{18}$$

then the solution for the problem (7)-(8) is $p^{co} = c$, and hence profits, for both firms, will be identically and deterministically equal to zero.

Proofs for the three Theorems are in Appendix II.

Notice that solution for problems (7) and (8) could be $p^{co} = c$ even under other competition regimes without imposing restriction (18). It would in fact suffice to suppose that, after detecting deviation, firm 2 chooses Bertrand competition, setting its price equal to $p_2 = c$. Firm 1 would then have zero profits regardless of its response: any $p_1 > c$ would in fact result in a zero market share, after condition *vi*.

3.2.2 Results for the dynamic game

Before introducing the main results concerning the infinitely repeated game, some preliminary notation is required:

1. the *discount factor* will be denoted as $\delta \in (0, 1)$;
2. the probability that firm 2's profit is be lower than the threshold profit $\widehat{\Pi}$ under collusion will be denoted as ϕ^c and is equal to $\int_{-\infty}^{\widehat{\Pi}} \widetilde{\Pi}^c(x)dx$. We will refer to $\int_{\widehat{\Pi}}^{+\infty} \widetilde{\Pi}^c(x)dx$ as $\widehat{\phi}^c$;
3. the probability that firm 2's profit will be lower than the lower bound when firm 1 deviates from collusion while 2 doesn't will be referred to as ϕ^d and is equal to $\int_{-\infty}^{\widehat{\Pi}} \widetilde{\Pi}_2^d(x)dx$. We will denote $\int_{\widehat{\Pi}}^{+\infty} \widetilde{\Pi}_2^d(x)dx$ as $\widehat{\phi}^d$.

The theorems presented above provide a solution for each of the three problems in the previous subsection. With them, at any t, we know what the possible outcomes of the industry will be depending on the firms' behavior. Through them, it's immediate to understand the outcomes of the repeated game.
The problem firm 1 faces is whether to collude or to cheat. The decision it will make will be based on the discounted sum of the *expected* two payoffs corresponding to either alternative, following a framework that has been quite popular in literature. As pointed out in the previous subsection, firm 2 will begin producing and pricing at collusive level and continue to do so until it realizes, through the tail test, firm 1 has cheated. If firm 2's profit should fall below the threshold level $\widehat{\Pi}$, then it will revert forever to competition. Modeling this may be sketched as follows.
At time $t = 0$, firm 1 will decide whether to collude or deviate, knowing 2 is colluding. Results will be obtained for the first alternative, being the second analogous. If firm 1 and 2 both collude, the problem they will face will be (5). Hence firm 1 will earn Π^c at time $t = 0$. At time $t = 1$, firm 1 will earn Π^c iff at $t = 0$ 2 has earned enough to think no cheating has taken place, i.e. iff $\Pi^c > \widehat{\Pi}$; otherwise, the industry will revert to competition and $\Pi_{11} = 0$. The probability firm 2 will earn enough given 1 colludes is $\widehat{\phi}^c$. If the game should finish after time $t = 1$, the expected profit firm 1 obtains is

$$\Pi_1 = \Pi^c + \delta\widehat{\phi}^c\Pi^c + \delta\phi^c 0 = \Pi^c + \delta\widehat{\phi}^c\Pi^c$$

It is now immediate to extend the result given above to the case of infinite iterations of the game and to both collusion and deviation cases:

Lemma 3.1 *Under collusion, firm 1's discounted expected profit is equal to:*

$$\Pi_1^c = \frac{\Pi^{\mathrm{c}}}{1-\delta\widehat{\phi}^{\mathrm{c}}}$$

Under deviation, 1's discounted sum of expected profits is:

$$\Pi_1^d = \frac{\Pi^{\mathrm{d}}}{1-\delta\widehat{\phi}^{\mathrm{d}}}$$

Hence, given the game is repeated an infinite number of times, firm 1 will deviate iff

$$\frac{\Pi^{\mathrm{d}}}{1-\delta\widehat{\phi}^{\mathrm{d}}} > \frac{\Pi^{\mathrm{c}}}{1-\delta\widehat{\phi}^{\mathrm{c}}} \tag{19}$$

The following section is aimed at analyzing the conditions under which collusion is found to be more profitable than deviation.

4 Analytical results and simulation

The analytical results we consider here are about the roles played by demand uncertainty, i.e. variance σ^2, and the threshold profit level $\widehat{\Pi}$.

