TAKEHOME EXAM ECON 7800 FALL 2000

ECONOMICS DEPARTMENT, UNIVERSITY OF UTAH

Problem 22. *4 points You are the contestant in a game show. There are four closed doors at the back of the stage. (Note: in the in-class exercise there were only three doors!) Behind one of the doors is a sports car, behind the other three doors are goats. The game master knows which door has the sports car behind it, but you don't. You have to choose one of the doors; if it is the door with the sports car, the car is yours.*

After you make your choice, say door A, the game master says: "I want to show you something." He opens one of the three other doors, let us assume it is door B, and it has a goat behind it. Then the game master asks: "Do you still insist on door A, or do you want to reconsider your choice?"

Date of exam Due Date is Thursday, November 30, 9:10 am.

Can you improve your odds of winning by abandoning your previous choice and instead selecting one of the two doors which the game master did not open? If so, by how much?

Answer. Again, you should condition on whether you had originally picked the right door or not. You picked the right door with probability $\frac{1}{4}$, and if you did and switch, you will lose. You picked the wrong door with probability $\frac{3}{4}$, and if you did and switch, you will win with probability $\frac{1}{2}$, since one of the two remaining doors must have the car behind it. Therefore your odds of winning if you switch are $\frac{1}{4}0 + \frac{3}{4}\frac{1}{2} = \frac{3}{8}$. □

Problem 23. *You are repeatedly flipping a fair coin.*

- **a.** *1 point Which sequence is in the long run more frequent: "heads, heads, heads"* or *"tails, heads, heads"? Or are they equally likely?*

Answer. They have equal likelihood to appear. For a formal proof, define the following random variables for every i: x_i is 1 if the ith coin throw is the beginning of a hhh-sequence, and 0 otherwise; and $y_i = 1$ if the ith coin throw is the beginning of a thh-sequence. Clearly, $\mathrm{E}[x_i] = \mathrm{E}[y_i] = 1/8$. The total number of hhh- and thh-sequences in the first $n+2$ throws is $\sum_{i=1}^{n} x_i$ and $\sum_{i=1}^{n} y_i$. The hhh and thh sequences are equally likely iff $\mathrm{E}[\sum x_i] = \mathrm{E}[\sum y_i]$. This equation does indeed hold; x_i is not indepedent of x_{i+1} or x_{i-1} etc., but the expected values of the sums are the sums of the expected values even if the random variables involved are not independent. □

- **b.** *3 points If you start flipping a coin and continue to flip until one of the two above sequences has occured, which sequence is more likely to come first, and what is* *the probability that it will come first? (It is recommended that you actually do this experiment a few times and count how often each sequence comes first.)*

Answer. They can have different probabilities to come first although in the long run both sequences will appear equally often, because their appearances are not independent of each other. If you start with a tail, then the tails, heads, heads sequence comes before the heads, heads, heads sequence. Therefore the heads heads heads sequence can appear first only if the first three throws all are heads, i.e., with probability 1/8. □

Problem 24. *6 points Consider a study in epidemiology where a number of people are diagnosed as having a certain disease during a certain calendar year, and then are followed until they die. It is a disease where nobody has ever lived more than $n \geq 1$ years after diagnosis, and where every day in the n years after diagnosis is approximately equally likely as a date of death. We are at the end of the year and therefore we only have the death dates of those who died in the same year that they were diagnosed. Assume the diagnosis date is also uniformly distributed over the year. To fix notation, say x is the date when they are diagnosed, which has a uniform distribution over the year, and z is the date of their death. Both dates are measured in years, so that $x = 1$ or $z = 1$ means the end of the year. Show that* $\mathrm{E}[x|z \leq 1] = 1/3$ *and* $\mathrm{E}[z|z \leq 1] = 2/3$.

Answer. $\mathrm{E}[x|z \le 1] = 1/3$ means: expected date of diagnosis is 1/3, i.e., 4 months, given that the person dies within one year of diagnosis. $\mathrm{E}[z|z \le 1] = 2/3$ means: expected date of death is 2/3, i.e., 8 months, given that the person dies within one year of diagnosis.

