FIRST MIDTERM ECON 7800 FALL 2000

ECONOMICS DEPARTMENT, UNIVERSITY OF UTAH

Problem 1. *3 points Draw a Venn Diagram which shows the validity of de Morgan's laws: $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$. If done right, the same Venn diagram can be used for both proofs.*

Answer. There is a proof in [HT83, p. 12]. Draw A and B inside a box which represents U, and shade A' from the left (blue) and B' from the right (yellow), so that $A' \cap B'$ is cross shaded (green); then one can see these laws. □

Problem 2. *2 points (Bonferroni inequality) Let A and B be two events. Writing $\Pr[A] = 1 - \alpha$ and $\Pr[B] = 1 - \beta$, show that $\Pr[A \cap B] \geq 1 - (\alpha + \beta)$. You are allowed to use that $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$ (Problem **??**), and that all probabilities are ≤ 1.*

Date of exam Tuesday, September 26, 2000, 9:10–10:30 pm in the usual classroom.

Answer.

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B] \le 1 \tag{1}$$

$$\Pr[A] + \Pr[B] \le 1 + \Pr[A \cap B] \tag{2}$$

$$\Pr[A] + \Pr[B] - 1 \le \Pr[A \cap B] \tag{3}$$

$$1 - \alpha + 1 - \beta - 1 = 1 - \alpha - \beta \le \Pr[A \cap B] \tag{4}$$

□

Problem 3. *2 points A and B are arbitrary events. Prove that the probability of B can be written as:*

$$\Pr[B] = \Pr[B|A]\Pr[A] + \Pr[B|A']\Pr[A'] \tag{5}$$

Answer. $B = B \cap U = B \cap (A \cup A') = (B \cap A) \cup (B \cap A')$ and this union is disjoint, i.e., $(B \cap A) \cap (B \cap A') = B \cap (A \cap A') = B \cap \emptyset = \emptyset$. Therefore $\Pr[B] = \Pr[B \cap A] + \Pr[B \cap A']$. Now apply definition of conditional probability to get $\Pr[B \cap A] = \Pr[B|A]\Pr[A]$ and $\Pr[B \cap A'] = \Pr[B|A']\Pr[A']$. □

Problem 4. *The first three questions here are discussed in* [Lar82, example 2.6.3 on p. 62]*: There is an urn with 4 white and 8 black balls. You take two balls out without replacement.*

- **a.** *1 point What is the probability that the first ball is white?*

Answer. 1/3 □

- **b.** *1 point What is the probability that both balls are white?*

Answer. It is Pr[second ball white|first ball white] Pr[first ball white] $= \frac{3}{3+8}\frac{4}{4+8} = \frac{1}{11}$. □

• **c.** *1 point What is the probability that the second ball is white?*

Answer. It is Pr[first ball white and second ball white]+Pr[first ball black and second ball white] =

$$= \frac{3}{3+8}\frac{4}{4+8} + \frac{4}{7+4}\frac{8}{8+4} = \frac{1}{3}. \tag{6}$$

This is the same as the probability that the first ball is white. The probabilities are not dependent on the order in which one takes the balls out. This property is called "exchangeability." One can see it also in this way: Assume you number the balls at random, from 1 to 12. Then the probability for a white ball to have the number 2 assigned to it is obviously $\frac{1}{3}$. □

• **d.** *1 point What is the probability that both of them are black?*

Answer. $\frac{8}{12}\frac{7}{11} = \frac{2}{3}\frac{7}{11} = \frac{14}{33}$ (or $\frac{56}{132}$). □

• **e.** *1 point What is the probability that both of them have the same color?*

Answer. The sum of the two above, $\frac{14}{33} + \frac{1}{11} = \frac{17}{33}$ (or $\frac{68}{132}$). □

Problem 5. *2 points A friend tosses two coins. You ask: "did one of them land heads?" Your friend answers, "yes." What's the probability that the other also landed heads?*

Answer. $U = \{HH, HT, TH, TT\}$; Probability is $\frac{1}{4}/\frac{3}{4} = \frac{1}{3}$. □

Problem 6. *AIDS diagnostic tests are usually over 99.9% accurate on those who do not have AIDS (i.e., only 0.1% false positives) and 100% accurate on those who have AIDS (i.e., no false negatives at all). (A test is called positive if it indicates that the subject has AIDS.)*