It is well known that in order for collusion to be preferred to cheating, the discount factor δ has to be large enough. Solving out condition (19) leads to the following inequality

$$\delta > \frac{\Pi_1^{\mathrm{d}} - \Pi^{\mathrm{c}}}{\Pi_1^{\mathrm{d}}\widehat{\phi}^{\mathrm{c}} - \Pi^{\mathrm{c}}\widehat{\phi}^{\mathrm{d}}} \equiv \delta^* \tag{20}$$

Relationship (20) states the well known result according to which unless the discount factor is higher than the threshold level δ^*, deviation will be preferred to collusion as a solution of the supergame $G(\delta)$. Anyway, relationship (20) considers a deeper result. The threshold δ^* is in fact a function of several variables, and particularly of σ^2 and $\widehat{\Pi}$: $\delta^* = \delta^*(\sigma^2, \widehat{\Pi})$: the threshold level δ^* is hence a frontier, in the $(\delta, \sigma^2, \widehat{\Pi})$ space, that divides the whole space into a collusion and deviation region.
Now, consider the following well-known results from probability analysis, that hold for a normally distributed random variable $\widetilde{Y} \sim N(\mu, \sigma^2)$:

1.
$$\frac{\partial}{\partial\sigma^2}\int_x^{+\infty} \widetilde{Y}(t)dt = \begin{cases} > 0 & \text{if } t > \mu \\ < 0 & \text{if } t < \mu \end{cases} \tag{21}$$

2.
$$\lim_{\sigma^2 \to +\infty} \int_x^{+\infty} \widetilde{Y}(t)dt = \frac{1}{2} \tag{22}$$

3.
$$\lim_{\sigma^2 \to 0} \int_x^{+\infty} \widetilde{Y}(t)dt = I[x > \mu] \tag{23}$$

Consider also the following, preliminary, notation:

$$\tau_i = \begin{cases} \frac{1}{4}(\frac{\mathrm{a-bc}}{\mathrm{2b}})^2 & \text{if } i = c \\ (1-\alpha^d)^2(\frac{\mathrm{a-bc}}{\mathrm{2b}})^2 & \text{if } i = d \end{cases}$$

$$\gamma = \frac{\Pi^{\mathrm{c}}(\widehat{\Pi} - \Pi_2^{\mathrm{d}})}{\sqrt{\tau_{\mathrm{c}}}} \frac{\sqrt{\tau_{\mathrm{d}}}}{\Pi_1^{\mathrm{d}}(\widehat{\Pi} - \Pi^{\mathrm{c}})}$$

Then:

THEOREM 4.1

For the supergame $G(\delta)$, if the following integral inequality holds

$$\widehat{\phi}^c \Pi_1^d - \widehat{\phi}^d \Pi^c > 0 \tag{24}$$

then:

1. *the threshold level δ^* is continuous and differentiable w.r.t. σ^2;*
2. *for any $\widehat{\Pi}$,*
$$\lim_{\sigma^2 \to +\infty} \delta^* = 2$$
3. *if $\widehat{\Pi} \in (\Pi_2^d, \Pi^c)$,*
   (a) $$\lim_{\sigma^2 \to 0^+} \delta^* = \frac{\Pi_1^{\mathrm{d}} - \Pi^{\mathrm{c}}}{\Pi_1^{\mathrm{d}}}$$
   (b) *δ^* is monotonically increasing in σ^2, i.e.*
   $$\frac{\partial}{\partial \sigma^2}\delta^* \equiv \delta_\sigma^* > 0 \text{ for any } \sigma^2$$
   (c) *there exists a $d > 0$ s.t.*
   $$\begin{cases} \delta^* < 1 & \text{if } \sigma^2 \in [0, d) \\ \delta^* \in (1, 2) & \text{if } \sigma^2 \in (d, +\infty) \end{cases}$$

4. *if* $\widehat{\Pi} < \Pi_2^d$ *- i.e. if the tail test is "mild" - consider the following two inequalities:*

$$\frac{(\widehat{\Pi} - \Pi^{\mathrm{c}})}{\sqrt{\tau_{\mathrm{c}}}} < \frac{(\widehat{\Pi} - \Pi_2^{\mathrm{d}})}{\sqrt{\tau_{\mathrm{d}}}} \tag{25}$$

$$\frac{\Pi_1^{\mathrm{d}}(\widehat{\Pi} - \Pi^{\mathrm{c}})}{\sqrt{\tau_{\mathrm{c}}}} < \frac{\Pi^{\mathrm{c}}(\widehat{\Pi} - \Pi_2^{\mathrm{d}})}{\sqrt{\tau_{\mathrm{d}}}} \tag{26}$$

   Then it holds that:

   (a) $\lim_{\sigma^2 \to 0^+} \delta^* = 1$

   (b) *if (25) doesn't hold and (26) does, then* $\delta^* > 1$ *for any* $\sigma^2 \in (0, +\infty)$*;*