Say x is the date when they are diagnosed, which has a uniform distribution over the year. Define y to be the time between diagnosis and death, which has a uniform distribution between 0 and n years. It is much easier to work with x and y than with x and z, since x and y are independent. We want to know $\mathrm{E}[x|x + y \le 1]$ and $\mathrm{E}[x + y|x + y \le 1]$. Now the joint distribution of x and y is a uniform distribution in the rectangle $(0, 0)$, $(1, 0)$, $(0, n)$, $(1, n)$. If we know that $x + y \le 1$ we just have the triangle from $(0, 0)$ to $(0, 1)$ to $(1, 0)$. For this conditional density it is easy to see that $\mathrm{E}[x|z \le 1] = 1/3$ and $\mathrm{E}[z|z \le 1] = 2/3$. □

Problem 25. *In this exercise we will write the bivariate normal density in its most natural form. For this we set the multiplicative "nuisance parameter" $\sigma^2 = 1$, i.e., write the covariance matrix as $\boldsymbol{\Psi}$ instead of $\sigma^2\boldsymbol{\Psi}$. This Problem is slightly different than the class notes because we are not assuming that the vector has zero mean.*

- **a.** *1 point Write the covariance matrix $\boldsymbol{\Psi} = \mathcal{V}[\begin{bmatrix} u \\ v \end{bmatrix}]$ in terms of the standard deviations σ_u and σ_v and the correlation coefficient ρ.*

- **b.** *1 point Show that the inverse of a 2×2 matrix has the following form:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1}$$

• **c.** *2 points Show that*

$$q^2 = \begin{bmatrix} u-\mu & v-\nu \end{bmatrix} \boldsymbol{\Psi}^{-1} \begin{bmatrix} u-\mu \\ v-\nu \end{bmatrix} \tag{2}$$

$$= \frac{1}{1-\rho^2}\Big(\frac{(u-\mu)^2}{\sigma_u^2} - 2\rho\frac{u-\mu}{\sigma_u}\frac{v-\nu}{\sigma_v} + \frac{(v-\nu)^2}{\sigma_v^2}\Big). \tag{3}$$

• **d.** *2 points Show the following quadratic decomposition:*

$$q^2 = \frac{(u-\mu)^2}{\sigma_u^2} + \frac{1}{(1-\rho^2)\sigma_v^2}\Big(v-\nu-\rho\frac{\sigma_v}{\sigma_u}(u-\mu)\Big)^2. \tag{4}$$

• **e.** *1 point Show that (4) can also be written in the form*

$$q^2 = \frac{(u-\mu)^2}{\sigma_u^2} + \frac{\sigma_u^2}{\sigma_u^2\sigma_v^2-(\sigma_{uv})^2}\Big(v-\nu-\frac{\sigma_{uv}}{\sigma_u^2}(u-\mu)\Big)^2. \tag{5}$$

• **f.** *1 point Show that $d = \sqrt{\det \boldsymbol{\Psi}}$ can be split up, not additively but multiplicatively, as follows: $d = \sigma_u \cdot \sigma_v\sqrt{1-\rho^2}$.*

• **g.** *1 point Using these decompositions of d and q^2, show that the density function $f_{u,v}(u,v)$ reads*

$$\frac{1}{\sqrt{2\pi\sigma_u^2}}\exp\Big(-\frac{(u-\mu)^2}{2\sigma_u^2}\Big)\cdot\frac{1}{\sqrt{2\pi\sigma_v^2}\sqrt{1-\rho^2}}\exp\Big(-\frac{\big((v-\nu)-\rho\frac{\sigma_v}{\sigma_u}(u-\mu)\big)^2}{2(1-\rho^2)\sigma_v^2}\Big). \tag{6}$$

Problem 26. *The evaluation of two intelligence tests, one at the beginning of the semester, one at the end, gives the following disturbing outcome: While the underlying intelligence during the first test was $z \sim N(100, 20)$, it changed between the first and second test due to the learning experience at the university. If w is the intelligence of each student at the second test, it is connected to his intelligence z at the first test by the formula $w = 0.5z + 50$, i.e., those students with intelligence below 100 gained, but those students with intelligence above 100 lost. (The errors of both intelligence tests are normally distributed with expected value zero, and the variance of the first intelligence test was 5, and that of the second test, which had more questions, was 4. As usual, the errors are independent of each other and of the actual intelligence.)*