- **a.** *3 points Assuming that 0.5% of the population actually have AIDS, compute the probability that a particular individual has AIDS, given that he or she has tested positive.*

Answer. A is the event that he or she has AIDS, and T the event that the test is positive.

$$\Pr[A|T] = \frac{\Pr[T|A]\Pr[A]}{\Pr[T|A]\Pr[A] + \Pr[T|A']\Pr[A']} = \frac{1 \cdot 0.005}{1 \cdot 0.005 + 0.001 \cdot 0.995} =$$
$$= \frac{100 \cdot 0.5}{100 \cdot 0.5 + 0.1 \cdot 99.5} = \frac{1000 \cdot 5}{1000 \cdot 5 + 1 \cdot 995} = \frac{5000}{5995} = \frac{1000}{1199} = 0.834028$$

Even after testing positive there is still a 16.6% chance that this person does not have AIDS. □

- **b.** *1 point If one is young, healthy and not in one of the risk groups, then the chances of having AIDS are not 0.5% but 0.1% (this is the proportion of the applicants to the military who have AIDS). Re-compute the probability with this alternative number.*

Answer.

$$\frac{1 \cdot 0.001}{1 \cdot 0.001 + 0.001 \cdot 0.999} = \frac{100 \cdot 0.1}{100 \cdot 0.1 + 0.1 \cdot 99.9} = \frac{1000 \cdot 1}{1000 \cdot 1 + 1 \cdot 999} = \frac{1000}{1000 + 999} = \frac{1000}{1999} = 0.50025.$$

□

Problem 7. *2 points A and B are two independent events with $\Pr[A] = \frac{1}{3}$ and $\Pr[B] = \frac{1}{4}$. Compute $\Pr[A \cup B]$.*

Answer. $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B] = \Pr[A] + \Pr[B] - \Pr[A]\Pr[B] = \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{2}$. □

Problem 8. *2 points Flip a coin two times independently and define the following three events:*

$$\begin{aligned} A &= \text{Head in first flip} \\ B &= \text{Head in second flip} \qquad (7) \\ C &= \text{Same face in both flips.} \end{aligned}$$

Are these three events pairwise independent? Are they mutually independent?

Answer. $U = \left\{ \begin{smallmatrix} HH & HT \\ TH & TT \end{smallmatrix} \right\}$. $A = \{HH, HT\}$, $B = \{HH, TH\}$, $C = \{HH, TT\}$. $\Pr[A] = \frac{1}{2}$, $\Pr[B] = \frac{1}{2}$, $\Pr[C] = \frac{1}{2}$. They are pairwise independent, but $\Pr[A \cap B \cap C] = \Pr[\{HH\}] = \frac{1}{4} \neq \Pr[A]\Pr[B]\Pr[C]$, therefore the events cannot be mutually independent. □

Problem 9. *4 points Assume the distribution of z is symmetric about zero, i.e., $\Pr[z < -z] = \Pr[z > z]$ for all z. Call its cumulative distribution function $F_z(z)$. Show that the cumulative distribution function of the random variable $q = z^2$ is $F_q(q) = 2F_z(\sqrt{q}) - 1$ for $q \geq 0$, and 0 for $q < 0$.*

Answer. If $q \geq 0$ then

$$F_q(q) = \Pr[z^2 \leq q] = \Pr[-\sqrt{q} \leq z \leq \sqrt{q}] \tag{8}$$
$$= \Pr[z \leq \sqrt{q}] - \Pr[z < -\sqrt{q}] \tag{9}$$
$$= \Pr[z \leq \sqrt{q}] - \Pr[z > \sqrt{q}] \tag{10}$$
$$= F_z(\sqrt{q}) - (1 - F_z(\sqrt{q})) \tag{11}$$
$$= 2F_z(\sqrt{q}) - 1. \tag{12}$$

□

Problem 10. *4 points* [Ame94, example 3.6.1 on p. 49] *Suppose y has density function*

$$f_y(y) = \begin{cases} 1 & \textit{for } 0 \leq y \leq 1 \\ 0 & \textit{otherwise.} \end{cases} \tag{13}$$

Obtain the density $f_q(q)$ of the random variable $q = y^2$.