   (c) *if (25) holds or, alternatively, if neither (25) nor (26) hold, then there always exists an* $e > 0$ *s.t.*

$$\begin{cases} \delta^* < 1 & \text{if } \sigma^2 \in [0, e) \\ \delta^* \in (1, 2) & \text{if } \sigma^2 \in (e, +\infty) \end{cases}$$

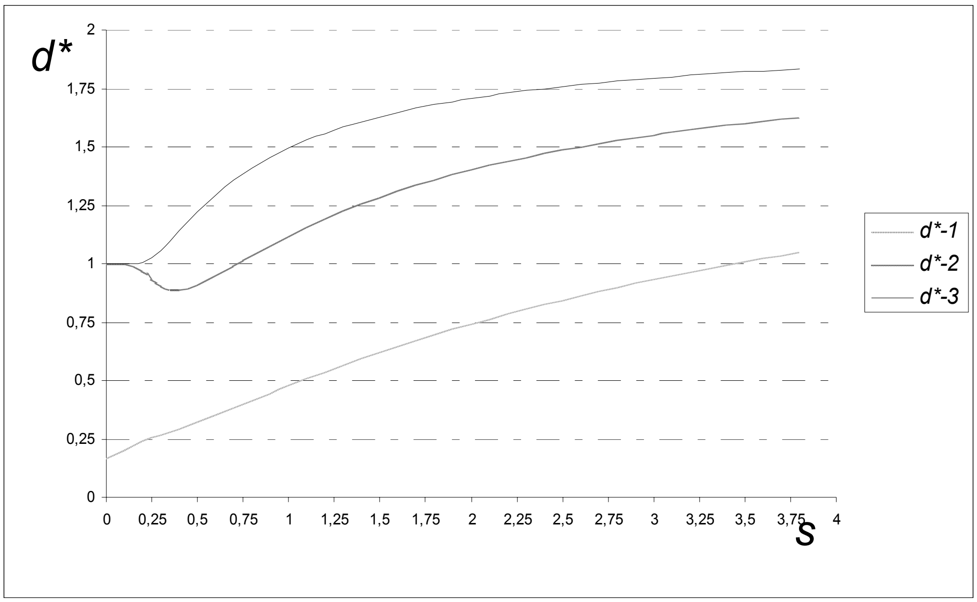
Proof is in Appendix II.

Theorem 4.1 states that:

1. the deterministic case, where the outcomes of the firms' strategies are observable, implies collusion to be possible either according to the well-known condition of result 3.a) or never, after result 4.a), when the tail test is mild and hence cheating is never detected and punished. A mild tail test under deterministic demand implies then that colluding is never rational;

2. the other results - 3.b), 3.c), 4.b), 4.c) - all state that for high demand volatility collusion will never be sustainable.

   The following graph illustrates the two possible shapes of the function δ^*, denoted as d^*, as σ, denoted as s, varies:



Simulation was made assuming the following data:

	$\widehat{\Pi}$	Π^c	Π_1^d	Π_2^d	τ_c	τ_d
d_1	900	1000	1200	600	100	16
d_2	400	1000	1200	600	100	16
d_3	400	1000	1300	660	100	16

We remind that δ^* is also a function, among the other parameters, of the threshold profit $\widehat{\Pi}$, constrained to be $\in [0, \Pi^c]$. This means that one may wonder what happens to the frontier when $\widehat{\Pi}$ is changed.
Analytical results concerning δ^*'s dependence on $\widehat{\Pi}$ can be obtained but this would be rather tedious, if easy. Anyway the following Lemma, which is only a partial response to the problem, holds:

Lemma 4.1

For the frontier $\delta^* = \delta^*(\widehat{\Pi})$, let

$$\frac{\partial}{\partial \widehat{\Pi}} \delta^* = \delta^*_{\widehat{\Pi}}$$

$$\sigma_{k1}^2 = \frac{(\Pi_2^{\mathrm{d}} - \Pi^{\mathrm{c}})^2}{2(\tau_{\mathrm{c}} - \tau_{\mathrm{d}})\log[\frac{1}{\gamma}]}$$

$$\sigma_{k2}^2 = \frac{(\Pi_2^{\mathrm{d}} - \Pi^{\mathrm{c}})^2}{2\tau_{\mathrm{c}}\log[\frac{1}{\gamma}]}$$

Then, if

$$\frac{\Pi_1^{\mathrm{d}}}{\sqrt{\tau_{\mathrm{c}}}} \frac{\sqrt{\tau_{\mathrm{d}}}}{\Pi^{\mathrm{c}}} > 1 \tag{27}$$

the following results hold:

1. $\delta^*(\widehat{\Pi})$ *is continuous and differentiable w.r.t.* $\widehat{\Pi}$*;*
2. *for the partial derivative* $\delta^*_{\widehat{\Pi}}$*:*
   (a) *if* $\sigma^2 > \sigma_{k1}^2$*, then* $\delta^*_{\widehat{\Pi}} < 0$ *for any* $\widehat{\Pi}$*;*
   (b) *if* $\sigma_{k2}^2 < \sigma^2 < \sigma_{k1}^2$*, then there exists a* $m_1 \in (0, \Pi_2^d)$ *s.t.*

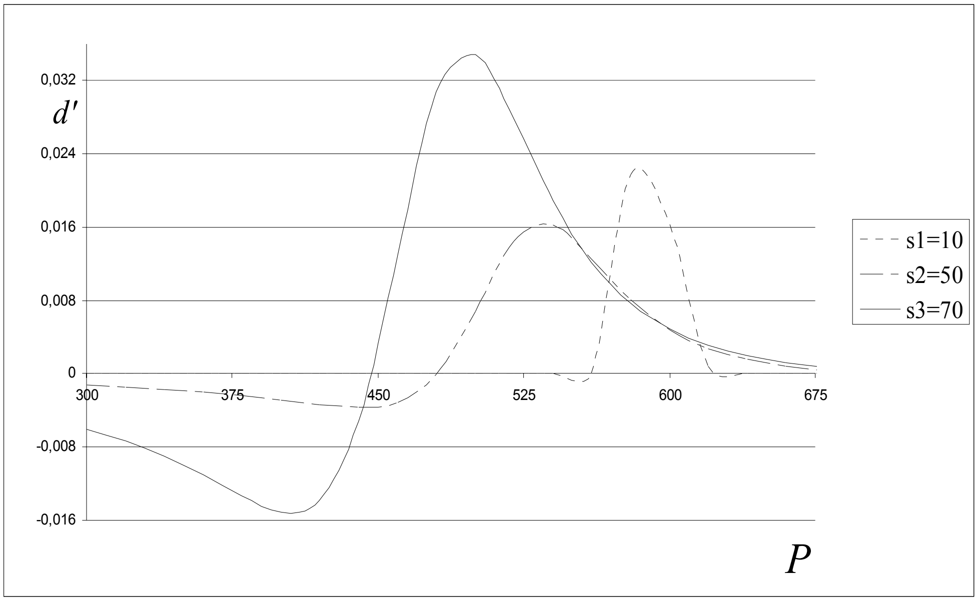
   $$\delta^*_{\widehat{\Pi}} = \begin{cases} > 0 & \text{if } \widehat{\Pi} > m_1 \\ < 0 & \text{if } \widehat{\Pi} < m_1 \end{cases}$$

   (c) *if* $\sigma^2 < \sigma_{k2}^2$*, then there exists a* $m_2 > \Pi_2^d$ *s.t.*

   $$\delta^*_{\widehat{\Pi}} = \begin{cases} > 0 & \text{if } \widehat{\Pi} > m_2 \\ < 0 & \text{if } \widehat{\Pi} < m_2 \end{cases}$$

Proof is omitted and, together with the other analytical results concerning $\delta^*_{\widehat{\Pi}}$. The Lemma states that, under condition (27), for any σ^2 there is an "optimal" level for the value of the threshold profit $\widehat{\Pi}$. Here "optimal" means that this is the threshold level that makes the collusion equilibrium set the widest. Lemma 4.1 states that the higher volatility, the less confident a firm can be when trying to detect cheating via a tail test (and vice versa), and hence when uncertainty is high, the best way to try and prevent from deviating is to employ a mild tail test. This lowers the threshold level towards zero as volatility increases.
If condition (27) does not hold, anyway, some further structure is required, and results are possibly the opposite as Lemma 4.1. In this case, we will provide results from simulation, in the following graph. It represents δ^*'s first derivative w.r.t. $\widehat{\Pi}$ - denoted as d' - as a function of $\widehat{\Pi}$ itself for several different levels of

σ, referred to as s. Notice that condition (27) doesn't hold:



Results have been obtained employing the following data:

	$\widehat{\Pi}$	Π^c	Π_1^d	Π_2^d	τ_c	τ_d
σ_1^2	10	1000	1200	600	7	1
σ_2^2	50	1000	1200	600	7	1
σ_3^2	70	1000	1200	600	7	1

The graph shows that as σ^2 increases, the "optimal" threshold profit decreases. This seems to confirm the initial intuition given by Theorem 4.2, according to which the more demand is volatile, the more employing a mild tail test pays in order to make collusion more sustainable. Actually, when condition (27) doesn't hold, there are some regions in the parameters space where our conclusions don't hold. This happens for instance whenever τ_i is very large. These and other results from simulation are available upon request.

