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The Transformation Problem: A Simple Presentation of the "New Solution"

INTRODUCTION

Volume I of *Capital* opens with the argument that the substance and quantitative determinant of value is labor time.[1] It goes on to argue that the quantity of value embodied in a commodity underlies the ratios with which it exchanges with other commodities. Marx found it unnecessary to introduce the complicating effects of capital for this discussion of the nature of and the creation of value. Later, in Volume III, Marx discussed the effects of capital on the exchange of commodities under capitalism. Competition among capitals tended to equalize the profit rates on capital advanced. The corresponding "prices of production" at which the commodities exchanged[2] were not proportional to the values created in production that he had argued determined prices.

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1 Marx actually argued that value was determined by the amount of socially necessary abstract human labor needed to produce the commodity. These adjectives reflect important parts of Marx's Labor Theory of Value, but the issue of the transformation problem that is the topic of this paper can be, and traditionally has been, treated by abstracting from these issues.

2 Marx called the prices that actually appear in the real capitalist world "market prices." "Prices of production" were the prices which equalized sectoral profit rates. In line with the classical theories of Smith and Ricardo, market prices gravitated around prices of production. For a fuller discussion of the relation among these prices, see Duménil and Lévy (1993). Prices of production are the "prices" of the transformation problem.

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Marx maintained that even when the effects of capital were taken into account, prices should be understood as redistributed values. By "redistributed values" Marx meant specifically that what he referred to as two "invariance conditions" must hold: that the sum of prices must be the same as the sum of values, and that the sum of profits must be the same as the sum of surplus values. By the late 1970s there was a fairly broad consensus among both defenders of Marx's Labor Theory of Value and its critics that an economically meaningful transformation of the type Marx envisioned was mathematically impossible.[3]

In the early 1980s the "new solution" was advanced.[4] New solution advocates observed that when the net product is used to sum prices and values, and when wages are considered to be paid in money rather than goods, Marx's invariance conditions are satisfied.

Critics responded that the new solution is merely a mathematical trick, and does not truly eliminate the problem. In this short paper I will argue that the new solution not only solves the transformation problem, but that its proposals are (i) the only economically sensible assumptions, and (ii) they are consistent with Marx's discussion of the issue.

MARX'S APPROACH TO A SOLUTION

Going "from values to prices of production" involves two issues. First, there is the redistribution of values. Marx's economic idea was that prices of production resulted from a redistribution or transfer of value[5] caused by the competition of capitals. And second, there is the issue that Marx already spoke about in Volume I, of the relation of values (measured in hours of labor) to prices (measured in dollars or ounces of gold). This latter relation exists for values and the direct prices[6] of Volume I, and it exists for redistributed values and prices of production in Volume III. The graphic figure 1 summarizes these relations. Here λ is

[3] In general. That is, it was only possible in exceptional circumstances such as equal organic compositions of capitals or a common profit rate of zero, which obviously are not the general conditions of capitalism.

[4] The "new solution" was rediscovered in the early 1980s by Gérard Duménil (1980, 1983) and, independently, by Duncan Foley (1982).

[5] Two other ways Marx sometimes used to express this idea are that no value is created in circulation (exchange), and that value created in one branch is realized in another.

[6] In line with some (but not all) recent work, I will use the term "direct prices" to refer to prices that are proportional to values, that is, the prices of Volume I.

the vector[7] of values of the various commodities created in production, d is the vector of direct prices, γ the vector of redistributed values, and p is the vector of prices of production. The price vectors are simple scaler multiples of their corresponding value vectors,

$$\tilde{d} = Direct\_price\_(a\_price\_form) = k\bar{\lambda},\_and \qquad [1]$$
$$\tilde{\gamma} = \text{Re}distributed\_values\_(a\_value\_form) = \tilde{p}/k$$