Answer. (1) Since y takes values only between 0 and 1 (inclusively), its square also takes values between 0 and 1 (inclusively), i.e., $A = \{q : 0 \leq q \leq 1\}$. (2) Express y, whose density we know, in terms of q, whose density we want to know: since y cannot have negative values, we can infer its values uniquely from q: $y = \sqrt{q}$. (The square-root symbol always denotes the nonnegative square root). (3) The derivative is $1/(2\sqrt{q})$. (4) Plugging $y = \sqrt{q}$ into the density function always gives the value 1 since $y = \sqrt{q}$ and $q \in A$ implies $0 \leq y \leq 1$. Therefore the density function is $f_q(q) = 1/(2\sqrt{q})$ for $0 \leq q \leq 1$ and 0 otherwise.

□

Problem 11. *You perform a Bernoulli experiment, i.e., an experiment which can only have two outcomes, success s and failure f. The probability of success is p.*

• **a.** *3 points You make 4 independent trials. Show that the probability that the first trial is successful, given that the total number of successes in the 4 trials is 3, is 3/4.*

Answer. Let $B = \{sfff, sffs, sfsf, sfss, ssff, ssfs, sssf, ssss\}$ be the event that the first trial is successful, and let $\{x{=}3\} = \{fsss, sfss, ssfs, sssf\}$ be the event that there are 3 successes, it has $\binom{4}{3} = 4$ elements. Then

$$\Pr[B|x{=}3] = \frac{\Pr[B \cap \{x{=}3\}]}{\Pr[x{=}3]} \tag{14}$$

Now $B \cap \{x{=}3\} = \{sfss, ssfs, sssf\}$, which has 3 elements. Therefore we get

$$\Pr[B|x{=}3] = \frac{3 \cdot p^3(1-p)}{4 \cdot p^3(1-p)} = \frac{3}{4}. \tag{15}$$

□

• **b.** *2 points Discuss this result.*

Answer. It is significant that this probability is independent of p. I.e., once we know how many successes there were in the 4 trials, knowing the true p does not help us computing the probability of the event. From this also follows that the outcome of the event has no information about p. The value 3/4 is the same as the unconditional probability if $p = 3/4$. I.e., whether we know that the true frequency, the one that holds in the long run, is 3/4, or whether we know that the actual frequency in this sample is 3/4, both will lead us to the same predictions regarding the first throw. But not all conditional probabilities are equal to their unconditional counterparts: the conditional probability to get 3 successes in the first 4 trials is 1, but the unconditional probability is of course not 1. □

Problem 12. *4 points Assume your data show that counties with high rates of unemployment also have high rates of heart attacks. Can one conclude from this that the unemployed have a higher risk of heart attack? Discuss, besides the "ecological fallacy," also other objections which one might make against such a conclusion.*

Answer. Ecological fallacy says that such a conclusion is only legitimate if one has individual data. Perhaps a rise in unemployment is associated with increased pressure and increased workloads among the employed, therefore it is the employed, not the unemployed, who get the heart attacks. Even if one has individual data one can still raise the following objection: perhaps unemployment and heart attacks are both consequences of a third variable (both unemployment and heart attacks depend on age or education, or freezing weather in a farming community causes unemployment for workers and heart attacks for the elderly).

□

Problem 13. *2 points You make two independent trials of a Bernoulli experiment with success probability θ, and you observe t, the number of successes. Compute the expected value of t^3.*

Answer. $\Pr[t = 0] = (1-\theta)^2$; $\Pr[t = 1] = 2\theta(1-\theta)$; $\Pr[t = 2] = \theta^2$. Therefore an application of (**??**) gives $\mathrm{E}[t^3] = 0^3 \cdot (1-\theta)^2 + 1^3 \cdot 2\theta(1-\theta) + 2^3 \cdot \theta^2 = 2\theta + 6\theta^2$. □

Problem 14. *2 points* [CT91, example 2.1.2 on pp. 14/15]*: The experiment has four possible outcomes; outcome $x{=}a$ occurs with probability 1/2, $x{=}b$ with probability 1/4, $x{=}c$ with probability 1/8, and $x{=}d$ with probability 1/8. Suppose we wish to determine the outcome of this experiment with the minimum number of questions. An efficient first question is "Is $x{=}a$?" This splits the probability in half. If the answer to the first question is no, then the second question can be "Is $x{=}b$?" The third question, if it is necessary, can then be: "Is $x{=}c$?" Compute the expected number of binary questions required.*

Answer. For each outcome write down the probability and the number of questions required in this scheme, then multiply the probability with the number of questions, and then add. Expected value is $\frac{7}{4}$. □

Problem 15. *Assume t is a geometric random variable with parameter p, i.e., it has the values $k = 1, 2, \ldots$ with probabilities*

$$p_t(k) = pq^{k-1}, \text{ where } q = 1 - p. \tag{16}$$

The geometric variable denotes the number of times one has to perform a Bernoulli experiment with success probability p to get the first success.