4.1 Results under multiplicative uncertainty

If demand follows the specification referred to as (10), then the following Theorem - viewable as a generalization of the results obtained so far for the additive case - holds:

THEOREM 4.2

If the total demand q_t follows, for any t, the multiplicative uncertainty specification as in equation (10), then (dropping the time subscript t):

1. $\widehat{\Pi^c} \sim N[\Pi^c, \sigma^2 (\Pi^c)^2]$;

2. $\widehat{\Pi_1^d} \sim N[\Pi_1^d, \sigma^2 (\Pi_1^d)^2]$;

3. $\widehat{\Pi_2^d} \sim N[\Pi_2^d, \sigma^2 (\Pi_2^d)^2]$,

with Π^c, Π_1^d, Π_2^d as in Theorems 3.1 and 3.2. Moreover:

1. $\tau_c > \tau_d$ *always*;

2. *equation (24) always holds.*

Moreover, the first three points of Theorem 4.1 hold. Also, if $\widehat{\Pi} < \Pi_2^d$:

1. *relationship (25) will always hold*;

2. *there always exists an $e > 0$ s.t.*

$$\begin{cases} \delta^* < 1 & \text{if } \sigma^2 \in [0, e) \\ \delta^* \in (1, 2) & \text{if } \sigma^2 \in (e, +\infty) \end{cases}$$

Also, for any σ^2, there exists a $\widehat{\Pi}^* > 0$ s.t. $\widehat{\Pi}^* = argmin\{\delta^*(\widehat{\Pi})\}$, with $\widehat{\Pi}^*$ monotonically increasing with σ^2.

The Proof is sketched in Appendix II. The results obtained here are almost the same as the ones holding for the additive uncertainty case, and hence no further comment is required. The only striking result, which is the opposite as what stated in Lemma 4.1, is that now the "optimal" threshold level increases as volatility increases. This means that *the higher volatility, the stricter the tail test is required to be to try and prevent the rival from cheating.*

5 Conclusions

This paper has developed a dynamic model of collusion under demand uncertainty, focusing on the role of demand volatility in determining the sustainability of tacit collusion. The work presents two main issues:

1. modeling market share under the smooth transition hypothesis;

2. deriving theorems aimed at establishing how market volatility affects tacit collusion.

Notice that the structure of the model is such that the two issues are actually independent of each other. Modeling market share in a different way would in fact result in a change in Theorems 3.1-3.3, but it wouldn't change the results (or, rather, their structure) provided by Theorems 4.1, 4.2 and Lemma 4.1.
As far as market share is concerned, to our knowledge this is the first attempt to model it under the smooth transition framework. Such a specification is based on a question that arises from the Italian experience of the telephonic service market. There exist in fact several companies selling the same product at different prices: the Bertrand framework, where the low-price firm gets all the market, seems inadequate to describe this situation.
As to the second set of conclusions, a central result in the model is that volatility affects both tacit collusion sustainability (via the impact on the discount factor) and the optimal tail test as well. We have shown that under quite general assumptions increases in volatility result in a narrower possibility for tacit collusion to be possible. In a game theoretic framework, this would mean that the equilibrium would be competition, i.e. zero profit for both firms. Also, we have derived the quite surprising result that at some times collusion is facilitated when deviations detection rule is mild, i.e. when the threshold profit is lower than the profit in case of the rival's deviation. This will hold more and more likely as uncertainty increases.
Actually, a possible extension of our assumptions is to allow α to be time-varying, i.e. to increase over time with a speed depending on the price difference $x(t)$:

$$\alpha' = \alpha'(x, p_{2t}, t)$$

This would lead to an optimal control problem, where the firm's objective would be to maximize, in case of deviation, its expected sum of discounted profits w.r.t. a vector of optimal prices changing over time. Such a problem remains for future work.