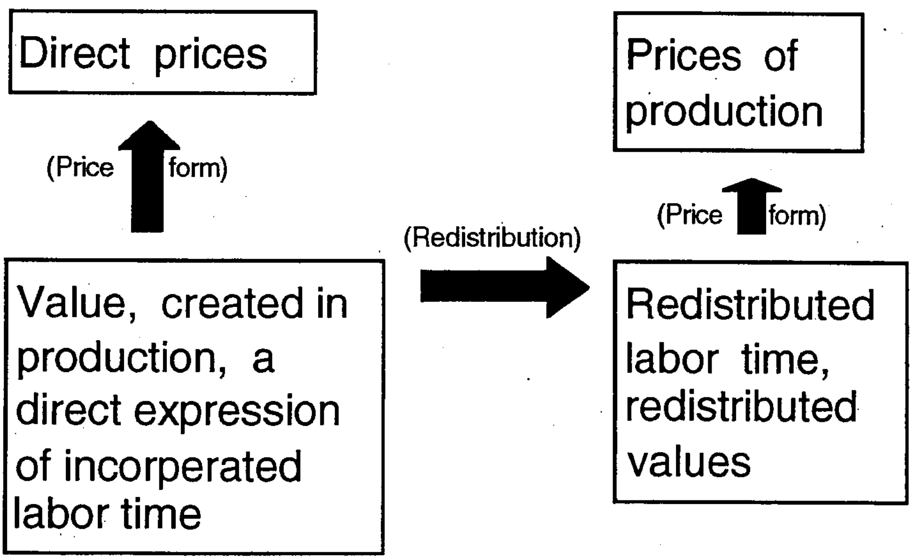
The word "redistribution" implies a process in which aggregates are preserved,

$$\sum redistributed\_values = \sum values \qquad [2]$$

The same condition could be expressed in price terms instead of value terms:

$$\sum prices = \sum direct\_prices \qquad [3]$$

Figure 1
Values, Direct Prices, Redistributed Values, and Prices of Production



[7] A left pointing arrow in the figure or the equations indicates a row vector, while a right pointing arrow indicates a column vector.

Unfortunately confusion has been introduced into the debate by traditionally referring to this condition as being that "the sum of values equals the sum of prices (of production)." Such loose language makes no sense dimensionally, since one side of the equation is measured in hours of labor, while the other side is measured in units of money (dollars, etc.). But although literally the phrase is nonsensical, it can be thought of simply as a label for a particular economic concept: The transformation of values into prices is a particular type of transformation, a transformation that involves a redistribution (of values).[8]

One of Marx's bedrock concerns was to explain the origin of profits, a category in the world of prices and money. Marx held that the origin of profits was surplus value, the value created by labor but not returned to the workers. Marx felt that the sum of the surplus values also was not changed in the process of redistribution, something I will expand on below. Analogously to figure 1, we can talk about the surplus value before and after the redistribution of values,[9] and the price forms of both of these value quantities. Marx's second requirement for the redistribution then could be represented symbolically as

$$\sum value\_forms\_of\_profits = \sum surplus\_values \qquad [4]$$

or equivalently in the price world, as

$$\sum profits = \sum price\_form\_of\_surplus\_values \qquad [5]$$

A FORMAL PRESENTATION OF THE TRANSFORMATION ENVISIONED BY MARX

The value and price equations that have been used traditionally are fairly well known, and for reasons of space will simply be presented here. I will consider the usual n good economy, with a single process of production for each good. We

8 This economic concept stands opposed to other conceivable types of transformations, for example, a transformation where the values proportional to the prices of production sum to a larger number than the original values, thus suggesting that value is created in the transformation (that is, as a result of circulation).

9 In standard Marxist terminology, surplus values appropriated and realized, respectively.

will write t_j to represent the newly expended labor (measured by time) per ton of output of good j, a_{ij} for the tons of good i used up to produce one ton of good j, and λ_i for the value per ton of good i. The value equation is

$$\bar{\lambda} = \bar{\lambda}A + \bar{t} \qquad [6]$$

which can (under not seriously restrictive assumptions) be solved for the lambda vector of values,

$$\bar{\lambda} = \bar{t}(I - A)^{-1} \qquad [7]$$

With w for the wage and r for profit, the price equation is:

$$\bar{p} = (1 + r)(\bar{p}A + w\bar{t}) \qquad [8]$$

This can be rewritten by collecting the price terms (but not solved for prices unless the wage or profit rate is known)

$$\bar{p} = w\bar{t}(\frac{1}{1+r}I - A)^{-1} \qquad [9]$$

At this point we can easily establish that there is always[10] a mathematical transformation between values and prices. Solving (6) for the t vector and substituting this into (9) gives:

$$\bar{p} = w\bar{\lambda}(I - A)(\frac{1}{1+r}I - A)^{-1} \qquad [10]$$

With this framework, we can clearly present what "the transformation problem" consists of. With the technical coefficients of the matrix A, the vector t, and w given, (7) determines the lambda vector and (9) determines the p vector and r.[11] Hence the relation between prices and values must satisfy (10), but that relation does not necessarily reflect the economics Marx put forward, that the transformation involved a redistribution of values. To capture this economic idea, Marx needed values and prices of production to also satisfy equations (3) and (5), or equivalently (2) and (4). Depending on how the economic ideas stated by (3) and (5) are interpreted mathematically, imposing these "invariance conditions" onto (10) may be mathematically impossible.

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[10] The existence of the matrix inverse is guaranteed by a simple economic assumption that the economy produces more than it consumes.

[11] There is a degree of homogeneity, so only $n-1$ of the p_j are determined, that is, the price vector is determined only up to a multiplicative constant.

Imposing either one of these two invariance conditions presents no mathematical problem, since the condition itself introduces an additional parameter into the system, the scaler k from equation (1) that links values and prices. Let the column vector x represent the goods produced. Then using (1), the traditional interpretation of the economic condition (3) becomes:

$$\bar{p}\bar{x} = k\bar{\lambda}\bar{x} \qquad [11]$$

This equation can be thought of mathematically as determining k (and economically as determining the value of a unit of money), so it will always be satisfied.

Now we must consider if (5) can hold simultaneously with (11). For this, we have to define surplus value. There has been a debate in the literature over how to do this. Because of space limitations, here I will just consider the way that seems economically appropriate, given that workers are paid wages and not bundles of goods.

Surplus value is the value created by the workers that is not returned to them. Workers are paid wages (not bundles of goods), and these wages give them a claim on goods. The scaler k in (1) takes value quantities to price quantities, so we will define the value quantity that corresponds to the money wage w by using k to go the other way,

$$\omega = w/k \qquad [12]$$

Note that ω has the units of value (hours) per unit of time, and so it is dimensionless. If we now define T as the total new value created, we can express surplus value as

$$s = (1-\omega)\bar{t}\bar{x} = (1-\omega)T \qquad [13]$$

or the equivalent relation in price terms,

$$ks = (k-w)\bar{t}\bar{x} = (k-w)T \qquad [14]$$

We will now calculate mathematical expressions for both sides of (5), and then investigate when they are equal. If we expand (14) into two terms, use the basic value equation (6) to solve for the t vector, and substitute that into one of the two terms from (14), we get

$$\sum price\_forms\_of\_surplus\_values = ks \qquad [15]$$
$$= (k-w)\bar{t}\bar{x} = k\bar{t}\bar{x} - w\bar{t}\bar{x} = k\bar{\lambda}\bar{x} - k\bar{\lambda}A\bar{x} - w\bar{t}\bar{x}$$

We will represent the other side of (5), total profits, simply as the total revenue of sales minus the costs of inputs,

$$\sum profits = \bar{p}\bar{x} - \bar{p}A\bar{x} - w\bar{t}\bar{x} \qquad [16]$$

Substituting (15) and (16) into (5) then gives an equation which must hold if (5) is to be true,

$$(\bar{p} - \bar{p}A - w\bar{t})\bar{x} = (k\bar{\lambda} - k\bar{\lambda}A - w\bar{t})\bar{x} \qquad [17]$$

The problem of determining if the two conservation equations can hold simultaneously has now been reduced to determining if, and under what circumstances, (17) will hold.