• **a.** *1 point Given a positive integer n. What is $\Pr[t>n]$? (Easy with a simple trick!)*

Answer. $t>n$ means, the first n trials must result in failures, i.e., $\Pr[t>n] = q^n$. Since $\{t > n\} = \{t = n + 1\} \cup \{t = n + 2\} \cup \cdots$, one can also get the same result in a more tedious way: It is $pq^n + pq^{n+1} + pq^{n+2} + \cdots = s$, say. Therefore $qs = pq^{n+1} + pq^{n+2} + \cdots$, and $(1-q)s = pq^n$; since $p = 1 - q$, it follows $s = q^n$. □

• **b.** *2 points Let m and n be two positive integers with $m < n$. Show that $\Pr[t=n|t>m] = \Pr[t=n-m]$.*

Answer. $\Pr[t=n|t>m] = \frac{\Pr[t=n]}{\Pr[t>m]} = \frac{pq^{n-1}}{q^m} = pq^{n-m-1} = \Pr[t=n-m]$. □

• **c.** *1 point Why is this property called the memory-less property of the geometric random variable?*

Answer. If you have already waited for m periods without success, the probability that success will come in the nth period is the same as the probability that it comes in $n - m$ periods if you start now. Obvious if you remember that geometric random variable is time you have to wait until 1st success in Bernoulli trial.

□

Problem 16. *2 points An exponential random variable t with parameter $\lambda > 0$ has the density $f_t(t) = \lambda e^{-\lambda t}$ for $t \geq 0$, and 0 for $t < 0$. Use this density to compute the expected value of t.*

Answer. $\operatorname{E}[t] = \int_0^\infty \lambda t e^{-\lambda t} dt = \int_0^\infty u v' dt = uv\Big|_0^\infty - \int_0^\infty u' v dt$, where $\begin{smallmatrix} u=t & v'=\lambda e^{-\lambda t} \\ u'=1 & v=-e^{-\lambda t} \end{smallmatrix}$. One can also use the more abbreviated notation $= \int_0^\infty u\, dv = uv\Big|_0^\infty - \int_0^\infty v\, du$, where $\begin{smallmatrix} u=t & dv'=\lambda e^{-\lambda t}\, dt \\ du'=dt & v=-e^{-\lambda t} \end{smallmatrix}$. Either way one obtains $\operatorname{E}[t] = -te^{-\lambda t}\Big|_0^\infty + \int_0^\infty e^{-\lambda t} dt = 0 - \frac{1}{\lambda} e^{-\lambda t}|_0^\infty = \frac{1}{\lambda}$. □

Problem 17. *Let x be uniformly distributed in the interval $[a, b]$, i.e., the density function of x is a constant for $a \le x \le b$, and zero otherwise.*

- **a.** *1 point What is the value of this constant?*

Answer. It is $\frac{1}{b-a}$ □

- **b.** *2 points Compute* $\operatorname{E}[x]$

Answer. $\operatorname{E}[x] = \int_a^b \frac{x}{b-a}\, dx = \frac{1}{b-a} \frac{b^2-a^2}{2} = \frac{a+b}{2}$ since $b^2 - a^2 = (b+a)(b-a)$. □

- **c.** *2 points Show that* $\operatorname{E}[x^2] = \frac{a^2+ab+b^2}{3}$.

Answer. $\operatorname{E}[x^2] = \int_a^b \frac{x^2}{b-a}\, dx = \frac{1}{b-a} \frac{b^3-a^3}{3}$. Now use the identity $b^3 - a^3 = (b-a)(b^2 + ab + a^2)$ (check it by multiplying out). □

- **d.** *2 points Show that* $\operatorname{var}[x] = \frac{(b-a)^2}{12}$.