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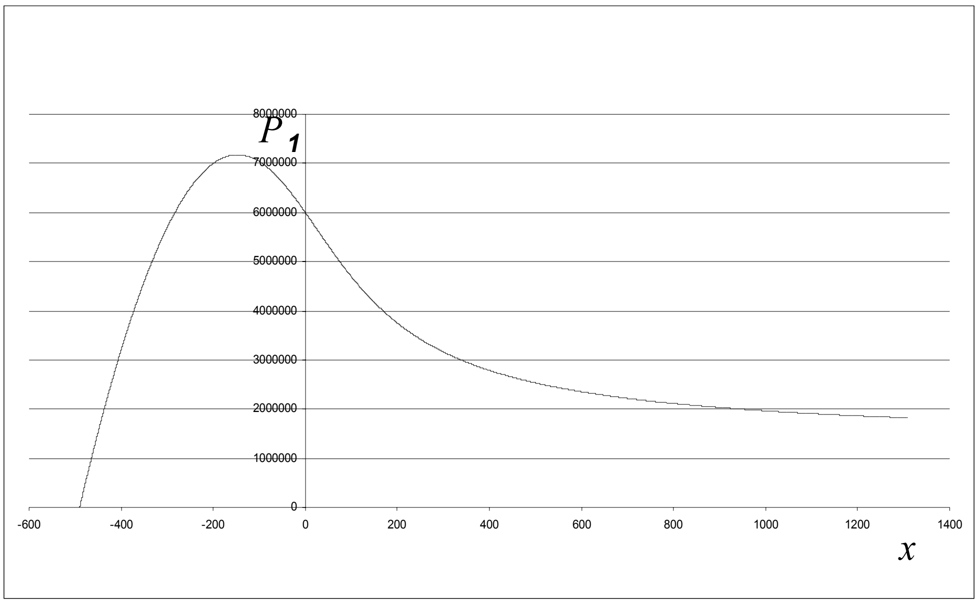
Appendix I: smooth transition market share

This Appendix considers some results from the simulation of the market share α; this is modeled, as in the rest of the paper, as being firm 1's market share.
We employed the following functional form:

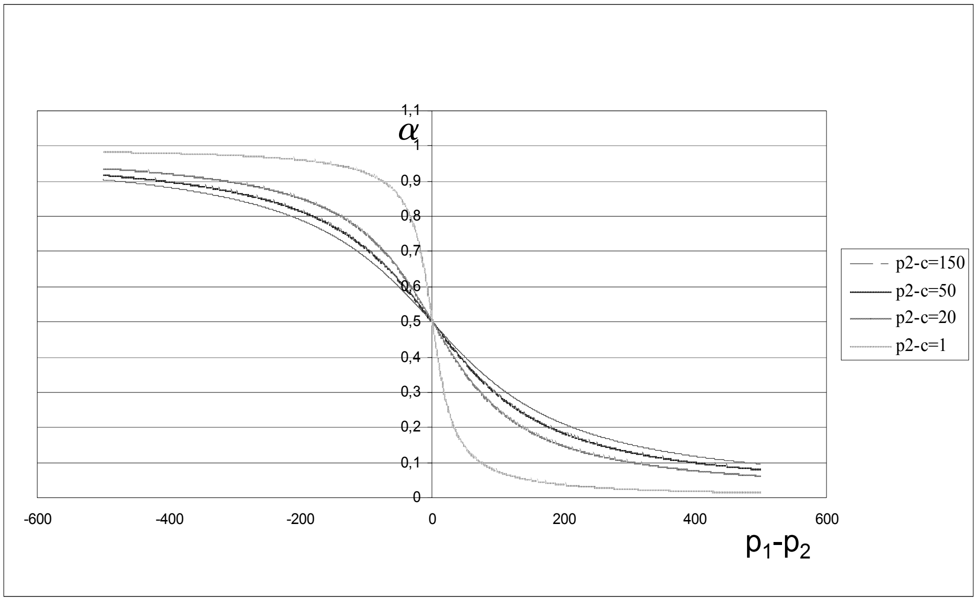
$$\alpha(x, p_2) = \frac{1}{2} - \frac{1}{\pi}\arctan\{mx[1 + \gamma exp(\frac{\delta}{(\mathrm{p}_2 - \mathrm{c})^2})]\} \tag{28}$$

It is immediate to verify that this functional specification satisfies all the properties required. Moreover, though less immediate, the optimum uniqueness condition is satisfied as well.
Condition (13) needs for some further comments. It is immediate to verify that a wide class satisfying this uniqueness requirement is the family of quasi-convex functions. Imposing quasi-convexity to Π_1^d would then be enough in order to ensure there is a unique critical point. Condition (14) and equation (5) - which will be presented in Appendix II when proving Theorem 3.2 - will then ensure this critical point to correspond to an $x < 0$. Its being a maximum would be a direct consequence

of quasi-convexity. Notice also that - if Π is the profit function and $\Pi\prime$ its first derivative w.r.t. the price - the sufficient condition for quasi-convexity based on the bordered Hessian here is ineffective. It is straightforward to prove that this condition would work iff $-(\Pi')^2 = 0$ had only one solution, which is indeed what required by relationship (13).
The following graph represents the profit Π_1^d, denoted as P as a function of the price difference x, employing the functional form (28) chosen for α. Notice how this functional specification for α *does result* in a quasi-convex profit function:



Notice that the parameters m, δ, γ are used to control the results from simulation. Also, there is actually no need of squaring firm 2's markup. The square power is to be regarded as another parameter employed to control the simulation.