- **a.** *3 points If x and y are the outcomes of the first and second intelligence test, compute $\mathrm{E}[x]$, $\mathrm{E}[y]$, $\mathrm{var}[x]$, $\mathrm{var}[y]$, and the correlation coefficient $\rho = \mathrm{corr}[x, y]$. Figure 1 shows an equi-density line of their joint distribution; 95% of the probability mass of the test results are inside this ellipse. Draw the line $w = 0.5z + 50$ into Figure 1.*

Answer. We know $z \sim N(100, 20)$; $w = 0.5z + 50$; $x = z + \varepsilon$; $\varepsilon \sim N(0, 4)$; $y = w + \delta$; $\delta \sim N(0, 5)$; therefore $\mathrm{E}[x] = 100$; $\mathrm{E}[y] = 100$; $\mathrm{var}[x] = 20 + 5 = 25$; $\mathrm{var}[y] = 5 + 4 = 9$; $\mathrm{cov}[x, y] = 10$; $\mathrm{corr}[x, y] = 10/15 = 2/3$. In matrix notation

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\Bigl[\begin{bmatrix} 100 \\ 100 \end{bmatrix}, \begin{bmatrix} 25 & 10 \\ 10 & 9 \end{bmatrix}\Bigr] \tag{7}$$

The line $y = 50 + 0.5x$ goes through the points $(80, 90)$ and $(120, 110)$. □

• **b.** *4 points Here is the general formula for the conditional mean of two jointly normal variables u and v with mean μ_u and μ_v, standard deviations σ_u and σ_v, and correlation coefficient ρ:*

$$\mathrm{E}[v|u = u] = \mu_v + \rho\frac{\sigma_v}{\sigma_u}(u - \mu_u). \tag{8}$$

Since $\rho = \frac{\mathrm{cov}[u,v]}{\sigma_u \sigma_v}$, (8) can als be written as follows:

$$\mathrm{E}[v|u = u] = \mathrm{E}[v] + \frac{\mathrm{cov}[u, v]}{\mathrm{var}[u]}(u - \mathrm{E}[u]). \tag{9}$$

Compute $\mathrm{E}[y|x=x]$ and $\mathrm{E}[x|y=y]$. The first is a linear function of x and the second a linear function of y. Draw the two lines representing these linear functions into Figure 1.

Answer.

$$\mathrm{E}[y|x=x] = 100 + \frac{10}{25}(x - 100) = 60 + \frac{2}{5}x \tag{10}$$

$$\mathrm{E}[x|y=y] = 100 + \frac{10}{9}(y - 100) = -\frac{100}{9} + \frac{10}{9}y. \tag{11}$$

The line $y = \mathrm{E}[y|x=x]$ goes through the points $(80, 92)$ and $(120, 108)$ at the edge of Figure 1; it intersects the ellipse where it is vertical. The line $x = \mathrm{E}[x|y=y]$ goes through the points $(80, 82)$ and

$(120, 118)$, which are the corner points of Figure 1; it intersects the ellipse where it is horizontal. The two lines intersect in the center of the ellipse, i.e., at the point $(100, 100)$.

□

• **c.** *2 points Another researcher says that* $w = \frac{6}{10}z + 40$, $z \sim N(100, \frac{100}{6})$, $\varepsilon \sim N(0, \frac{50}{6})$, $\delta \sim N(0, 3)$. *Is this compatible with the data?*

Answer. Yes, it is compatible: $\mathrm{E}[x] = \mathrm{E}[z] + \mathrm{E}[\varepsilon] = 100$; $\mathrm{E}[y] = \mathrm{E}[w] + \mathrm{E}[\delta] = \frac{6}{10}100 + 40 = 100$; $\mathrm{var}[x] = \frac{100}{6} + \frac{50}{6} = 25$; $\mathrm{var}[y] = \left(\frac{6}{10}\right)^2 \mathrm{var}[z] + \mathrm{var}[\delta] = \frac{63}{100}\frac{100}{6} + 3 = 9$; $\mathrm{cov}[x, y] = \frac{6}{10}\mathrm{var}[z] = 10$. □

• **d.** *4 points A third researcher asserts that the IQ of the students really did not change. He says* $w = z$, $z \sim N(100, 5)$, $\varepsilon \sim N(0, 20)$, $\delta \sim N(0, 4)$. *Is this compatible with the data? Is there unambiguous evidence in the data that the IQ declined?*