Answer. $\operatorname{var}[x] = \operatorname{E}[x^2] - (\operatorname{E}[x])^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} = \frac{4a^2+4ab+4b^2}{12} - \frac{3a^2+6ab+3b^2}{12} = \frac{(b-a)^2}{12}$. □

Problem 18. *5 points Assuming that not only the expected value but also the variance exists, derive the Weak Law of Large Numbers, which can be written as*

$$\lim_{n\to\infty} \Pr\big[|\bar{y}_n - \mathrm{E}[y]|\geq\delta\big] = 0 \text{ for all } \delta > 0, \tag{17}$$

from the Chebyshev inequality

$$\Pr[|x-\mu|\geq k\sigma] \leq \frac{1}{k^2} \qquad \text{where } \mu = \mathrm{E}[x] \text{ and } \sigma^2 = \mathrm{var}[x] \tag{18}$$

Answer. From nonnegativity of probability and the Chebyshev inequality for $x = \bar{y}$ follows $0 \leq \Pr[|\bar{y} - \mu|\geq\frac{k\sigma}{\sqrt{n}}] \leq \frac{1}{k^2}$ for all k. Set $k = \frac{\delta\sqrt{n}}{\sigma}$ to get $0 \leq \Pr[|\bar{y}_n - \mu|\geq\delta] \leq \frac{\sigma^2}{n\delta^2}$. For any fixed $\delta > 0$, the upper bound converges towards zero as $n \to \infty$, and the lower bound is zero, therefore the probability iself also converges towards zero. □

Problem 19. *1 point Construct a sequence of random variables $y_1, y_2 \dots$ with the following property: their cumulative distribution functions converge to the cumulative distribution function of a standard normal, but the random variables themselves do not converge in probability. (This is easy!)*

Answer. One example would be: all y_i are independent standard normal variables.

□

Problem 20. *3 points Explain verbally clearly what the law of large numbers means, what the Central Limit Theorem means, and what their difference is.*

Problem 21. *3 points For two random variables x, y, their* covariance *is defined as*

$$\operatorname{cov}[x, y] = \operatorname{E}\Big[\big(x - \operatorname{E}[x]\big)\big(y - \operatorname{E}[y]\big)\Big]. \tag{19}$$

Using this definition, prove the following formula:

$$\operatorname{cov}[x, y] = \operatorname{E}[xy] - \operatorname{E}[x]\operatorname{E}[y]. \tag{20}$$

Write it down carefully, you will lose points for unbalanced or missing parantheses and brackets.

Answer. Here it is side by side with and without the notation $\operatorname{E}[x] = \mu$ and $\operatorname{E}[y] = \nu$:

$$\begin{aligned} \operatorname{cov}[x, y] &= \operatorname{E}\Big[(x - \operatorname{E}[x])(y - \operatorname{E}[y])\Big] \\ &= \operatorname{E}\Big[xy - x\operatorname{E}[y] - \operatorname{E}[x]y + \operatorname{E}[x]\operatorname{E}[y]\Big] \\ &= \operatorname{E}[xy] - \operatorname{E}[x]\operatorname{E}[y] - \operatorname{E}[x]\operatorname{E}[y] + \operatorname{E}[x]\operatorname{E}[y] \\ &= \operatorname{E}[xy] - \operatorname{E}[x]\operatorname{E}[y]. \end{aligned} \qquad \begin{aligned} \operatorname{cov}[x, y] &= \operatorname{E}[(x - \mu)(y - \nu)] \\ &= \operatorname{E}[xy - x\nu - \mu y + \mu\nu] \\ &= \operatorname{E}[xy] - \mu\nu - \mu\nu + \mu\nu \\ &= \operatorname{E}[xy] - \mu\nu. \end{aligned} \tag{21}$$

□

Maximum number of points: 68.

References

[Ame94] Takeshi Amemiya, *Introduction to statistics and econometrics*, Harvard University Press, Cambridge, MA, 1994. 6

[CT91] Thomas M. Cover and Joy A. Thomas, *Elements of information theory*, Series in Telecommunications, Wiley, New York, 1991. 9

[HT83] Robert V. Hogg and Elliot A. Tanis, *Probability and statistical inference*, second ed., Macmillan, 1983. 1

[Lar82] Harold Larson, *Introduction to probability and statistical inference*, Wiley, 1982. 2

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