There are two conditions that have been discussed over the years under which (17) will hold: the case when the profit rate goes to zero, and the case of "equal organic compositions of capitals." Neither case represents a general condition of capitalism, and hence neither can be considered a solution to the transformation problem.

The important error in all the above is that equation (11) involves essentially the type of double counting that all economists are trained to avoid when calculating the GDP. Marx was aware of this problem, and verbally indicated what was necessary to avoid the transformation problem.[12]

> Considering the calculation as a whole, to the same extent that the profits of one sphere of production go into the cost price of another, to that extent these profits have already been taken into account for the overall price of the final end product and cannot appear on the profit side twice. They appear on this side only because the commodity in question was itself an end-product, so that its price of production does not go into the cost price of another commodity (Marx 1981, p. 260).
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> To apply this method of reckoning to the total social product, we have to make certain rectifications, since, considering the whole society, the profit contained in the price of flax, for instance, cannot figure twice, not as both part of the price of the linen and as the profit of the flax producers (Marx 1981, p. 260).

12 Ehrbar and Glick (1987: 298) credit Duménil with pointing out these and another related passage.

I will next unpack the economic content of this issue.

Double Counting—
Marx's Flax and Linen Example

Assume that 4 units of labor are spent to produce 1 pound of flax, and 1 pound of flax with 2 units of labor are used to produce 1 yard of linen. Assume further that the monetary unit, $1, represents exactly the value produced by one unit of labor, and the workers receive a wage of $0.50 per labor unit. Consider further a *reproduction scheme* for this economy, that is, a table showing what each industry must produce so that all industries have appropriate inputs. The important point here is that we are concerned with values before they are redistributed to equalize profit rates, though we will express the numbers in the price form of non-redistributed values, that is, what we have called direct prices.

	Constant Capital	Variable Capital	Surplus Value	Total Value
1 lb. of flax	0	2	2	4
1 yd. of linen	4	1	1	6

For direct prices, we see that the rate of profit in the flax industry is 100 percent (2/2); while the rate of profit in the linen industry is 20 percent (1/5).

Now we will consider the redistribution of value and the transformation to prices of production. If the direct prices calculated above obtained, capital would observe the unequal rates of profit and move into the branch with the high profit rate (flax), expand output there, and thus drive down the price. Generally the opposite would happen in the other branch, but to simplify the presentation we will keep the price in the other industry (linen) fixed.[13] Flax no longer sells at the price form of the value created in production (direct price), 4, but instead at a price (of production) of 3. This reduces the profits of the flax industry from 2 to 1. While the price of the output in the linen industry remains unchanged, its profits now rise to 2 because of the lower price of flax. Thus this price change equalizes the rates of profit in both industries at 50 percent. The following reproduction scheme, now given in the price form of the transformed values, the prices of production, reflects this change.

13 As we have discussed above, prices can be determined only up to a multiplicative constant.

	Constant Capital	Variable Capital	Profit	Total Value
1 lb. of flax	0	2	1	3
1 yd. of linen	3	1	2	6

Now consider the two desired conservation relations.

The simplicity of this example allows one to see plainly that the surplus value has merely been redistributed. The sum of the price form of the surplus values created in production is 3, and the sum of the price form of the redistributed surplus values, the profits, is 3. Half of the surplus value created by the flax workers is not realized by the flax industry but instead by the linen industry.

	Price form of Surplus Value created	Profits Realized
Flax Industry	2	1
Linen Industry	1	2

The mechanism of this transfer to the linen industry is the lower price of flax. The transfer satisfies conservation law 5, leaving the sum of profits equal to the sum of the price forms of the surplus values. Note in passing that if the workers spend their total wages of 3 on linen alone, they can still buy the same amount, 1/2 the linen produced.[14]

The above example captures the economics of what Marx had in mind, the transfer of surplus values due to competition, in such a way that the result was an equalized rate of profit between industries.

So it appears that we have run into the transformation problem. Conservation law (3), the sum of the prices equals the sum of the direct prices, when interpreted to imply aggregation by the gross product, does not hold. The post redistribution table shows the prices of production sum to 9, while the first table gives the sum of the direct prices as 10.