Answer. This is not compatible. This scenario gets everything right except the covariance: $\mathrm{E}[x] = \mathrm{E}[z] + \mathrm{E}[\varepsilon] = 100$; $\mathrm{E}[y] = \mathrm{E}[z] + \mathrm{E}[\delta] = 100$; $\mathrm{var}[x] = 5 + 20 = 25$; $\mathrm{var}[y] = 5 + 4 = 9$; $\mathrm{cov}[x, y] = 5$. A scenario in which both tests have same underlying intelligence cannot be found. Since the two conditional expectations are on the same side of the diagonal, the hypothesis that the intelligence did not change between the two tests is not consistent with the joint distribution of x and y. The diagonal goes through the points $(82, 82)$ and $(118, 118)$, i.e., it intersects the two horizontal boundaries of Figure 1. □

Problem 27. *Assume* $y_i \sim \mathrm{NID}(0, \sigma^2)$ *(i.e., normally independently distributed) with unknown* σ^2. *The obvious estimate of* σ^2 *is* $s^2 = \frac{1}{n}\sum y_i^2$.

• **a.** *2 points Show that* s^2 *is an unbiased estimator of* σ^2*, is distributed* $\sim \frac{\sigma^2}{n}\chi^2_n$*, and has variance* $2\sigma^4/n$*. You are allowed to use the fact that a* χ^2_n *has variance* $2n$*, which is equation* (**??**)*.*

Answer.

$$\mathrm{E}[y_i^2] = \mathrm{var}[y_i] + (\mathrm{E}[y_i])^2 = \sigma^2 + 0 = \sigma^2 \tag{12}$$

$$z_i = \frac{y_i}{\sigma} \sim \mathrm{NID}(0, 1) \tag{13}$$

$$y_i = \sigma z_i \tag{14}$$

$$y_i^2 = \sigma^2 z_i^2 \tag{15}$$

$$\sum_{i=1}^n y_i^2 = \sigma^2 \sum_{i=1}^n z_i^2 \sim \sigma^2 \chi^2_n \tag{16}$$

$$\frac{1}{n}\sum_{i=1}^n y_i^2 = \frac{\sigma^2}{n}\sum_{i=1}^n z_i^2 \sim \frac{\sigma^2}{n}\chi^2_n \tag{17}$$

$$\mathrm{var}\Big[\frac{1}{n}\sum_{i=1}^n y_i^2\Big] = \frac{\sigma^4}{n^2}\,\mathrm{var}[\chi^2_n] = \frac{\sigma^4}{n^2}2n = \frac{2\sigma^4}{n} \tag{18}$$

□

• **b.** *4 points Show that this variance is at the same time the Cramer Rao lower bound. Here is the Cramer Rao inequality: Assume* $y_1, \ldots, y_n$ *are* n *independent* *observations of a random variable y whose density function depends on the unknown parameter θ and satisfies certain regularity conditions. Write this density function as $f_y(y;\theta)$ and let t be any unbiased estimator of θ. Then*

$$\operatorname{var}[t] \geq \frac{1}{n \operatorname{E}[\left(\frac{\partial}{\partial\theta} \log f_y(y;\theta)\right)^2]} = \frac{-1}{n \operatorname{E}[\frac{\partial^2}{\partial\theta^2} \log f_y(y;\theta)]}. \tag{19}$$

Answer.

$$\ell(y,\sigma^2) = \log f_y(y;\sigma^2) = -\frac{1}{2}\log 2\pi - \frac{1}{2}\log\sigma^2 - \frac{y^2}{2\sigma^2} \tag{20}$$

$$\frac{\partial \log f_y}{\partial\sigma^2}(y;\sigma^2) = -\frac{1}{2\sigma^2} + \frac{y^2}{2\sigma^4} = \frac{y^2-\sigma^2}{2\sigma^4} \tag{21}$$

Since $\dfrac{y^2-\sigma^2}{2\sigma^4}$ has zero mean, it follows

$$\operatorname{E}[\left(\frac{\partial \log f_y}{\partial\sigma^2}(y;\sigma^2)\right)^2] = \frac{\operatorname{var}[y^2]}{4\sigma^8} = \frac{1}{2\sigma^4}. \tag{22}$$