Last, we point out that our *smooth*, continuous representation for the market share α embeds the case of splitting consumers. The splitting consumers case in is fact a collection of points in the (x, α) space, and by manipulating the parameters of the functional specification of α (provided they are enough), this collection of points can be embedded in the continuous space defined by α itself.

Appendix II: proofs of theorems and lemmata

Proof of Theorem 3.1

The result of problem (5) is standard in tacit collusion literature - see for example Koutsoyiannis (1992) or Tirole (1988).

Proof of Theorem 3.2

Theorem 2.2 considers two results: a non-closed form expression for q^d, and results on the nature of the price cutting, i.e. whether firm 1, under deviation, chooses to reduce or increase its own price w.r.t. the collusive solution p^c. Proof will be aimed at showing the latter.
The first derivative of firm 1's profit w.r.t. price p_1 is equal to

$$\begin{aligned}\frac{\partial}{\partial \mathrm{p}_1}\Pi_1 = \quad & \alpha'(p_1 - c)\{a - b[\alpha p_1 + (1-\alpha)p^c]\} + \alpha\{a - b[\alpha p_1 + (1-\alpha)p^c]\} + \\ & + \alpha(p_1 - c)(-b\alpha - b\alpha' p_1 + b\alpha' p^c)\end{aligned}$$

and it is a continuous function being it a transform of the continuous functions α and its first derivative. Optimum uniqueness ensures it has only one zero. Being:

$$\lim_{p_1 \to c^+} \frac{\partial}{\partial p_1}\Pi_1 = \alpha(c)\{a - b[\alpha c + (1-\alpha)p^c] > 0\}$$

and

$$\frac{\partial}{\partial p_1}\Pi_1|_{p_1=p^c} = \alpha'(0)(a - bp^c) + \frac{1}{2}(a - bp^c) + \frac{1}{2}(-\frac{1}{2}b) < 0 \tag{29}$$

due to condition (14), the zero of the first derivative must be between c - the lower bound for p_1 - and p^c; hence, $p^d \in (c, p^c)$ and therefore, by definition of α, $\alpha^d > \frac{1}{2}$. This, together with relationship (29) which states the suboptimality of p^c for problem (6), proves this part of the theorem. Last, notice that the industry's total profit is at most equal to $2\Pi^c$. Hence, $\Pi_1 + \Pi_2 \leq 2\Pi^c$ always. In case of deviation, Theorem 3.2 ensures that $\Pi_1^d > \Pi^c$, and therefore $\Pi_2^d < \Pi^c$.

Proof of Theorem 3.3

Following the same passages in the previous proof, it is immediate to extend the results of Theorem 2.2 to the case of any $p_2 > c$. Hence, for any $p_2 > c$, the solution for problem (7) will be $c < p_1^{co} < p_2$. The same will hold for problem (8) by symmetry. This means that for any $p_1 > c$ and $p_2 > c$, the two reaction functions $p_2^{co}(p_1)$ and $p_1^{co}(p_2)$ will not intersect.
When $p_2 \to c^+$, a well-known result of limit theory states that $p_1^{co} = c$ - see Pagani and Salsa (1994) for details. As the same will hold for p_2^{co}, the two reaction functions will have a common point in c. This will be the only intersection point and therefore the two reaction functions will determine the solution $p_1^{co} = p_2^{co} = p^{co} = c$. Q.E.D.

Proof of Theorem 4.1

Recall the definition of the threshold level δ^* in equation (20):

$$\frac{\Pi_1^{\mathrm{d}} - \Pi^{\mathrm{c}}}{\Pi_1^{\mathrm{d}}\widehat{\phi}^{\mathrm{c}} - \Pi^{\mathrm{c}}\widehat{\phi}^{\mathrm{d}}}$$

Consider then the following variables:

$$\widetilde{\phi}_\sigma^i = \frac{\partial\widehat{\phi}^{\mathrm{i}}}{\partial\sigma^2} = \frac{1}{2\sqrt{2\pi}(\sigma^2)^{\frac{3}{2}}} \frac{\widehat{\Pi} - \Pi^{\mathrm{i}}}{\sqrt{\tau_{\mathrm{i}}}} \exp\{-\frac{1}{2\sigma^2}\frac{(\widehat{\Pi} - \Pi^{\mathrm{i}})^2}{\tau_{\mathrm{i}}}\}$$

with

$$\rho = -\frac{(\widehat{\Pi} - \Pi_2^{\mathrm{d}})^2}{2\tau_{\mathrm{d}}} + \frac{(\widehat{\Pi} - \Pi^{\mathrm{c}})^2}{2\tau_{\mathrm{c}}}$$