How can such a discrepancy come about when the only thing that occurred in the economy was that the profit realized in flax, was reduced by 1 and the profit realized in linen increased by 1, the type of transfer Marx envisioned as effecting the equalization of profit rates? The mystery is solved by the observation that the flax entered the linen as an input. The reduction of the flax profit reduced not only the flax output price but also the price of inputs to linen. One can say that if one counts the prices aggregated by the gross output, the reduction in the profits of flax was counted

[14] The fact that the production of flax does not require constant capital inputs is a particularity of this example which is not essential to the argument.

twice, since it was not only reflected in the output price of flax but also in the output price of linen because of the reduction in the price of the inputs to linen.

While there is a divergence between the sum of the prices and the direct prices of the gross product, the net product of this economy maintains a price equal to its direct price (price form of its value), 6. This is the key economic point that I must now generalize.

I begin by introducing the notation

$$Net\_product = (I - A)\bar{x} = \bar{y} \qquad [18]$$

With this notation I can now generalize the result observed in the flax/linen example. Instead of (11), I formalize the economic idea of (3) by aggregating over the net product,

$$\bar{p}\bar{y} = k\bar{\lambda}\bar{y} \qquad [19]$$

Now (19) and (17) serve as the mathematical formalization of the two desired conservation laws, (3) and (5). Noting that the third terms on both sides of (17) are the same, one can see by inspection that whenever one of these equalities holds, so will the other.

With this understanding of the economics, both conservation equations necessarily hold for any technology and any level of output. Once one incorporates Marx's suggestions of eliminating double counting, the two conservation equations become mathematically equivalent, and the transformation problem ceases to exist.

It is worth reflecting on the significance of the change in the mathematical structure of the problem that results from this different economic interpretation. The two conservation equations (3) and (5) are no longer mathematically independent. If one begins from the relevant economics, there is nothing particularly disturbing about this. The two conservation equations were intended by Marx to capture the idea that the process of transformation was in fact a redistribution of value, not to establish some particular structure for a mathematical problem. Historically the first people who tried to formalize Marx's model interpreted the ideas by aggregating on the gross product, a procedure which necessarily leads to the two conservation equations becoming independent constraints. Since then it has become an assumption that Marx had two independent constraints in mind. As there is only one free parameter involved, k, the problem then is necessarily overdetermined, and cannot be solved. This was the basis for concluding that the transformation procedure, as a process that satisfied Marx's conservation equations, was impossible. As we have seen, however, the economic process of redistribution which Marx intended to model, when handled properly to avoid double counting, mandates that the two conservation equations not be interpreted as the historically used independent constraints.

CONCLUSION

Both values and prices of production can be determined from the technical data of input/output coefficients, required new labor time, and the wage, and this establishes a necessary mathematical relation between them. Marx put forward the additional economic idea that the "transformation" involved a redistribution of values. Early mathematical formalizations of this conservation or invariance idea treated it as involving two independent restrictions, which one can show cannot (in general) hold simultaneously. An economically appropriate treatment of the two constraints, however, results in their being mathematically equivalent, and hence does not give rise to the historically discussed "transformation problem."

REFERENCES

Duménil, Gérard. 1980. *De La Valeur aux Prix de Production.* Paris: Economica.

Duménil, Gérard. 1983. Beyond the Transformation Riddle: A Labor Theory of Value. *Science and Society* 47(4), 427–450.

Duménil, Gérard and Dominique Lévy. 1993. *The Economics of the Profit rate: Competition, Crises, and Historical Tendencies in Capitalism.* Aldershot, England: Edward Elgar.

Foley, Duncan. 1982. The Value of Money, the Value of Labor Power and the Marxian Transformation Problem. *Review of Radical Political Economics* 14(2), 37–47.

Glick, M. and H. Ehrbar. 1987. The Transformation Problem: An Obituary. *Australian Economic Papers* December, 294–317.

Marx, Karl. 1977 (1867). *Capital.* Volume One. New York: Vintage.

______. 1981 (1895). *Capital.* Volume 3. London: Penguin.

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