Alternatively, one can differentiate one more time:

$$\frac{\partial^2 \log f_y}{(\partial\sigma^2)^2}(y;\sigma^2) = -\frac{y^2}{\sigma^6} + \frac{1}{2\sigma^4} \tag{23}$$

$$\mathrm{E}[\frac{\partial^2 \log f_y}{(\partial\sigma^2)^2}(y;\sigma^2)] = -\frac{\sigma^2}{\sigma^6} + \frac{1}{2\sigma^4} = \frac{1}{2\sigma^4} \tag{24}$$

(25)

This makes the Cramer Rao lower bound $2\sigma^4/n$. □

Problem 28. *Our universal set U consists of patients who have a certain disease. We will explore the causal effect of a given treatment with the help of three events, T, C, and S, the first two of which are* counterfactual, *compare* [Hol86]. *These events are defined as follows: T consists of all patients who would recover if given treatment; C consists of all patients who would recover if not given treatment (i.e., if included in the control group). The event S consists of all patients actually receiving treatment. The average causal effect of the treatment is defined as* $\Pr[T] - \Pr[C]$.

- **a.** *2 points Show that*

$$\Pr[T] = \Pr[T|S]\Pr[S] + \Pr[T|S'](1 - \Pr[S]) \tag{26}$$

and that

$$\Pr[C] = \Pr[C|S]\Pr[S] + \Pr[C|S'](1 - \Pr[S]) \tag{27}$$

Which of these probabilities can be estimated as the frequencies of observable outcomes and which cannot?

Answer. This is a direct application of (**??**). The problem here is that for all $\omega \in C$, i.e., for those patients who do not receive treatment, we do not know whether they would have recovered if given treatment, and for all $\omega \in T$, i.e., for those patients who do receive treatment, we do not know whether they would have recovered if not given treatment. In other words, neither $\Pr[T|S]$ nor $\mathrm{E}[C|S']$ can be estimated as the frequencies of observable outcomes. □

• **b.** *2 points Assume now that S is independent of T and C, because the subjects are assigned randomly to treatment or control. How can this be used to estimate those elements in the equations (26) and (27) which could not be estimated before?*

Answer. In this case, $\Pr[T|S] = \Pr[T|S']$ and $\Pr[C|S'] = \Pr[C|S]$. Therefore, the average causal effect can be simplified as follows:

$$\begin{aligned}
\Pr[T] - \Pr[C] &= \Pr[T|S]\Pr[S] + \Pr[T|S'](1 - \Pr[S]) - \Pr[C|S]\Pr[S] + \Pr[C|S'](1 - \Pr[S]) \\
&= \Pr[T|S]\Pr[S] + \Pr[T|S](1 - \Pr[S]) - \Pr[C|S']\Pr[S] + \Pr[C|S'](1 - \Pr[S]) \\
&= \Pr[T|S] - \Pr[C|S'] \qquad (28)
\end{aligned}$$

□

• **c.** *2 points Why were all these calculations necessary? Could one not have defined from the beginning that the causal effect of the treatment is* $\Pr[T|S] - \Pr[C|S']$*?*

Answer. $\Pr[T|S] - \Pr[C|S']$ is only the empirical difference in recovery frequencies between those who receive treatment and those who do not. It is always possible to measure these differences, but these differences are not necessarily due to the treatment but may be due to other reasons. □

-

Problem 29. *8 points The daily levels of the exchange rate of the Pound Sterling (£) against the US \$, taken at noon in New York City, from 1990 to the present, are published at*

www.federalreserve.gov/releases/H10/hist/dat96_uk.txt

and the data from 1971–1989 are at

www.federalreserve.gov/releases/H10/hist/dat89_uk.txt

Download these data, make a high quality plot, import the plot into your wordprocessor, and write a short essay describing what you see.

Answer. Figure 2 plots the daily levels of the exchange rate. It has a lot of detail which one can only see if one magnifies the plot on the pdf-reader. Similar graphs are in [WH97, p. 67] and [Gut94, Figure 14.1 on p. 370].

The following description of what you see in an exchange rate graph borrows heavily from [Gut94, p. 369].