The continuity and differentiability w.r.t. σ^2 follows from $\widetilde{\phi}^i$ and $\widetilde{\phi}_\sigma^i$'s being continuous δ^*'s being a continuous transform via equation (24).
Result 2) follows from equation (22).
If $\widehat{\Pi} \in (\Pi_2^d, \Pi^c)$, then after equation (23)

$$\lim_{\sigma^2 \to 0^+} \widetilde{\phi}_\sigma^c = 1 \qquad \lim_{\sigma^2 \to 0^+} \widetilde{\phi}_\sigma^d = 0$$

This proves result 3.a). Noticing that $\widetilde{\phi}_\sigma^d > 0$ and $\widetilde{\phi}_\sigma^c < 0$, given that

$$\delta_\sigma^* = \frac{\Pi_1^{\mathrm{d}} - \Pi_{\mathrm{c}}}{(\widetilde{\phi}^{\mathrm{c}}\Pi_1^{\mathrm{d}} - \widetilde{\phi}^{\mathrm{d}}\Pi^{\mathrm{c}})^2}(\widetilde{\phi}_\sigma^d\Pi^c - \widetilde{\phi}_\sigma^c\Pi_1^d) = \omega^2(\widetilde{\phi}_\sigma^d\Pi^c - \widetilde{\phi}_\sigma^c\Pi_1^d)$$

result 3.b) follows immediately. Continuity and monotonicity between $\frac{\Pi_1^{\mathrm{d}} - \Pi^{\mathrm{c}}}{\Pi_1^{\mathrm{d}}} < 1$ and 2 imply - via the zeroes theorem - that there must be one only point between 0 and $+\infty$ s.t. $\delta^* = 1$. This, together with increasing monotonicity, proves result 3.c).

If $\widehat{\Pi} < \Pi_2^d$, after equation (23)

$$\lim_{\sigma^2 \to 0^+} \widetilde{\phi}^i_\sigma = 1$$

result 4.a) follows immediately. Being moreover $\widetilde{\phi}^i_\sigma > 0$, the sign of δ^*_σ is ambiguous, in that it may have or not some zeroes depending on the solutions w.r.t. σ^2 of the equation δ^*_σ, which is equivalent to

$$\gamma \exp(\frac{\rho}{\sigma^2}) = 1$$

If both (25) and (26) hold, then the equation doesn't admit a solution in R^+, which means $\delta^*_\sigma > 0$ always. This proves result 4.b) following the same argument as for result 3.c). Otherwise, monotonicity implies a unique, positive solution which may be either a minimum or a maximum for δ^*. Being

$$\lim_{\sigma^2 \to +\infty} \delta^*_\sigma = 0^+$$

always, provided either (25) or (26) doesn't hold, it follows that the only zero for the first derivative δ^*_σ must be a minimum. Having δ^* a minimum between 1 and 2, result 4.c) follows by continuity.
Q.E.D.

Proof of Theorem 4.2

The Proof here is just sketched, a full version being available upon request but easily obtainable by the proof of Theorem 4.1.
Notice first that $\tau_c = (\sigma\Pi^c)^2 > \tau_d = (\sigma\Pi^d_2)$. As

$$\frac{\widehat{\Pi} - \Pi^{\mathrm{c}}}{\sigma\Pi^{\mathrm{c}}} < \frac{\widehat{\Pi} - \Pi^{\mathrm{d}}_2}{\sigma\Pi^{\mathrm{d}}_2}$$

it is straightforward to verify that $\widehat{\phi^c} > \widehat{\phi^d}$. This implies (24) to hold always, and particularly for any level of the threshold profit $\widehat{\Pi}$.
The extension of point 4 of Theorem 4.1 is immediate if one considers that inequality (25), which here is

$$\{\frac{\widehat{\Pi} - \Pi^{\mathrm{c}}}{\sqrt{\tau_{\mathrm{c}}}} = (\frac{\widehat{\Pi}}{\Pi^{\mathrm{c}}} - 1)\} < \{\frac{\widehat{\Pi} - \Pi^{\mathrm{d}}_2}{\sqrt{\tau_{\mathrm{d}}}} = (\frac{\widehat{\Pi}}{\Pi^{\mathrm{d}}_2} - 1)\}$$

will always hold.
Last, the extension of Lemma 4.1 follows from finding out the minimum for $\delta^*_{\widehat{\Pi}}$. This is a function of σ, and its first derivative w.r.t. σ^2 is found to be always higher than zero.