Secular trend: In long run exchange rates reflect a country's competitiveness in the international hierarchy of nations. When a country manages to strengthen its competitive position in the world market, its external accounts improve and its currency appreciates (e.g. Germany, Japan). The reverse happens if a country faces gradual erosion (Great Britain, United States). Therefore you see steady runs of gradual linear advances or declines over many years.

Business cycle: Woven around this secular trend are cycles of 4–7 years. "This pattern suggests that exchange-rate movements trigger counteracting adjustments in goods and assets markets. But these effects take time to unfold, and in the meantime foreign-exchange markets overshoot. The overshooting sets the stage for the next phase of the cycle, when it has finally begun to turn around such economic fundamentals as inflation and the direction of macroeconomic policy."

Is there an even shorter cycle due to inventories and the time it takes to find new suppliers?

Shorter-term exchange rate fluctuations lasting a few weeks or months are due to expectations and speculation: "At times expectational biases are widely differentiated, and the markets move sideways. But most of the time we can see pronounced price movements in one direction reflecting widely shared market sentiments. These speculative "runs" usually last a few weeks or months before being temporarily interrupted by even shorter countermovements. Because runs outweigh corrections, they reinforce whatever phase of the currency cycle we are in."

Daily variability: Despite these regularities, exchange rates are very volatile in the short run: they often fluctuate 1–2% per day.

Then there are several complete changes in regime which are due to institutional changes in the monetary system. In August 1971, when Nixon abolished the convertibility of the dollar, at the beginning of 1993, with increasing European monetary integration, and at the beginning of 1999, with the introduction of the euro.

□

Maximum number of points: 56.

References

[Gut94] Robert Guttmann, *How credit money shapes the economy*, Sharpe, Armonk, 1994. 13

[Hol86] Paul W. Holland, *Statistics and causal inference*, JASA **81** (1986), no. 396, 945–960. 11

[WH97] Mike West and Jeff Harrison, *Bayesian forecasting and dynamic models*, second ed., Springer-Verlag, 1997. 13

Economics Department, University of Utah, 1645 Campus Center Drive, Salt Lake City, UT 84112-9300, U.S.A

E-mail address: `ehrbar@econ.utah.edu`

URL: `http://www.econ.utah.edu/ehrbar`

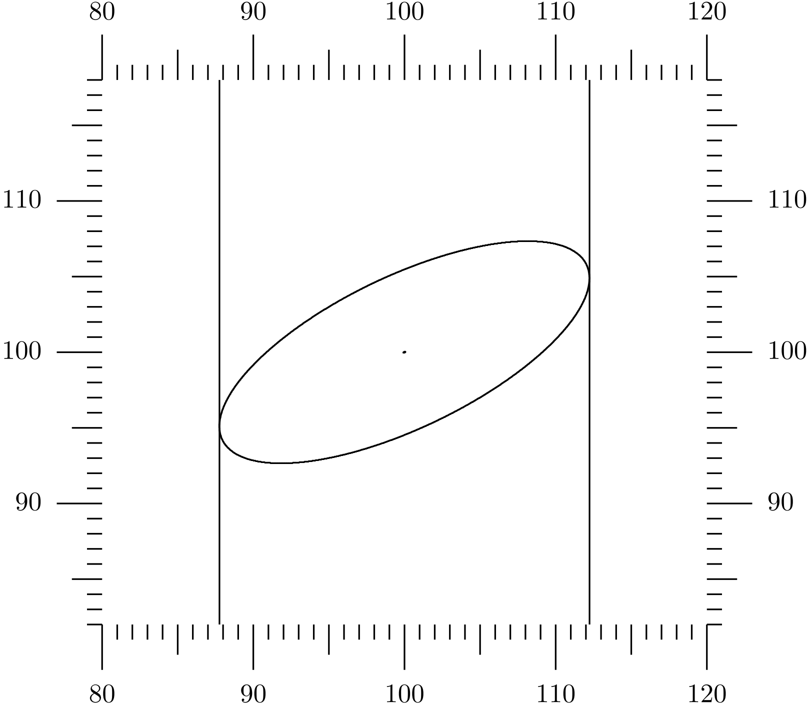
80
90
100
110
120
110
100
90
110
100
90
80
90
100
110
120

FIGURE 2. Exchange Rate of Pound in terms of Dollar